ORBIFOLD KÄHLER GROUPS RELATED TO MAPPING CLASS GROUPS

PHILIPPE EYSSIDIEUX AND LOUIS FUNAR

Abstract. We construct certain orbifold compactifications of the moduli stack of pointed stable curves over $\mathbb{C}$ and study their fundamental groups by means of their quantum representations. This enables to construct interesting Kähler groups and to settle most of the candidates for a counter-example to the Shafarevich conjecture on holomorphic convexity proposed in 1998 by Bogomolov and Katzarkov, using TQFT representations of the mapping class groups.

2010 MSC Classification: 14 H 10, 32 Q 30 (Primary) 32 J 25, 32 G 15, 14 D 23, 57 M 07, 20 F 38, 20 G 20, 22 E 40 (Secondary).

Keywords: Kähler Orbifolds, Kähler Groups, Mapping class group, Moduli space of Curves, Deligne-Mumford stacks, Topological Quantum Field Theory.

Contents

1. Introduction and statements

Let $\mathrm{Mod}(\Sigma_g^n)$ be the mapping class group of the genus g orientable surface Σ_g^n with n punctures (or marked points) and $\mathrm{PMod}(\Sigma_g^n)$ denote the *pure* mapping class group of those classes which fix *pointwise* the punctures. Then $\mathrm{PMod}(\Sigma_g^n)$ and $\mathrm{Mod}(\Sigma_g^n)$ occur as the fundamental groups of the analytification $\mathcal{M}_g^{n\ an}$ of the moduli stacks of curves and $[S_n\backslash\mathcal{M}_g^{n\ an}]$, respectively, where S_n acts on $\mathcal{M}_g^{n\ an}$ permuting the markings. These stacks are separated, smooth, Deligne-Mumford and their moduli spaces are quasiprojective, but non proper except in the trivial case $g = 0, n = 3$. Alternatively, $\mathrm{PMod}(\Sigma_g^n)$ is the orbifold topological fundamental group of the coarse moduli space $M_g^{n\ an}$ of the stack $\mathcal{M}_g^{n\ an}$, which is the usual moduli space of curves.

Constructing stacky compactifications of such objects can be used as an intermediate step to construct interesting Kähler groups [19, 20] and in the present case, to recast some of the basic ideas of Teichmüller level structures [7, 9, 43].

For every integer $p \geq 2$ we consider the subgroup $\mathrm{Mod}(\Sigma_g^n)[p] \subset \mathrm{Mod}(\Sigma_g)$ generated by the p-th powers of Dehn twists along simple closed curves in Σ_g^n. Note that $\mathrm{Mod}(\Sigma_g^n)[p] \subset \mathrm{PMod}(\Sigma_g^n)$ is a normal subgroup of $\mathrm{Mod}(\Sigma_g)$ and in fact a characteristic one. More generally, there are only finitely many, say $N_{g,n}$ conjugacy classes of Dehn twists, or equivalently, distinct orbits of essential simple closed curves on Σ_g^n under the mapping class group action.

Date: February 21, 2024.

The research of P. E. was partially supported by the ANR project Hodgefun ANR-16-CE40-0011-01.

For instance $N_g = \lfloor \frac{g}{2} \rfloor + 1$, $N_{g,1} = g$. Simple closed curves orbits are determined by the homeomorphism type of the complementary subsurface, which might be either connected for non-separating curves or disconnected and hence determined by the set/pair of genera of its two components, for separating curves. Fix an enumeration of these homeomorphism types starting with the non-separating one. For each vector $\mathbf{k} = (k_0, k_1, k_2, \ldots, k_{N_{g,n}-1})$ we define $\mathrm{Mod}(\Sigma_g^n)[\mathbf{k}]$ as the (normal) subgroup generated by k_0-th powers of Dehn twist along non-separating simple closed curves and k_j-th powers of Dehn twists along simple closed curves of type j. As a shortcut we use $\mathrm{Mod}(\Sigma_g^n)[k; m]$ for $\mathbf{k} = (k, m, m, m \ldots, m)$ and $\mathrm{Mod}(\Sigma_g^n)[k; -]$ for $\mathbf{k} = (k;)$, where k_i are absent for $i > 0$.

The groups $\mathrm{Mod}(\Sigma_g)/\mathrm{Mod}(\Sigma_g)[p]$ were studied in [24, 31] and $\mathrm{Mod}(\Sigma_g^1)/\mathrm{Mod}(\Sigma_g^1)[p]$ in [26, 37]. In [2] the authors proved that they are virtually Kähler groups when g.c.d.$(p, 6) = 1$, by considering the Deligne-Mumford compactifications of the moduli spaces of curves for some level structures for which they are smooth projective varieties (see e.g. [6]).

