Non-compact 3-manifolds proper homotopy equivalent to geometrically simply connected polyhedra and proper 3-realizability of groups

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Abstract

The principal result of this paper is a homotopy criterion for detecting the tameness of non-compact 3-manifolds which extends the one worked out by L.Funar and T.L.Thickstun for open 3-manifolds. A group is properly 3-realizable if it is the fundamental group of a compact 2-polyhedron whose universal covering is proper homotopy equivalent to a 3-manifold. As a consequence of the main result a properly 3-realizable group which is also quasi-simply filtered has *pro-(finitely generated free) fundamental group at infinity* and *semi-stable ends*. Conjecturally the quasi-simply filtration assumption is superfluous. Using these restrictions we provide the first examples of finitely presented groups which are not properly 3-realizable, for instance large families of Coxeter groups.

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1 Introduction

In [18, 21] the authors proved that an open 3-manifold is simply connected at infinity if it has the proper homotopy type of a *weakly geometrically simply connected* polyhedron. The simple connectivity at infinity is a strong tameness condition for open 3-manifolds which, roughly speaking, expresses the fact that each end is collared by a 2-sphere.

The main concern of this paper is to give a similar homotopy criterion for detecting the tameness in the case of 3-manifolds *with boundary*. The relevant tameness conditions have to be changed accordingly, in order to take into account the boundary behavior. Instead of the simple connectivity at infinity we will consider the so-called *missing boundary manifold* condition introduced by Simon (see [40]), while the weak geometric simple connectivity has to be replaced by the stronger pl-geometric simple connectivity, to be defined below.

Despite the fact that there are similarities in the proofs with [18, 21], the case where manifolds have *non-compact boundary components* presents some new and interesting features. Working in this more general context opens the possibility to find applications to geometric group theory. Specifically, we obtain necessary conditions for a finitely presented group to act freely co-compactly on a simply connected 2-complex having the proper homotopy type of a 3-manifold. In particular we find explicit examples of groups which do not have this property.

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We start with definitions of various tameness conditions that will be used in the sequel. We will introduce first the concepts which are relevant in the study of open manifolds and then the enhanced ones arising when we work with manifolds with boundary.

Definition 1.1. *A polyhedron P is* weakly geometrically simply connected *(*wgsc*) if it admits an exhaustion by compact connected sub-polyhedra $P_1 \subset P_2 \subset \cdots$ such that $\pi_1(P_n) = 0$, for all n.*

The wgsc property for polyhedra is the piecewise-linear counterpart of the geometric simple connectivity of open manifolds. Specifically, let us recall (following [37]) that:

Definition 1.2. *The* non-compact manifold W *is* geometrically simply connected *(*gsc*) if it admits a proper handlebody decomposition without index one handles.*

It is an immediate consequence of Smale's theory that an open manifold of dimension at least 5 which is wgsc is actually gsc. Moreover, in dimension 3 the wgsc condition implies that the manifold is simply connected at infinity. The main results of [18, 21] state that an open 3-manifold which is proper homotopy equivalent to a wgsc polyhedron is wgsc and thus simply connected at infinity. Therefore one can homotopically detect the simple connectivity at infinity of 3-manifolds.

The wgsc property is not so useful anymore if we look upon the tameness of 3-manifolds *with boundary*. We will give below examples of non-compact wgsc 3-manifolds which are not tame in a suitable sense.

In the realm of manifolds with boundary the relevant tameness condition that will replace the simple connectivity at infinity is the following:

Definition 1.3. *A manifold W is a* missing boundary manifold *(also called* almost compact*) if there exists a compact manifold with boundary M and a closed subset $A \subset \partial M$ of the boundary (not necessarily a sub-complex) such that W is homeomorphic to $M - A$.*

Interesting examples of manifolds which are not missing boundary manifolds can be found in [38, 43].

The antecedent of the papers [18, 21] is the paper [37] of Poenaru where the geometric simple connectivity is already defined and used for non-compact manifolds with boundary.

Poenaru has proved in [37] that an open 3-manifold is simply connected at infinity if the product with a closed n-ball (for some $n \geq 2$) is a geometrically simply connected manifold with boundary. One might expect therefore that the analogous statement be true for the more general case of non-compact 3-manifolds. However, we will have to consider products of non-compact manifolds and disks, namely manifolds with corners. It is then natural to look for the piecewise-linear analogue of the geometric simple connectivity of manifolds with boundary. Specifically, we set:

Definition 1.4. *A polyhedron P is* pl-gsc *if it admits an exhaustion by compact connected sub-polyhedra $P_1 \subset P_2 \subset \cdots$ such that $\pi_1(P_n) = 0$ and $\pi_1(A, A \cap P_n) = 0$, for every connected component A of $\overline{P_{n+1} - P_n}$ and all n. Equivalently, the map induced by inclusion $\pi_1(A \cap P_n) \to \pi_1(A)$ is surjective for all A as above and all n. For a pair of spaces $X \supset Y$ we set $\Pi(X, Y) = \pi_1(\overline{X - Y}, \overline{X - Y} \cap Y)$.*

This definition is consistent with the previous ones since, by using Smale's theorem again, a non-compact manifold of dimension $n \geq 6$ is pl-gsc iff it is gsc. Moreover the gsc and pl-gsc are equivalent for open manifolds without any dimensional restrictions. The pl-gsc is stronger than the wgsc (see proposition 2.1 below) for 3-manifolds with boundary.

Let us introduce very briefly, for the sake of completeness, some end invariants of non-compact spaces which will be used in the sequel. Standard references are [2, 29] where these notions are studied in detail.

Given the sequence of homomorphisms $A_{i-1} \leftarrow A_i$, called *bonding morphisms*, one builds up the *tower of groups* $A_0 \leftarrow A_1 \leftarrow \cdots$. A *pro-isomorphism* between the towers $A_0 \leftarrow A_1 \leftarrow \cdots$ and $B_0 \leftarrow B_1 \leftarrow \cdots$ is given by two sequences of morphisms $B_{j_{2n+1}} \to A_{i_{2n+1}}$ and $A_{i_{2n}} \to B_{j_{2n}}$ where $0 = i_1 < j_1 < j_2 < i_2 < i_3 < j_3 < j_4 < i_4 < \cdots$, which are commuting with the respective compositions of bonding morphisms in the two towers. A pro-isomorphism class of towers of groups is called a *pro-group*.

Definition 1.5. *A pro-group is said* pro-(finitely generated free) *if it has a representative tower in which all groups involved are finitely generated free groups.*

Definition 1.6. *The tower $A_0 \leftarrow A_1 \leftarrow \cdots$ is* telescopic *if the following conditions are satisfied:*

1. *A_i are free groups of finite basis D_i, for all i;*
2. *$D_i \subset D_{i+1}$ and the differences $D_{i+1} - D_i$ are finite for all i;*
3. *the bonding morphisms are the obvious projections.*

It was shown in ([26]) that if a pro-group is pro-(finitely generated free) and has a representative tower with surjective bonding maps, then it has a representative telescopic tower (i.e., a tower in which both conditions hold simultaneously).

Pro-groups arise in topology by means of towers associated to exhaustions of non-compact spaces.

Definition 1.7. *If X is a polyhedron then a proper map $\omega : [0, \infty) \to X$ is called a* proper ray*. Two proper rays define the same* end *if their restrictions to the subset of natural numbers are properly homotopic. An end is called* semi-stable *if every two proper rays defining that end are actually properly homotopic; one also says that the two rays define the same strong end.*

Given now a proper base ray ω in X and an exhaustion $C_1 \subset C_2 \subset \cdots \subset X = \cup_{i=1}^{\infty} C_i$ by compact sub-polyhedra we can associate a tower of groups

$$\pi_1(X, \omega(0)) \leftarrow \pi_1(X - C_1, \omega(1)) \leftarrow \cdots$$

where the bonding morphisms are induced, on one hand, by the inclusions of spaces and on the other hand, by the change of base points which are slid along the ray ω restricted to integral intervals.

