Southeast Asian Bulletin of Math. manuscript No.
(will be inserted by the editor)

On the spectrum obtained from packing balls on Riemann manifolds

Louis Funar[1] ⋆,⋆⋆, Renata Grimaldi[2]

[1] Institut Fourier, BP 74, Mathématiques, UMR 5582, Univ.Grenoble I, 38402 Saint-Martin-d'Hères Cedex, France, e-mail: `funar@fourier.ujf-grenoble.fr`

[2] Dipartimento di Matematica ed Applicazioni, Universita di Palermo, Facolta di Ingegneria, Viale delle Scienze, 90128 Palermo, Italia,
e-mail: `grimaldi@dipmat.math.unipa.it`

Received: date / Revised version: date

Abstract One considers the set of radii (called spectrum) of topological ball packings on a Riemannian manifold and computes explicitly such spectra for 2-packings of flat tori. We show that the volume is determined by the spectrum and that any manifold has a metric whose 2-spectrum is that of the sphere.

Key words Packing, injectivity radius, critical value, diameter, volume.
Mathematics Subject Classification (1991): 53 C 23, 52 C 17, 52 C 15.

1 Introduction

1.1 The aim of this paper is to consider some metric invariants for manifolds which are related to the basic geometric invariants like the **curvature**, **diameter** and **volume**. It is known that the latter alone does not suffice to characterize Riemannian spaces in general. Recent results brought to the attention some invariants related to the ball packings on length spaces (see [2,7,4,5]). These authors considered for instance the q-th **packing radius** $pack_q(M)$ of the metric space M, which is the greater radius a set of q equal and disjoint metric balls can have in M. One of their major recognition result states then ([7]) *that a n-manifold of sectional curvature ≥ 1 and $pack_{n-1}(M) > \frac{\pi}{4}$ is diffeomorphic to the sphere S^n.* Such results for

⋆ *Present address (September 1, 1999 - April 1, 2000):* Department of Mathematics, Tokyo Institute of Technology, 2-12-1, O-okayama, Meguro-ku, Tokyo, 152-8551 Japan, e-mail: funar@math.titech.ac.jp

⋆⋆ Partially supported by a Canon grant and GNSAGA.

low values of the parameter q in $pack_q(M)$ should have an interesting counterpart for very large q, since as mentioned in [7] these invariants seem to be also responsible for the **global shape** of the manifold. The results below try to comfort such a viewpoint.

Our strategy is to consider also packings with non-necessarily equal balls. The space of n-packings in the metric space (M, d) is the subset $TC_n \subset M \times \mathbf{R}_+^n$, of those n-tuples of $(p_i, r_i)_{i=1,n}$ for which the metric balls $B(p_i, r_i) = \{x; d(x, p_i) < r_i\}$ are disjoint, where d states for the distance function. We will be concerned henceforth with the case of domains in Riemannian manifolds endowed with the induced metric structure.

1.2 Definition. *The image $C_n(M) \subset \mathbf{R}_+^n$ under the projection of TC_n on the second factor is the (extended)* **packing spectrum** *C_n of order n of M.*

1.3 *Remark.* When $n = 1$ the spectrum so defined would be $\mathbf{R}_+$, but one sets by convention $C_1(M) = [0, d]$, where d is the diameter.

It seems that one cannot read from $C_n(M)$ (for small values of n) too much things about the topology of M. One introduces then some related spectra, which might be responsible for the **small scale geometry** of the manifold, by asking the metric balls be genuine topological balls:

1.4 Definition. *Set C_*^b (respectively C_*^s) for the subset obtained when we ask that the closed metric balls $cl(B(p_i, r_i))$ (respectively $B(p_i, r_i)$), and all smaller copies $B(p_i, r)$ for $r < r_i$ be homeomorphic to the standard closed (respectively open) ball.*

If M has non-empty boundary, then as the radius of a ball grows, it might happen that the ball reaches the boundary but the topology of the closed ball $cl(B(p, r)$ does not change. However $B(p, r) \cap \partial M \neq \emptyset$, hence this ball contributes to C_*^b but not to C_*^s.