Our first main result is:

Theorem 1.1. *Let $g, n \in \mathbb{N}$ such that $2 - 2g - n < 0$, $(g, n) \neq (1, 1), (2, 0)$.*

For every $\mathbf{k} \in \mathbb{N}_{>0}^{N_{g,n}}$ the quotient $\mathrm{PMod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[\mathbf{k}]$ is the fundamental group of a compact Kähler orbifold $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{k}]$ compactifying $\mathcal{M}_g^{n\ an}$ whose coarse moduli space is the moduli space $\overline{M_g^{n\ an}}$ of stable n-punctured curves of genus g.

If $p \geq 5$ is odd, $\overline{\mathcal{M}_g^{n\ an}}[p]$ is uniformizable, hence $\mathrm{PMod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[p]$ is a Kähler group, i.e. it occurs as the fundamental group of a compact Kähler manifold.

If $p \geq 10$ is even and $(g, n, p) \neq (2, 0, 12)$, $\overline{\mathcal{M}_g^{n\ an}}[2p, p/\mathrm{g.c.d.}(p, 4)]$ is uniformizable and $\mathrm{Mod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[2p, p/\mathrm{g.c.d.}(p, 4)]$ is a Kähler group.

Performing first the $\mathbf{k}$-th root stack construction on the moduli stack $\overline{\mathcal{M}_g^{n\ an}}$ of n-punctured stable curves of genus g and further the canonical stack construction of Vistoli [1, 28, 54] provides us with a smooth proper Deligne Mumford stacks $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{k}]$. After the completion of this article we learned about an alternative interpretation of $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{k}]$ as the stack of twisted stable curves (see [12]). A Van Kampen argument shows that $\mathrm{PMod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[\mathbf{k}]$ is the fundamental group of the compact Kähler orbifold $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{k}]$. To prove that it is actually Kähler we will use the uniformizability criterion of [18] by showing that a suitable linear representation of it has infinite image while its restriction to isotropy groups is injective. Although a representation with finite image will still be convenient for settling the uniformisability, the infiniteness will be essential in the second part of the paper. The representations we consider are the so-called Reshetikhin-Turaev TQFT representations of mapping class groups, which define local systems on the Kähler orbifolds $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{k}]$, for a suitable choice of $\mathbf{k}$ as in the statement, depending on a natural number p which is the level of the TQFT. The explicit study of the images of the isotropy groups under quantum representations of mapping class groups will then prove our theorem. Note that quantum representations of these groups cannot be injective unless their images are non-arithmetic subgroups of higher rank Lie groups (see [27]). It is not clear to the authors whether the uniformizability criterion could also be verified by some convenient homological representations associated to characteristic coverings (see [6, 7, 49]) in order to give an alternate proof of our claim.

Our construction was motivated by its relation to certain complex projective surfaces proposed by Bogomolov and Katzarkov [8] as potential counterexamples to the Shafarevich conjecture on holomorphic convexity that remained unsettled up to now. These surfaces are obtained in the following way. Let $\pi : S \to C$ be a smooth projective surface fibered over a curve such that the regular fibers have genus $g \geq 2$ and the singular fibers are reduced and have only nodes. Let $N \in \mathbb{N}^*$ be a positive integer. Perform a base change along a covering $C' \to C$ with uniform ramification index N at the images of the singular fibers. Then, for most N, and if π is stable, the surface $S \times_C C'$, which has a finite number of singular points of type A_{N-1} lying over the nodes in the singular fibers, has a quasi-étale cover $S(N)$ which is a smooth projective surface inheriting from π a fibration over some curve.

Our second main result, which was actually our main motivation and is to our knowledge the first application of Quantum Topology to the theory of fundamental groups of projective manifolds, hence to uniformization in several complex variables, settles the question raised in [8] in all but a few explicit cases, as follows:

Theorem 1.2. *If $N \notin \{1, 2, 3, 4, 6, 8, 12, 16, 20, 24\}$ the universal covering space of the Bogomolov-Katzarkov surface $S(N)$ is holomorphically convex. Moreover, if $\pi : S \to C$ is a stable curve over C, then it is Stein.*

To prove this theorem we first note that quantum representations give rise to representations of $\pi_1(S(N))$, the reason being that $S(N)$ maps non trivially to $\overline{\mathcal{M}_g^{1\ an}}[N]$. Let π_g denote the fundamental group of the closed orientable surface of genus g. The image I_N of the fundamental group of a general fiber F, say of genus g, of this fibration in $\pi_1(S(N))$ is what Bogomolov and Katzarkov call a Burnside type quotient of π_g, and is an infinite group if N is large enough. In [21] the authors proved that the Stein property will follow from the existence of a finite dimensional semi-simple representation of $\pi_1(S(N))$ with suitable infiniteness properties on the fibers of the morphism π, the simplest one being that the restriction to I_N is infinite. The only *obvious* linear representations of $\pi_1(S(N))$ with infinite image have finite image when restricted to I_N. However, and this observation was our starting point, the recent result [37] translates into the fact that quantum representations restricted to I_N have an infinite image for large enough odd N. Eventually, a remarkable property of TQFT representations, namely their functoriality when passing to subsurfaces, enables us to settle the question raised by [8].