Definition 1.8. *The* (fundamental) pro-group at infinity *of X based at ω, denoted $\pi_1^{\infty}(X, \omega)$, is the pro-group associated to the tower of groups*

$$\pi_1(X, \omega(0)) \leftarrow \pi_1(X - C_1, \omega(1)) \leftarrow \cdots$$

Two rays defining the same strong end yield isomorphic pro-groups. In particular, if the end is semi-stable, the pro-group at infinity is an invariant of the end, and then also called the (fundamental) pro-group of the end. *The end is called simply connected at infinity (or π_1-trivial) if the associated pro-group is pro-isomorphic to a tower of trivial groups.*

Remark 1.1. *There are alternative equivalent definitions of the semi-stability, in particular the one used in Siebenmann's thesis: an end is semi-stable if its fundamental pro-group has a representative tower with* surjective *bonding morphisms (see also [25]). For the sake of completeness we recall that an end is called* stable *if there exist some representative tower in which all bonding morphisms are* isomorphisms. *Examples of Davis show that the ends of universal coverings of finite complexes might be* not stable*, although it is not known whether they should be always semi-stable. Notice that sometimes in literature one uses the terms π_1-stable, π_1-semi-stable etc. for the corresponding notions introduced above. As already observed above, we can infer from [26] that a semi-stable end having pro-(finitely generated free) fundamental pro-group at infinity admits a representative* telescopic tower *for that fundamental pro-group at infinity.*

We introduce now a family of 3-manifolds which is, in some sense, the smallest one containing the missing boundary 3-manifolds and allowing manifolds to have infinitely many boundary components. These manifolds will be the right analog of the open manifolds which are simply connected at infinity in the pl-gsc context.

Before to proceed let us recall that a compact 0-dimensional subset C is *tame* (or tamely embedded) in $\mathbb{R}^n$ if there exists a homeomorphism of $\mathbb{R}^n$ sending C into a subset of $\mathbb{R} \times \{0\} \subset \mathbb{R}^n$. It is well-known that perfect (i.e. without isolated points) compact 0-dimensional separable topological spaces are homeomorphic to the Cantor space. Hence the tameness condition above is mostly relevant for Cantor subsets of $\mathbb{R}^n$. Notice that there exist wild Cantor sets in any $\mathbb{R}^n$, with $n \geq 3$, while Cantor sets in $\mathbb{R}^2$ are tame, by a classical theorem of Bing ([3]).

Definition 1.9. *A* standard model *is a 3-manifold with boundary V constructed as follows. Let $\{B_i\}_{i\in I}$ be a collection of pairwise disjoint 3-balls in the interior* $\mathrm{int}(B)$ *of the 3-ball whose radii go to* 0 *and whose limit set L is a tame* 0*-dimensional subset disjoint from ∂B. Let $X \supset L$ be a tame* 0*-dimensional subset of* $\mathrm{int}(B)$ *which is disjoint from* $\mathrm{int}(B_i)$*, for all $i \in I$, and $T \subset \partial B \cup \cup_{i\in I} \partial B_i$. Then we put $V = B - (X \cup T \cup_{i\in I} \mathrm{int}(B_i))$. Manifolds of this form where $T \cap \partial B = \emptyset$ were called* ragged cells *by Brin and Thickstun in ([6], p.9-10).*

Remark 1.2.
1. *The open simply connected 3-manifolds V which are simply connected at infinity can be described are the manifolds of the form $S^3 - X$, where X is a tame* 0*-dimensional compact subset of B^3. Alternatively, V can be written as the ascending union of compact simply connected sub-manifolds, i.e. holed balls, by the Poincareé Conjecture (see [21, 44]).*
2. *A simply connected missing boundary 3-manifold V is homeomorphic to $M - T$, where M is a simply connected compact 3-manifold and T is a closed subset of ∂M (see e.g. [44]). By the Poincaré Conjecture there is a finite set of pairwise disjoint balls B_i, $i \in I$ such that $V = B - (\cup_{i\in I} \mathrm{int}(B_i) \cup T)$ and T is a closed subset of $\partial B \cup \cup_{i\in I} \partial B_i$. Thus standard models V with finite I correspond precisely to simply connected missing boundary manifolds. Actually any standard model can be obtained by making connected sums of (possibly infinitely many) simply connected missing boundary manifolds.*

Remark 1.3. *Another characterization of standard models was given by Brin and Thickstun (see[6], Full End Description Theorem (b), p.10), as follows. Modulo the Poincaré Conjecture, the set of simply connected end 1-movable 3-manifolds coincide with that of standard models.*

Remark 1.4. *The boundary of a standard model consists of 2-spheres and open planar surfaces. Each end has pro-(finitely generated free) fundamental group at infinity. In fact the complementary of an unknotted ball in a 1-ended standard model is homotopy equivalent to the complementary of a finite graph, namely a holed handlebody. Thus its fundamental group is a finitely generated free group. Moreover, each end of a standard model is semi-stable.*

The principal result of this paper is the following extension of the result of [21, 18] to arbitrary non-compact 3-manifolds, as follows:

Theorem 1.1. *A non-compact 3-manifold which is proper homotopy equivalent to a pl-gsc polyhedron is homeomorphic to a standard model. In particular, each end is semi-stable and its fundamental pro-group at infinity is pro-(finitely generated free).*

The plan of the paper is as follows. In the next section one shows that the pl-gsc condition for a 3-manifold implies that the manifold is standard. Section 3 is devoted to the proof of the main theorem by using some of the methods developed in [37, 18]. The last section contains applications to geometric group theory. We state a conjectural characterization of properly 3-realizable groups and prove it for a special class of groups. In particular we find examples of groups which are not properly 3-realizable.

Proviso: In order to simplify some arguments we will use in the sequel the fact that there are no fake homotopy disks in dimension 3, as the Poincaré conjecture has been settled by Perelman in [34, 35] (see a detailed and self-contained exposition of Perelman's proof in [33]).

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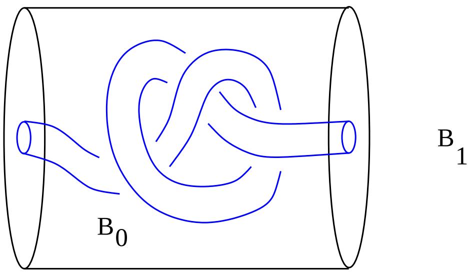
2 Wgsc and gsc in dimension 3

Let W^3 be a 3-manifold with boundary which is wgsc. Thus W^3 is the ascending union of compact simply connected 3-manifolds with boundary. The boundary components of a simply connected compact manifold are 2-spheres. In fact if one boundary had nontrivial genus then some nontrivial boundary loop would give a nontrivial class in the fundamental group of the 3-manifold. Let us cap the boundary spheres by 3-balls. We then obtain a simply connected closed 3-manifold. The Poincaré conjecture states then the latter is a 3-sphere. In particular, any compact simply connected 3-manifold with boundary is standard i.e. homeomorphic to a closed 3-ball with finitely many open 3-balls deleted. We call it a holed ball.

In the case of open manifolds of dimension $n \geq 5$ it is well-known, as a consequence of Smale's proof of the generalized Poincaré conjecture, that wgsc and gsc are equivalent. This is equally true in dimension 3 if one assumes the manifolds irreducible, and more generally for non-compact manifolds admitting boundary, when each boundary component is compact. However, this equivalence fails for more general non-compact manifolds with non-compact boundary. In dimension 3 the wgsc condition is essential in order to derive the simple connectivity at infinity and thus the tameness of the respective open manifold. First, we have to understand whether the wgsc is sufficient for tameness in the non-compact case.

Consider the manifold $Wh = \cup_{n\geq 0} B_n$ where B_n are closed 3-balls such that the inclusion $B_n \subset B_{n+1}$ is isotopic to the embedding of 3-balls given in the figure below. Actually we can consider any knotted properly embedded arc $A \subset B_1$ and B_0 a regular neighborhood of the arc A in B_1.



Proposition 2.1. *The manifold Wh is wgsc,* $\mathrm{int}(Wh) = \mathbb{R}^3$ *and* $\partial Wh = \mathbb{R}^2 \cup \mathbb{R}^2$*. However, Wh is not a missing boundary manifold and in particular, it is not homeomorphic to* $\mathbb{R}^2 \times [0, 1]$.

Proof. It is clear that Wh is wgsc being the union of 3-balls. Moreover, its interior is the ascending union of open 3-balls and thus it is homeomorphic $\mathbb{R}^3$ by an old result of Morton Brown (see [7]). Each boundary component is also the ascending union of closed 2-disks, each included in the interior of the next, and thus homeomorphic to $\mathbb{R}^2$ by the same result.

Moreover, Wh is P^2-irreducible being an ascending union of balls. We can thus apply Tucker's criterion ([42]) which asserts that a P^2-irreducible 3-manifold is a missing boundary manifold iff the complement of any of its compact sub-complexes has finitely generated fundamental group. We consider $B_0 \subset Wh$. Then $Wh - B_0 = \cup_{n\geq 1} \overline{B_n - B_{n-1}}$ and thus $\pi_1(Wh - B_0) = \Gamma *_{\mathbb{Z}} \Gamma *_{\mathbb{Z}} \Gamma *_{\mathbb{Z}} * \cdots$, where $\Gamma = \pi_1(\overline{B_1 - B_0})$.

