Khovanov homology in terms of immersed curves

Exercises for Lecture 1

1. Consider the second mutant pair of knots given in the lecture. Show that both the given knot and its mutant have the same Seifert genus. **Hint:** Use the fact that the knots are alternating, and apply high-powered tools.

2. Show that $\langle K \rangle$ is unchanged under Reidemeister moves RII and RIII, and using this prove that $\langle K \rangle$ is a well defined invariant up to multiplication by units.

3. Given a diagram for K with a choice of orientation, use the writhe of the knot (that is, a sum of the signs of the crossings using a right-hand rule) to remove the *up to units* part of the invariance of $\langle K \rangle$. **Note:** By using $\langle K \rangle$ and the writhe together in this way, you effectively recover the Jones polynomial $V_K(t)$. See [10].

4. Prove that $\langle T \rangle$ and $\langle T^\mu \rangle$ agree as elements of the skein module $\mathcal{S}$. **Further reflection:** You might ponder why the pair of crossingless tangles provides a basis. One approach is to consider the Temperley-Lieb algebra, which is an interesting object in its own right, and its relationship to $\mathcal{S}$. See [11]

5. Consulting the figure drawn in the lecture, prove that in $\mathrm{Cob}_{/\ell}$ the torus has value $2 \in \mathbf{k}$.

6. This final exercise completes the example constructing a homotopy equivalence between chain complexes associated with tangles that differ by a RI move.

(a) Verify that $G \circ F = \mathrm{id}$.

(b) Complete the proof that $F \circ G \simeq \mathrm{id}$ by establishing the case $i = 1$. **Note:** We did the case $i = 0$ at the end of lecture.

As I prepare these exercises, I am aware there is a strong possibility I will spend less time on the example associated with this exercise during lecture than I would like. As such, it may be worthwhile to work through the entire homotopy equivalence for this exercise. For consultation, the solution is on page 1457–1458 of [12].

This first lecture, being introductory, took us on a somewhat meandering tour. I collected the references mentioned, listed in the order they appeared in lecture, below.

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