CAT(0) METRICS ON CONTRACTIBLE MANIFOLDS

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ABSTRACT. We prove that an open manifold M of dimension at least 5 which admits a complete CAT(0) polyhedral metric is pseudo-collarable, its fundamental group at infinity is strongly perfectly semistable and has vanishing Chapman-Siebenmann obstruction $\tau_\infty(M)$. Moreover, this implies that M is topologically arborescent, when $n \geq 6$. Conversely, any PL arborescent polyhedron is PL homeomorphic to a CAT(0) cubical complex.

1. INTRODUCTION

1.1. **Context.** The Cartan–Hadamard theorem in Riemannian geometry can be accentuated in two parts:

(1) Nonpositive sectional curvature (a local condition) together with simple connectivity implies global nonpositive curvature (i.e. Alexandrov's CAT(0) condition).
(2) The Riemannian manifold in question is in particular diffeomorphic to an Euclidean space.

With the increase of interest in Alexandrov's coarse curvature notions (motivated chiefly by the work of Burago, Perelman, Shioya, Gromov and others) it was noticed that while the first part holds quite generally for metric length spaces [12], the second part of the Cartan–Hadamard seemed to break down in the topological and polyhedral categories. When revitalizing the interest in CAT(0) geometry for his work on hyperbolic groups, Gromov therefore prominently asked in the eighties for other open manifolds which can be endowed with complete CAT(0) metrics.

Gromov also noticed that this question should be asked for geodesically complete metrics (an assumption we restrict to throughout), as every manifold with boundary can be given a smooth non-complete metric of curvature < 0 (and also a metric of curvature > 0) using the h-principle. In this setting, a CAT(0) manifold is necessarily contractible.

A first answer to this question was provided by Davis and Januszkiewicz [26], who proved the existence of nontrivial CAT(0) n-manifolds, $n \geq 5$, using Gromov's own hyperbolization construction, combined with the Cannon–Edwards criterion. Soon after, Ancel and Guilbault [5] extended the picture by showing that the interior of any compact contractible manifold of dimension $n \geq 5$ can be given a complete CAT(-1) geodesic metric.

On the other hand, already examples of CAT(0) manifolds constructed by Davis and Januszkiewicz have fundamental groups at infinity not stable, and are therefore not compactifiable, giving us two disjoint sources for CAT(0) manifolds. A complete understanding of CAT(0) manifolds remained elusive.

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1.2. **CAT(0) metrics.** It is understood that throughout this paper any CAT(0) metric which we define on a given topological space has the property that the metric and usual topologies agree.

Recall that a metric space (X, d) is *geodesic* (also called a length space or an inner metric space) if every two points of it can be joined by a minimizing geodesic, namely a curve whose length equals the distance between the points. The length of the continuous path $\gamma : [0, 1] \to X$ is defined as

$$\sup_{r, 0=t_0<t_1<\cdots t_r<t_{r+1}=1} \sum_{j=0}^{r} d(\gamma(t_j), \gamma(t_{j+1}))$$

A geodesic triangle in (X, d) satisfies the CAT(κ) inequality if the geodesic comparison triangle with sides of the same length within the simply connected curvature κ Riemannian surface has distances between pairs of boundary points larger than those between corresponding pairs of points in the initial triangle. Moreover, the geodesic metric space (X, d) is CAT(0) if every geodesic triangle, which for $\kappa > 0$ has perimeter less than $\frac{2\pi}{\sqrt{\kappa}}$, satisfies the CAT($\kappa$) inequality.

It is well-known that a simply connected CAT(0) space is contractible. Conversely, we would like to know which contractible spaces admit CAT(0) metrics. In this paper we will only consider this question for manifolds instead of arbitrary spaces and more specifically, *contractible* topological manifolds M of dimension $n \geq 5$.

Classical results show that every open contractible m-manifold M is triangulable, namely there exists a *locally finite simplicial complex* Δ homeomorphic to M (see e.g. [44], Annex B, p. 300, Annex C, p.315). The CAT(κ) metrics which we consider on M are supposed to be *polyhedral*, namely there exists a suitable triangulation Δ such that every cell of Δ, when equipped with the induced metric, is isometric to the convex hull of a finite set of points in the hyperbolic or Euclidean space of curvature $\kappa \leq 0$ (see [9], I.7, Def. 7.37). For instance the *piecewise flat equilateral metric* is the length metric obtained when simplices or cubes are Euclidean and have all their edges of the same (unit) length.

Throughout this paper all polyhedral metrics are supposed to have only *finitely many* isometry classes of cells. We shall point out that our arguments are more general, it suffices that the restriction of the metric to every cell of Δ be a piecewise analytic Riemannian metric whose curvature is bounded above and below by two constants independent on the cell and that the distorsion of cells be uniformly bounded.

Note that we only consider topological manifolds endowed with polyhedral CAT(κ) metrics which are *complete geodesic* metric spaces. Explicit examples can be obtained from a locally finite triangulation of M whose simplices are endowed with constant curvature metrics and have finite distorsion, by ([9], I.7A.13 and I.3.7).

1.3. **Pseudo-collarability.** The goal of this note is to give a topological characterization of CAT(0) manifolds for dimensions ≥ 6. The key notion was introduced by Guilbault in [33]:

Definition 1. An open manifold M is *pseudo-collarable* if it admits an exhaustion $M = \cup_{j=1}^{\infty} M_j$ by compact manifolds M_j such that $M_j \subset \mathrm{int}(M_{j+1})$ and the inclusion $\partial M_j \hookrightarrow M - \mathrm{int}(M_j)$ is a homotopy equivalence, for every $j \geq 1$.

Note that there exist open contractible n-manifolds which are not pseudo-collarable, for every $n \geq 5$.

1.4. **The Chapman-Siebenmann obstruction τ_∞ for pseudo-collarable manifolds.** The obstruction τ_∞ was defined by Siebenmann in [56] and Chapman and Siebenmann in [19].

Definition 2. Let $\varepsilon(M)$ denote the end of a one ended manifold M and $U(\varepsilon(M))$ be a system of open neighborhoods of infinity, namely having compact complement. The *attenuation* of the Whitehead functor is:

$$\mathrm{Wh}^1(\varepsilon(M)) = \underset{\leftarrow}{\lim}{}^1 \left(\mathrm{Wh}(\pi_1(U))\right)_{U\in U(\varepsilon(M))}$$

where $\lim^1$ denotes the first derived limit and $\mathrm{Wh}(\pi_1(K))$ denotes the Whitehead group of the fundamental group $\pi_1(K)$.

Let M be an open contractible pseudo-collarable manifold. Consider a compact manifold exhaustion $M_i \subset \mathrm{int}(M_{i+1})$ of M, such that the inclusion map $\partial M_i \to M_{i+1} - \mathrm{int}(M_i)$ is a homotopy equivalence. Then the Chapman-Siebenmann obstruction $\tau_\infty(M) \in \mathrm{Wh}^1(\varepsilon(M))$ is the image of the sequence (τ_i), where τ_i is the image of the Whitehead torsion $\tau(M_{i+1} - \mathrm{int}(M_i), \partial M_i) \in \mathrm{Wh}(\pi_1(M_{i+1}))$ into $\mathrm{Wh}(\pi_1(M - \mathrm{int}(M_i)))$. See section 2.1 for details.

1.5. **Collapsibility and arborescence.** Complete CAT(0) metrics are strongly convex, namely there is an unique midpoint associated to any two points of the space. Rolfsen proved in [50, 51] that the only compact manifolds of dimension $n \leq 3$ admitting a strongly convex metric are homeomorphic to the ball. This implies that complete open CAT(0) manifolds of dimension $n \leq 3$ are homeomorphic to $\mathbb{R}^n$. Thus Whitehead 3-manifolds cannot be endowed with complete CAT(0) metrics.

If one looks more generally upon complexes instead of manifolds White (see [67]) proved that a 2-complex admits a strongly convex metric if and only if it is collapsible. As such the result cannot be extended to higher dimensions, as there exist (non rectilinear) triangulations of the 3-cell which are not collapsible (see [8]).

A triangulation of some n-manifold is *PL* if the link of every vertex is PL homeomorphic to a $(n-1)$-sphere. A simply connected manifold M endowed with a PL triangulation for which the associated polyhedral (piecewise flat or hyperbolic) metric is CAT(0) is homeomorphic to $\mathbb{R}^n$, by a classical theorem of Stone ([58]). Thus CAT(0) polyhedral metrics on exotic manifolds are subjacent to non PL triangulations.

Consider two finite simplicial complexes X and Y such that $X = Y \cup e$, where e is a cell whose boundary ∂e intersects Y along the complement of a single facet (face of maximal dimension). We say that X *elementary collapses* on Y, or X is an *elementary expansion* of Y. Moreover, a simplicial collapse or expansion is a sequence of elementary collapses or expansions, respectively.

A triangulable topological manifold will be called *PL collapsible* in the sequel if it admits a triangulation which is collapsible, i.e. it collapses to a point. We emphasize that this triangulation is *not* required to be a PL triangulation.

Let B^n be the closed unit n-dimensional ball and $B^{n-1}_+, B^{n-1}_- \subset \partial B^n$ the upper and lower hemispheres of the boundary sphere. We denote by X^n the n-skeleton of the CW complex X. Collapsibility extends readily to CW-complexes (see [21]), as follows:

Definition 3. Let X and Y be finite CW complexes such that $X = Y \cup e^n \cup e^{n-1}$, where e^n and e^{n-1} are open cells of dimension n and $n-1$ respectively, such that there exists a continuous map $\phi : B^n \to X$ with the following properties:

(1) ϕ is a characteristic map for attaching the cell e^n, namely $\phi|_{\mathrm{int}(B^n)}$ is a homeomorphism of $\mathrm{int}(B^n)$ onto e^n and $\phi(\partial B^n) \subset Y^{n-1}$;

(2) $\phi|_{B_+^{n-1}} : B_+^{n-1} \to X$ is a characteristic map for the cell e^{n-1}, namely $\phi|_{\mathrm{int}(B_+^n)}$ is a homeomorphism of $\mathrm{int}(B_+^n)$ onto e^{n-1} and $\phi(\partial B_+^n) \subset Y^{n-2}$;
(3) $\phi(B_-^{n-1}) \subset Y^{n-1}$.

We then say that X *elementary collapses* on Y, or X is an *elementary expansion* of Y. Moreover, a collapse/expansion of CW complexes is a sequence of elementary collapses/expansions.

The two notions of collapsibility described above agree for polyhedra, when they are endowed with the induced CW complex structures. For instance, if a CW complex is regular, namely all characteristic maps of its cells are embeddings, then it is triangulable, i.e. homeomorphic to a simplicial complex. Note however that there exist finite CW complexes which are not triangulable.

Note that in Definition 3 we can drop the cellularity requirements if the characteristic map is assumed to be PL.

We have a more general related concept introduced in [11], as follows:

Definition 4. Let X and Y be compact Hausdorff topological spaces such that $X = Y \cup e^n \cup e^{n-1}$, where e^n and e^{n-1} are open cells of dimension n and $n-1$ respectively, such that there exists a continuous map $\phi : B^n \to X$ with the following properties:

(1) ϕ is an attaching map for the cell e^n, namely $\phi|_{\mathrm{int}(B^n)}$ is a homeomorphism of $\mathrm{int}(B^n)$ onto e^n;
(2) $\phi|_{B_+^{n-1}} : B_+^{n-1} \to X$ is an attaching map for the cell e^{n-1}, namely $\phi|_{\mathrm{int}(B_+^n)}$ is a homeomorphism of $\mathrm{int}(B_+^n)$ onto e^{n-1};
(3) $\phi(B_-^{n-1}) \subset Y$.

We say that X has a topological *elementary collapses* on Y, or X is a topological *elementary expansion* of Y. Moreover, a topological collapse/expansion of topological spaces is a sequence of topological elementary collapses/expansions.

The attaching maps in the definition of a topological collapse are not necessarily characteristic maps for a CW complex, namely they cannot be made cellular. In particular, even if Y is a CW complex, the space X obtained by expansion is not necessarily a CW complex.

A compact topological manifold will be called *topologically collapsible* in the sequel if it topologically collapses to a point.

Recall that the topological *mapping cylinder* $M(f)$ of a *continuous* map $f : M \to N$ is the topological space given by $M \times [0,1] \cup_{(x,1)\sim f(x)} N$. When M and N have an additional structure which is preserved by f, then $M(f)$ inherits that structure, in general. For instance if M, N are simplicial complexes and f is a simplicial map, then $M(f)$ has a natural simplicial structure and moreover $M(f)$ collapses simplicially onto N. In the same way, if M, N are finite CW complexes and f is a cellular map then $M(f)$ is a CW complex and moreover $M(f)$ collapses onto N.

However, if M, N are finite CW complexes but f is just a continuous map, then a priori we cannot define a natural CW complex structure on $M(f)$. We still have a decomposition into cells of $M(f)$ by putting together the cells of $M \times [0,1]$ and those of N. For every cell e of M we have a topological collapse sending the cell $e \times [0,1]$ of $M \times [0,1]$ onto the image of $e \times \{1\}$ within $M(f)$. Using these topological collapses in reversing order of the cell dimensions provides a topological collapse of $M(f)$ onto the subspace N.

Remark 1. Asume that we have a topological elementary collapse of X on Y. If ϕ is the attaching map from Definition 4 then the mapping cylinder $M(\phi|_{B_-^{n-1}})$ of its restriction $\phi|_{B_-^{n-1}} : B_-^{n-1} \to Y$ is homeomorphic to X. The natural retract of the mapping cylinder $M(\phi|_{B_-^{n-1}})$ onto Y provides a strong deformation retract $X \to Y$.

Remark 2. It is known (see [6]) that there exist polyhedra which are homeomorphic to balls and hence topologically collapsible but which are not PL collapsible. Specifically, if Σ is a PL homology n-sphere with nontrivial fundamental group and M is the complement of a PL ball embedded into the join $S^p * \Sigma$ (or, equivalently the p-th iterated suspension of Σ) whose closure is disjoint from S^p, then M is not PL collapsible, although for $p \geq 1$ the double suspension theorem of Cannon implies that M is homeomorphic to a ball ([6], Main Proposition).