The cylinder $\partial B_0 \setminus (B_0 \cap \partial Wh)$ has fundamental group $\mathbb{Z}$ generated by a meridian. Assume that $\pi_1(\overline{B_1 - B_0})$ is also generated by the class of this meridian. Let us glue together $\overline{B_1 - B_0}$ with $\overline{B_1' - B_0'}$ where now $B_0' \subset B_1'$ are two concentric solid cylinders. The result of the gluing is the complement $S^3 - N(K)$ of a tubular neighborhood of a knot K obtained by closing up the extremities of the arc A. It also follows that $\pi_1(S^3 - N(K)) = \Gamma *_{\mathbb{Z}} \mathbb{Z} \cong \Gamma$ is generated by the class of the meridian. Since $H_1(S^3 - N(K)) = \mathbb{Z}$, this implies that $\pi_1(S^3 - N(K))$ is isomorphic to $\mathbb{Z}$ and, by the well-known theorem of Papakyriakopoulos, K should be a trivial knot. But this implies that the embedding $B_0 \subset B_1$ is unknotted, contradicting our assumptions. This shows that Γ is not generated by the image of a meridian.

Suppose now that the infinite product $\Gamma_\infty = \Gamma *_{\mathbb{Z}} \Gamma *_{\mathbb{Z}} \cdots$ is finitely generated. Each generator is a finite word in the set of generators of infinitely many copies of Γ and hence belongs to some finite amalgamated product, say $\Gamma_k = \Gamma *_{\mathbb{Z}} \Gamma *_{\mathbb{Z}} \cdots_{\mathbb{Z}} \Gamma$, which contains k factors. We can therefore suppose that all generators of Γ_∞ belong to some Γ_n, for some finite n. Consider then $\Gamma_{n+1} = \Gamma_n *_{\mathbb{Z}} \Gamma$. The Van der Waerden structure theorem for amalgamated products shows that Γ_{n+1} contains nontrivial elements $ab \neq 1$, with $a \in \Gamma_n$ and $b \in \Gamma \setminus \mathbb{Z}$. Thus Γ_{n+1} cannot be generated by using only elements of Γ_n, which contradicts our assumptions.

This proves that $\pi_1(Wh - B_0) = \Gamma_\infty$ is infinitely generated. In particular Wh is not a missing boundary manifold since it does not satisfy Tucker's criterion. □

Thus the wgsc condition is too weak to imply tameness when the manifold has a non-compact boundary. We will now turn towards a stronger gsc condition, which seems more appropriate in our context:

Proposition 2.2. *If a non-compact 3-manifold W^3 is gsc (or pl-gsc) then it is homeomorphic to a standard model.*

Proof. When W^3 is 1-ended there is a simple proof where we can apply directly Tucker's criterion from [42]. In this case W^3 is the ascending union of simply-connected compact sub-manifolds P_j and thus, by the Poincaré conjecture, of holed balls. We cap off all sphere boundary components of W^3. Then W^3 is P^2-irreducible, because there are no embedded essential 2-spheres nor embedded P^2 in a 3-ball.

We can assume that $P_n - P_n \cap \partial W \subset \mathrm{int}(P_{n+1})$. Let us denote $\mathrm{Fr}(P_n) = \partial P_n - P_n \cap \partial W$. Then $\mathrm{Fr}(P_n) = \overline{P_{n+1} - P_n} \cap P_n$. We can further suppose that $\mathrm{Fr}(P_n)$ and ∂P_n are connected for all n. Since one obtains P_{n+k} from P_n by adding a number of 2-handles (for any $k \geq 1$) it follows that the homomorphism induced by the inclusion $\pi_1(\mathrm{Fr}(P_n)) \to \pi_1(\overline{P_{n+k} - P_n})$ is a surjection.

We claim now that:

Lemma 2.1. *The map $\pi_1(\mathrm{Fr}(P_n)) \to \pi_1(\overline{W - P_n})$ induced by inclusion is surjective.*

Proof. When W is gsc this is a consequence of the fact that $\pi_1(\overline{W - P_n})$ is a direct limit of $\lim_{k\to\infty} \pi_1(\overline{P_{n+k} - P_n})$. In other words the class of a loop in $\overline{W - P_n}$ is represented by a loop contained into some $\overline{P_{n+k} - P_n}$, by compactness and thus it belongs to the image of $\pi_1(\mathrm{Fr}(P_n))$.

The claim above holds true also when W^3 is supposed to be pl-gsc. By Van Kampen we have

$$\pi_1(\overline{P_{n+2} - P_n}) = \pi_1(\overline{P_{n+1} - P_n}) *_{\pi_1(\mathrm{Fr}(P_{n+1}))} \pi_1(\overline{P_{n+2} - P_{n+1}})$$

Since $\pi_1(\mathrm{Fr}(P_n))$ surjects onto $\pi_1(\overline{P_{n+1} - P_n})$ and $\pi_1(\mathrm{Fr}(P_{n+1}))$ surjects onto $\pi_1(\overline{P_{n+2} - P_{n+1}})$ we obtain that $\pi_1(\mathrm{Fr}(P_n))$ surjects onto $\pi_1(\overline{P_{n+2} - P_n})$. By induction $\pi_1(\mathrm{Fr}(P_n))$ surjects onto $\pi_1(\overline{P_{n+k} - P_n})$ for all k and we conclude as above. □

Therefore the algebraic assumptions in Tucker's theorem are satisfied, namely:

Lemma 2.2. *The complement of any compact sub-polyhedron K of W has finitely generated fundamental group.*

Proof. This is clear when K is one of the compacts P_n exhausting W. Let n be such that $K \subset P_n$. Using Van Kampen we obtain:

$$\pi_1(\overline{W - K}) = \pi_1(\overline{W - P_{n+1}}) *_{\pi_1(\mathrm{Fr}(P_{n+1}))} \pi_1(\overline{P_{n+1} - P_n}) *_{\pi_1(\mathrm{Fr}(P_n))} \pi_1(\overline{P_n - K})$$

From the surjectivity claim above one derives that the map $\pi_1(\overline{P_n - K}) \to \pi_1(\overline{W - K})$ is surjective and thus $\pi_1(\overline{W - K})$ is finitely generated, for any compact K. □

We derive then from [42] that W^3 is a missing boundary manifold. Since it is also simply connected it should be homeomorphic to a standard model. This proves the proposition in the one-ended case.

Let us now consider the case of a multi-ended manifold W^3. Without loss of generality we can assume that P_n are connected and $P_n \cap \partial W \subset \mathrm{int}(P_{n+1} \cap \partial W)$, for all n.

The boundary components of P_n are 2-spheres. Thus each component of ∂W is either a 2-sphere or an ascending union of compact planar surfaces and hence an open planar surface. Moreover we will discard the compact components of ∂W, which are 2-spheres, and hence suppose that all boundary components are planar surfaces.

Lemma 2.3. *We can assume that, for every n, each connected component of $P_n \cap \partial W$ consists of a closed disk D with several disjoint open disks $\mathrm{int}\delta_i$ removed from it such that the circles $\partial\delta_i$ are essential in ∂W.*

Proof. Suppose that P_n intersects the planar component F of ∂W in a compact planar surface homeomorphic to $D - \cup_i \mathrm{int}\delta_i$ so that $\partial\delta_j$ is not essential in F. Then there exists an embedding of δ_j in F with boundary $\partial\delta_j$. We can therefore define P'_n to be the result of adding a 2-handle along $\partial\delta_j$ (with the natural framing) to P_n and next P'_{n+1} to be the first P_m which contains P'_n. Remove then the terms $P_n, P_{n+1}, \ldots, P_{m-1}, P_m$ from our initial exhaustion and replace them by the sequence P'_n, P'_{n+1}.

We can continue this procedure until all boundary loops of P_n become essential and all terms of the exhaustion have the required property.

Eventually, observe (using Van Kampen) that all transformations we made preserve the property $\Pi(P_{j+1}, P_j) = 0$, for all consecutive terms of the considered exhaustions. Thus the lemma follows. □

Consider a connected component S of $\mathrm{Fr}(P_n)$, which is contained into some connected component of ∂P_n. By adding 2-handles we do not connect components which were disconnected before and thus each component S of $\mathrm{Fr}(P_n)$ determines a unique component $B(S)$ of $\overline{P_{n+1} - P_n}$.

Lemma 2.4. *The homomorphism $\pi_1(S) \to \pi_1(B(S))$ induced by inclusion is surjective.*

Proof. Clear. □

Let T denote $\partial B(S) \setminus S$ and T^* denote T deprived of those components which are 2-spheres.

