ON KNOTS HAVING THE SAME INVARIANTS UP TO A CERTAIN DEGREE

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Abstract. Two links are k-equivalent if their Vassiliev invariants in degree at most k coincide. Examples of pairs of k-equivalent knots were constructed by Stanford: cut open a knot by a transversal plane, insert an element of the $k+1$-th term of the lower central series of the pure braid group (on the corresponding number of strands) and glue back the two pieces. Then the resulting knot is k-equivalent to the initial one. We prove that the converse holds true: two k-equivalent knots can be obtained one from the other by a Stanford move.

1. Introduction

A knot invariant is of finite type and degree $\leq k$ if its natural extension to singular knots vanishes for singular knots with $\geq k+1$ double points.

Two knots K and L are (Vassiliev) $V-k$-equivalent if $v(K) = v(L)$ holds for all Vassiliev invariants of degree $\leq k$. It has been shown by Gusarov in [3] (using a slightly different, but equivalent definition of finite type invariants) that k-equivalence classes of knots, under the connected sum form a finitely generated Abelian group.

In [4, 5, 6] Stanford defined some k-moves which preserve the k-th equivalence class of the knot. A Stanford k-move relates two knots K and L if there is an element $x \in LCS_{(k+1)}(P_n)$, and some transversal plane cutting open K such that, after inserting x and glueing back we obtain L. The question we want to answer in these notes is whether the converse affirmation is also true. Let us call the two links (Stanford) $S-k$-equivalent if there is a k-move relating them. The main result (see [7]) is:

THEOREM 1.1. *Two $V-k$-equivalent knots can be obtained one from the other by using one k-move.*

This result is also a corollary of a theorem due to Habiro ([2]).

2. Proof of the theorem

2.1. **The plan.** Let us make a few notations:

$LCS_k(G)$ is the lower central series of a group G.

$I_G \subset \mathbf{Z}G$ is the augmentation ideal.

S_* is the set of singular objects with $*$ double points, of a given kind i.e. knots, braids, etc.

F^* is the Vassiliev filtration obtained from the desingularization of singular objects from S_*.

$\mathcal{K}$ states for the set of knots (modulo isotopy).

$\mathcal{K}/V_k$ states for the equivalence classes of knots modulo $V-k$-equivalence.

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PROPOSITION 2.1. *Stanford's k-equivalence is an equivalence relation on the set of links.*

PROPOSITION 2.2. *The set $\mathcal{K}/S_k$ of knots modulo $S-k$-equivalence forms a group (under the connected sum of knots).*

PROPOSITION 2.3. *If two links are $S-k$-equivalent then they are $V-k$-equivalent.*

PROPOSITION 2.4. *If two knots are $V-k$-equivalent then they are $S-k$-equivalent.*

2.2. **Proof of Proposition 1.** Before to proceed let us make some comments on the Markov theorem. Any link can be identified with a closed braid in infinitely many ways. As the Reidemeister moves act transitively on the set of diagrams associated to the same link, the theorem of Markov provides the moves relating two braids having the same closure.

It is useful in applications to have an improved version of Markov's theorem, which is folklore (after J.Birman, H.Morton). The proof we present here is due to S.Kamada. This result provides a rearrangement of the elementary moves in the Markov theorem.

PROPOSITION 2.5. *If the closed braids $\overline{x}$ and $\overline{y}$ are equivalent then there exists a sequence of elementary moves relating them $x = x_0 \to x_1 \to ... \to x_k \to y_k \to y_{k-1} \to ... \to y_0 = y$, such that, for each j*

- *$x_j \to x_{j+1}$ is either a conjugation or a stabilization.*
- *$y_{j+1} \to y_j$ is either a conjugation or a destabilization (the inverse of a stabilization).*

Proof. Define a *generalized stabilization* to be the transformation $x \in B_n$ goes to $xw^{-1}b_n^{\varepsilon}w \in B_{n+1}$, $w \in B_n$, $\varepsilon \in \{-1, 1\}$. It is easier to work with them and also these are not far from usual stabilization: a generalized stabilization $x \to x'$ can be decomposed as $x \to y$ a conjugation, $y \to z$ a stabilization, and $z \to x'$ a conjugation. In fact if $x' = xw^{-1}b_n^{\varepsilon}w$ we set $y = wxw^{-1}$ and $z = wxw^{-1}b_n^{\varepsilon}$.