In this paper we will focus on the noncompact case. The right analog of collapsibility in this case is the following notion:

Definition 5. A noncompact Hausdorff space is topologically *arborescent* if it is obtained from a point by an infinite sequence of topological elementary expansions. Moreover, a noncompact CW complex is called PL *arborescent* if it is obtained from a point by an infinite sequence of elementary expansions.

The Euclidean space is the simplest example of a PL arborescent polyhedron.

1.6. **The main results.**

Theorem 1. *An open contractible n-manifold M, $n \geq 5$ which admits a CAT(0) complete polyhedral metric is pseudo-collarable, it has strongly perfectly semistable fundamental group at infinity and vanishing Chapman-Siebenmann obstruction $\tau_\infty(M)$. Moreover, if $n \geq 6$, then M is topologically arborescent.*

Theorem 2. *A locally finite PL arborescent polyhedral complex is PL homeomorphic to a cubical complex which is CAT(0) when endowed with the piecewise flat equilateral metric.*

The fact that a finite cubical complex which is CAT(0) with respect to the equilateral piecewise flat metric is collapsible was established in [1].

As an immediate consequence interiors of compact contractible manifolds are homeomorphic to CAT(0) cubical complexes, when $n \geq 4$. This should be compared with [5], where Ancel and Guilbault proved that there are CAT(-1) metrics on these manifolds.

In essence, pseudo-collarability guarantees an exhaustion by contractible manifolds as well as a sufficiently "nice" structure at infinity (cf. Lemma 7). A simpler notion is the notion of *weakly geometrically contractible manifolds*, which abandons the structure at infinity and describes open contractible manifolds that can be exhausted by compact contractible manifolds. Section 8, which explores different topological notions related to pseudo-collarability, reveals a hierarchy:

$$\text{compactifiable} \subsetneq \text{pseudo-collarable} \subsetneq \text{geometrically contractible} \subsetneq \text{general.}$$

1.7. **Comments and questions.** We formulate here two questions, for which affirmative answers might bridge the gap between the two main theorems.

Question 1. *Consider a compact PL n-manifold W with boundary, $n \geq 5$, which is homeomorphic to the topological mapping cylinder $M(f)$ of an acyclic map $f : M \to N$ between closed PL homology spheres. Suppose that the kernel of the* *map induced by f at fundamental groups level is the normal closure of a finitely generated perfect group. Then, is the pair (W, N) homeomorphic to a polyhedron pair (P, Q) such that P PL collapses onto Q?*

In other words we ask whether any topological collapse between PL manifolds as in the statement can be made PL if we accept to replace the initial triangulations with non PL triangulations. A particular case of this question asks whether a PL manifold which topologically collapses to a point is homeomorphic to a polyhedron which PL collapses. This is known to hold by the construction of an arc spine for any compact contractible manifold M by Ancel and Guilbault in [4]. Specifically, if $n \geq 5$ and M is a compact contractible n-manifold, then there exists a map $f : \partial M \to [0, 1]$ such that M is homeomorphic to the topological mapping cylinder $M(f)$. One constructs first a codimension one homology sphere $\Sigma \subset \partial M$ providing a surjective map at fundamental group level, so that $\partial M - \Sigma \times (0, 1)$ is the disjoint union of two acyclic manifolds A and B. Then the map f sends A into 0, $\Sigma \times \{t\}$ into t and B into 1. In particular we can refine the triangulation of M such that the map f becomes simplicial. Now, we can define the simplicial mapping cylinder $C(f)$ of f, which is a simplicial complex collapsing onto $[0, 1]$ and hence to a point. By a result of Cohen (see [20]) the simplicial mapping cylinder $C(f)$ is homeomorphic to the topological mapping cylinder $M(f)$ and in particular to M.

A weaker version concerns the case of open manifolds and reads as follows:

Question 2. *Consider an open contractible n-manifold W, $n \geq 5$, which is a mapping telescope, namely the union of PL manifolds with disjoint interiors, each one homeomorphic to the topological mapping cylinder of some acyclic map between closed PL homology spheres. Suppose that the kernels of the induced maps at fundamental groups level are normal closures of finitely generated perfect groups. Then is the manifold W homeomorphic to a polyhedron which is PL arborescent?*

In the absence of the requirement that the CAT(0) metric be polyhedral we expect the following related question.

Question 3. *An open contractible n-manifold W, $n \geq 5$, admits a CAT(-1) complete length metric if and only if it is pseudo-collarable, it has perfectly semistable fundamental group at infinity and the Chapman-Siebenmann obstruction τ_∞ vanishes. Moreover this is so if only if W is homeomorphic to a topologically collapsible polyhedron?*

2. Preliminaries

2.1. **Obstructions.** According to [33] the manifold M is *inward tame at infinity*, if for arbitrarily small neighborhoods of infinity U there exist homotopies $H : U \times [0, 1] \to U$ with H_0 being identity and $H_1(U)$ having compact closure. Alternatively, M is inward tame if and only if arbitrarily small neighborhoods of infinity U are *finitely dominated*, namely there exist finite complexes K and maps $u : U \to K$ and $d : K \to U$ such that $d \circ u$ is homotopic to the identity of U.

The projective class group functor $\widetilde{K}_0$ associates to a group π the abelian group $\widetilde{K}_0(\pi)$ of stable isomorphism classes of finitely generated projective left modules over $\mathbb{Z}[\pi]$. Wall proved in [62, 63] that each finitely dominated CW complex X determines a class $\sigma(X) \in \widetilde{K}_0(\pi_1(X))$ which vanishes if and only if X has the homotopy type of a finite complex.

We denote by Wh the Whitehead functor which associates to a group π the abelian group $\widetilde{K}_1(\pi)/\pi = K_1(\pi)/(\pm\pi)$, namely the quotient of $GL(\mathbb{Z}[\pi])$ by the subgroup generated by the elementary matrices, elements of π and $-\mathbf{1}$. Note that the subgroup generated by the elementary matrices coincides with the derived subgroup of $GL(\mathbb{Z}[\pi])$.

It is well-known that for every homotopy equivalence of finite CW complexes $f : K \to L$ there exists an element $\tau(f) \in \mathrm{Wh}(\pi_1(L))$, called the Whitehead torsion of f, which vanishes if and only if f is a simple homotopy equivalence.

The previous obstructions have natural extensions to the case of infinite complexes. Let F be one of the two functors above. If X is a topological space and X_i denote its connected components we set $F(X) = \oplus_i F(\pi_1(X_i))$. Note that base points are irrelevant as F sends inner automorphisms into identity maps. For a complex X one defines the limit of the F functor at the ends $\varepsilon(X)$ of X as the projective limit:

$$F(\varepsilon(X)) = \lim_{\leftarrow}(F(\pi_1(X - C))_{C\subset X, C \text{ compact}}$$

Set also $F^1(\varepsilon(X))$ be the first derived functor of projective limit applied to the inverse system $(F(\pi_1(X - C)))_{C\subset X, C \text{ compact}}$, also called the *attenuation* of F. Recall that the derived limit of an inverse sequence $G_0 \overset{p_1}{\leftarrow} G_1 \overset{p_1}{\leftarrow} G_2 \overset{p_1}{\leftarrow} \cdots$ is the quotient:

$$\lim_{\leftarrow}{}^1(G_i, p_i) = \frac{\prod_{i=0}^{\infty} G_i}{\langle (x_i - p_{i+1}(x_{i+1}))_{i\in\mathbb{Z}_+}, x_i \in G_i\rangle}$$

Note that $F^1(\varepsilon(X))$ vanishes if and only if the inverse system $(F(\pi_1(X - C))_{C\subset X, C \text{ compact}}$ is equivalent to an inverse system with surjective bonding maps.

Let now M be an open manifold which we suppose to be inward tame at infinity. Choose a compact manifold exhaustion $M_i \subset \mathrm{int}(M_{i+1})$ of M. Define $\sigma_\infty(M) \in \widetilde{K}_0(\varepsilon(M))$ to be the class of $(\sigma(M - \mathrm{int}(M_i))_{i\in\mathbb{Z}_+}$. This is a well-defined and independent on the chosen exhaustion (see [19]).

Let τ_i denote the image of the Whitehead torsion $\tau(M_{i+1} - \mathrm{int}(M_i), \partial M_i)$ into $\mathrm{Wh}(\pi_1(M - \mathrm{int}(M_i)))$ by the map induced by the inclusion $M_{i+1} - \mathrm{int}(M_i) \hookrightarrow M - \mathrm{int}(M_i)$.

The Chapman-Siebenmann obstruction $\tau_\infty(M) \in \mathrm{Wh}^1(\varepsilon(M))$ is the image of $(\tau_i)_{i\in\mathbb{Z}_+} \in \prod_{i=1}^{\infty} \mathrm{Wh}(\pi_1(M - \mathrm{int}(M_i))$ in the quotient $\mathrm{Wh}^1(\varepsilon(M))$.

Note that in [56] there is a more general definition of the obstructions σ_∞ and τ_∞ for proper homotopy equivalences of locally finite complexes, while the one in [19] mainly concerns Q-manifolds.

2.2. **Semistability.** Recall that the *inverse limit* of an inverse sequence of groups

$$G_0 \overset{p_1}{\leftarrow} G_1 \overset{p_2}{\leftarrow} G_2 \overset{p_3}{\leftarrow} \cdots$$

is defined as:

$$\lim_{\leftarrow}(G_i, p_i) = \{(x_i)_{i\in\mathbb{Z}_+} \in \prod_{i=0}^{\infty} G_i ; p_{i+1}(x_{i+1}) = x_i,\ i \in \mathbb{Z}_+\}$$

The inverse limit is an invariant of the pro-equivalence class of the inverse system. Here the sequence (G_i, p_i) and (H_i, q_i) are pro-equivalent, if, after passing to subsequences (namely replacing some p_i by a composition of arrows) and reindexing there exist homomorphisms $H_{i+1} \to G_{i+1}$ and $G_{i+1} \to H_i$ producing commutative diagrams:

$$\begin{array}{ccccccc} G_i & & \longleftarrow & & G_{i+1} & & \longleftarrow \\ & \nwarrow & & \swarrow & & \nwarrow & \\ \longleftarrow & & H_i & & \longleftarrow & & H_{i+1} \end{array}$$

An inverse sequence is *stable* if it is pro-equivalent to a constant sequence (H, id). An inverse sequence is *semistable* if it is pro-equivalent to an inverse sequence (H_i, q_i) where all bonding morphisms q_i are surjective.

In [33] the author introduced another meaningful related notion, as follows. An inverse sequence is *perfectly semistable* if it is pro-equivalent to an inverse sequence (H_i, q_i) where H_i are finitely presented, the bonding morphisms q_i are surjective and $\ker q_i$ are perfect. Note that in the case of perfectly semistable sequence each $\ker q_i$ is the normal closure of a finitely generated subgroup (see [33], Lemma 2), but this subgroup might not be perfect.

Further we define an inverse sequence to be *strongly perfectly semistable* if it is pro-equivalent to an inverse sequence (H_i, q_i) where H_i are finitely presented, the bonding morphisms q_i are surjective, $\ker q_i$ are perfect and for all $i \leq j$ the subgroup $\ker(q_i \circ q_{i+1} \circ \cdots \circ q_j) \subset H_i$ is the normal closure in H_i of a finitely generated perfect subgroup.

For a one ended open manifold M we consider the inverse system $\pi_1(\varepsilon(M))$ of fundamental groups of $\pi_1(M - K)$, where K are compact subcomplexes of M. The refined semistability conditions above extend then accordingly to open manifolds by requiring $\pi_1(\varepsilon(M))$ to fulfill them.

The main result of [35] is the following characterization of pseudo-collarable manifolds:

Proposition 1 ([35])**.** *An open manifold M of dimension $n \geq 6$ is pseudo-collarable if and only if it satisfies the following conditions:*

(1) M is inward tame at infinity;
(2) $\pi_1(\varepsilon(M))$ is perfectly semistable;
(3) Wall's finiteness obstruction $\sigma_\infty(M) \in \widetilde{K}_0(\varepsilon(M))$ vanishes.

2.3. **Homology manifolds.** Let G be an abelian group.

Definition 6. Let X be a locally compact topological space X with finite cohomological dimension over G. Then X is a (generalized) *homology G-manifold* with boundary of dimension n if

$$H_i(X, X - \{x\}; G) \cong 0, \text{if } i \neq n$$

and for each $x \in X$ the group $H_n(X, X - \{x\}; G)$ is either isomorphic to G or 0. The *boundary* ∂X of a homology G-manifold with boundary of dimension n consists of the set of points $x \in X$ for which $H_n(X, X - \{x\}; G)$ vanishes.

Homology manifolds are the same as homology $\mathbb{Z}$-manifolds. An example of a homology manifold that is not a topological manifold is the suspension of a homology sphere that is not a sphere.

3. Plus constructions

A classical construction by Quillen gives a way to kill normal subgroups of the fundamental group of a CW complex while keeping the homology unaltered, as follows:

Proposition 2. *Suppose that X is a finite CW complex and $\pi_1(X) \to \pi$ is a surjective homomorphism onto the finitely presented group π perfect kernel. Then there exists a CW complex Y and a continuous map $f : X \to Y$, unique up to homotopy equivalence, which induces the given epimorphism $\pi_1(C) \to \pi$ at the level of fundamental groups and is a $\mathbb{Z}[\pi]$-homology equivalence.*

The space Y is said to be obtained by the plus construction out of X and the given epimorphism; sometimes it will be denoted by X^+.

The plus construction could also be performed in other categories, e.g. for topological manifolds. To this purpose we need the following:

Definition 7. The compact cobordism (W, M, N) of topological manifolds is a plus cobordism if

(1) the inclusion $N \hookrightarrow W$ is a simple homotopy equivalence;
(2) the map $\pi_1(M) \to \pi_1(W)$ induced by inclusion is surjective;
(3) the inclusion $N \hookrightarrow W$ is a $\mathbb{Z}[\pi_1(W)]$-homology equivalence.

Following Hausmann ([38], section 3), we have:

Proposition 3. *Given a closed topological manifold M of dimension $n \geq 5$ and a surjective homomorphism $\pi_1(M) \to \pi$ onto a finitely presented group with perfect kernel, then there exists an unique plus cobordism (W, M, N) such that the map induced by the inclusion $M \to W$ is $\pi_1(M) \to \pi$, up to homeomorphism relative to M.*

Moreover N has the homotopy type of the Quillen plus construction M^+. If M had a PL or smooth structure, then W is unique up to PL homeomorphism and diffeomorphism rel M, respectively.