Lemma 2.5. *We have $\chi(S) \geq \chi(T^*)$.*

Proof. Then T^* is obtained from gluing $B(S) \cap \partial W$ with $B(S) \cap \mathrm{Fr}(P_{n+1})$ along $B(S) \cap \partial \mathrm{Fr}(P_{n+1})$. The previous lemma shows that $(\overline{P_{n+1} - P_n}) \cap P_n \cap \partial W = \partial(P_n \cap \partial W)$. Thus ∂T^* contains the circles from ∂S and possibly also some other circles coming from $B(S) \cap \partial W - (\partial \mathrm{Fr}(P_{n+1}) \cup \partial S)$. Recall now that S and T^* are 2-spheres with several disks removed from it. We found therefore that the number of boundary components of T^* is at least as large as the number of boundary components of S.

Now all other components of T^* should be closed surfaces of non-positive Euler characteristic and so $\chi(T^*) \leq \chi(S)$. □

Set further $B^*(S)$ for the result of capping off all those 2-spheres on the boundary of $B(S)$ not containing S by 3-balls.

Lemma 2.6. *$B^*(S)$ is homeomorphic to $S \times [0,1]$.*

Proof. Theorem 1.1 from ([9]) states that, given a compact connected 3-manifold B and S a non-void compact connected 2-manifold properly contained in ∂B such that $\chi(S) \geq \chi(B)$ and $\pi_1(S) \to \pi_1(B)$ is onto then both S and $\overline{\partial B - S}$ are strong deformation retracts of B. Moreover, ([8], Theorem 3.4) improved the conclusion, namely under the extra assumption that S is not a projective plane and B has no fake homotopy disks then B is homeomorphic to $S \times [0,1]$.

Since $\chi(B^*(S)) = \frac{1}{2}(\chi(T^*) + \chi(S))$ the previous lemma shows that $\chi(S) \geq \chi(B)$.

Moreover, we infer from lemma 2.4 that $\pi_1(S) \to \pi_1(B^*(S))$ is surjective and hence, by the previous cited results of Brown and Crowell, $B^*(S)$ is homeomorphic to $S \times [0,1]$. □

Let us consider now the collection of 2-spheres arising in $\partial P_n \setminus \mathrm{Fr}(P_n)$ for all n. This is a proper collection of embedded 2-spheres, namely each compact intersects only finitely many of them. We cut W open along this collection of spheres and then cap off the boundary components by 3-balls. The result is an infinite collection of 3-manifolds W_i such that W is the infinite connected sum of the W_i's. Moreover each W_i is a gsc (respectively a pl-gsc) 3-manifold and inherits from W an exhaustion $P_{n,i}$ such that $\partial P_{n,i}$ has no sphere components. If $S_{n,i,p}$ are the components of $\mathrm{Fr}(P_{n,i})$ within W_i then $B^*(S_{n,i,p})$ are disjoint and their union is all of $\overline{P_{n+1,i} - P_{n,i}}$. Therefore lemma 2.6 shows that $\overline{P_{n+1,i} - P_{n,i}}$ is homeomorphic to $\mathrm{Fr}(P_{n,i}) \times [0, 1]$. Lemma 1 from ([42]) states that a 3-manifold admitting such an exhaustion is a missing boundary manifold. Thus each W_i is a simply connected missing boundary manifold and thus homeomorphic to a standard model as that described in Remark 1.2.2. As the infinite proper connected sum of such standard models is a standard model the result follows. □

3 Proof of the Theorem 1.1

3.1 Outline

Let us introduce the concept of geometric Dehn exhaustibility, which seems slightly weaker than the gsc.

Definition 3.1. *A non-compact manifold with boundary W^3 is geometrically Dehn exhaustible if there exist a sequence of compact simply connected 3-manifolds with boundary M_n endowed with triangulations, the simplicial immersions $p_n : M_n \to M_{n+1}$ and the* generic smooth immersions *$f_n : M_n \to W$ fulfilling the properties:*

1. *f_n and p_n are compatible i.e. $f_{n+1} \circ p_n = f_n$, for all n;*
2. *$f_n(M_n)$ exhausts W^3;*
3. *$\Pi(M_{n+1}, p_n(M_n)) = 0$ for all n.*

Remark 3.1. *For a generic immersion $f : M^3 \to W^3$ the set of double points (at source) $M_2(f)$ is a compact 3-dimensional sub-manifold of M^3, the set of triple points $M_3(f)$ is finite and there are no higher multiplicities i.e. $M_4(f) = \emptyset$, where $M_k(f) = \{x \in M^3; \mathrm{card} f^{-1}(f(x)) \geq k\}$. We can actually take this properties as defining the genericity in the sequel.*

First we have the following:

Proposition 3.1. *If a 3-manifold W^3 is proper homotopy equivalent to a pl-gsc polyhedron then W^3 is geometrically Dehn exhaustible.*

Remark 3.2. *One can define similarly the geometric Dehn exhaustibility for non-compact manifolds with boundaries in any dimension n. Moreover, if dimension $n \neq 3$ then one drops the genericity assumption. Then the result of the previous proposition extends, as stated, to any dimension n. The proof in dimensions $n \geq 4$ is easier, as one can see from [19], where the case of open manifolds was treated. As we will not use this higher dimensional extension anymore in this paper we leave the details to the reader.*

The second step uses in an essential way the fact that the dimension is 3 and we are not aware of an extension to all dimensions.

Proposition 3.2. *If the non-compact 3-manifold W^3 is geometrically Dehn exhaustible then it is pl-gsc.*

End of proof of Theorem 1.1. Propositions 3.1 and 3.2 above settle our claim. In fact, altogether they imply that a 3-manifold W^3 proper homotopy equivalent to a pl-gsc polyhedron should be gsc and so, by proposition 2.2, W^3 is homeomorphic to a standard model. By remark 1.4, W^3 has semi-stable ends with pro-(finitely generated free) pro-groups. □

Remark 3.3. *From the Poincaré conjecture and [6] simply connected non-compact 3-manifold whose ends are semi-stable and have pro-(finitely generated free) fundamental groups at infinity are homeomorphic to standard models. However, if the semi-stability condition is dropped then we can find also manifolds which are not standard models.*

3.2 Proof of Proposition 3.1

Let W^3 be a non-compact 3-manifold which is proper homotopy equivalent to the pl-gsc polyhedron P.

We say that the polyhedron Q is an *enlargement* of W^3 if there exist an embedding $i : W \hookrightarrow Q$ and a proper retraction $r : Q \to W$ such that $r \circ i$ is the identity.

Lemma 3.1. *It suffices to prove the proposition in the case when W^3 has a pl-gsc enlargement P.*

Proof. We already proved the similar result for the wgsc property in ([18], Lemma 2.1 p.18). The idea is to consider the mapping cylinder of a homotopy equivalence $W \to P$, which admits a strong deformation retraction onto W which makes it an enlargement of it. It is immediate that P being pl-gsc implies that the mapping cylinder is also pl-gsc. □

Assume then that P is a pl-gsc enlargement of W^3 and $i : W \to P$ and $r : P \to W$ are the associated embedding and retraction. Thus there exists an exhaustion $X_0 \subset X_1 \subset X_2 \subset \cdots$ by finite sub-complexes of P with the property that $\pi_1(X_j) = \Pi(X_{j+1}, X_j) = 0$, for all $j \geq 0$. By restricting r to the 3-skeleton of P we can suppose that P is of dimension 3 (and actually 3-full, see [18], Definition 2.2 p.19), while r is a non-degenerate simplicial map.

Recall that Poenaru defined an equivalence relation $\Psi(r) \subset M^2(r) \cup \mathrm{Diag}(P)$, where $M^2(r) \cup \mathrm{Diag}(P) = \{(x, y) \in P \times P; r(x) = r(y)\}$, which is associated to an arbitrary non-degenerate simplicial map $r : P \to W$. Specifically $\Psi(r)$ is the *smallest equivalence relation* such that the induced map $\overline{r} : P/\Psi(r) \to W$ is an *immersion*.

Recall that $f : P \to W$ is an *immersion* if it has *no singularities*, i.e. points $z \in P$ such that $z \in \sigma_1 \cap \sigma_2$ for distinct simplices σ_1, σ_2 of P such that $f(\sigma_1) = f(\sigma_2)$.

There is an explicit formula for $\Psi(f)$ in [36]. But there exists a simple procedure for constructing $\Psi(f)$ recurrently: pick a singular point z of f and two distinct simplices σ_i of the same dimension with $z \in \sigma_1 \cap \sigma_2$ and $f(\sigma_1) = f(\sigma_2)$. Identify σ_1 and σ_2 (this is called a folding map) and call the quotient P', which is endowed with a simplicial map $f' : P' \to W$. Continue this procedure for the singularities of P' and so on. If P is finite this procedure stops when we obtain an immersion. If P is infinite, then it is shown in [36] that an appropriate choice of folding maps yields an infinite ascending sequence of equivalence relations whose union is $\Psi(f)$.