One might not ask that all smaller balls $B(p_i, r)$, $r < r_i$ be topological balls. In this case, for fixed p, the set of admissible r might be non-connected; jumps appear when one reaches non-essential critical points of the distance function to p.

1.5 *Remark.* The balls $B(p, r)$ contributing to C_n^b are those satisfying $r < c(p)$ where $c(p)$ is the first positive critical value of the function $d(p, *)$. According to ([3], p.361-363), if the distance function $d(p, *)$ has no critical values in $[0, r]$ then $cl(B(p, r))$ is a topological ball.

1.6 The main question we want to address here is *to what extent the knowledge of the packing spectra for all n determines the isometry type of the manifold.* This is weaker than asking about *the relevance of the q-packing radii sequence for the recognition problem,* because the spectra contain more information about the metric geometry. We should mention that really interesting results in the vein of [7] have been obtained by restricting the set of spaces under consideration to those which satisfy some nice curvature conditions (e.g. bounded from bellow by a positive constant). We work instead in a general context but ask for infinitely many measurements of metric invariants. Notice that it should be a close connection between the packing spectra, the packing radii and the length spectrum. The packing radii have been proved an useful tool in the presence of positive curvature and the length spectrum in the case of negatively curved manifolds (see e.g. [8] where it is proved that *the marked length spectrum is a complete invariant for non-positive curved surfaces*). We expect therefore that the packing spectrum be useful in all cases.

Acknowledgements Part of this work was done during first author's visit at University of Palermo and then at Tokyo Institute of Technology, whose support and hospitality are gratefully acknowledged. We are thankful to the referee for pointing out an error in the previous version of this paper.

2 Examples

We denote by $\mathbf{R}_+^n$ the open cone of n-tuples of positive reals.

2.1 Dimension 1. Consider the circle S^1 of length l. Then $C_n^b = int(C_n^s)$, and

$$C_n = C_n^s = \{(r_1, ..., r_n) \in \mathbf{R}_+^n; r_1 + r_2 + ... + r_n \leq l/2\}.$$

2.2 Consider M an interval of length d. Then, for $n \geq 3$ we have:

$$C_n = C_n^b = \{(r_1, ..., r_n) \in \mathbf{R}_+^n; \exists i \neq j \text{ such that } r_i + r_j + 2 \sum_{k \neq i,j} r_k \leq d\},$$

and

$$C_2 = \{(r_1, r_2) \in \mathbf{R}_+^2; r_1 + r_2 \leq d\}.$$

Moreover

$$C_n^s = \{(r_1, ..., r_n) \in \mathbf{R}_+^n; r_1 + r_2 + ... + r_n \leq d/2\}.$$

2.3 The subsets $C_*^* \subset \mathbf{R}_+^n$ have a conical structure i.e. $\lambda C_*^* \subset C_*^*$ for $0 < \lambda \leq 1$.

2.4 2-packings. For any closed manifold M of diameter d one has

$$C_2(M) = \{(r_1, r_2) \in \mathbf{R}_+^2; r_1 + r_2 \leq d\}.$$

Proof: Consider two disjoint balls $B(p_i, r_i)$, $i = 1, 2$. Then $d(p_1, p_2) \leq d$ but $d(p_1, p_2) \geq r_1 + r_2$. Further, if we have two points a and b sitting at distance d one from the other then the balls $B(a, r_1)$ and $B(b, r_2)$ are disjoint, for each $r_1 + r_2 \leq d$. □

2.5 *Remark.*The difficulty in computing the subsets C_n begins with $n = 3$. We have an obvious inclusion

$$C_3(M) \subset \{(r_1, r_2, r_3) \in \mathbf{R}^3_+; r_1 + r_2 + r_3 \leq 3d/2\},$$

but the precise description might be rather tricky, even for nice metrics, like the flat ones. The qualitative estimates of the sets $C_n(M)$ for $n \geq 4$ are even more delicate since they are directly related to the curvature properties of the manifold.