This construction can be realized also by embedded codimension zero cobordisms in a given manifold with boundary (see [36]) and all these constructions could be performed by asking $N \hookrightarrow W$ be a homotopy equivalence with a prescribed torsion in $\mathrm{Wh}(\pi_1(W))$ (see [59]).

We can relax the plus cobordism definition following Guilbault (see e.g. [33]), to a notion essential for pseudo-collarable manifolds:

Definition 8. The compact cobordism (W, M, N) of topological manifolds is a one sided h-cobordism if the inclusion $N \hookrightarrow W$ is a homotopy equivalence.

It is a well-known consequence of a result of Hausmann (see [39], Lemma 2.0, [23], Lemma 2.5., [33], Lemma 6) that the map $\pi_1(M) \to \pi_1(W)$ induced by inclusion is surjective with perfect kernel and $N \hookrightarrow W$ is a $\mathbb{Z}$-homology equivalence, i.e. $H_*(W, M) = 0$.

4. Tameness of metric spheres

4.1. **Subanalytic geometry.** The theory of subanalytic sets originates in the work of Lojasiewicz [46] and was elaborated by Gabrielov [31], Hironaka [40, 41], Hardt [37] and Shiota [54]. We recall here some key ingredients of subanalytic geometry following [7, 41, 46, 54] and refer to these for details. Let V be an analytic manifold, most often the Euclidean space. A set $X \subseteq V$ is a subanalytic subset of V if any point of V has some neighborhood U and finitely many proper real analytic maps f_i, g_i defined on real analytic manifolds with values in U such that

$$X \cap U = \bigcup_{i=1}^{p} \mathrm{Im}(f_i) - \mathrm{Im}(g_i).$$

Semianalytic sets, i.e. subsets of analytic manifolds locally defined by a finite set of analytic equalities or inequalities are subanalytic. A set X is subanalytic if it is locally the projection of a semianalytic set. Locally finite unions and products, intersections and set differences of subanalytic sets are subanalytic sets. The closure, interior and the frontier of a subanalyic set are subanalytic. The set of components of a subanalytic set is locally finite and each connected component is still subanalytic. These properties allow us to define subanalytic subsets of subanalytic sets, which are themselves subanalytic. A map is said subanalytic if its graph is subanalytic in the product. Inverse images of subanalytic sets by subanalytic maps and images of subanalytic sets by proper analytic maps are still subanalytic. A result of Hironaka actually tells us that any closed subanalytic set is the image of an analytic manifold by some proper analytic map.

We will need in the sequel the following result of Tamm ([60], Prop. 1.3.9):

Proposition 4. *Let $\check{\varphi}_A(f)(x) = \inf\{f(u); u \in \varphi^{-1}(x) \cap A\}$, where $f : M \to \mathbb{R}$, $\varphi : N \to M$ is a subanalytic function, A a subanalytic set and $\varphi : \overline{A} \to M$ is proper. Then for any subanalytic function $f : M \to \mathbb{R}$, the function $\check{\varphi}_A(f)$ is subanalytic.*

The subanaliticity was defined above for subsets X of some given analytic manifold V. We now say that $X \subset V$ is *locally subanalytic* if it has some open neighborhood $V' \subseteq V$ such that X is subanalytic in V'. Compact locally subanalytic sets are subanalytic but noncompact locally subanalytic sets are not analytic if they are not closed. All results above about subanalytic sets have immediate reformulations in the local subanalytic case.

4.2. **Locally subanalytic sets and polyhedra.** The main property of subanalytic sets needed in this paper is their triangulability. There are several far reaching generalizations including the simultaneous subanalytic triangulability of locally finite collections of subanalytic sets see [7, 37, 41, 46]. The simplest version states that a closed subanalytic set is homeomorphic to a polyhedron and in particular it is a CW complex and hence an ANR.

A basic example of a locally subanalytic set of $\mathbb{R}^n$ is a PL embedded polyedron. We can also provide a subanalytic structure on every finite dimensional polyhedron by considering a proper PL embedding into an Euclidean space of large dimension. In this respect PL maps between polyhedra are locally subanalytic maps. In the reverse direction, note that there exist analytic maps between analytic manifolds which are not piecewise linearizable, i.e. PL with respect to some triangulations up to homeomorphisms at the source and target.

We record here for further use the following version of the triangulation result alluded above, following Hardt (see [37]), Thm. 2) along with Shiota and Yokoi (see [53], Cor.4.3):

Proposition 5. *For every locally closed and locally subanalytic subset X of $\mathbb{R}^n$ there exists some locally subanalytic isotopy of $\mathbb{R}^n$ which transforms X into a polyhedron in $\mathbb{R}^n$. Moreover, this polyhedron is unique up to PL homeomorphism.*

4.3. **Metric spheres are subanalytic.** Let us recall after [9] some properties of the metric on a M_κ-polyhedral complex K, abridged metric polyhedron. Such a complex is the result of successive gluings of convex cells by means of isometries of their faces, where each cell is the convex hull of a finite set of points in the hyperbolic or Euclidean space of curvature $\kappa \leq 0$. By passing to a subdivision we can assume that the complex is simplicial.

There are two natural pseudometrics that can be defined. Each cell inherits a natural metric and K is the quotient of the disjoint union of all cells by the equivalence relation induced by gluing. The *quotient pseudometric* is the infimal total length of paths in the union of cells which project onto a path in the quotient. A particularly convenient collection of paths is defined as follows. An *m-string* in K joining two points a and b is a sequence of $m+1$ points starting with a and ending with b such that two consecutive points belong to some simplex. The *intrinisc pseudometric* between two points is the shortest length of strings joining them, where the distance between consecutive points is the hyperbolic/Euclidean distance within the corresponding simplex. Since the cells are convex in M the pseudometric above is a metric and coincides with the quotient metric on M.

An m-string is called *taut* if there is no simplex containing three consecutive points and the concatenation of two consecutive segments is a gedesic segment in the union of the two simplices that contain them.

Under the assumption that there are only finitely many isometry classes of cells in K one proved in [9] that every metric polyhedron K is a complete geodesic length space. Moreover, geodesic paths are given by taut strings and the number of points of a taut string is commensurable with the distance between the two points. This implies that for each $x, y \in M$ there is some $m = m(x, y)$ such that the distance $d(x, y)$ is the minimal length of all taut m-strings joining them.

Proposition 6. *If M is a metric polyhedron with finitely any isometry classes of cells, then the distance function is subanalytic.*

Proof. By ([9], Thm. I.7.19) M is a complete geodesic space. A taut m string determines then a sequence of m simplices each simplex containing two consecutive points. Let us consider all taut m-strings x_i joining a and b, with $x_0 = a, x_m = b$ associated with a fixed sequence of simplices. Each intermediary point of the string should belong to some simplex contained in the boundary of some simplex from the fixed sequence. Fixing these boundary simplices A_i too, the corresponding taut strings have to minimize the function $f(a, b, (x_i)) = \sum_{i=0}^{m-1} d(x_i, x_{i+1})$, over all configurations $x_i \in A_i$, $1 \leq i \leq m - 1$.

Note that this function $f : M \times M \times \prod_{i=1}^{m-1} A_i$ is subanalytic, since the distance function on a geodesic simplex of constant curvature is analytic. Consider the projection map $\varphi : M \times M \times \prod_{i=1}^{m} A_i \to M \times M$, which is obviously subanalytic. Then Proposition 4 implies that the shortest length of taut strings with given constraints $x_i \in A_i$, $1 \leq i \leq m - 1$ is a subanalytic function.

By ([9], Cor.I.7.30) there is a constant α such that every taut m string on M has length at least $\alpha m - 1$. Therefore, if a and b belong to two relatively compact open subsets of M there are only finitely many simplex sequences which can contain a geodesic between a and b. Since there are only finitely many configurations of simplices possible and the infimum of a finite set of subanalytic functions is still subanalytic it follows that the distance function is subanalytic. □

4.4. **Geodesic contraction.** The *geodesic contraction* is the map $c_r : M - \mathrm{int}(B(p, r)) \to \partial B(p, r)$ sending a point $q \in M - \mathrm{int}(B(p, r))$ into the point $c_r(q) \in \partial B(p, r)$ lying on the geodesic segment joining p and q which is at distance r from p. Its restriction to a metric sphere provides the geodesic contraction map $c_{R,r} : \partial B(p, R) \to \partial B(p, r)$, for any $r < R$. The goal of this section is to prove:

Proposition 7. *Let M be a metric polyhedron with finitely many isometry classes of cells whose polyhedral metric is CAT(0). Then the fibers of the geodesic contraction $c_{R,r} : \partial B(p, R) \to \partial B(p, r)$ are acyclic ANR.*

Before to proceed with the proof we need the following key lemma:

Lemma 1. *Let M be a metric polyhedron with finitely many isometry classes of cells whose polyhedral metric is CAT(0). Then the geodesic retraction map $\partial B(p, R) \to \partial B(p, r)$ is subanalytic.*

Proof. The retraction map is well-defined and continuous since the space is CAT(0). Since the distance is subanalytic, the metric balls $B(p, R)$ and metric spheres $\partial B(p, R)$ are subanalytic subsets and hence the product $\partial B(p, R) \times \partial B(p, r) \subset M \times M$ is subanalytic. The graph of the geodesic retraction map is identfied with:

$$\{(x, y) \in \partial B(p, R) \times \partial B(p, r); d(x, y) = R - r\}$$

By Proposition 6 the distance function on $M \times M$ is subanalytic and its restriction to an analytic subset is still subanalytic. It follows that the graph of the geodesic retraction is subanalytic, as claimed. □

Proof of Proposition 7. The metric spheres in M are subanalytic subsets of the polyhedron M and hence within some Euclidean space where M has a proper PL embedding. The inverse image of one point of a subanalytic set by a subanalytic map is a subanalytic set of the Euclidean space. By Proposition 5, this inverse image is homeomorphic to a polyhedron and hence an ANR, as claimed.

Moreover, the map $c_{R,r}$ has acyclic point inverses (see [26], Proof of Thm. 3d.1 and [61], Cor. 2.10) and hence it is a degree one map between homology manifolds with the homology of a sphere, for all $r < R$. This implies that point inverses of the geodesic contraction $c_{R,r}$ are acyclic ANR. □

5. Necessary conditions for CAT(0) metrics

The aim of this section is to prove the first part of Theorem 1, which we restate here for completeness:

Theorem 3. *An open topological n-manifold M with $n \geq 5$ carrying a complete polyhedral CAT(0) metric satisfies the following conditions:*

1. *M is pseudo-collarable;*
2. *The Chapman-Siebenmann obstruction $\tau_\infty(M) \in \mathrm{Wh}^1(\varepsilon(M))$ vanishes;*
3. *$\pi_1(\varepsilon(M))$ is strongly perfectly semistable.*

Note that the result is true also when $n \leq 4$, because polyhedral homology manifolds of dimension at most 3 are actually manifolds.

5.1. **CAT(0) metrics necessitate pseudo-collarability.** We first argue that the condition of pseudo-collarability is necessary.

Proposition 8. *An open topological n-manifold M, with $n \geq 5$, which admits a polyhedral CAT(0) metric is pseudo-collarable.*

This result is known for $n \geq 6$: it follows from Remark 5 of [33] and the main Theorem in [35]. Here is an alternative proof, covering the case $n = 5$, as well.

Proof. The closed metric ball centered at $p \in M$ of radius r is denoted by $B(p, r)$. Its interior $\mathrm{int}(B(p, r))$ is the open metric ball of radius r.

Now, Alexandrov proved ([10], Prop. 8.2-8.3 and [61], Prop. 2.) that any geodesic space endowed with a CAT(0) metric which is a homology manifold also has the geodesic extension property, namely every geodesic segment can be extended to a bi-infinite geodesic (see also [9], ch. II.5, Prop. 5.12). This property is also called geodesic completeness. Therefore the metric sphere of radius r coincides with the frontier $\partial B(p, r)$, namely with the set of points $q \in B(p, r)$ such that $\mathrm{int}(B(q, \varepsilon)) \cap (M \setminus B(p, r)) \neq \emptyset$, for every $\varepsilon > 0$.

Lemma 2. *The metric sphere $\partial B(p, r)$ is homotopy equivalent to $\overline{M \setminus B(p, r)}$.*

Proof. Indeed $\partial B(p, r)$ is a strong deformation retract of $\overline{M \setminus B(p, r)}$ (see [9], ch. II.2, Prop. 2.4.(4)). □

We shall see next that there exists a manifold approximation of $\partial B(p,r)$ sharing the same property.

Lemma 3. *When the CAT(0) metric is polyhedral, the metric sphere $\partial B(p,r)$ is an ANR homology $(n-1)$-manifold having the homology of a $(n-1)$-sphere.*

Proof. According to ([61], Prop. 2.7) $B(p,r)$ is a homology n-manifold with boundary, whose boundary $\partial B(p,r)$ as homology manifold coincides with the metric sphere sphere of radius r and hence with its frontier. According to Mitchell theorem from [48] the metric sphere $\partial B(p,r)$ is a homology $(n-1)$-manifold. Since $B(p,r)$ has the homology of a n-ball, its boundary $\partial B(p,r)$ has the homology of a $(n-1)$-sphere.

We already noted in the proof of Proposition 7 that the metric sphere $\partial B(p,r)$ is homeomorphic to a polyhedron and hence an ANR, as a consequence of Propositions 5 and 6, when there are only finitely many isometry classes of cells.

Let us give below a proof which does not require finitely many ismetry classes of cells. The metric ball $B(p,r)$ is convex, namely the geodesic segment determined by two points of it is contained in $B(p,r)$. Note that every closed cell e of a polyhedral $CAT(k)$ simplicial complex is convex and its metric coincides with the restriction of the intrinsic metric. Therefore the intersection $B(p,r)\cap e$ with a closed cell e of Δ is a convex subset of e. A compact convex set in $\mathbb{R}^n$ or $\mathbb{H}^n$ is homeomorphic to a ball D^m, of dimension $m\leq n$. It follows that $B(p,r)\cap e$ is homeomorphic to a ball. There is no loss of generality in assuming that $B(p,r)$ intersects nontrivially the interior of e in a set of maximal dimension.