Morover a conjugation and a generalized stabilization could "commute" each other as follows. Let $x_0 \to x_1$ be a conjugation, $x_1 \to x_2$ be a generalized stabilization. Then there exists some y such that $x_0 \to y$ is a generalized stabilization and $y \to x_2$ is a conjugation. Indeed the hypothesis is that $x_1 = w^{-1}x_0w$, $x_2 = x_1t^{-1}b_n^{\varepsilon}t$. We take then $y = x_0wt^{-1}b_n^{\varepsilon}tw^{-1}$.

The main argument is the equivalence of a sequence having the shape down-up with another one up-down (with respect to the braid index):

LEMMA 2.1. *Let $x_0 \to x_1$ be a generalized destabilization and $x_1 \to x_2$ be a generalized stabilization. Then there exist y, z such that $x_0 \to y$ is a generalized stabilization, $y \to z$ is a conjugation, and $z \to x_2$ is a generalized destabilization.*

Proof. We can write $x_0 = x_1w^{-1}b_n^{\varepsilon}w \in B_{n+1}$, $x_2 = x_1t^{-1}b_n^{\delta}t \in B_{n+1}$, where $x_1, w, t \in B_n$, $\varepsilon, \delta \in \{-1, +1\}$. We put then $y = x_0(b_nt)^{-1}b_{n+1}^{\delta}(b_nt) \in B_{n+2}$, $z = b_{n+1}^{-1}yb_{n+1} \in B_{n+2}$. We have therefore

$$\begin{aligned}
z &= b_{n+1}^{-1}x_0(b_nt)^{-1}b_{n+1}^{\delta}(b_nt)b_{n+1} = \\
&= b_{n+1}^{-1}x_1w^{-1}b_n^{\varepsilon}wt^{-1}b_n^{-1}b_{n+1}^{\delta}b_ntb_{n+1} = \\
&= b_{n+1}^{-1}x_1w^{-1}b_n^{\varepsilon}wt^{-1}b_{n+1}b_n^{\delta}b_{n+1}^{-1}tb_{m+1} = \\
&= x_1w^{-1}b_{m+1}^{-1}b_m^{\varepsilon}b_{n+1}wt^{-1}b_n^{\delta}t = \\
&= x_1w^{-1}b_nb_{n+1}^{\varepsilon}b_n^{-1}wt^{-1}b_n^{\delta}t = \\
&= x_1t^{-1}b_n^{\delta}t(b_n^{-1}wt^{-1}b_n^{\delta}t)^{-1}b_{n+1}^{\varepsilon}(b_n^{-1}wt^{-1}b_n^{\delta}t)
\end{aligned}$$

□

A straightforward recurrence yields the result. □

LEMMA 2.2. *Consider $y \in B_N$ which is obtained by a sequence of generalized stabilizations from $x \in B_n$ and $h \in LCS_k(P_n)$. Then there exists $g \in LCS_k(B_N)$ such that gy is obtained by a sequence of generalized stabilizations from hx.*

Proof. It suffices to do that for one stabilization: $y = a^{-1}xab_n^{\varepsilon}$. Then $a^{-1}hxab_n^{\varepsilon} = (a^{-1}ha)a^{-1}xab_n^{\varepsilon}$ and we take $g = a^{-1}ha$. ☐

Final argument in the proof of the Proposition 1: We have to check that the $S-k$ equivalence is transitive. Take then three links $K_1 = \overline{hx}$, $K_2 = \overline{x} = \overline{y}$, $K_3 = \overline{gy}$, where h and g are from the LCS_{k+1}. By the strengthened Markov lemma there exists some braid $z \in B_N$ which is obtainable by sequences of generalized stabilizations from both x and from y. According to the previous lemma there exist $h', g' \in LCS_{k+1}(P_N)$ such that $h'z$ and hx (and respectively $g'z$ and gy) are still related by a sequence of generalized stabilizations, hence $K_1 = \overline{h'z}$, $K_3 = \overline{g'z}$.