2.6 Rectangles. Let $D = D_{a,b}$, for $a \leq b$, be the $a \times b$ planar rectangle. If $na \leq b$, $n \in \mathbf{Z}$ then $C^s_k(D) = (0, a/2]^k$, for $k \leq n$. If $a \leq b \leq 2a$ then

$$C^s_2(D_{a,b}) = \{(r_1, r_2) \in \mathbf{R}^2_+; r_1 + r_2 \leq a + b - \sqrt{2ab},\ 0 < r_1, r_2 \leq a/2\}.$$

Proof: Let us call a configuration of balls extremal if we cannot increase simultaneously all radii. Since the spectra are conical it suffices to determine the extremal configurations. The extremal configuration with two balls in the rectangle are of two types: either two balls sitting in opposite corners, and touching each other, or else one of maximum radius and the second one in a corner. In the first case one derives that the radii fulfill

$$r_1 + r_2 = a + b - \sqrt{2ab},$$

while the second case yields the restrictions

$$r_1 = a/2,\ r_2 \leq r_e = \frac{a + 2b - 2\sqrt{2ab}}{2}.$$

These prove the claim. □

2.7 Let T be a planar triangle of angles α_j and edges l_j. Then

$$C^s_2(T) = \{(r_1, r_2) \in \mathbf{R}^2_+;\ r_1(1 + \coth(\alpha_j)) + r_2(1 + \coth(\alpha_k)) \leq l_i, \\ \forall i \neq j \neq k \neq i,\ 0 < r_1, r_2 \leq r, \}$$

where r is the incircle radius.

Contrasting to this situation (see also 2.3) for manifolds without boundary one has:

2.8 Set c_M for the (infimum on p of the) first positive critical value of the distance function (to some point p). Then for a closed homogeneous manifold M one has

$$C^b_n(M) = C_n(M) \cap \{r_j < c_M, \forall j = 1, n\}.$$

Moreover the observation for 2.5. extends more generally to:

2.9 If $n \leq \frac{d}{c_M}$, where d is the diameter of the homogeneous M then $C^b_n(M) = (0, c_M)^n$.

Proof: Since the manifold is homogeneous one can pack n disjoint balls of radius c_M having their centers along a diameter. □
Actually $\frac{d}{c_M}$ is a very rough approximation for the range of triviality of $C_n^b(M)$. One can choose instead the maximal number $n(M, c_M)$ of disjoint balls of radius c_M which can be packed in M.

2.10 If i_M denotes the injectivity radius of the manifold M then $i_M \leq c_M$. In fact the critical points of the distance function are contained in the cut locus (see e.g. [3], p.361). Recall that the point x is critical for the function $d(p, *)$ if, for any vector v in the tangent space at x, there exists a geodesic segment from x to p whose tangent vector at x makes an angle $\alpha \leq \frac{\pi}{2}$ with v. For instance, if i_M equals half the length of shortest closed geodesic then $i_M = c_M$.

3 2-packings for flat tori

3.1 2-packings for homogeneous manifolds. If M is a flat torus then

$$C_2^b(M) = \{(r_1, r_2);\ r_1 + r_2 \leq d,\ 0 < r_1, r_2 < \frac{i_M}{2}\},$$

where d is the diameter and i_M the injectivity radius (of any of its points).

In fact for flat tori the injectivity radius equals half the length of the shortest geodesic loop, and one uses 2.10.

3.2 Theorem. Let T_τ be the flat torus $\mathbf{C}/\mathbf{Z}\oplus\mathbf{Z}\tau$. One can always choose τ such that $Re(\tau) > 0$ and $Im(\tau) \in [0, 1)$. Let us assume that $|\tau|^2 + |1-\tau|^2 \geq 1$. Then the following holds:

$$C_2^b(T_\tau) = \{r_1 + r_2 \leq \frac{|1-\tau|}{2\sin(\arg(\tau))},\ 0 < r_1, r_2 < \frac{1}{2}\min(|\tau|, 1, |1-\tau|)\}.$$

Proof: Using the previous claim one has to determine the diameter $d(T_\tau)$ and the injectivity radius i_{T_τ}.

3.3 The diameter of T_τ. If $|\tau|^2 + |1-\tau|^2 \geq 1$ then

$$d(T_\tau) = \frac{|1-\tau|}{2\sin(\arg(\tau))}.$$

Suppose that $|\tau|^2 + |1-\tau|^2 \leq 1$.