In the case of the retraction map $r : P \to W$ the main property of this equivalence relation is that it induces (for a simplicial complex P which is 3-full) a simplicial isomorphism $\overline{r} : P/\Psi(r) \to W$ (see ([37], p.442, Proposition A)). Recall that P has the nice exhaustion X_j. The main idea in [37] is to use the compactness in order to see that *any* compact $K \subset W$ is covered by some $X_k/\Psi(r|_{X_k})$ such that the immersion $r_k : X_k/\Psi(r|_{X_k}) \to W$ has no double points in K. Specifically, K is covered by some $r(X_{j_1})$ (since r is proper) and $r^{-1}(r(X_{j_1}) \subset X_{j_2}$ by compactness. We can think of K as $i(K) \subset i(W) \cap X_{j_2}$. Obviously if $(x, y) \in M^2(r)$ are double points of r and $x \in i(K)$ then $y \in X_{j_2}$. The properness of r and the fact that $\Psi(r)$ contains all double points of r imply that there exists j_3 such that

$$\Psi(r|_{X_{j_3}})|_{X_{j_2}} = \{(x, y) \in X_{j_2} \times X_{j_2}; r(x) = r(y)\}$$

Then $k = j_3$ is convenient for us, since

$$K = i(K) \subset r(X_{j_1}) \subset X_{j_1}/\Psi(r|_{X_{j_3}}) \subset X_{j_3}/\Psi(r|_{X_{j_3}})$$

and

$$\overline{r|_{X_{j_3}}} : X_{j_3}/\Psi(r|_{X_{j_3}}) \to W$$

is an immersion whose double points are far from X_{j_1}. In particular $K \cap M_2(\overline{r|_{X_{j_3}}}) = \emptyset$.

We consider above the indices n_i such that for each i, if we put $j_1 = n_i$ then $j_3 = n_{i+1}$.

We then have the families of simplicial immersions

$$p_i : X_{n_i}/\Psi(r|_{X_{n_i}}) \to X_{n_i}/\Psi(r|_{X_{n_{i+1}}}) \hookrightarrow X_{n_{i+1}}/\Psi(r|_{X_{n_{i+1}}})$$

where the first map is the obvious projection, while the second one is the natural inclusion. Notice that $p_i(X_{n_i}/\Psi(r|_{X_{n_i}})) = X_{n_i}/\Psi(r|_{X_{n_{i+1}}})$. According to ([37], p.422, Theorem (II)) we have surjective homomorphisms

$$\pi_1(X_{n_i}) \to \pi_1(X_{n_i}/\Psi(r|_{X_{n_{i+1}}})) \text{ and } \pi_1(X_{n_{i+1}}) \to \pi_1(X_{n_{i+1}}/\Psi(r|_{X_{n_{i+1}}})$$

and hence

$$\pi_1(X_{n_{i+1}}/\Psi(r|_{X_{n_{i+1}}})) = \pi_1(X_{n_i}/\Psi(r|_{X_{n_{i+1}}})) = 0$$

The same argument works also in the relative case and thus

$$\Pi(X_{n_{i+1}}/\Psi(r|_{X_{n_{i+1}}}), X_{n_i}/\Psi(r|_{X_{n_{i+1}}})) = 0$$

Further recall that we also have the collection of immersions induced by r, namely $\overline{r_i} = \overline{r|_{X_{n_i}}}$ that are compatible with the p_i.

The only problem with the family of spaces $Y_i = X_{n_i}/\Psi(r|_{X_{n_i}})$ is that these are just simplicial complexes and not manifolds with boundary. Nevertheless, we can replace Y_i by the regular neighborhood $M_i = \Theta(Y_i, \overline{r_i})$ associated to the immersion $\overline{r_i}$, viewed as generically immersed in W^3. The construction of a regular neighborhood associated to an immersion can be found in [28] and one can consult ([18], p.21). The data consisting of the 3-manifolds M_i (endowed with induced triangulations from Y_i), induced simplicial embeddings from p_i and immersions induced from r_i (which are generic) verifies the conditions required in definition 3.1 and thus W^3 is geometrically Dehn exhaustible. □

3.3 Proof of Proposition 3.2

The key-point is the following enhanced Dehn-type lemma:

Lemma 3.2. *Let W^3 be a simply connected non-compact 3-manifold with boundary, $K \subset W$ a compact subset and $B \subset W$ a holed ball. Assume that there exist a compact simply connected 3-manifold with non-empty boundary X^3, an embedding $i : K \cup B \hookrightarrow X$ and a generic immersion $f : X \to W$ such that*

1. *$f \circ i$ is the natural inclusion $K \cup B \hookrightarrow W$;*
2. *$M_2(f) \cap (K \cup B) = \emptyset$; and*
3. *$\Pi(X, i(B)) = 0$.*

Then there exists a compact simply connected *sub-manifold with boundary $U \subset W$ such that $K \cup B \subset U$ and $\Pi(U, B) = 0$.*

Proof. The proof is a slight improvement of that given in ([37], p.433-439). Compact, not necessarily connected, 3-manifolds Y having ∂Y a nonempty union of 2-spheres are said to have property S. Then compact submanifolds $Y \subset W$ have property S if and only if they are simply connected.

By [39] any compact connected 3-manifold with nonempty boundary which does not possess 2-sheeted coverings has property S.

We can assume that $f(X) \subset W$ is a sub-manifold. Consider $f(X) \subset W$ and remark that $\Pi(f(X), B) = 0$. If $f(X)$ does not admit 2-sheeted coverings then take $U = f(X)$ and we are done.

If $f(X)$ has a 2-sheeted coverings $X_1 \to f(X)$ then observe that the map $f : X \to f(X)$ has a lift $f_1 : X \to X_1$, because $\pi_1(X) = 0$. Consider then a 2-sheeted covering of X_2 of $f_1(X)$, if it exists, setting the stage for an inductive tower construction. We consider X_{j+1} to be a 2-sheeted covering of $f_j(X) \subset X_j$ so that the map $f_j : X \to X_j$ extends to $f_{j+1} : X \to X_{j+1}$. We remark that the extensions f_j are generic immersions. Moreover, $\Pi(f_j(X), f_j(B)) = 0$.

This tower construction should end up in finitely many steps. We reproduce here the argument in [37], p.438-439 for completeness. If $h : A \to B$ is a map set $M^2(h) = \{(x, y) \in A \times A; x \neq y, h(x) = h(y)\}$. It is immediate that all inclusions

$$M^2(f) \supset M^2(f_1) \supset M^2(f_2) \supset \cdots M^2(f_k) \supset \cdots$$

are strict. Since all f_j are generic immersions the sets $M^2(f_j)$ are compact manifolds with boundary. Further $M^2(f_{j+1})$ is both open and closed in $M^2(f_j)$ and thus it has at least one component less than $M^2(f_j)$. Therefore climbing the tower the number of connected components decreases strictly proving the claim. This means that eventually we obtain a map $f_k : X \to X_k$ such that $f_k(X)$ has property S and $\Pi(f_k(X), f_k(B)) = 0$.

Since the composition of $f_j|_{K\cup B}$ with the covering $p_j : f_j(X) \to f_{j-1}(X)$ is an embedding it follows that $f_j|_{K\cup B}$ are embeddings for all j. Moreover, $M_2(p_j|_{f_j(X)}) \cap f_j(K) = \emptyset$.

The next step is to descend the tower step by step. In order to be able to do that we need the following:

Lemma 3.3. *Let W^3 be a non-compact 3-manifold with boundary, $K \subset W$ a compact subset and $B \subset W$ a holed ball. Assume that there exist a compact 3-manifold X^3 with property S, an embedding $i : K \cup B \hookrightarrow X$ and a generic immersion $f : X \to W$ such that*

1. *$f \circ i$ is the natural inclusion $K \cup B \hookrightarrow W$;*
2. *$M_2(f) \cap (K \cup B) = \emptyset$;*
3. *$M_3(f) = \emptyset$; and*
4. *$\Pi(X, i(B)) = 0$.*

Then there exists a compact *sub-manifold with boundary $U \subset W$ having property S such that $K \cup B \subset U$ and $\Pi(U, B) = 0$.*

Proof. Following ([37], p.436-437) we consider the set S of double points of f that involve one point from X and the other from ∂X; this is a 2-dimensional sub-manifold.

Suppose that $\partial S \neq \emptyset$. Any double circle from $\partial S \subset \partial X$ bounds a 2-disk d whose interior is disjoint from the other circles in ∂S. Since $K \cap \partial S = \emptyset$ then either $K \cap d = \emptyset$ or else $K \cap \partial X \subset \mathrm{int}(d)$. We claim that there exists a minimal double circle γ from ∂S such that $K \cap d = \emptyset$. Here minimal means that the disk d does not contain other points of ∂S. In fact, if we choose a minimal circle γ_0 with the disk d_0 that does not satisfy all requirements above then $K \cap \partial X \subset \mathrm{int}(d_0)$. Consider now the complementary disk d_1 of d_0 in the respective connected component of ∂X, which has to be a sphere. If d_1 contains other components of ∂S take a minimal one γ and its associated 2-disk d. Otherwise take $\gamma = \gamma_0$ with the associated disk $d = d_1$.