2.3. **Proof of Proposition 2.** Notice that it is essential here to consider knots and not arbitrary links. Denote by $\overset{S,n}{\sim}$ the $S-n$-equivalence.

LEMMA 2.3. *If $K_1 \overset{S,n}{\sim} K_1'$ and respectively $K_2 \overset{S,n}{\sim} K_2'$ then $K_1\sharp K_2 \overset{S,n}{\sim} K_1'\sharp K_2$.*

Proof. We can assume that all knots involved in the lemma are closures of braids in B_k. By using conjugations we can assume that the permutations associated to these braids are the same as that of the element $t_k = b_{k-1}^{-1}b_{k-1}^{-1}...b_1^{-1}$. In fact all complete cycles are conjugated in the permutation group. Thus we have $K_1 = \overline{x_1t_k}$, $K_1' = \overline{h_1x_1t_k}$, $K_2 = \overline{x_2t_k}$, $K_1 = \overline{h_2x_2t_k}$, where $h_1, h_2 \in LCS_{n+1}(P_k)$.

An immediate inspection shows that $\overline{xt_k}\sharp\overline{yt_k} = \overline{xt_{2k}^{-k}yt_{2k}^{k+1}}$, and so $K_1\sharp K_2 = \overline{x_1t_{2k}^{-k}x_2t_{2k}^{k+1}}$, $K_1'\sharp K_2' = \overline{h_1x_1t_{2k}^{-k}h_2x_2t_{2k}^{k+1}} = \overline{h_1t_{2k}^{-k}h_2t_{2k}^{k}x_1t_{2k}^{-k}x_2t_{2k}^{k+1}}$, because x_1 and $t_{2k}^{-k}h_2t_{2k}^{k}$ commute each other, modulo $LCS_{n+1}(P_{2k})$. But now $h_1t_{2k}^{-k}h_2t_{2k}^{k} \in LCS_{n+1}(P_{2k})$, and the claim follows. ☐

It remains to show that the monoid structure induced by $\sharp$ is a group one, which is equivalent to:

LEMMA 2.4. *Every equivalence class in $\mathcal{K}/S_k$ has an inverse.*

Proof. It suffices to prove that for a $S-k$-trivial knot K there exists some knot K' such that $K\sharp K'$ is $S-(k+1)$-trivial. Since any knot is $S-1$-trivial an inductive use of this claim proves the lemma.

Let now $K = \overline{xt_k}$, where $h \in LCS_{k+1}(P_n)$ and $\overline{hxt_k}$ is the unknot. We claim that $K' = \overline{ht_k}$ verifies the previous claim. More generally, for any $x \in B_N$ and $h \in LCS_{k+1}(P_n)$ $\overline{xt_k}\sharp\overline{ht_k}$ is $S-(k+1)$-equivalent to $\overline{hxt_k}$.

In fact we have, for all j:

$$\overline{ht_{2k}^{-j}xt_{2k}^{j+1}} = \overline{t_{2k}^{-j-1}xt_{2k}^{j+1}ht_{2k}} = \overline{[t_{2k}^{-j-1}xt_{2k}^{j+1}, h]ht_{2k}^{-j-1}xt_{2k}^{j+2}},$$

hence by transitivity we get $\overline{hxt_{2k}^{1}}$ is $S-(k+1)$-equivalent to $\overline{ht_{2k}^{-k}xt_{2k}^{k+1}}$. But the latter is isotopic to $\overline{xt_k}\sharp\overline{ht_k}$, and we are done. ☐

2.4. **Proof of Proposition 3.** The following well-known lemma holds for any group G:

LEMMA 2.5. *If $x \in LCS_k(G)$ then $x - 1 \in I_G^k$.*

Proof. By recurrence on k. Consider $x \in LCS_n(G)$ and $y \in G$. Then $[x, y] - 1 = (xy - yx)x^{-1}y^{-1} = ((x-1)(y-1) - (y-1)(x-1))x^{-1}y^{-1} \in I^{n+1}(G)$. Also if $x - 1 \in I_g^n$ and $y - 1 \in I_G^n$ then $xy - 1 = x(y-1) + (x-1) \in I_G^n$, and $x^{-1} - 1 = -x^{-1}(x-1) \in I_G^n$. ☐

If $J^k_{B_n}$ denotes the 2-sided ideal of $\mathbf{Z}B_n$ generated by $I^k_{P_n} \subset P_n \subset B_n$, then we have the following simple interpretation of the Vassiliev filtration on braids:

LEMMA 2.6. *We have* $F^k(B_n) = J^k_{B_n}$.

Proof. The desingularization of a crossing can be written in $\mathbf{Z}B_n$ (and not $\mathbf{Z}P_n$!) as the difference $b_i - b_i^{-1} = (b_i^2 - 1)b_i^{-1}$, where the b_i ($i = 1, 2, ..., n-1$) are the usual generators of the braid group B_n. Then all crossings of $x \in S_dB$ yield by desingularization $\delta(x) = y_0(b^2_{i_1} - 1)y_1(b^2_{i_2} - 1)y_2...(b^2_{i_d} - 1)y_d$, where all elements $y_i \in B_n$. Since x was a singular braid we have $y_0y_1...y_d \in B_n$. We can write:

$$\delta(x) = \left(\prod_{j=0}^{d-1} y_0y_1...y_j(b^2_{i_{j+1}} - 1)(y_0y_1...y_j)^{-1}\right) y_0y_1...y_d.$$

Now each term $y_0y_1...y_j(b^2_{i_{j+1}} - 1)(y_0y_1...y_j)^{-1} = y_0y_1...y_jb^2_{i_{j+1}}(y_0y_1...y_j)^{-1} - 1$, has the form $x - 1$, with $x \in P_n$. This proves that $F^k(B_n) \subset J^k_{B_n}$.

If J is an ideal and $x - 1, y - 1 \in J$ we derive that $xy - 1 = (x - 1)(y - 1) + x - 1 + y - 1 \in J$. Hence, in order to prove the reverse inclusion $I_{P_n} \subset F^1(B_n)$ it suffices to show that $x - 1 \in F^1(B_n)$ for x running over a set of generators of P_n, for instance $x_{ij} = b_{j-1}b_{j-2}...b_{i+1}b_i^2b^{-1}_{i+1}...b^{-1}_{j-1}$. But $x_{ij} - 1 = b_{j-1}b_{j-2}...b_{i+1}(b_i^2 - 1)b^{-1}_{i+1}...b^{-1}_{j-1}$, so that $x_{ij} - 1 \in IG^{(1)}$. Since both filtrations are ideals in the group algebras, this argument shows also that $J^k_{B_n} \subset F^k(B_n)$. □

Now the Proposition 3 follows: if $K = \overline{x}$ and $L = \overline{yx}$, where $x \in B_n$, $y \in LCS_{k+1}(P_n)$ then $x(y - 1) \in F_{k+1}B_N$, hence $K - L \in F^{k+1}\mathcal{K}$.

2.5. **Proof of Proposition 4.** We consider the composition, $\mathbf{Z}\mathcal{K} \to \mathbf{Z}[\mathcal{K}/V_k] \to \mathcal{K}/V_k$ where the second arrow is the $\mathbf{Z}$-linear extension of the identity map between the two groups. Roughly speaking $\mathcal{K}/V_k$ is the dual of the group of additive Vassiliev invariants of degree $\leq k$. We have a similar map $\mathbf{Z}\mathcal{K} \to \mathbf{Z}[\mathcal{K}/S_k] \to \mathcal{K}/S_k$, and our task is to identify these two projections map. It is sufficient to prove the the kernel of the former projection is contained in the kernel of the second one.