- if $|\tau| \geq |1-\tau|$ then

$$d(T_\tau) = \frac{|\tau|}{2\cos(\arg(\tau))},$$

- and if $|\tau| \leq |1-\tau|$ then

$$d(T_\tau) = \frac{|1-\tau|}{2\cos(\arg(1-\tau))}.$$

Proof: Take two points at distance d on T_τ. The parallelogram of vertices $0, \tau, 1, 1+\tau$ is a fundamental domain for the action of the lattice $\mathbf{Z} \oplus \mathbf{Z}\tau$ on the plane. One can suppose that one of the two points is the origin 0. The diagonal joining τ to 1 is the smallest of two and divides the parallelogram into two equal triangles. By symmetry one can assume that the other point belongs to the triangle T of vertices $0, 1, \tau$. One seeks then for the point in T which maximizes the minimum of the three distances at the vertices of the triangle. The way we chose τ implies that T has two acute angles with respect to the horizontal axis.

Assume first that the third angle (at τ) is also acute, which is equivalent to ask that $|\tau|^2 + |1-\tau|^2 \geq 1$ holds. Then there exists an unique point in plane situated at equal distance from all three vertices, and this point lies in T. Thus the diameter is the distance from this point to the origin. An elementary computation yields the desired formula for the diameter.

The other alternative is when $|\tau|^2 + |1-\tau|^2 \leq 1$ holds, and so the third angle is greater than $\frac{\pi}{2}$. Then the point sitting at equal distance from the vertices lies outside the triangle T. However a simple argument shows that the point in T solving the minimax problem is the unique point of the segment $[0, 1]$ which is at equal distance from τ and the other endpoint of the longest edge among $[\tau, 0]$ and $[\tau, 1]$. Again an elementary check proves the claim.□

One has now to compute the length of the smallest closed geodesic for flat tori, or alternatively, the smallest non-zero Euclidean distance between two points of the lattice $\mathbf{Z} \oplus \mathbf{Z}\tau$. This is a classical problem equivalent to the computation of the non-zero infimum a positive definite bilinear form over the integers. There is no close formula for a general binary form, but only estimations (due to Minkowski) of the infimum, and an algorithm permitting to compute it in finitely many steps. Our result below, based on a geometric argument, is slightly more general than the classical one for reduced binary forms (see [1], Theorem II, p.33):

3.4 The injectivity radius. If $|\tau|^2 + |1-\tau|^2 \geq 1$ then

$$i_{T_\tau} = \frac{1}{2} \min(|\tau|, 1, |1-\tau|)\}.$$

Proof: Let consider the points $1, \tau, \tau-1, -1, -\tau, 1-\tau$ which we label $v_1, ..., v_6$. The half-lines $[0v_i$ and $[0v_{i+1}$ determine an angular sector in the plane, which we denote by A_i. Here is to be understood that the indices take values in $\mathbf{Z}/6\mathbf{Z}$. The first remark is that the vectors v_i and v_{i+1} generate all the lattice points in the sector they define:

3.5 One has $\mathbf{Z} \oplus \mathbf{Z}\tau \cap A_i = \mathbf{Z}_+ v_i \oplus \mathbf{Z}_+ v_{i+1}$.

Proof: It is sufficient to check it for A_1. If $m + n\tau \in A_1$ then $n \geq 0$. Assume that $m < 0$. Then $arg(m + n\tau) = \frac{nIm(\tau)}{nRe(\tau)+m} > arg(\tau)$ would contradict the fact that $m + n\tau \in A_1$.□

3.6 If two vectors v and w make an acute angle then $|kv+nw| \geq \min(|v|, |w|)$, for $k, n \in \mathbf{Z}_+$.

The proof is obvious. □
Now this lemma shows that the smallest distance from the origin, in each acute angle sector is obtained for one of the two base vectors. The angles around the origin are the same as those of the triangle T, which are then less than $\frac{\pi}{2}$. This leads to the claimed value for the injectivity radius. □
The advantage of this method is that it generalizes immediately to the higher dimensional cases:

3.7 Flat (acute-angled) n-tori. Let $T_\Lambda = \mathbf{R}^n/\Lambda$, where Λ is a lattice of basis $v_1, v_2, ..., v_n$. Assume that the simplex spanned by the vectors v_i has all its face angles acute, or equivalently that the following conditions are fulfilled:

$$(v_i, v_j) \geq 0, \ i \neq j, \ \ (v_i - v_j, v_i - v_k) \geq 0, \ i \neq j \neq k \neq i,$$

where (,) denotes the inner product in $\mathbf{R}^n$. Then

$$i_{T_\Lambda} = \min(|v_i|, |v_i - v_j|, i \neq j \in \{1, ..., n\}).$$

Proof: Using the obvious reflections of the lattice one divides the Euclidean space into infinite cones, which are spanned by the vectors $v_i - v_j$ and v_j, for $i \neq j \in \{1, ..., n\}$. In each infinite cone the lattice points can be expressed as linear combinations with positive integers coefficients from the basis vectors spanning the cone (the analogue of 3.4). The angle between two spanning vectors is less than $\frac{\pi}{2}$, hence 3.5 shows that in each cone the minimum length is reached on its respective basis vectors.□

4 The volume and the spectrum

In this section one proves that (a subset of) the metric spectrum determines the volume for domains in homogeneous manifolds:

4.1 Theorem. Let D_1 and D_2 be domains in a homogeneous manifold M. If their spectra agree then $vol(D_1) = vol(D_2)$.

Proof: Since the isometry group of M is transitive it follows that the volume of the radius r ball is a function $f(r)$ depending only on r (and M). Consider then the function $f : C_n(M^d) \longrightarrow \mathbf{R}_+$ given by $f(r_1, ..., r_n) = f(r_1) + ... + f(r_n)$ which is the volume of the packed balls. Consider next $vol_n(M) = sup_{(r_i) \in C_n} f(r_1, ..., r_n)$. The claim follows from:

4.2 We have $\sup_n vol_n(M) = vol(M)$.

Proof: This result is valid more generally, without the homogeneity assumption, by setting $vol_n(M)$ for the maximal volume of a n-packing. Roughly speaking it says that we can approximate as close as we want the volume of a domain by a packing with balls (see also a similar argument in [2], p.20-24).

4.3 Let $H \subset M$ be compact, or such that the scalar curvature is bounded. There exists a constant $c_H > 0$ such that: for any domain $D \subset H$ there exists some $r > 0$ and a packing with radius r balls $b_1, b_2, ..., b_k \subset D$ satisfying

$$vol(b_1) + vol(b_2) + ... + vol(b_k) \geq c_H vol(D).$$

Proof: Let consider a maximal packing of D using only radius r balls. Then the balls having the same centers and radius $2r$ will cover D, since otherwise one contradicts the maximality. We derive therefore

$$\sum_{i=1}^{n} vol(B(p_i, r)) \leq vol(D) \leq \sum_{i=1}^{n} vol(B(p_i, 2r)),$$

where n is the number of balls, and p_i are the centers. However, for r close to zero one has

$$\frac{vol(B(p,2r))}{vol(B(p,r))} \sim \frac{\omega(2r)}{\omega(r)} \frac{1-(2r)^2 S_{2r}/6 + O(r^3)}{1 - r^2 S_r/6 + O(r^3)},$$

where S_ρ is the integral scalar curvature on the radius ρ ball and $\omega(r)$ is the volume of the Euclidean ball of radius r. If the scalar curvature is bounded there exists some $r_0(H)$ such that for $r \leq r_0(H)$ one has $\frac{vol(B(p,2r))}{vol(B(p,r))} \leq 2\frac{\omega(2r)}{\omega(r)}$. In particular the volume of the packing is at least $\frac{\omega(r)}{2\omega(2r)} vol(D)$, and so c_H is bounded below by a universal constant. □

Now the proof of 4.2 is clear: take a packing with disjoint balls in M, whose reunion is P_1 and the volume covered is at least $c_M vol(M)$. Take further another packing P_2 covering at least $c_M vol(M - P_1)$, and continue. At the n^{th} step the remaining domain not covered yet has volume less than $(1 - c_M)^n vol(M)$. This ends the proof. As the balls can be chosen small enough the claim is true for C_*^b and C_*^s as well. □