If the disk $d \subset \partial X$ is made of double points then consider the associated paired disk $\overline{d}$ such that $f(d) = f(\overline{d})$ and $\overline{d} \subset X$; one splits X along $\overline{d}$ and this operation is far from K since the double points locus is disjoint from K. After splitting, the component containing K has one less components in ∂S.

If the disk d is not made of double points then consider the circle $\overline{\gamma} \subset \mathrm{int}(X)$ that is paired with γ i.e. such that $f(\gamma) = f(\overline{\gamma})$. Let us attach a 2-handle H to X along $\overline{\gamma}$. We define $\overline{f} : X \cup H \to W$ by sending the core of the 2-handle H homeomorphically onto the 2-disk d. Since $K \cap d = \emptyset$ the new map $\overline{f}$ can be made a generic immersion without double points in K.

Using repeatedly the procedure above we obtain a manifold X as above with the property that $\partial S = \emptyset$. Thus the double locus $M_2(f)$ of f is a manifold of codimension zero in X whose boundary S is made of spheres which are grouped into pairs, each pair consisting of one sphere in $\mathrm{int}(X)$ and the other one in ∂X. Consider a connected component $Y \subset M_2(f)$ containing an interior sphere from S. We split X along Y and get a manifold Z having only spheres as boundary, one connected component being Y. Consider the connected component X' of $Z - Y$ that contains K.

By continuing this process we reduce the number of components of $M_2(f)$. Eventually we obtain a manifold X' for which $f : X' \to W$ is an embedding and its boundary is made of spheres, as claimed.

Let us follow these steps by taking care of the modifications made on the pair $(X, i(B))$. Cutting along an embedded disk d of double points does not affect $\Pi(X, B)$ since d is included in a boundary component, and thus it is a subset of the sphere, and so it is unknotted.

Changing X by adding a new 2-handle along a circle disjoint from B also preserves $\Pi(X, i(B))$. Thus we can suppose that all steps from above can be performed until we obtain $\partial S = \emptyset$.

Now, the main modification is when we throw away (one component of) the codimension zero manifold with boundary $N = M_2(f) \subset X$, whose boundary is made of spheres from ∂X and spheres inside $\text{int}(X)$.

The Poincaré conjecture implies that N is a holed sphere, like X. We have to prove that $\Pi(\overline{X-N}, B) = 0$. First, the spheres from $\overline{X-N} \cap N$ are in the interior of X. All such spheres are separating, since otherwise a non-separating interior sphere will create a nontrivial loop in X, contradicting its simple connectivity.

Recall that we have a surjection $\pi_1(\text{Fr}(B)) \to \pi_1(\overline{X-B})$. By using Van Kampen in the decomposition $\overline{X-B} = \overline{X-(N\cup B)}) \cup N$ it follows that the inclusion induces a surjection $\pi_1(\text{Fr}(B)) \to \pi_1(\overline{X-(N\cup B)})$, and thus $\Pi(\overline{X-N}, B) = 0$. This proves the lemma. □

We apply lemma 3.3 to the last floor of our tower by taking as X the manifold $f_k(X)$ which has property S, the generic immersion f to be the restriction of the double covering $p_k : X_k \to X_{k-1}$ to $f_k(X)$ which has no triple points and $W = f_{k-1}(X)$ Then we obtain a sub-manifold $X'_{k-1} \subset f_{k-1}(X)$ which has property S and contains $f_{k-1}(K \cup B)$ and $\Pi(X'_{k-1}, f_{k-1}(B)) = 0$. We are now able to apply lemma 3.3 to the restriction of the double covering $p_{k-1}|_{X'_{k-1}}$ to obtain a compact sub-manifold $X'_{k-2} \subset f_{k-2}(X)$ which has property S, it contains $f_{k-2}(K\cup B)$ and satisfies $\Pi(X'_{k-2}, f_{k-2}(B)) = 0$. This way we can descend inductively all steps of the tower and at the bottom level we find a sub-manifold $X'_0 \subset W$ with property S such that $K \cup B \subset W$ and $\Pi(X'_0, B) = 0$. Since W is simply connected $U = X'_0$ is also simply connected and has the required properties. □

End of the proof of proposition 3.2. Let $K_0 \subset K_1 \subset \cdots$ be a compact exhaustion of a 3-manifold W^3 which is geometrically Dehn exhaustible. By lemma 3.2, K_0 is contained into a simply connected compact sub-manifold X_1. Applying lemma 3.2 to K_1 and $B = X_1$ we find a simply connected compact sub-manifold $X_2 \supset K_1 \cup X_1$ such that $\Pi(X_2, X_1) = 0$. By inductive application of the lemma 3.2 to K_j and $B = X_j$ one obtains then a simply connected compact sub-manifold $X_{j+1} \supset K_j \cup X_j$ such that $\Pi(X_{j+1}, X_j) = 0$ for every j. Therefore the exhaustion X_j of W^3 verifies the conditions of definition 1.4 and hence W^3 is pl-gsc. □

3.4 Comments on the proof

The subject of this section is to explain why do we need to develop the additional gsc/pl-gsc machinery instead of simply using the wgsc.

The wgsc analogues of the propositions above hold for non-compact manifolds in any dimension by [19], and in particular if W^3 is a Dehn exhaustible non-compact 3-manifold then W^3 is wgsc.

In [18, 21] it was proved that *open* 3-manifolds W^3 which are proper homotopy equivalent to wgsc polyhedra are simply connected at infinity and thus they are obtained by deleting a 0-dimensional tame subset from the sphere S^3, according to Edwards and Wall ([16, 44]). The proof given there can be adjusted with minor modifications in order to cover the case where W^3 is non-compact but each boundary component is compact.

Assume then that we have a wgsc 1-ended 3-manifold W^3 with boundary. Let U be the complement of the planar components of ∂W. Observe that, if W^3 is Dehn exhaustible then U is also Dehn exhaustible. In particular, we can apply the results from [37, 21] in order to obtain that U is wgsc and hence U is simply connected at infinity.

Since U is 1-ended, U is homeomorphic to $\mathbb{R}^3$ with a collection of closed 3-balls removed. This is because in the one-ended case the second homology of W^3 (and hence $\pi_2(W)$) is represented by the spheres on the boundary of W^3. Thus, after gluing to U a collection of 3-balls to kill these spheres, we get an open contractible 3-manifold which is simply connected at infinity and hence $\mathbb{R}^3$.

The problem we face now is that there is *not only one way* to attach a plane to $\mathbb{R}^3$ in order to get a boundary component. In fact many examples can be obtained by using the so-called Artin-Fox arcs (see [17]). Recall that Artin-Fox (or wild) arcs are topological embeddings of the unit interval in $\mathbb{R}^3$ which are not topologically equivalent to the standard embedding. In [17] one might find instructive examples of wild arcs, including arcs whose complements are simply connected. We can associate to any Artin-Fox arc λ in $\mathbb{R}^3$ a manifold with boundary $W_\lambda \subset \mathbb{R}^3$ such that the "normal" arc to W_λ at the origin is isotopic to λ. Specifically, consider an arc $A \subset S^3$ that is wild at only one point p and has simply connected complement. The arc $A - p$ is tamely embedded in $\mathbb{R}^3 = S^3 - p$. We thicken $A - p$ to a 3-cell B in $\mathbb{R}^3$. Then its complement $M = \mathbb{R}^3 - \mathrm{int}(B)$ is a manifold with boundary that has $\mathrm{int}(M) = \mathbb{R}^3$ and $\partial M = \mathbb{R}^2$. However, M is not homeomorphic to $\mathbb{R}^3_+$ and actually it is not a missing boundary manifold.

As a consequence we cannot use information about U alone in order to derive the tameness of W.

It is amazing to observe that the only surfaces with finitely generated fundamental groups yielding exotic compactifications are actually the planes, as explained by Tucker in [43]. More examples and exotic constructions can be found in [38, 42].

Notice also that in any dimension $n \geq 4$ there is only one way to add an $\mathbb{R}^{n-1}$ boundary to a $\mathbb{R}^n$, up to diffeomorphism.

Moreover, if one obtained that W^3 were wgsc, this would still be not enough in order to conclude that W^3 is tame (e.g. a standard model). The example Wh above (see Proposition 2.1) shows that one needs to consider whether the manifold W^3 is gsc and not only wgsc and this is more subtle.