Definition 2.1. *A relator of order* n *and length* m *is an element of* $\mathbf{Z}\mathcal{K}$ *having the form*

$$\overline{(x_1 - 1)(x_2 - 1)...(x_m - 1)yt_k},$$

where n_i *is the greater natural such that* $x_i \in LCS_{n_i}(P_k)$, *and* $n = \sum_i n_i$, $y \in P_k$.

LEMMA 2.7. $F^m(\mathcal{K})$ *is generated by relators of length* m *and order* $\geq m$.

Proof. This is equivalent to show that the projection map induced by the closure yields a surjective map $F^m(B_\infty) \to F^m(\mathcal{K})$. Any singular knot with $m + 1$ double points is the closure of a singular braid with $m + 1$ double points. This generalization of Alexander's theorem is due to Birman ([1]). Conjugating by a braid we can suppose the permutation associated is the same as that of t_k. The identification of $F^m(B_k)$ with $J^m_{B_k}$ made above sends $F^m(P_k)$ isomorphically on $I^m_{P_k}$. □

We remark that the kernel of $\mathbf{Z}\mathcal{K} \to \mathbf{Z}[\mathcal{K}/S_k]$ is generated by relators of length 1 and order $\geq k$. The kernel of the second projection $\mathbf{Z}[\mathcal{K}/S_k] \to \mathcal{K}/S_k$ is generated by the composite combinations (which we call relators) of the form $K_1\sharp K_2 - K_1 - K_2$ (corresponding to the additivity of Vassiliev invariants).

We have therefore to prove that:

PROPOSITION 2.6. *Any relator of order $\geq n$ is a linear combination (over $\mathbf{Z}$) of relators of length 1 and order $\geq n$ and of composite relators.*

Proof. Set C_n be the span of all relators length 1 and order $\geq n$ and of composite relators. Suppose on the contrary that there is one relator of order $\geq n$ which does not belong to C_n. Choose one such of minimal length m, and among those with minimal length, one with maximal order. Such one should exist because, if the order is large enough then the element is in C_n. In fact there exists one $x_i \in LCS_n(P_N)$ and the relator can be rewritten as a sum of elements of the form as $\overline{w(x_i-1)yt_k} = \overline{(wx_iw^{-1}-1)yt_k}$, but now $wx_iw^{-1} \in LCS_n(P_k)$ hence all these are relators of length 1 and order $\geq n$.

Suppose this relator is $\overline{(x_1-1)...(x_m-1)yt_k} = \overline{(x_1-1)...(x_m-1)yt_{2k}}$, $x_j, y \in P_k$.

We can interchange (x_i-1) and $(x_{i+1}-1)$ modulo relators of shorter length or greater order, using

$$(x_i-1)(x_{i+1}-1)-(x_{i+1}-1)(x_i-1) = x_ix_{i+1}-x_{i+1}x_i = ([x_i,x_{i+1}]-1)+([x_i,x_{i+1}]-1)(x_{i+1}x_i-1).$$

Meantime (x_m-1) can be interchanged with y modulo relators of greater order, since

$$(x_m-1)y - y(x_m-1) = ([x_m,y]-1)yx_m.$$

Using a conjugation we have that $\overline{(x_1-1)...y(x_m-1)t_{2k}} = \overline{((t_{2k-1}^{-1}x_mt_{2k}-1)(x_1-1)...(x_{m-1}-1)yt_{2k}}$. Applying these transformations repeatedly we derive that the initial relator is equal (modulo relators of smaller length or greater order) to $\overline{((t_{2k-1}^{-k}x_mt_{2k}^{k}-1)(x_1-1)...(x_{m-1}-1)yt_{2k}}$ which equals $\overline{(x_1-1)(x_2-1)...(x_{m-1}-1)yt_k}\sharp\overline{(x_1-1)t_k}$. The latter is a linear combination of composite relators, and the proposition is proved. □

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