5 Prescribing 2-packing spectra

5.1 Theorem. For any closed n-manifold M there exists a metric with respect to which one has $C_2^b(M) = C_2^b(S^n)$.

Proof: Let us consider $p \in M$ and $B \subset M$ be a small open ball around p. The manifold with boundary $M - B$ has a $(n-1)$-dimensional spine $K \subset M - B$ on which it collapses. Moreover K should be connected. Consider a metric tubular neighborhood of K in M, consisting of the points having distance at most ε to K. For small enough ε this is a regular neighborhood of K. Since $M - B$ is also a regular neighborhood of K there exists an ambient isotopy carrying the metric tubular neighborhood into $M - B$. One induces then a metric on $M - B$ for which $M - B = \{x \in M; d(x, K) \leq \varepsilon\}$. In this metric all points of ∂B are at distance ε far from K.

Consider now the ball B, which inherits a boundary metric structure $g_{\partial B}$ from $M - B$. One extends $g_{\partial B}$ all over B by using a cone construction:

$g_B^A = f(r)g_{\partial B} + A\,d\,r^2$. For general differentiable $f(r)$ the metric g_B is smooth everywhere but at the origin. One chooses $f(r) = r$ for $r \geq \delta > 0$, and vanishing fast enough at 0 in order to make the metric smooth at p, and A very large, in comparison to the size of the boundary ∂B.

One use a similar metric blowing-up at K. Let N_λ be the metric tubular neighborhood around K of radius $\lambda\varepsilon$. If s is the local parameter describing the distance to K, we can decompose locally the initial metric $g_{M-B} = g_\lambda + d\,s^2$, where $g_\lambda = g|_{\partial N_\lambda}$, by using the fact that $M - B - K$ is an open cylinder. Let now deform the metric radially: set $g_{M-B}^A = h(s)g_\lambda + A\,d\,s^2$, where $h(s) = 1$, for $s \geq \delta > 0$ is decreasing fast enough to make g_{M-B}^A smooth at the points of K. One glue the two metrics g_B^A and $g_{M-B}^{A/\varepsilon}$ and get a metric on M with the following properties:

- geodesics emanating from p are radial and have length A.
- from each point of ∂B there exists an unique minimal geodesic to a point of K which realizes the distance to K, and so it has length A.

Using these properties it follows that for large enough A the diameter of M is attained by $d = d(p,q) = 2A$ for any point $q \in K$, since they are all at the same distance from p. Moreover the open balls $B(p,\mu)$ and $B(q, d-\mu)$ are disjoint for $\mu \geq A$, thereby settling our claim. In fact the boundary of $B(p, d-\mu)$ is the same as $\partial N_{d-\mu}$ and $B(q, d-\mu) \subset N_{d-\mu}$. □

5.2 *Remark.*The metrics constructed this way are highly non-generic.

5.3 Theorem. Suppose that $C_2^b(M) = C_2^b(S^n)$ for a complete generic metric on M, and some (generic) metric of convex body on S^n. Then M is homeomorphic to S^n.

Proof: For a generic metric on M, for each point p there exists at most one point q such that $d(p,q) = d$, where d is the diameter. Actually one might have supposed only that this set is finite (since it should be a spine of the 1-holed manifold, hence connected).

By hypothesis there exists points p_n, q_n such that the balls $B(p_n, d - \frac{1}{n})$ and $B(q_n, \frac{1}{n})$ are disjoint topological balls. By extracting a subsequence one finds the limit points p_∞ and q_∞ in M which satisfy $d(p_\infty, q_\infty) = d$. The genericity assumption implies that $d(p_\infty, x) < d$ for any $x \in M - \{q_\infty\}$.