Consider a polyhedral cell f which is contained in $B(p,r)\cap e$. For the sake of simplicity we can suppose that the curvature is $\kappa=0$ and f is obtained from e by a homotethy with center p, where p is a point in the interior of f. The radial projection from the point p induces a homeomorphism $\phi:\partial(B(p,r)\cap e)\to\partial f$. We denote by $F=\phi(\partial B(p,r)\cap e)$ and $J=\phi(B(p,r)\cap\partial e)$. It follows that ∂f is the union $A\cup B$. Now, J is the union of $J_i=\phi(B(p,r)\cap\partial e_i)$, where e_i denote the facets of e. Note that J_i are images of convex subsets of the facets e_i and hence they are convex subsets within the facets f_i of f.

We will show that there exists a straightening isotopy of ∂f sending J into a polyhedron. To this purpose we prove by induction on m that there is a straightening isotopy of ∂f such that the intersections of the image of J with every m-cell in ∂f is a polyhedron. When $m=1$, the convex subsets of segments are also segments and hence the claim automatically holds. To prove the induction step we observe that given a maximal dimension convex subset of a simplex which intersects the simplex boundary along polyhedra can be isotoped rel the boundary to the convex hull of the union of these polyhedra and possibly some additional points for dimensions reasons. Then the corresponding straigthening isotopies can be extended from the m-skeleton of some $(m+1)$-cell to the $(m+1)$-cell, by the Alexander trick.

It follows that the closed complement F of J within ∂f is homeomorphic to a polyhedron embedded into ∂f. In particular $\partial B(p,r)\cap e$ is a CW complex and hence an ANR. As $B(p,r)$ intersects nontrivially only finitely many cells, $\partial B(p,r)$ is homeomorphic to a finite union of polyhedra which pairwise intersect along subpolyhedra and hence a polyhedron itself. This implies that $\partial B(p,r)$ is a CW complex and hence an ANR. □

Next we will extend the result of Ferry (see [28, 29]) to an approximation theorem of resolvable generalized homology manifolds in codimension one. We have first:

Lemma 4. *For generic radii r the ANR homology manifold $\partial B(p,r)$ admits a resolution.*

Proof. Quinn's resolution theorem ([64]) states that there exists a locally defined obstruction invariant in $1 + 8\mathbb{Z}$ which detects precisely when a generalized homology sphere has a resolution. As above, the intersection $B(p,r) \cap e$ with a closed cell e of Δ is convex. Furthermore, since the metric is piecewise smooth, then for generic r the frontier of $B(p,r) \cap e$ is a convex hypersurface. The later contains an open dense set which is a piecewise linear submanifold. Therefore $\partial B(p,r)$ contains manifold points. In particular Quinn's obstruction is trivial. □

We now want to prove that there exist arbitrarily close approximations of the generalized homology manifold $\partial B(p,r)$, for generic r, by locally flat topological submanifolds of M. Namely, there exists by Lemma 4 a closed $(n-1)$-manifold S endowed with a surjective cell-like map $g : S \to \partial B(p,r)$.

Lemma 5. *For generic r and for every $\varepsilon > 0$ there exists a topologically flat embedding $h_\varepsilon : S \to M$, such that $h_\varepsilon(S)$ is ε-close to $\partial B(p,r)$. Moreover, there exists an ambient homotopy $H : M \times [0,1] \to M$, with the property that $H_1 = \mathrm{id}$, H_t is a homeomorphism for any $t > 0$, $H_\varepsilon(h_1(S)) = h_\varepsilon(S)$ and $H_0(h_1(S)) = \partial B(p,r)$.*

Proof. Observe that $\partial B(p,r)$ is separating M. Consider the ENR $M' = B(p,r) \cup \partial B(p,r) \times [0,1] \cup M - \mathrm{int}(B(p,r))$, which is a generalized homology manifold. There is a proper cell-like map $q : M' \to M$ which collapses $\partial B(p,r) \times [0,1]$ to $\partial B(p,r)$. Further M' admits a resolution, as it contains manifolds points; namely there exists a proper cell-like map $p : P \to M'$ from a manifold P. Let $f : P \to M$ be the composition $q \circ p$. Then f is a proper cell-like map. By a classical result of Siebenmann (see [55], Approximation Thm. A) f is a limit homeomorphism. This means that there exists a level preserving cell-like map $F : P \times [0,1] \to M \times [0,1]$ such that $F(x,t) = (f_t(x), t)$, where $f_t : P \to M$ for $0 \leq t < 1$ are homeomorphisms ε-close to f and $f_1 = f$.

As $\partial B(p,r)$ is a codimension one compact polyhedron, the combinatorics of the intersections with a triangulation subjacent to the polyhedral complex Δ, and hence its homeomorphism type, will not change in a small neighborhood of the generic r. It follows that $\partial B(p,r) \times (-\varepsilon, \varepsilon)$ is embedded in M and hence it is a manifold. The product map $g \times \mathrm{id} : S \times (\varepsilon, \varepsilon) \to \partial B(p,r) \times (\varepsilon, \varepsilon)$ is proper and cell-like. Since both polyhedra are topological manifolds, $g \times \mathrm{id}$ is a limit homeomorphism, by the result of Siebenmann cited above. By the same argument map $p|_{p^{-1}(\partial B(p,r) \times (\varepsilon,\varepsilon))}$ is also a limit homeomorphism. Therefore there exists a codimension zero embedding $g_\varepsilon : S \times (\varepsilon, \varepsilon) \to P$.

It follows that $f_{1-\varepsilon} \circ g_\varepsilon(S \times \{0\})$ is a locally flat approximation of $\partial B(p,r)$ (see also ([29], Thm.1). We put $h_\varepsilon = f_{1-\varepsilon} \circ g_\varepsilon$. The ambient homotopy H is constructed from F, by identifying P and M by means of F_0.

An alternative proof using an argument provided to us by the referee is as follows. The uniqueness part of Quinn's resolution theorem ([64]) implies that $C = p^{-1}(\partial B(p,r) \times [0,1])$ is homeomorphic to $S \times [0,1]$ by a homeomorphism which identifies $p|_C$ to $g \times \mathrm{id}_{[0,1]}$. Thus an approximation of $q \circ p : P \to M$ by a homeomorphism $h : P \to M$ will take a bicollared copy of S arbitrarily closed to $\partial B(p,r)$. □

Let V be the closure of the unbounded component of $M \setminus h_1(S)$. It follows that for all $j \geq 1$ we have:

$$\pi_j(V, h_1(S)) \cong \pi_j(H_t(V), H_t(h_1(S))), \text{ for all } t > 0.$$

As $h_1(S)$ has codimension one and H_1 is a hereditary homotopy equivalence we can pass to the limit $t \to 0$ to obtain:

$$\pi_j(V, h_1(P)) \cong \pi_j(\overline{M \setminus B(p,r)}, \partial B(p,r)) = 0.$$

This shows that V is a manifold pseudo-collar, as claimed.

□

5.2. **CAT(0) metrics need trivial Chapman-Siebenmann obstruction.** Any locally compact ANR, in particular a manifold M with a CAT(0) metric has a $\mathcal{Z}$-compactification (see [3], Ex.6, Rk.1). It follows that the product $M \times Q$ with the Hilbert cube Q has a $\mathcal{Z}$-compactification. The main theorem of [19] shows that $\tau_\infty(M \times Q) = 0$. On the other hand τ_∞ is a proper homotopy invariant and hence $\tau_\infty(M) = 0$.

5.3. **Strong perfect semistability.** By Proposition 7 the fibers of the geodesic contraction $c_{R,r} : \partial B(p,R) \to \partial B(p,r)$ are acyclic ANR, when the CAT(0) metric is polyhedral.

We shall now use the following extension of a result from ([23], Prop.4.8):

Lemma 6. *Suppose that $f : N_1 \to N_2$ is an acyclic map between $(n-1)$-dimensional homology manifolds without boundary such that $f^{-1}(y)$ is an ANR for every $y \in N_2$. Assume moreover that N_1 is an ANR. Then the kernel $\ker f_*$ of the map $f_* : \pi_1(N_1) \to \pi_1(N_2)$ induced by f at the level of fundamental groups is the normal closure of a finitely generated perfect group.*

Proof. The proof is similar to that presented in [23] for topological manifolds but we include it here for the sake of completeness. For every $y \in N_2$ the preimage $f^{-1}(y)$ being an ANR by our assumptions admits an open neighborhood $U \subset N_1$ which deformation retracts onto $f^{-1}(y)$. As f is surjective, since acyclic, the collection of open sets U is a covering of N_1. As N_1 is compact we can extract a finite cover $\{U_i\}$ corresponding to the points $y_i \in N_1$. We can join a base point $z_0 \in N_1$ with $f^{-1}(y_i)$ by means of pairwise disjoint arcs which only intersect $\cup_i f^{-1}(y_i)$ at their end points. Denote by Y the union of $\cup_i f^{-1}(y_i)$ with these arcs based at z_0.

Then Y is a compact acyclic ANR. By a theorem of West ([66]) any compact metrizable ANR is homotopy equivalent to a finite cell-complex and hence each $\pi_1(f^{-1}(y_i)$ is finitely presented and hence $\pi(Y)$ is finitely presented and perfect.

Moreover, $\ker f_*$ is normally generated by the image of $\pi_1(Y)$ under the map $\pi_1(Y) \to \pi_1(N_1)$ induced by the inclusion $Y \hookrightarrow N_1$. Thus $\ker f_*$ is the normal closure of a finitely generated perfect group. □

We obtained so far a compact exhaustion of M by homology manifolds for which the fundamental pro-group at infinity is given by a sequence of surjective homomorphisms, each bonding map having its kernel normally generated by a finitely generated perfect subgroup.

The homology manifolds arising as boundaries admit arbitrarily close approximations by locally flat topological submanifolds of M, by Lemma 5. Using notation from this lemma, we know that for generic pairs $r_1 > r_2$ there exist topologically flat embeddings $h_\varepsilon : S_i \to M$ such that $h_\varepsilon(S_i)$ are ε-close to $\partial B(p,r_i)$ and an ambient homotopy $H : M \times [0,1] \to M$ such that $H_1 = id$, H_ε is a homeomorphism for every $\varepsilon > 0$ and $H_\varepsilon(h_1(S_i)) = h_\varepsilon(S_i)$, while $H_0((h_1(S_i)) = \partial B(p,r_i)$. Let Z be the manifold bounded by $h_1(S_1) \sqcup h_1(S_2)$ and V the closure of the unbounded component of $M - h_1(S_2)$. Recall from the last lines of the proof of Proposition 8 that V is a pseudo-collar.

As H_1 is a hereditary homotopy equivalence we have an identification between the sequence of maps:

$$\pi_1(\partial B(p,r_1)) \to \pi_1(B(p,r_1) - \mathrm{int}(B(p,r_2))) \to \pi_1(M - \mathrm{int}(B(p,r_2))) \to \pi_1(\partial B(p,r_2))$$

and

$$\pi_1(h_1(S_1)) \to \pi_1(Z) \to \pi_1(V) \to \pi_1(h_1(S_2))$$

Therefore the map $\pi_1(h_1(S_1)) \to \pi_1(h_1(S_2))$ is surjective and its kernel is normally generated by a finitely generated perfect group. Thus we can replace replace metric spheres by their topologically flat approximation, while keeping the same fundamental pro-group at infinity for the associated exhaustions.

This implies that M is strongly perfectly semistable.

6. Exhaustions of pseudo-collars by plus cobordisms

6.1. **Exhaustions of pseudo-collars with trivial τ_∞.** We start by recalling the following result of Guilbault:

Proposition 9 ([33]). *An open manifold is pseudo-collarable if and only if it is union of one-sided h-cobordisms with disjoint interiors.*

This section aims at refining this characterization as follows:

Proposition 10. *An open manifold is pseudo-collarable and its Chapman-Siebenmann obstruction $\tau_\infty(M) \in \mathrm{Wh}^1(\varepsilon(M))$ vanishes if and only if it is the union of one-sided h-cobordisms with trivial torsion and disjoint interiors.*

The if part is trivial, as unions of one-sided cobordisms with vanishing torsion have trivial Chapman-Siebenmann obstruction.

For the converse we first record, following [33, 35]:

Lemma 7. *If the open contractible manifold M is pseudo-collarable then there exists an exhaustive filtration M_i, $i \geq 0$ of M with the following properties:*

(1) M_i are compact contractible manifolds;
(2) the inclusion maps $\overline{M \setminus M_j} \hookrightarrow \overline{M \setminus M_i}$ for $j > i$ induce surjections at the level of fundamental groups, and
(3) the inclusions $\partial M_i \hookrightarrow \overline{M \setminus M_i}$ induce isomorphisms at the level of fundamental groups.

Recall now from [33] that any pseudo-collar W can be written as the union of 1-sided h-cobordisms W_i with disjoint interiors. This means that W_i is a cobordism with left boundary J_i and right boundary J_{i+1}, so that $J_1 = \partial W$, with the property that $J_i \subset W_i$ is a homotopy equivalence. The 1-sided h-cobordism W_i is said to be a *plus cobordism* (see [49, 52]) if the inclusion $J_i \subset W_i$ is a simple homotopy equivalence, namely the torsion $\tau(W_i, J_i)$ vanishes in the Whitehead group $Wh(\pi_1(J_i))$. One key property needed in the construction below is the following:

Lemma 8. *A pseudo-collar manifold W is the union of plus cobordisms with disjoint interiors if (and only if) $\tau_\infty(W) = 0$.*

Proof. Our proof follows closely the one given in [19] for Q-manifolds. We start with:

Lemma 9. *Let N be a closed $(n-1)$-manifold, $n \neq 4$ and $\mu \in \mathrm{Wh}(\pi_1(N))$. Then there is a decomposition of $N \times [0,1] = Z_1 \cup Z_2$ into two h-cobordisms Z_1 and Z_2 with disjoint interiors such that*

$$\tau(Z_1, N \times \{0\}) = \mu, \ \tau(Z_2, Z_1 \cap Z_2) = -\mu$$

Proof. There exist h-cobordisms Z_1 and Z_2 with prescribed torsions. Their composition then has trivial torsion:

$$\tau(Z_1 \cup Z_2, N \times \{0\}) = \tau(Z_1, N \times \{0\}) + \tau(Z_2, Z_1 \cap Z_2) = 0$$

By the topological s-cobordism theorem $Z_1 \cup Z_2$ is homeomorphic to $N \times [0,1]$. □

Assume now that we have a filtration M_i of the pseudo-collar M with the property that $M_{i+1} - \mathrm{int}(M_i)$ are one sided h-cobordisms (see e.g. [33], Prop.2). Let $\tau_i \in \mathrm{Wh}(M - \mathrm{int}(M_i))$ denote the image of $\tau(M_{i+1} - \mathrm{int}(M_i), \partial M_i) \in \mathrm{Wh}(\pi_1(M_{i+1} - \mathrm{int}(M_i))$ in the group $\mathrm{Wh}(M - \mathrm{int}(M_i))$, by means of the inclusion induced homomorphism.