The difference with respect to the open case is that although there is only one isotopy class of embeddings of a codimension zero ball inside the *interior* of another ball, there are at least as many different isotopy classes of embeddings of one 3-ball into another 3-ball as knots. The gsc condition amounts to asking for ascending unions of (holed balls) with extra *unknottedness* assumptions for the terms of the exhaustion. Fortunately, in dimension 3 we can algebraically express the unknottedness of balls, in terms of fundamental groups of pairs.

4 Applications to discrete groups

The second aim of this paper is to obtain necessary conditions for a finitely presented group be properly 3-realizable, that lead conjecturally to a complete characterization. Lasheras introduced and studied this class of groups in [26, 12, 11]. Recall that:

Definition 4.1. *A finitely presented group Γ is properly 3-realizable (abbreviated* P3R *from now on) if there exists a compact 2-dimensional polyhedron X with fundamental group Γ such that the universal covering $\widetilde{X}$ is proper homotopy equivalent to a 3-manifold W^3.*

Remark 4.1. *Here and henceforth we will consider only* infinite *groups Γ and thus the associated 3-manifolds W^3 appearing in the definition above will be* non-compact. *Notice that, in general, the 3-manifolds W^3 will also have* non-compact *boundary.*

Remark 4.2. *In the definition of a P3R group one does not claim that* any *compact 2-dimensional polyhedron X with fundamental group Γ has its universal covering proper homotopy equivalent to a 3-manifold. However one proved in ([1], Proposition 1.3) that given a P3R group G then for* any *2-dimensional compact polyhedron X of fundamental group G the universal covering of the wedge $X \bigvee S^2$ is proper homotopy equivalent to a 3-manifold.*

Recall the following classical theorem of embedding up to homotopy due to Stallings. Let P be a finite CW-complex of dimension k, let M be a PL-manifold of dimension m and let $f\colon P \to M$ be a c-connected map. If $m - k \geq 3$ and if $c \geq 2k - m + 1$ then there exists a compact sub-polyhedron $j\colon Q \hookrightarrow M$ and a homotopy equivalence $h\colon P \to Q$ such that jh is homotopic to f. This was generalized in [10] to the non-compact situation by replacing the connectivity with the proper connectivity. Thus the proper homotopy type of a locally finite CW-complex X of dimension n is represented by a closed sub-polyhedron of R^{2n-c} if X is properly c-connected.

In particular, the universal covering $\widetilde{X}$ of an *arbitrary* compact 2-polyhedron X^2 is proper homotopy equivalent to a 4-manifold, because any 2-polyhedron embeds, up to *proper homotopy*, into $\mathbb{R}^4$. Therefore P3R groups are singled out among the set of all finitely presented groups by the fact that the universal covering $\widetilde{X}$ of some compact polyhedron X with given $\pi_1(X)$ is proper homotopy equivalent to a particular 4-manifold, namely the product of a 3-manifold with an interval.

Remark 4.3. *Fundamental groups of compact 3-manifolds are obviously P3R, but there exist also P3R groups which are* not *3-manifold groups. For instance, any ascending HNN extension of a finitely presented group is P3R ([26], see also other explicit examples in [12]). Moreover, given any infinite finitely presented groups G and H their direct product $G \times H$ is P3R (according to [11]). Further amalgamated products of P3R groups (and HNN extensions) over finite groups yield P3R groups (see [13]).*

Definition 4.2. *A finitely presented group has* semi-stable ends *if there exist a compact polyhedron X with given fundamental group whose universal covering has semi-stable ends.*

Remark 4.4. *If a group has semi-stable ends then the universal covering of* any *compact polyhedron with given fundamental group has semi-stable ends. Although there exist spaces whose ends are not semi-stable, there are still not known examples of finitely presented groups (i.e. universal coverings of compact polyhedra) without semi-stable ends (see also [30, 23]).*

Definition 4.3. *The* (fundamental) pro-group at infinity of a finitely presented group *is the pro-group at infinity of the universal covering of a compact polyhedron with given fundamental group. This depends of course on the base ray (and thus only on the end if it is semi-stable), but not on the the particular compact polyhedron we chose.*

The main source of examples of P3R groups is the paper ([26]) of Lasheras where it is proved that *a one-ended finitely presented group which is semi-stable and whose fundamental pro-group at infinity is pro-(finitely generated free) is P3R.* In particular, any one-ended finitely presented group Γ which is simply connected at infinity (and hence automatically semi-stable at infinity) is P3R.

We expect the following to be a complete characterization of this class of groups:

Conjecture 1 (3-dimensional homotopy covering conjecture)**.** *A finitely presented group is P3R iff each one of its ends is semi-stable and has pro-(finitely generated free) fundamental pro-group.*

In this paper we give evidence in the favor of this conjecture, by proving it in the case when the group under consideration satisfies an additional hypothesis related to the geometric simple connectivity. In order to explain this we have to introduce, following Brick, Mihalik ([5]) and Stallings ([41]), the following tameness condition for groups and spaces.

Definition 4.4. *A space X is* quasi-simply filtered *(i.e. qsf) if for any compact $C \subset X$ there exists a connected and simply connected compact K together with a map $f : K \to X$ such that $f(K) \supset C$ and $f|_{f^{-1}(C)} : f^{-1}(C) \to C$ is a homeomorphism.*

A finitely presented group Γ is qsf if there exists a (equivalently, for every) compact polyhedron P with fundamental group Γ such that the universal covering $\tilde{P}$ is qsf.

Remark 4.5. *The condition qsf is a rather mild assumption on finitely presented groups. There are still no known examples of groups which do not have the qsf property and most classes of known groups, as hyperbolic, semi-hyperbolic, automatic, tame combable etc., are qsf.*

We can state now our main result in this section:

Theorem 4.1. *If a finitely presented group is P3R and qsf then all its ends are semi-stable and have pro-(finitely generated free) fundamental group at infinity.*

Remark 4.6. *We do not know whether all finitely presented groups which have semi-stable ends and pro-(finitely generated free) fundamental groups at each end are actually qsf. Notice that one-ended groups with stable end having an element of infinite order should be either simply connected at infinity or pro-$\mathbb{Z}$ at infinity, by a theorem of Wright (see [22], Theorem 16.5.6). Thus they are P3R by the above cited result of Lasheras.*

Remark 4.7. *Cardenas announced that 1-ended groups which are P3R and semi-stable have actually pro-(finitely generated free) pro-group at infinity, as an application of the Brin-Thickstun structure theorem ([6]). In fact 3-manifolds with semi-stable ends are homeomorphic to standard models. Notice that for P3R groups having semi-stable ends is equivalent to being qsf.*

Remark 4.8. 1. *The homotopy covering conjecture implies the well-known covering conjecture in dimension 3 which states that the universal covering of an irreducible closed 3-manifold M^3 with infinite fundamental group is simply connected at infinity. In fact the universal covering $\widetilde{M}$ is an open contractible 3-manifold (thus one-ended) which is semi-stable and has pro-(finitely generated free) fundamental pro-group at infinity. This implies that there exists an exhaustion by compact sub-manifolds C_i such that $\pi_1(\widetilde{M} - C_i)$ are finitely generated free. Tucker's criterion from [42] implies that the manifold $\widetilde{M}$ is a missing boundary manifold and thus it is homeomorphic to* $\mathrm{int}(N^3)$, *for a suitable compact 3-manifold N^3 with boundary. By the contractibility of the universal covering each component of ∂N^3 is homeomorphic to a 2-sphere and this implies that* $\mathrm{int}(N^3)$ *(and so $\widetilde{M}$) is simply connected at infinity.*

2. *Conversely, it is immediate that the universal covering conjecture implies the homotopy covering conjecture* for closed 3-manifold groups *because open 3-manifolds which are simply connected at infinity are semi-stable and have pro-(finitely generated free) pro-group at infinity (in fact trivial pro-group!).*

3. *Notice that the universal covering $\widetilde{X}$ of a compact 2-polyhedron X can never be proper homotopy equivalent to an open (simply connected) 3-manifold M^3. In fact, the Poincaré duality would give us that the third cohomology group with compact support $H^3_c(\widetilde{X})$ is isomorphic to $H^3_c(M) = H_0(M) = \mathbb{Z}$, which is impossible, as* $\dim(\widetilde{X}) = 2$.

