Exercise 1. Prove that $\widehat{HF}(M) = 0$ if and only if $HF^+(M) = 0$.

The next exercise is a slightly harder version of the previous one.

Exercise 2. Let $\widehat{F}\colon \widehat{HF}(M) \to \widehat{HF}(N)$ and $F^+\colon HF^+(M) \to HF^+(N)$ be linear maps such that the diagram

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \widehat{HF}(M) & \longrightarrow & HF^+(M) & \xrightarrow{U} & HF^+(M) & \longrightarrow & \widehat{HF}(M) & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle \widehat{F}} & & \downarrow{\scriptstyle F^+} & & \downarrow{\scriptstyle F^+} & & \downarrow{\scriptstyle \widehat{F}} & & \\ \dots & \longrightarrow & \widehat{HF}(N) & \longrightarrow & HF^+(N) & \xrightarrow{U} & HF^+(N) & \longrightarrow & \widehat{HF}(N) & \longrightarrow & \dots \end{array}$$

commutes. Then $\widehat{F}$ is an isomorphism if and only if F^+ is an isomorphism.

Exercise 3. Let M be a three-manifold with boundary which has g pairwise disjoint properly embedded discs $D_1, \dots, D_g$ such that $M \setminus (D_1 \cup \dots \cup D_g)$ is homeomorphic to a ball. Then M is a genus g handlebody.

Exercise 4. Let S be a surface with boundary. Then $S \times [-1, 1]$ is homeomorphic to a handlebody. What is its genus?

Exercise 5. Let $\Sigma \subset S^3$ be a standardly embedded genus g surface. Show that Σ decomposes S^3 in two handlebodies and draw the corresponding Heegaard diagram.

Exercise 6. Which three-manifold is described by a genus g Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ such that α_i is isotopic to β_i for all $i = 1, \dots, g$?

Exercise 7. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a pointed Heegaard diagram for a three-manifold M. Show that the group of the periodic domains of $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is isomorphic to $H_2(M; \mathbb{Z})$.

Exercise 8. Regard $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. Given $1 \leq q < p$ we define the *lens space* $L(p, q)$ as the quotient of S^3 by the equivalence relation

$$(z_1, z_2) \sim (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi i q}{p}} z_2).$$

Draw a genus one Heegaard diagram for $L(p, q)$.

Exercise 9. Prove that S^3, $S^2 \times S^1$ and the lens speces $L(p, q)$ are the only three-manifolds with a genus one Heegaard splitting.