By hypothesis $\tau_\infty(M) = 0$ and hence there exist $(\mu_1, \mu_2, \ldots) \in \prod_{i=1}^{\infty} \mathrm{Wh}(M - \mathrm{int}(M_i))$, such that for every i

$$\mu_i - p_i(\mu_{i+1}) = \tau_i$$

where $p_i : \mathrm{Wh}(M - \mathrm{int}(M_{i+1})) \to \mathrm{Wh}(M - \mathrm{int}(M_i))$ is the induced homomorphism.

The only reason to consider this obstruction is the fact that although the group maps are surjective the corresponding maps between the Whitehead groups is not necessarily surjective.

Recall that $\partial M_i \hookrightarrow M - \mathrm{int}(M_i)$ is a homotopy equivalence. Let then $\mu_i' \in \mathrm{Wh}(\partial M_i)$ be a class whose image by the inclusion induced homomorphism is $\mu_i \in \mathrm{Wh}(M - \mathrm{int}(M_i))$. The previous lemma gives us a decomposition of a collar $\partial M_i \times [0,1] \subset M_{i+1} - \mathrm{int}(M_i)$ as the union of two h-cobordisms $Z_i^1 \cup Z_i^2$ with disjoint interiors, such that

$$\tau(Z_i^1, \partial M_i) = \tau_i', \ \tau(Z_i^2, Z_i^1 \cap Z_i^2) = -\tau_i'$$

We set now $M_i' = M_i \cup Z_i^1$. Then $\partial M_i' \hookrightarrow M_{i+1}' - \mathrm{int}(M_i')$ is a homotopy equivalence.

By the formula of the torsion of a composition we have:

$$\begin{aligned} \tau(M_{i+1}' - \mathrm{int}(M_i'), \partial M_i') &= \tau(Z_i^2, \partial M_i) + \tau(M_{i+1} - \mathrm{int}(M_i), \partial M_i) + j_*\tau(Z_{i+1}^1, \partial M_{i+1}) \\ &= -\mu_i + \tau_i + p_{i+1}(\mu_{i+1}) = 0 \end{aligned}$$

where j_* is the map induced from inclusion $Z_{i+1}^1 \hookrightarrow M - \mathrm{int}(M_i)$. It follows that M is the union of one-sided h-cobordisms with trivial torsion. □

6.2. **Exhaustions by plus cobordisms.** The aim of this section is to provide the following key ingredient in the proof of Theorem 1:

Proposition 11. *An open contractible n-manifold W, $n \geq 6$, which is pseudo-collarable, has strongly perfectly semistable fundamental group at infinity and has a vanishing Chapman-Siebenmann obstruction τ_∞ is the union of plus cobordisms with disjoint interiors. Moreover, the plus cobordisms are homeomorphic to mapping cylinders of acyclic maps between the boundaries.*

Remark 3. Chapman and Siebenmann considered in [19] the notion of *mapping telescope* of an inverse sequence of compact metric spaces $f_i : X_{i+1} \to X_i$, by sewing together the mapping cylinders $M(f_i)$ along their naturally identified boundaries. Proposition 11 states that W^n is a mapping telescope of an inverse sequence of acyclic maps with X_0 being a point and X_i closed $(n-1)$-manifolds.

Lemma 10. *Assume that the open manifold M is strongly perfectly semistable at infinity and union of plus cobordisms with disjoint interiors. Then there exists a compact exhaustion M_i such that $M_{i+1} - \mathrm{int}(M_i)$ are one sided h-cobordisms with trivial torsion and moreover each $\ker(\pi_1(\partial M_{i+1}) \to \pi_1(\partial M_i))$ is the normal closure in $\pi_1(\partial M_{i+1})$ of some finitely generated perfect subgroup.*

Proof. We start with a compact exhaustion M_i such that $\partial M_i \hookrightarrow M_{i+1} - \mathrm{int}(M_i)$ are homotopy equivalences. We can change the exhaustion by passing to subsequences and relabelling such that the the kernel $\ker(\pi_1(\partial M_{i+1}) \to$

$\pi_1(\partial M_i))$ is the normal closure in $\pi_1(\partial M_{i+1})$ of some finitely generated perfect subgroup, while keeping the property that $\partial M_i \hookrightarrow M_{i+1} - \mathrm{int}(M_i)$ are homotopy equivalences. We are then in the situation of lemma 8. We alter the decomposition into one sided h-cobordisms by adjoining h-cobordisms. However these changes preserve the fundamental groups involved and hence the new decomposition fulfills all required conditions.

By hypothesis there exists a compact exhaustion M_i of M such that the kernel $\ker(\pi_1(\partial M_{i+1}) \to \pi_1(\partial M_i))$ is the normal closure in $\pi_1(\partial M_{i+1})$ of some finitely generated perfect subgroup. We can change the exhaustion by passing to subsequences and relabelling such that the fundamental inverse sequence

$$\cdots \leftarrow \pi_1(M - \mathrm{int}(M_i)) \leftarrow \pi_1(M - \mathrm{int}(M_{i+1})) \leftarrow \pi_1(M - \mathrm{int}(M_{i+1})) \leftarrow \cdots$$

has surjective bonding maps and consists of finitely generated groups with perfect kernels which are normal closures of finitely generated perfect subgroups. Then ([33], Lemma 8) provides a new compact exhaustion whose associated inverse sequence matches a subsequence of the inverse sequence above and moreover has the additional property that inclusion maps induce isomorphisms $\pi_1(\partial M_i) \to \pi_1(M - \mathrm{int}(M_i))$. By the generalized $(n-3)$-neighborhoods theorem ([33], Thm. 5 and Lemma 10) we can find another compact exhaustion M_i such that the inverse sequence does not change while $\pi_k(M - \mathrm{int}(M_i), \partial M_i) = 0$, for $k = 1, 2, \ldots, n-3$. Eventually the proof of [35], Thm.1) shows that we can alter the exhaustion whitout changing the fundamental groups such that $\pi_k(M - \mathrm{int}(M_i), \partial M_i) = 0$, for $k \leq n-2$ and hence obtaining a pseudo-collar whose system of fundamental groups matches a subsequence of the inverse sequence representing the fundamental group at infinity. □

Thus $\partial M_i \hookrightarrow M_{i+1} - \mathrm{int}(M_i)$ are simple homotopy equivalences.

6.3. **Mapping cylinders.** Before to proceed, recall that the *extended mapping cylinder* $M^e(f)$ of a continuous map $f : M \to N$ is the union $M \times [0,1] \cup_{(x,1) \sim (f(x),1)} N \times [1,2]$. Further, we will need the following key result from ([23], Thm. 5.2):

Proposition 12. *Suppose that (M, N_1, N_2) is an n-dimensional cobordism, $n \geq 6$, such that $N_2 \hookrightarrow M$ is a homotopy equivalence and the kernel of the map induced by inclusion $\pi_1(N_1) \to \pi_1(M)$ is the normal closure in $\pi_1(M)$ of a finitely generated perfect group. Then there exists an acyclic map $f : N_1 \to N_2'$ to a closed manifold N_2' such that M is homeomorphic to $M^e(f) \cup_{N_2' \times \{1\}} M'$, where $M^e(f)$ is the extended mapping cylinder of f and $(M', N_2' \times \{1\}, N_2)$ is an h-cobordism.*

This shows that every cobordism $M_{i+1} - \mathrm{int}(M_i)$ obtained from an exhaustion as provided by Lemma 10 is homeomorphic to the composition of the mapping cylinder C_i of some acyclic map $\partial M_{i+1} \to N_i$ composed with an h-cobordism Z_i with boundary $\partial Z_i = N_i \sqcup \partial M_i$. Here N_i is some closed manifold homotopy equivalent to ∂M_i.

Now, after identifying the groups $\pi_1(N_i)$, $\pi_1(\partial M_i)$, $\pi_1(M_{i+1} - \mathrm{int}(M_i))$ and $\pi_1(Z_i)$ we have in $\mathrm{Wh}(\pi_1(\partial M_i))$:

$$\tau(M_{i+1} - \mathrm{int}(M_i), \partial M_i) = \tau(C_i, N_i) + \tau(Z_i, \partial M_i)$$

On the other hand

$$\tau(M_{i+1} - \mathrm{int}(M_i), \partial M_i) = 0$$

because $M_{i+1} - \mathrm{int}(M_i)$ is a plus cobordism.

We will use now the following:

Lemma 11. *If $f : M \to N$ is an acyclic map whose extended mapping cylinder $M^e(f)$ is a manifold, then the retraction map $\pi : M^e(f) \to N$ is a simple homotopy equivalence and hence $\tau(M^e(f), N) = 0$.*

Proof. We use Chapman's simple homotopy type of compact ANR spaces (see [16]). Every metrizable topological manifold has the proper homotopy type of a locally finite simplicial complex (see [44], Thm. 4.1.3, p.123). Thus the product of a compact topological manifold and the Hilbert cube Q is a compact Hilbert cube manifold.

According to a deep result of ([65], Thm. 2 and section 4, [57], Thm. 3.4), the retraction map $\pi \times id_Q : M^e(f) \times Q \to N \times Q$ is homotopic to a homeomorphism of Hilbert cube manifolds. Further, Chapman's theorem ([17, 18]) says that π is a simple homotopy equivalence. As the inclusion $N \hookrightarrow M^e(f)$ is a homotopy inverse for the retraction map, we derive that $\tau(M^e(f), N) = 0$.

Alternatively, we can use the fact that a cell-like map between compact ANR's is a simple homotopy equivalence (see [45], Thm. 4.3, following Chapman and West). The retraction map $M^e(f) \to N$ is obvious cell-like, as the point preimage of $y \in N$ is the wedge of a cone over $f^{-1}(y)$ and a segment. □

Now $\tau(C_i, N_i) = 0$ and hence $\tau(Z_i, \partial M_i) = 0$. By the s-cobordism theorem Z_i is homeomorphic to a product and hence $M_{i+1} - \mathrm{int}(M_i)$ is homeomorphic to a mapping cylinder C_i.

Note that we don't know if we can perturb the acyclic map f to an acyclic PL map having the same extended mapping cylinders, up to a homeomorphism; if this were true, then its mapping cylinder would be homeomorphic to a simplicial mapping cylinder and hence it would PL collapse onto ∂M_i.

6.4. **Acyclic maps.** We will need later the description of f following [23] and [22]. Let $Q_{i+1} \subset \pi_1(\partial M_{i+1})$ be a finitely generated perfect subgroup whose normal closure within $\pi_1(\partial M_{i+1})$ is $\ker(\pi_1(\partial M_{i+1}) \to \pi_1(\partial M_i))$. By Hausmann's trick there exists Q^*_{i+1} a finitely presented perfect group of deficiency 0 equipped with a surjection onto Q_{i+1} (see [38], section 2.1) . Let D_{i+1} be a 2-dimensional complex associated to a balanced presentation of Q^*_{i+1}, which is then acyclic. We embed D_{i+1} within ∂M_{i+1} such that the induced map on fundamental groups is the composition $Q^*_{i+1} \to Q_{i+1} \subset \pi_1(\partial M_{i+1})$. Then the boundary $\partial \mathrm{N}_{\partial M_{i+1}}(D_{i+1})$ of a regular neighborhood $\mathrm{N}_{\partial M_{i+1}}(D_{i+1})$ of D_{i+1} within ∂M_{i+1} is a codimension one homology sphere.

Now, the inclusion induces an isomorphism

$$\pi_1(\partial \mathrm{N}_{\partial M_{i+1}}(D_{i+1})) \to \pi_1(\mathrm{N}_{\partial M_{i+1}}(D_{i+1}))$$

Indeed any loop in $\mathrm{N}_{\partial M_{i+1}}(D_{i+1})$ based at a point of the boundary can be homotoped out of D_{i+1}, by general position. Since $\mathrm{N}_{\partial M_{i+1}}(D_{i+1}) - D_{i+1}$ is homeomorphic to $\partial \mathrm{N}_{\partial M_{i+1}}(D_{i+1}) \times [0, 1)$ we can further homotope the loop onto $\partial \mathrm{N}_{\partial M_{i+1}}(D_{i+1})$, proving that the map above is surjective. Further, a null-homotopy 2-disk mapped properly into $(\mathrm{N}_{\partial M_{i+1}}(D_{i+1}), \partial \mathrm{N}_{\partial M_{i+1}}(D_{i+1}))$ can be homotoped off D_{i+1} by general position and hence into $\partial \mathrm{N}_{\partial M_{i+1}}(D_{i+1})$. This yields the injectivity of the homomorphism above.

We can further homotope the map $D_{i+1} \to \mathrm{N}_{\partial M_{i+1}}(D_{i+1})$ to an embedding of D_{i+1} in the boundary, namely such that its image is $D^*_{i+1} \subset \partial \mathrm{N}_{\partial M_{i+1}}(D_{i+1})$, and the induced map on fundamental groups is an isomorphism.

Eventually we consider a collar $\partial \mathrm{N}_{\partial M_{i+1}}(D_{i+1}) \times [0, 1] \subset \partial M_{i+1}$ of the boundary and a Cantor set $C \subset [0, 1]$. The sets $D^*_{i+1} \times C$ form the set of nondegenerate elements of a upper semi-continuous decomposition $\mathcal{U}$ of ∂M_{i+1}. The associated quotient space $\partial M_{i+1}/\mathcal{U}$ is the quotient by the equivalence relation induced by $\mathcal{U}$, namely two points are identified if and only if they belong to the same set of the decomposition.