Remark 4.9. *Let us consider the universal covering $\widetilde{M^3}$, of a 3-manifold M^3 with boundary. If the boundary is a union of spheres then $\widetilde{M}$ is obtained from the universal covering of a closed 3-manifold (obtained by capping off boundary spheres by balls) by deleting out a collection of disjoint balls. Assume that the boundary is non-trivial i.e. not a union of 2-spheres. Then M^3 is Haken and thus, by Thurston's theorem, it is a geometric 3-manifold. Let us moreover assume that M^3 is* atoroidal *i.e. there are no $\mathbb{Z} \oplus \mathbb{Z}$ embedded in $\pi_1(M)$ other than peripheral subgroups coming from boundary torus components. Then Thurston's geometrization theorem tells us that M^3 is hyperbolic. Therefore the universal covering $\widetilde{M}$ is obtained geometrically by deleting out a collection of horoballs from the hyperbolic 3-space. In particular the pro-group at infinity of $\widetilde{M}$ is pro-(finitely generated free) and its ends are semi-stable. Thus the conjecture holds true for fundamental groups of atoroidal 3-manifolds with non-trivial boundary. A similar but more involved discussion shows that it holds true for all 3-manifolds with non-trivial boundary (since these are geometric).*

Remark 4.10. *The homotopy covering conjecture implies that all 1-relator groups are P3R. This is already known to hold for 1-relator finitely ended groups (see [14]). In fact, 1-relator groups are semi-stable at infinity (see [32]) and it was proved in [14] Proposition 2.7 that their pro-groups at infinity are pro-(finitely generated free). Notice that 1-relator groups are also qsf (see [31]). Recently, Lasheras and Roy ([27]) have extended the results in [14] to a class of groups which contains all 1-relator groups.*

It is presently unknown but quite plausible that any finitely presented group which is qsf, semi-stable and has pro-(finitely generated free) pro-groups at infinity is P3R.

The homotopy covering conjecture admits an (a priori stronger) restatement as follows:

Conjecture 2. *Given a finitely presented P3R group then the universal covering of a compact 2-dimensional polyhedron with that fundamental group is proper homotopy equivalent to a standard model.*

Remark 4.11. *The equivalence between the two conjectures stated in this paper is a consequence of the proper homotopy classification of 3-manifolds with semi-stable ends and pro-(finitely generated free) pro-group at infinity. Simply connected non-compact 3-manifolds with semi-stable ends (and more generally, with 1-movable ends) were classified by Brin and Thickstun (see [6], Full End Description Theorem, p.10). Details are left to the reader.*

As an application of theorem 4.1 we will obtain explicit examples of groups which are not $P3R$, as follows.

Theorem 4.2. *Let Γ be one of the following:*

1. *the fundamental group of a finite non-positively curved complex which is a homology n-manifold ($n \geq 3$), but not a topological manifold.*
2. *the right angled Coxeter group associated to a flag complex L whose geometric realization is a closed combinatorial n-manifold ($n \geq 3$) and $\pi_1(L)$ is not a free group.*

Then Γ is not P3R.

In particular many Coxeter groups are *not* P3R. Similar examples were announced by Cardenas.

4.1 Proof of Theorem 4.1

We first prove:

Proposition 4.1. *If the finitely presented group G is P3R and qsf then there exists a 2-polyhedron X with fundamental group G such that $\widetilde{X}$ is pl-gsc* and *proper homotopy equivalent to a 3-manifold W^3.*

Proof. Since G is qsf then for *any* polyhedron Y with fundamental group G its universal covering $\widetilde{Y}$ is qsf (see [5]). Take Y to be a closed 5-manifold with fundamental group G. Then $\widetilde{Y}$ is an open 5-manifold. One proved in ([20], Proposition 3.2) that any open simply connected manifold of dimension at least 5 which is qsf is actually gsc, as consequence of general transversality results. It follows that $\widetilde{Y}$ is gsc. We triangulate Y and get an equivariant triangulation of $\widetilde{Y}$. Then the triangulated $\widetilde{Y}$ is a pl-gsc polyhedron. The pl-gsc property is preserved when passing to the 2-skeleton. This means that the 2-skeleton Z of the triangulation of Y has the property that $\widetilde{Z}$ is pl-gsc.

It was proved in ([1], Proposition 1.3), as an application of Whitehead's theorem, that given a P3R group G then for *any* 2-dimensional compact polyhedron X of fundamental group G the universal covering of the wedge $X \bigvee S^2$ is proper homotopy equivalent to a 3-manifold. In particular this holds when taking the 2-polyhedron Z from above and thus $\widetilde{Z \bigvee S^2}$ is proper homotopy equivalent to a 3-manifold. Moreover, $\widetilde{Z \bigvee S^2}$ is made of one copy of $\widetilde{Z}$ with infinitely many S^2's attached on it. In particular if $\widetilde{Z}$ is pl-gsc then it is immediate that $\widetilde{Z \bigvee S^2}$ is also pl-gsc. Then $X = Z \bigvee S^2$ has the required properties. □

End of the proof of Theorem 4.1. Let assume that we have a group G which is both P3R and qsf. The previous proposition shows that there exists some 2-polyhedron X such that $\widetilde{X}$ is pl-gsc and also proper homotopy equivalent to some 3-manifold W^3. Looking the other way around we can apply Theorem 1.1 to the 3-manifold W^3 (since it is proper homotopy equivalent to a pl-gsc polyhedron) and obtain that W^3 is homeomorphic to the standard model. In particular, W^3 has semi-stable ends and its pro-groups at infinity are pro-finitely generated free, as claimed. By the proper homotopy invariance of these end invariants $\widetilde{X}$ has the same properties. This proves Theorem 4.1. □

4.2 Proof of Theorem 4.2

First, recall that groups acting properly cellularly and co-compactly on a CAT(0)-complex are wgsc and qsf (see [20, 31]). Thus Coxeter groups and fundamental groups of finite non-positively curved complexes are qsf.

Let us consider X a finite non-positively curved complex. We will make use of the the criterion given in [4] for the semi-stability, which also provides a way to understand the pro-group at infinity. The link of a vertex in X can be given a piecewise spherical metric. Let p be a point of the link of some vertex. The set of points of the link which are at distance at least $\frac{\pi}{2}$ from p is called the punctured link. The punctured link deformation retracts onto the maximal sub-complex of the link that it contains. The main theorem of [4] states that if the links and the punctured links of X are connected then $\widetilde{X}$ is has a semi-stable end.

If X is a homology n-manifold both the links and the punctured links have the same k-homology as the $(n-1)$-sphere, for $k \leq n-2$. In particular they are connected. On the other hand there is at least one vertex v of x whose link is not simply connected, since the complex X is not a topological manifold. The fundamental group of the link is then perfect non-trivial and thus it cannot be a free group.

In [4] it is defined the Morse subdivision of $\widetilde{X}$ as a geodesic subdivision induced by adding the critical points of the distance to a fixed base point. Let $\widetilde{X}_{>r}$ be the maximal sub-complex contained in the complement of the ball of radius r in the Morse subdivision of $\widetilde{X}$. Since the distance is a Morse function on a CAT(0)-complex and the links are connected it is proved in [4] that the inverse system

$$\pi_1(\widetilde{X}_{>0}) \leftarrow \pi_1(\widetilde{X}_{>1}) \leftarrow \pi_1(\widetilde{X}_{>2}) \leftarrow \cdots$$

has surjective bonding maps i.e. the end is semi-stable. Taking the base point to be a lift of the vertex v it follows that the pro-group at infinity cannot be pro-free because $\widetilde{X}_{>0}$ deformation retracts onto the link of v, and thus the first term is a non-free group. Therefore $\widetilde{X}$ has a semi-stable end which is not pro-free. Since $\pi_1(X)$ is qsf it follows from the main theorem that it cannot be P3R.

The second part follows along the same lines. The topology at infinity of Coxeter groups was described in [15]. Recall that the right angled Coxeter group W_L associated to the flag complex L is generated by the vertices of L and the relations correspond to commutativity of adjacent vertices and the fact that these generators are of order two. Moreover W_L acts on the Davis complex properly and cellularly. The Davis complex is a flag cubical complex and thus a CAT(0) complex. Thus W_L is qsf (see also [31]).

There is a natural filtration of the end defined by iterated neighborhoods of some vertex (see [15]). If L is a closed connected combinatorial manifold then W_L has one semi-stable end and the inverse sequence of fundamental groups is as follows (see also ([22], Theorem 16.6.1)):

$$G \leftarrow G * G \leftarrow G * G * G \leftarrow \cdots$$

where $G = \pi_1(L)$ and each bonding map is a projection annihilating the last factor. Thus if L has dimension at least 2 and G is not free then the fundamental group at infinity is not pro-free. The main theorem implies then that W_L cannot be P3R. This settles theorem 4.2.

Remark 4.12. *We can infer from Remark 1.4 that the higher homotopy groups at infinity $\pi_k^{\infty}(W)$ vanish for any standard model W and $k \geq 3$. In particular this furnishes another practical tool for proving that a qsf finitely presented group G is not P3R. Notice however that this is a consequence of the fact that ends are semi-stable and pro-(finitely generated free).*

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