Let $\alpha = inj\ rad(q_\infty) > 0$. Then $B(q_\infty, \alpha)$ is a topological ball containing q_∞. In particular $d(p_\infty, \partial B(q_\infty, \alpha)) = \beta > 0$. Consider then $\varepsilon < \alpha/2$, and $n \geq 2/\alpha$ large enough in order to satisfy $d(p_n, p_\infty) < \varepsilon$ and $d(q_n, q_\infty) < \varepsilon$. From the triangle inequality one derives that $d(p_n, \partial B(q_\infty, \alpha)) \leq \varepsilon + d - \alpha < d - \frac{1}{n}$. This shows that $\partial B(q_\infty, \alpha) \subset B(p_n, \frac{1}{n})$. Using then Schoenflies theorem one derives that the sphere $\partial B(q_\infty, \alpha)$ bounds a ball in $B(p_n, \frac{1}{n})$. On the other hand $\partial B(q_\infty, \alpha)$ is the boundary of another ball $B(q_\infty, \alpha)$, and their reunion is M. Standard results in manifold topology imply that M is homeomorphic to a sphere. □

5.4 A similar argument was used in ([3], Thm.1.12, p.363) for proving that a manifold for which the distance function $d(p, *)$ has only one critical point is homeomorphic to a sphere.

5.5 *Remark.*The small enough balls in a smooth manifold are smooth balls. If $B(p,r)$ is a topological ball for which the concentric balls $B(p,r')$ for $r' < r$ are smooth balls, then $B(p,r)$ is smoothable. This shows that a smooth manifold as in 5.3 is actually a twisted sphere.

We didn't use the full power of the equality of spectra, but just the fact that there exist balls with radius arbitrary close to the diameter. Therefore our result states the stability of the topological shape in the presence of a stable 1-packing radius:

5.6 If M has a generic metric with $pack_1(M) = pack_1(S^n)$ then M is a twisted sphere.

5.7 *Remark.* In this respect the diameter sphere theorem of Grove and Shiohama ([6]) shows that a much stronger stability holds (i.e. $pack_1(M) > \frac{1}{2}pack_1(S^n)$ implies M is a twisted sphere) in the presence of a lower bound for the sectional curvature $sec(M) \geq 1$.

5.8 *Remark.* The genericity condition in 5.3. could be replaced by any other condition implying that the set $A(p)$ of points at distance d from p consists in finitely many points.

5.9 Questions.

- It seems that, given $N \geq 2$ and a closed n-manifold M, there exists a metric on M such that $C_k^b(M) = C_k^b(S^n)$, for all $k \leq N$.
- Whether the condition $C_k^b(M) = C_k^b(S^n)$, holding for some generic metric, and all $k \leq N$, implies that M and S^n are actually Lipschitz ε_k-close to each other, for some constants ε_k fulfilling $\lim_k \varepsilon_k = 0$? This would be a step towards the recognition problem for the standard metric of S^n.
- What happens when one relax the equality of spectra to that of packing radii i.e. $pack_k(M) = pack_k(S^n)$, for $k \leq N$? Does then follow that in the generic situation M is homeomorphic to S^n, or better, that it is Lipschitz close to S^n ? If one adds conditions like $sec(M) \geq 1$ then extra flexibility is expected, for instance that an inequality like $pack_k(M) > c_k pack_k(S^n)$ implies M is a sphere, where $c_k \in (2^{-1/n}, 1)$ are some constants to be determined.

References

1. Cassels, J.W.S.: *An introduction to the geometry of numbers*, Classics in Math., Springer-Verlag, 1997.
2. Gromov, M., Lafontaine, J., Pansu, P.: *Structures métriques pour les variétés riemannienes*, Cedic/Fernand Nathan, Paris, 1981.
3. Grove, K.: Critical point theory for distance functions, *Proceedings Symposia Pure Math., AMS,* (R.Greene, S.T.Yau, Ed.) vol **54**, part 3, 357-385 (1993).
4. Grove, K., Markorvsen, S.: Curvature, triameter and beyond, *Bull. A.M.S.* **27**, 261-165 (1992).

5. Grove, K., Markorvsen, S.: New extremal problems for the Riemannian recognition program via Alexandrov geometry, *Journal A.M.S.*, **8**, 1-28 (1995).
6. Grove, K., Shiohama, K.: A generalized sphere theorem, *Ann.of Math.*, **106**, 201-211 (1977).
7. Grove, K., Wilhelm, F.: Hard and soft packing radius theorems, *Ann.of Math.*, **142**, 213-237 (1995).
8. Otal, J.P.: Le spectre marqué des longueurs des surfaces à courbure negative, *Ann.of Math.*, **131**, 151-162 (1990).