We have then:

Proposition 13 ([23] Thm.4.3, [22], section 2, [24]). *The space $\partial M_{i+1}/\mathcal{U}$ is a topological manifold and the extended mapping cylinder $M^e(f_i)$ of the quotient map $f_i : \partial M_{i+1} \to \partial M_{i+1}/\mathcal{U}$ is a topological manifold.*

Proof of Proposition 11. By Proposition 12 and the discussion above it follows that $M_{i+1} - \mathrm{int}(M_i)$ is homeomorphic to the extended mapping cylinder $M^e(f_i)$ of an acyclic map $f_i : \partial M_{i+1} \to \partial M_i$, which is a quotient map as described in Proposition 13.

Since it is acyclic f_i is a $\mathbb{Z}[\pi_1(\partial M_i)]$ homology equivalence. On the other hand f_i factors as $\partial M_{i+1} \hookrightarrow M_{i+1} - \mathrm{int}(M_i) \to \partial M_i$, where the second map is the strong deformation retract of the extended mapping cylinder $M^e(f_i)$ onto its target boundary ∂M_i. This implies that the inclusion $\partial M_{i+1} \hookrightarrow M_{i+1} - \mathrm{int}(M_i)$ is a $\mathbb{Z}[\pi_1(\partial M_i)]$ homology equivalence, and hence this cobordism is a plus cobordism. □

6.5. **Arborescence of unions of plus cobordisms with strongly perfectly semistable group at infinity.** We found above that M is endowed with an exhaustion by plus-cobordisms, i.e. an ascending filtration by compact contractible submanifolds $M_i \subset M$ such that $M_{i+1} - \mathrm{int}(M_i)$ is a plus cobordism for every $i \geq 0$.

The main result of the previous section states that each plus cobordism $\overline{M_{i+1} - \mathrm{int}(M_i)}$ is topologically collapsible. If any of the Questions 1 or 2 has an affirmative answer, then M should be homeomorphic to a PL manifold which is PL arborescent.

7. CAT(0) METRICS FROM COLLAPSES

7.1. **Basic results and techniques.** Following [26], it is easy to construct CAT(0) metrics on regular neighborhoods of trees using Gromov's hyperbolization technique. To this end, we use metrics along Whitehead's collapsibility (cf. [68]) as a more direct and suitable (but much less elegant) alternative to Gromov's hyperbolization technique. We recall two critical criteria:

Lemma 12 (cf. [27]). *Consider a locally $CAT(\kappa)$ and locally compact metric length space X.*

(a) Cartan–Hadamard theorem. *If $\kappa \leq 0$ and X is simply connected, then X is $CAT(\kappa)$.*
(b) Bowditch criterion. *If $\kappa > 0$, and every closed curve of length $\leq {}^{2\pi}/_{\kappa}$ can be monotonously contracted to a point, then X is $CAT(\kappa)$.*

7.2. **Links and the Gromov-Alexandrov lemma.** Recall that the *star* and *link* of a face σ in a simplicial complex Σ are the subcomplexes

$$\mathrm{st}_\sigma \Sigma := \{\theta;\ \text{there exists } \tau \in \Sigma \text{ such that } \sigma \subseteq \tau \in \Sigma \text{ and } \theta \subseteq \tau\}$$

$$\mathrm{lk}_\sigma \Sigma := \{\theta \in \mathrm{st}_\sigma \Sigma;\ \theta \cap \sigma = \emptyset\}$$

Although these definitions of stars and links also make sense for any cell complex, we wish to emphasize that the combinatorial definition used below in the cubical case is slightly different. Specifically let $(P, \leqslant)$ be a poset, namely a partially ordered set. For $x \in P$ we denote by $\bigwedge x$ the order ideal and by $\bigvee x$ the filter, namely:

$$\bigwedge x = \{y \in P; y \leqslant x\},\ \bigvee x = \{y \in P;\ y \geqslant x\}$$

If $x \in P$ we define the *link* of the element x in P to be the poset

$$\mathrm{lk}_x P = \bigvee x - \{x\}$$

More generally, if $A \subset P$ is a sub-poset, then we set:

$$\mathrm{lk}_A P = \bigcap_{x \in A} \mathrm{lk}_x P$$

A poset P is called cubical if each order ideal $\bigwedge x$ is the product of the poset I associated to the interval

$$I = \{0, 1, [0,1]; 0 < [0,1], 1 < [0,1]\}$$

Also recall that a poset is called a lattice if every two elements of it have a least upper bound and a greatest lower bound.

To a cell complex X one associates the *face poset* $P(X)$, whose elements are faces (or cells) of X with respect to the inclusion order relation. One usually use $\widehat{P(X)}$ to denote the poset obtained by adjoining one minimal element and one maximal element to $P(X)$, elements which can be thought of as $\emptyset$ and X itself. With this definition we see that a cubical complex X is a cell complex whose face poset $P(X)$ is a cubical poset and $\widehat{P(X)}$ is a lattice, namely every couple of elements has a *least upper bound* and a *greatest lower bound*. We therefore define for an arbitrary cell complex X and σ a face of X the combinatorial link as being the poset:

$$\mathrm{lk}_\sigma X = \mathrm{lk}_\sigma P(X)$$

In the simplicial case the combinatorial definition matches the geometric one given above, in the sense that the face poset associated to the geometric link coincides with the combinatorial link. For notation simplicity we will use the same symbol to denote a simplicial complex and the corresponding poset.

If Σ is a decomposition of a facewise smooth length space, then $\mathrm{lk}_\sigma \Sigma$ carries a natural facewise spherical length metric.

Lemma 13 (Gromov–Alexandrov lemma; cf. [9]). *If Σ is a locally finite facewise constant curvature κ length space. If the link of every face in σ in Σ has a CAT(1) link, then Σ is locally $CAT(\kappa)$.*

The piecewise flat metric on a cube complex is the length metric obtained by endowing each cube with a metric making it isometric with an unit Euclidean cube.

In the case of cubical complexes the Gromov criterion from above reads:

Lemma 14 (Gromov–criterion for cubical complexes; cf. [9]). *The piecewise flat metric on a locally finite cube complex Σ is locally CAT(0) if and only if the link of every vertex of Σ is flag.*

A cube complex which has every vertex link a simplicial flag complex. will be called *nonpositively curved*.

7.3. **Comparison CAT(0) metrics on arborescent complexes.** The following is a slight improvement of Theorem 2.

Theorem 4. *Let C be any locally finite arborescent simplicial complex. Then there exists a cubical complex C' that is PL homeomorphic to C such that the piecewise flat metric on C' is CAT(0).*

Let $C = \bigcup_{n=1}^{\infty} C_n$ be the ascending union of subcomplexes obtained one from another by finitely many disjoint elementary expansions (reverse collapses):

$$\{\mathit{point}\} = C_0 \nearrow C_1 \nearrow \cdots C_{n-1} \nearrow C_n \nearrow \cdots C$$

Then C_n are convex subsets of C, when the later is endowed with the CAT(0) metric obtained by pullback from C'.

Proof. The proof is by a simple induction, constructing the desired facewise hyperbolic CAT(0)-metric along elementary expansions. The induction step is provided by the following result:

Proposition 14. *Let X be a finite simply connected nonpositively curved cube complex. Let Γ be a PL k-disk which is a subcomplex of X. Then there exists:*

(1) a cubical complex X' which contains X as a subcomplex.
(2) a cubical complex Δ which is a PL $(k+1)$-disk containing the subcomplex Γ in its boundary $\partial\Delta$

such that the cubical complex $X' = X \cup_\Gamma \Delta$ obtained by gluing Δ to X along Γ is nonpositively curved.

□

7.4. **Extensions of nonpositively curved complexes.** Let $X' = X \cup_\Gamma \Delta$, where $\Delta = \Gamma \times [0, 1]$. Our Proposition 14 would immediately follow if the links of vertices of Δ within X were simplicial flag complexes. However, this might not be true and to achieve it we should allow the cubical complex Δ be suitably modified, while preserving the condition that Δ is a PL $(k+1)$-disk containing Γ.

To this purpose, we check the flag condition at each vertex of Γ. If it is not satisfied at some vertex v, then we define a sequence of modifications eventually producing a new simplicial complex L'_v out of the link $\mathrm{lk}_v X$, such that L'_v is now flag. The final step is to prove that there is some finite cube complex Δ with the property that $\mathrm{lk}_v X'$ is isomorphic to L'_v, for any vertex v of Γ and flag for all the other vertices of Δ.

In order to construct Δ out of the links L'_v, for v vertex of Γ, let us introduce more notation. We call an *extension* of a simplicial/cubical complex S to be any simplicial/cubical complex containing it. We are concerned with the following problem: given a cubical complex X and a collection of simplicial complexes $L_\sigma, \sigma \in X$, whether there exists an extension X' of X such that $\mathrm{lk}_\sigma X' = L_\sigma$, for any σ?

To this purpose we define a *local extension* of X to be a collection $(L_\sigma, \sigma \in X)$ of *extensions* of the links $\mathrm{lk}_\sigma X$ of all proper faces σ of X. This means that the embedding of $\mathrm{lk}_\sigma X$ into L_σ is part of the data. Moreover, for any faces $\tau \subsetneq \sigma$ of X, we have defined simplicial embeddings maps which will be assimilated to inclusions:

$$L_\sigma \hookrightarrow L_\tau, \text{ if } \tau \subset \sigma.$$

If $\tau \subset \sigma$ are faces of X and X' is any extension of X, then $\mathrm{lk}_\tau \sigma$ has a natural embedding in $\mathrm{lk}_\tau X$: the poset $\mathrm{lk}_\tau \sigma$ is the poset associated to the face of $\mathrm{lk}_\tau X$ corresponding to σ. Therefore we can see $\mathrm{lk}_\tau \sigma$ as a face of the simplicial complex $\mathrm{lk}_\tau X$ and in particular as a face of any extension L_τ of $\mathrm{lk}_\tau X$.

Definition 9. A local extension $(L_\sigma, \sigma \in X)$ of a simplicial/cubical complex X is said to be *coherent* if it commutes with the passage from faces to links, namely for every faces $\tau \subset \sigma$ of X the following coherence equation holds:

$$\mathrm{lk}_{\mathrm{lk}_\tau \sigma}(L_\tau) = L_\sigma,$$

where $\mathrm{lk}_\tau \sigma$ is seen as a face of L_τ.

Note that $\mathrm{lk}_\tau \sigma \subset \mathrm{lk}_\tau X$, so that its image into L_τ is already part of the data. The coherence equation then says that the image of the inclusion $L_\sigma \hookrightarrow L_\tau$ is uniquely determined by the other data.

Definition 10. We say that a cubical complex Δ is a *combinatorial neighborhood* of Γ if every k-face of Δ either intersects Γ or is contained in a unique minimal face intersecting Γ.

To any extension $X' = X \cup_\Gamma \Delta$ of X we can associate a local extension, called its *restriction to* X consisting of the collection of links $L_\sigma = \mathrm{lk}_\sigma X'$, along with the natural embeddings $L_\sigma X \hookrightarrow L_\sigma$ and $L_\sigma \hookrightarrow L_\tau$, when $\tau \subset \sigma$ are faces of X. The key ingredient in globalizing a local extension is the following lemma:

Lemma 15. *A local extension of a cubical complex X is coherent if and only if it is the restriction of a global extension which is a combinatorial neighborhood of a subcomplex of X.*

Proof. The coherence is satisfied by any extension X' of X, since by the definition of the links we have:

$$\mathrm{lk}_{\mathrm{lk}_\tau \sigma}(\mathrm{lk}_\tau X') = \mathrm{lk}_\sigma X'$$

For the nontrivial implication consider a coherent local extension $(L_\tau, \tau \in X)$. Define the cubical cells $C_\alpha(\tau)$ of dimension $1 + \dim \sigma + \dim \alpha$, where α is a face of L_τ. For fixed τ these cubes are glued together such that

$$C_\alpha(\tau) \cap C_\beta(\tau) = C_{\alpha\cap\beta}(\tau), \text{ for } \alpha, \beta \in L_\tau,$$

where $C_\emptyset(\tau) = \tau$. The resulted complex $S(\tau)$ should be the closed star of the face τ in the hypothetical global extension and its underlying space is:

$$S(\tau) = \bigcup_{\alpha \subset L_\tau} C_\alpha(\tau)$$

Recall that for $\tau \subseteq \sigma$ we have an inclusion $L_\sigma \subseteq L_\tau$. However the natural map between the stars $S(\sigma) \subseteq S(\tau)$ is not the tautological one.

It is enough to define the embedding $S(\sigma) \subseteq S(\tau)$ by sending $C_\alpha(\sigma)$ isometrically onto $C_{\alpha * \mathrm{lk}_\tau \sigma}(\sigma)$, where denotes the join. Note that the image of α in L_τ is contained in $\mathrm{lk}_{\mathrm{lk}_\tau \sigma} L_\tau$, by the coherence equations, so that indeed $\alpha * \mathrm{lk}_\tau \sigma$ is a face of L_τ.

We then define

$$X' = \bigcup_{\tau \in X} S(\tau)/\sim$$

where the equivalence relation $\sim$ identifies $C_\alpha(\tau)$ and $C_{\alpha * \mathrm{lk}_\tau \sigma}(\sigma)$ if $\tau \subset \sigma$. Here we identified $\alpha \in L_\sigma$ to its image $\alpha \in L_\tau$. This complex is well-defined if and only if we have the following commutative diagrams of inclusions:

$$\begin{array}{ccc} S(\sigma) & \to & S(\tau_1) \\ \downarrow & & \downarrow \\ S(\tau_1) & \to & S(\theta) \end{array}$$

for every triple of faces $\theta \subseteq \tau_i \subseteq \sigma$, $i = 1, 2$. But this is a consequence of the fact that, under the above assumptions we have:

$$\mathrm{lk}_\theta \tau_1 * \mathrm{lk}_{\tau_1} \sigma = \mathrm{lk}_\theta \tau_2 * \mathrm{lk}_{\tau_2} \sigma$$

Moreover, the coherence of the local extension insures that after gluing we indeed have $\mathrm{lk}_\sigma X' = L_\sigma$, as desired. Moreover, X' is a combinatorial neighborhood of a subcomplex of X, by construction. $\square$

To see how a global extension is constructed out of a local extension consider the following example which starts from a genuine extension $X \cup_\Gamma \Delta$, where Δ is a product, as in the figure below. Let then L'_σ be the stellar subdivision of $\mathrm{lk}_\sigma X'$.

Then the collection $(L'_\sigma, \sigma \subset X)$ is a coherent local extension of X and the global extension associated to it is drawn below:

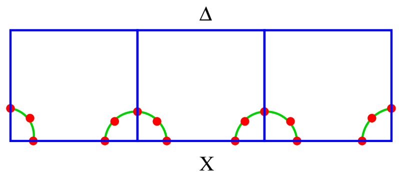

Figure 1. A local extension obtained by stellar subdivision of links in an extension

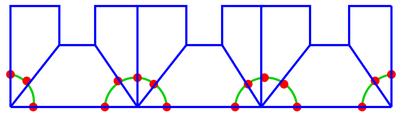

Figure 2. The extension obtained from the coherent local extension

This construction then can be used for modifying the links of a given extension. Specifically, we have:

Lemma 16. *Assume that the cubical complex X is flag and let $\mathcal{Z} = (L_\tau, \tau \in X)$ be a coherent local extension. Then there is another coherent local extension $\mathcal{W} = (L'_\tau, \tau \in X)$ such that:*

(1) every L'_τ is obtained by subdividing faces of L_τ which are not in $\mathrm{lk}_\tau(X)$;
(2) every link L'_τ is flag.
(3) the cubical complex X' whose restriction is $\mathcal{W} = (L'_\tau, \tau \in X)$ is a combinatorial neighborhood of a subcomplex of X. Moreover, every vertex link of X' is also flag.

Before to proceed, recall that a *stellar subdivision* of a simplex splits it into cones with a common vertex over the faces, as in the figure 3 below:

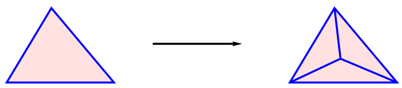

Figure 3. One stellar subdivision of a face

Further, a *derived subdivision* of a simplicial complex Δ is the composition of stellar subdivisions on the faces of Δ in reverse order of inclusion, starting with the maximal faces. Eventually, the *relative derived subdivision* (A', B) of a pair (A, B) of polyhedral complexes with $B \subset A$ is the result of stellar subdivisions of all faces of A which are not faces of B in reverse order of inclusion, starting with the maximal faces. Thus faces of B are not affected and $B \subset A'$ as in the picture below where we consider the pair (A, B) formed by a 2-cell and one 1-cell:

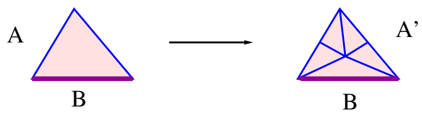

Figure 4. Relative stellar subdivision of a pair (A,B)

The following lemma is immediate:

Lemma 17. *Consider (A, B) a pair of simplicial complexes such that B is flag. Then the relative derived subdivision (A', B) is flag.*

Proof of Lemma 16. For every L_τ we perform a relative derived subdivision and call it L'_τ. Lemma 16 shows that flag properties of links of L'_τ are satisfied at all faces intersecting X. Indeed, the claim is immediate at links of vertices of X and descends to faces of such links.

Further note that for every τ in X, every face of L'_τ intersects $\mathrm{lk}_\tau(X)$ in a unique maximal face (which can be empty). This implies that the local extension $\mathcal{W} = (L'_\tau, \tau \in X)$ is coherent and hence there is a cubical complex X' whose restriction is $\mathcal{W} = (L'_\tau, \tau \in X)$. By Lemma 15 the cubical complex X' is a combinatorial neighborhood of a subcomplex of X.

By above, all links $\mathrm{lk}_F X'$ of faces F contained in X, in particular vertices of X, are flag. It remains therefore to check the links of vertices of X' which don't belong to X. To this purpose we will use the following description of their links:

Lemma 18. *Let o be a vertex of X' which is not a vertex of X. If F is the minimal face intersecting X and containing o, then $\mathrm{lk}_o X'$ is the free join of $\mathrm{lk}_o F$ and $\mathrm{lk}_F X'$.*

Proof. It suffices to observe that every face of $\mathrm{lk}_o X'$ is in the star of $\mathrm{lk}_o F$. If there were a face F' violating that property then, by the combinatorial neighborhood property, F' would intersect X. This contradicts the minimality of F. □

Now, $\mathrm{lk}_F X'$ is the link of $\mathrm{lk}_{F\cap X} F$ in $\mathrm{lk}_{F\cap X} X'$ and we noted above that $\mathrm{lk}_{F\cap X} X'$ is flag. Since links of flag complexes are flag complexes we derive that $\mathrm{lk}_F X'$ is flag.

As $\mathrm{lk}_o F$ is a simplex and $\mathrm{lk}_F X'$ is flag, their free join is a flag complex as well. Henceforth, by Lemma 18, $\mathrm{lk}_o X'$ is flag for any vertex o of X' which is not a vertex of X. This ends the proof of Lemma 16. □

Remark 4. One single stellar subdivision is enough to make links of vertices of X flag. However, if we stop here we might obtain new vertices in the corresponding extension X' which have not flag links. In the picture below we have a vertex v of a square 2-cell of Γ, which is a facet of the 3-cube $\Box$ of Δ. The link $\mathrm{lk}_v \Box \subset \mathrm{lk}_v \Delta$ is a (spherical) 2-simplex σ. Suppose that we aim at stellar subdividing this 2-simplex, as in figure 3. Then the 3-cube $\Box$ from $\mathrm{st}_v \Delta$ is replaced by a complex $Q(\Box, v)$ consisting of the union of three 3-cubes as in figure 5.

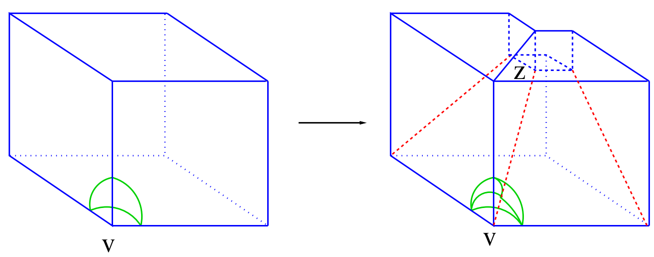

Figure 5. One stellar subdivision might create a new vertex whose link is not flag

We can see from the picture that the link of the newly created vertex z of is not flag. To remedy that we have to continue stellar subdividing all cells of σ which are not cells of $\mathrm{lk}_v X$, in decreasing order of their dimensions. Specifically, in the situation above we should continue to subdivide σ as indicated on figure 4. In this picture the horizontal edge of the 2-simplex σ in $\mathrm{lk}_v \Delta$ belongs also to $\mathrm{lk}_v X$ and hence it will be left untouched. Therefore the 3-cube $\Box$ from $\mathrm{st}_v \Delta$ is now replaced by the complex consisting of the union of five 3-cubes as in figure 6.

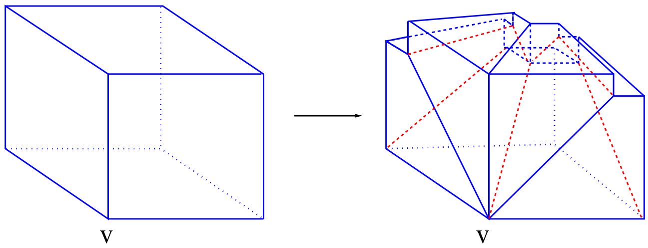

Figure 6. A relative derived subdivision of a 3-cube

The *cubical stellar subdivision* of a complex A at a face σ is obtained by removing σ, and gluing in the star attached along the mapping cylinder of the inclusion map (see [2]). The *relative cubical derived subdivision* of a pair (A, B) of cubical complexes is then obtained by performing cubical stellar subdivisions on faces of A which are not in B, in reverse order of inclusion.

There is a simpler way to obtain the extension discussed here, if we are given the global extension. We can turn this directly into the desired nonpositively curved extension without a detour through local extensions. Consider X a nonpositively curved cubical complex, and $X' = X \cup \Delta$ a cubical extension. Then the above lemma, on a global level, can be understood using a cubical derived subdivision. Now, we modify the extension $(X' = X \cup \Delta, X)$ as follows:

1. Perform a relative cubical derived subdivision.
2. Remove all cubes not incident to X (which remained unaffected in the subdivision process).

We obtain then the desired non-positively curved cubical extension.

Proof of Proposition 14. The result is a direct consequence of Lemmas 15, 16 and Gromov-Alexandrov's Lemma 14. □

7.5. **Proof of Theorem 4.** Consider a sequence of reverse collapses $C_{n-1} \nearrow C_n$, $C_0 = \{point\}$, such that $C = \bigcup_{n=1}^{\infty} C_n$. Assume that we constructed CAT(0) cubical complexes C_i', $0 \leq i \leq n$ such that C_i' are PL homeomorphic to C_i and moreover C_j' are convex in C_i', for $j \leq i$.

Consider the next elementary expansion $C_n \nearrow C_{n+1}$, where δ denotes the k-cell attached and set $\gamma = \delta \cap C_n$. As γ is a PL $(k-1)$-disk, its image Γ within C_n' by the PL homeomorphism $C_n \to C_n'$ is also a PL $(k-1)$-disk. By Proposition 14 there is a PL k-disk Δ containing Γ embedded within $\partial\Delta$ such that $C_{n+1}' = C_n' \cup_\Gamma \Delta$ is a CAT(0) cubical complex. It then follows that the PL homeomorphism $C_n' \to C_n$ extends to $C_{n+1}' \to C_{n+1}$. Then the claim follows by induction on n.

To make sure that C_n' is convex within C_{n+1}', we first note that any cubical cell of C_{n+1}' intersects C_n' along a face. Therefore C_n' is locally convex, with respect to the piecewise flat metric on C_{n+1}'. Moreover C_n' is connected and it is well-known that a connected simply connected locally convex subcomplex of a CAT(0) cubical complex is convex, e.g. as a consequence of ([9], Prop II.4.41). Eventually C_n is convex in C as well, as the CAT(0) metric on C_n won't be changed when we adjoin new cells.

8. Variations

8.1. **More tameness conditions.**

Definition 11. An open manifold M is *weakly geometrically k-connected* (see [30]) if $M = \cup_{j=1}^{\infty} K_j$, where $K_j \subset \mathrm{int}(K_{j+1})$, for $j \geq 1$, is an exhaustion by compact k-connected PL manifolds. When $k = \infty$ we use the term weak geometric contractibility.

It is obvious that CAT(0) polyhedra are weakly geometrically contractible. It suffices to consider any exhaustion by metric balls, which are convex. To guarantee the filtration is a filtration by manifolds, one merely has to to pass to the regular neighborhoods of these geometric balls to obtain the desired filtration.

Definition 12. An end is of type F_k (respectively F) if it admits arbitrarily small clean neighborhoods with the homotopy type of a CW complex having finite k-skeleton (respectively finitely many cells).

This generalizes the Tucker condition explored in [47] which requires that the complement of any compact subpolyhedron has finitely generated fundamental group, i.e. is of type F_1.

8.2. **Weak geometric contractibility is not sufficient.** The aim of this section is to construct examples of open weakly geometrically contractible manifolds which are neither semistable nor with end of type F_1.

Definition 13. An open manifold W has *injective* ends if it admits an ascending compact exhaustion by submanifolds K_j with the property that the maps induced by inclusions $\pi_1(\partial_* K_j) \to \pi_1(K_{j+1} - \mathrm{int}(K_j))$ and $\pi_1(\partial_* K_{j+1}) \to \pi_1(K_{j+1} - \mathrm{int}(K_j))$ are injective. Here $\partial_* K$ denotes an arbitrary connected component of ∂K. The ends of W are *strictly injective* if none of the maps above are surjective.

It is well-known (see e.g. [32], [34], ex.4.17) that:

Lemma 19. *An open manifold with strictly injective ends is not semistable.*

Our goal now is to construct geometrically contractible manifolds with strictly injective ends. To this purpose we introduce more terminology.

We say that the nontrivial pair $H \subset G$ of finitely presented groups is *tight* if the normal closure of H within G is H itself, i.e. there is no proper normal subgroup of G containing H. The pair is nontrivial if $H \subset G$ is proper. The group G is *superperfect* if $H_1(G) = H_2(G) = 0$. Observe that G is superperfect if and only if $G = \pi_1(K)$, where K is a finite complex whose integral homology is that of a point.

Lemma 20. *Given a nontrivial tight pair $H \subset G$ of superperfect finitely presented groups, there exists an open weakly geometrically contractible manifold W with a contractible compact exhaustion K_j such that the maps $\pi_1(\partial_* K_j) \to \pi_1(K_{j+1} - \mathrm{int}(K_j))$ and $\pi_1(\partial_* K_{j+1}) \to \pi_1(K_{j+1} - \mathrm{int}(K_j))$ are given by the proper inclusions $H \subset G$.*

Proof. A classical result of Kervaire ([43]) states that G is the fundamental group of a homology sphere Σ^n of dimension $n \geq 5$ if (and only) if G is finitely presented and superperfect. Let $H = \pi_1(K)$ be fundamental group of an acyclic k-complex, $k \geq 2$. Choose $n \geq 2k+1$, in order to be able to embed $K \to \Sigma^n$ such that the map induced by inclusion $\pi_1(K) \to \pi_1(\Sigma^n)$ corresponds to the inclusion $H \hookrightarrow G$. Consider two such embeddings K_1 and K_2, which by transversality could be assumed to be disjoint. Let N_1 and N_2 denote disjoint regular neighborhoods of K_1 and K_2 within Σ^n.

Using general position we derive that $\pi_1(\Sigma - \mathrm{int}(N_1 \sqcup N_2)) \cong \pi_1(\Sigma) = G$ and $\pi_1(\partial N_i) \cong \pi_1(N_i - K_i) \cong \pi_1(N_i) = H$. Moreover, the map $\pi_1(\partial N_i) \to \pi_1(\Sigma)$ induced by the inclusion is identified with the embedding $H \hookrightarrow G$. If H is weakly acyclic then $X = \Sigma - \mathrm{int}(N_1 \sqcup N_2)$ has the homology of a spherical cylinder.

Now, since ∂N_1 is a homology sphere of dimension at least 4, it bounds a compact contractible manifold M. Then, the result of gluing $M \cup X$ is acyclic and simply connected and hence contractible. By recurrence we find that $K_j = M \cup X \cup X \cdots \cup X$, where X occurs j-times, is also contractible. Therefore the open manifold $W = M \cup X \cup X \cdots$ is weakly geometrically contractible and the exhaustion K_j satisfies all the requirements. $\square$

Lemma 21. *Any finite superperfect group H is contained in a superperfect group G to form a nontrivial tight pair. In particular, this is the case for the binary icosahedral group $\langle a, b | a^5 = b^3 = (ab)^2 \rangle \cong SL_2(\mathbb{F}_5)$, $SL(2, \mathbb{F}_p)$, for odd prime p, or more generally any finite perfect balanced group.*

Proof. Any finite group is contained into some S_n which is contained into $Sp(2n, \mathbb{F}_q)$. The finite symplectic group $Sp(2n, \mathbb{F}_2)$ is simple (hence perfect) for $n \geq 4$ and has trivial Schur multiplier so that it is superperfect. The finite symplectic groups $PSp(2n, \mathbb{F}_q)$ are simple for $n \geq 4$ and have Schur multiplier $\mathbb{Z}/2\mathbb{Z}$, when q is odd, so that $Sp(2n, \mathbb{F}_q)$ is the universal central extension of $PSp(2n, \mathbb{F}_q)$. Therefore it is superperfect. Any proper normal subgroup of $Sp(2n, \mathbb{F}_q)$ should be contained in the center, so that the pair obtained is tight and nontrivial.

Note that $SL_2(\mathbb{F}_p)$, for odd prime p are perfect and admit balanced presentations (see [14]). Thus their presentation 2-complexes are acyclic since their Schur multiplier is trivial, by an old theorem of Schur.

Alternatively any finite group is contained in the Thompson group V, which is finitely presented, simple and superperfect ([42]). Moreover V is non co-Hopfian, as it contains copies of V corresponding to stabilizers of dyadic intervals of the circle, when we identify V with a group of piecewise linear dyadic bijections of the circle. Therefore these inclusions $V \hookrightarrow V$ are nontrivial tight pairs of infinite groups. $\square$

Remark 5. More examples of superperfect groups are 1-relator torsion-free groups (Lyndon's theorem) and perfect finitely presented groups of deficiency zero, in particular Higman's groups, whose presentation complexes are acyclic. Other finite examples are $SL_2(\mathbb{F}_8)$, $SL_2(\mathbb{F}_{32})$, $SL_2(\mathbb{F}_{64})$, $SL_2(\mathbb{F}_{27})$, $SL_2(\mathbb{F}_5) \times SL_2(\mathbb{F}_5)$, $\widehat{A_7}$, etc (see [15]). More recently the Burger-Mozes examples (see [13]) of simple finitely presented torsion-free groups acting on products of trees can be written as amalgamated product of free groups over free subgroups. A classical theorem of Whitehead ([68]) states that whenever we have aspherical spaces X and Y such that $\pi_1(X \cap Y) \to \pi_1(X)$ and $\pi_1(X \cap Y) \to \pi_1(Y)$ are injective, then $X \cup Y$ is aspherical. This proves that Burger-Mozes simple groups are superperfect.

References

[1] K. Adiprasito and B. Benedetti, *Collapsibility of CAT(0) spaces*, Geom. Dedicata 206 (2020), 181–199. 5

[2] K. Adiprasito and B. Benedetti, *Subdivisions, shellability and collapsibility of products*, Combinatorica 37 (2017), No. 1, 1–30. 26

[3] F. Ancel and C. R. Guilbault, *$\mathcal{Z}$-compactifications of open manifolds*, Topology 38 (1999), 1265–1280. 15

[4] F. Ancel and C. R. Guilbault, *Compact contractible n-manifolds have arc spines ($n \geq 5$)*, Pacific J. Math. 168 (1995), 1–10. 6

[5] F. Ancel and C. R. Guilbault, *Interiors of compact contractible n-manifolds are hyperbolic* ($n \geq 5$), J. Differential Geom. 45(1997), 1–32. 1, 5

[6] I.Berstein, M.Cohen and R.Connelly, *Contractible, non-collapsible products with cubes*, Topology 17 (1978), 183–187. 5

[7] E. Bierstone and V. Milman, *Semianalytic and subanalytic sets*, Publ. Math. I.H.E.S. 67 (1988), 5–42. 9, 10

[8] R.H.Bing, *Some aspects of the topology of 3-manifolds related to the Poincaré conjecture*, Lectures in Modern Math. II, T.L.Saaty, Ed., New York, 1964. 3

[9] M. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Grund. math. Wissenschaften, 319, Springer, 1999. 2, 10, 11, 12, 21, 26

[10] V.N.Berestovskii and I.G. Nikolaev *Multidimensional generalized Riemann spaces*, Geometry IV, (Yu. G. Reshetnyak Ed.), Encyclopedia Math. Sciences no 70, Springer Verlag, 1993. 12

[11] J.Bracho and L.Montejano, *The scorpions: examples in stable non collapsibility and in geometric category theory*, Topology 30 (1991), 541–550. 4

[12] D. Burago, Y. Burago and S. Ivanov, A course in metric geometry, Graduate Studies in Mathematics, vol. 33. A. M. S. , Providence, RI, 2001. 1

[13] M. Burger and S. Mozes, *Finitely presented simple groups and products of trees*, C. R. Acad. Sci. Paris Ser. I Math. 324 (1997), no. 7, 747–752. 28

[14] C. M. Campbell and E. F. Robertson, *A deficiency zero presentation for* $\mathrm{SL}(2,\ p)$, Bull. London Math. Soc. 12 (1980), 17–20. 28

[15] C. M. Campbell, E. F. Robertson, T. Kawamata, I. Miyamoto and P. D. Williams, *Deficiency zero presentations for certain perfect groups*, Proc. Roy. Soc. Edinburgh Sect. A 103 (1986), 63–71. 28

[16] T.A.Chapman, *Simple homotopy theory for ANR's*, General Topology Apllications 7 (1977), 165–174. 19

[17] T.A.Chapman, *Topological invariance of Whitehead torsion*, Amer. J. Math. 96 (1974), 488-497. 19

[18] T.A.Chapman, *Compact Hilbert cube manifolds and the invariance of Whitehead torsion*, Bull.Amer. Math. Soc. 79 (1973), 52–56. 19

[19] T.A.Chapman and L.C. Siebenmann, *Finding a bounadry for a Hilbert cube manifold*, Acta Math. 137 (1996), 171–208. 2, 7, 15, 16, 17

[20] M. M. Cohen, *Simplicial structures and transverse cellularity*, Ann. Math. 85 (1967), 218–245. 6

[21] M. M. Cohen, A course in simple homotopy theory, Springer Verlag, 1973. 3

[22] R.J.Daverman, *Embedding phenomena based upon decomposition theory: wild Cantors sets satisfying strong homogeneity properties*, Proc. Amer. Math. Soc. 75 (1979), 177–182. 19, 20

[23] R. J. Daverman, F.C. Tinsley, *Lamination, finitely generated perfect groups and acyclic mappings*, Michigan Math. J. 33(1986), 343–351. 9, 15, 18, 19, 20

[24] R. J. Daverman, F.C. Tinsley, *Controls on the plus construction*, Michigan Math. J. 43 (1996), 389–416. 20

[25] R.J.Daverman, G.A.Venema, Embeddings in manifolds, *Graduate Studies in Math.* 106, Amer. Math. Soc., Providence Rhode-Island, 2009.

[26] M. W. Davis, and T. Januszkiewicz, *Hyperbolization of polyhedra*, J. Differential Geom. 34(1991), 347-388. 1, 12, 20

[27] M. W. Davis and G. Moussong, *Notes on nonpositively curved polyhedra*, Low dimensional topology (Eger, 1996/Budapest, 1998) 8 (1999): 11-94. 20

[28] S. Ferry, *Homotoping ε-maps to homeomorphisms*, Amer. J. Math. 101 (1979), 567–582. 13

[29] S. Ferry, *On the Ancel-Cannon theorem*, Topology Proceedings 17 (1992), 41–58. 13, 14

[30] L. Funar and S. Gadgil, *On the geometric simple connectivity of open manifolds*, I. M. R. N. , no. 24, 2004, 1193-1248. 27

[31] A. M. Gabrielov, *Projections of semi-analytic sets*, Functional Analysis Appl., 2 (1968), 282–291. 9

[32] R. Geoghegan, Topological methods in group theory, Graduate Text Math., 243, Springer 2008. 27

[33] C. R. Guilbault, *Manifolds with non-stable fundamental groups at infinity*, Geom. Topol. 4 (2000), 537–579. 2, 6, 8, 9, 12, 16, 17, 18

[34] C. R. Guilbault, *Ends, shapes and boundaries in manifold topology and geometric group theory*, arxiv 1210. 6741. 27

[35] C. R. Guilbault and F. C. Tinsley, *Manifolds with non-stable fundamental groups at infinity III*, Geom. Topol. 4 (2006), 541–556. 8, 12, 16, 18

[36] C. R. Guilbault and F. C. Tinsley, *Spherical alterations of handles: embedding the manifold plus construction*, Alg. Geom. Topol. 13 (2013), 35–60. 9

[37] R. M. Hardt, *Triangulation of subanalytic sets and proper light subanalytic maps*, Invent. Math. 38 (1977), 207–217. 9, 10

[38] J.C.Hausmann, *Homological surgery*, Ann. Math. 104 (1976), 573–584. 9, 19

[39] J.C.Hausmann, *Manifolds with a given homology and fundamental group*, Comment. Math. Helv. 53 (1978), 113–134. 9

[40] H. Hironaka, *Subanalytic sets*, NumberTheory,Algebraic Geometry and Commutative Algebra in honour of Akizuki, Tokyo, Kinokuniya, 1973, p. 453–493. 9

[41] H. Hironaka, *Triangulations of algebraic sets*, Proc. Sympos. Pure Math., vol. 29, Amer. Math. Soc, Providence, R.I., 1975, 165–185. 9, 10

[42] C. Kapoudjian, *Virasoro-type extensions for the Higman-Thompson and Neretin groups*, Q. J. Math. 53 (2002), no. 3, 295–317. 28

[43] M. A. Kervaire, *Smooth homology spheres and their fundamental groups*, Trans. Amer. Math. Soc. 144 (1969), 67–72. 27

[44] R. C. Kirby and L. C. Siebenmann, Foundational essaas on topological manifolds, smoothings and triangulations, Ann. Math. Studies, Princeton Univ. Press, 1977. 2, 19

[45] R.C.Lacher, *Cell-like mappings and their generalizations*, Bull. Amer. Math. Soc. 83 (1977), 495–552. 19

[46] S. Lojasiewicz, *Sur la gÃl'omÃl'trie semi- et sous-analytique*, Annales Institut Fourier 43 (1993), 1575–1595. 9, 10

[47] M. L. Mihalik and S. T. Tschantz, *Tame combings of groups*, Trans. Amer. Math. Soc. 349 (1997), no.10, 4251–4264. 27

[48] W.B.R. Mitchell, *Defining the boundary of a homology manifold*, Proc. Amer. Math. Soc. 110 (1990), 509–513. 13

[49] D. Quillen *Cohomology of groups.* Actes Congr. internat. Math. 1970, 2 (1971), 47–51. 16

[50] D. Rolfsen, *Strongly convex metrics in cells*, Bull. Amer. Math. Soc. 74(1968), 171-175. 3

[51] D. Rolfsen, *Characterizing the 3-cell by its metric*, Fundamenta Math. 68(1970), 215-223. 3

[52] J. Rolland. *Some Results on Pseudo-Collar Structures on High-Dimensional Manifolds*, arXiv:1502. 04338. 16

[53] M. Shiota and M. Yokoi, *Triangulations of subanalytic sets and locally subanalytic manifolds*, Trans. Amer. Math. Soc. 286 (1984), 727–750. 10

[54] M. Shiota, *Piecewise linearizations of real valued subanalytic functions*, Trans. Amer. Math. Soc. 286 (1984), 727–750. 9

[55] L.C.Siebenmann, *Approximating cellular maps by homeomorphisms*, Topology 11 (1972), 271–294. 14

[56] L.C.Siebenmann, *Infinite simple homotopy types*, Indagartiones Math. 32 (1970), 479–495. 2, 7

[57] L.C.Siebenmann, L'invariance topologique du type simple d'homotopie (d'apres Chapman et Edwards), Seminaire Bourbaki 1972/1973, no. 428, 186–209. 19

[58] D. A. Stone, *Geodesics in piecewise linear manifolds*, Trans. Amer. Math. Soc. 215(1976), 1-44. 3

[59] Yang Su and ShengKui Ye, *Fundamental groups, homology equivalences and one-sided h-cobordisms*, Sci. China Mathematics, 58(2015), 2003-2014. 9

[60] M. Tamm, *Subanalytic sets in the calculus of variations*, Acta Math., 146 (1981), 167–199. 10

[61] P. Thurston, *CAT(0) 4-manifolds possessing a single tame point are Euclidean*, J. Geom. Analysis 6 (1996), 475–494. 12, 13

[62] C.T.C.Wall, *Finiteness conditions for CW complexes*, Ann. Math. 8 (1965), 55–69. 6

[63] C.T.C.Wall, *Finiteness conditions for CW complexes II*, Proc. Ro. Soc. Ser. A 295(1966), 129–139. 6

[64] S. Weinberger, *Homology manifolds*, Handbook of geometric topology, 1085–1102, North-Holland, Amsterdam (2002) 14

[65] J.E.West, *Mapping cylinders of Hilbert cube factors*, General topology Applications 1(1971), 111-125. 19

[66] J.E. West, *Mapping Hilbert cube manifolds to ANR's: A solution of a conjecture of Borsuk*, Ann. Math. 106 (1977), 1–18. 15

[67] W.White, *A 2-complex is collapsible if and only if it admits a strongly convex metric*, Fundamenta Math. 6(1968), 23–29. 3

[68] J. H. C. Whitehead, *On the asphericity of regions in a 3-sphere*, Fund. Math. 32 (1939), 149–166. 20, 28

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