If π is not stable, or N is even, then $S(N)$ may be only a non developable orbisurface. If $S(N)$ is not developpable, its universal covering space is a complex analytic Deligne-Mumford stack, and we say it is Stein, resp. holomorphically convex, if such is its moduli space (a normal complex analytic space with only quotient singularities).

We believe the examples constructed here to be extremely interesting from the point of view of uniformization in several complex variables. First of all, there are not so many examples of infinite Kähler groups or local systems. The fact that the examples in [8] had an unsettled status has been a major stumbling block of that part of Non Abelian Hodge Theory for 25 years. It does not seem to be a trivial task to settle the Shafarevich and the Toledo conjectures for $\overline{\mathcal{M}_g}[p]$. We also conjecture Kazhdan's property (T) for its fundamental group since it would follow from Kazhdan's property (T) for the mapping class group. We hope to come back on these questions in future work.

It is an important open question whether one can construct a complex projective surface whose fundamental group is infinite and has no complex linear finite dimensional representation with infinite image. For all but finitely many p, one knows that $\mathrm{Mod}(\Sigma_g)/\mathrm{Mod}(\Sigma_g)[p]$ is infinite but the proof uses a linear representation [24]. This representation can be lifted to a representation of $\mathrm{PMod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[p]$ with infinite image. Hence our construction gives infinite Kähler groups but will not solve that question. The fact that the representation comes from a Reshetikhin-Turaev TQFT is however striking from an algebro-geometric perspective since it is not known to the authors if these representations are motivic or rigid.

Outline. In the next Section we introduce quantum representations of mapping class groups, compute the orders of Dehn twists and prove they factor through $\mathrm{Mod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[p]$ if p is odd and through $\mathrm{Mod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[2p, p/g.c.d(4,p)]$ if p even. We prove that these quantum representations have infinite images in most cases. In Section 3 we define the Mumford-Deligne stack $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{k}]$ and show that it is an orbifold with the fundamental group $\mathrm{PMod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[\mathbf{k}]$. The results on the quantum representations are further used in Section 4 to show that the orbifolds $\overline{\mathcal{M}_g^{n\ an}}[p]$ are uniformizable, thereby proving Theorem 1.1. In Section 5, we explain the construction of the Bogomolov-Katzarkov surfaces by starting with a non isotrivial fibration of a smooth complex projective surface, namely the pull-back of the universal curve $\overline{\mathcal{M}_g^{1\ an}}$ along some holomorphic maps from a smooth algebraic curve into $\overline{\mathcal{M}_g^{an}}$ and then taking the cartesian product with $\overline{\mathcal{M}_g^{1\ an}}[p]$. We show that the Bogomolov-Katzarkov surfaces obtained in that manner are uniformizable Kähler orbifolds, when the orbifolds $\overline{\mathcal{M}_g^{1\ an}}[p]$ and $\overline{\mathcal{M}_g^{1\ an}}[2p, p/\mathrm{g.c.d}(4,p)]$ for even p respectively, are uniformizable. Section 6 contains a proof of Theorem 1.2 answering in the affirmative the Shafarevich conjecture on holomorphic convexity for these surfaces in most cases, by analyzing the images of their fundamental groups under the quantum representations above and using [21] (we actually only need the reductive surface case proved in [34]).

All algebraic varieties and stacks will be over $\mathbb{C}$ and we will mainly think of them through their analytification as complex-analytic objects.

Acknowledgements. The authors are indebted to J. Aramayona, M. Boggi, P. Godfard, L. Katzarkov, B. Klingler, J. Marché and C. Simpson for helpful discussions.

2. Quantum mapping class group representations

2.1. **Outline.** In this Section we analyse a series of finite dimensional representations of the mapping class groups, which arises in the Reshetikhin-Turaev construction of TQFTs. These representations factor through quotients of the form $\mathrm{PMod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[\mathbf{k}]$, where $\mathbf{k}$ depends on the level p of the TQFT and the boundary colors (see below). The main question addressed here is to find those levels $\mathbf{k}$ with the property that the corresponding representations of mapping class groups of punctured surfaces restrict to infinite representations of (possibly punctured) surface subgroups. After settting the stage with an informal description of the skein TQFT in Section 2.2 we discuss first the so-called spaces of conformal blocks acted by these representations. We give in Section 2.3 a complete description of those genus zero conformal blocks which are trivial, by means of explicit combinatorial computations. These are key ingredients of the computation of the order of a Dehn twist in terms of various parameters involved, which is given in Section 2.4. Using previous results on the infiniteness of the image of the mapping class groups we are finally able to settle our question in Proposition 2.22.

2.2. **The setting of the skein TQFT.** A 3d TQFT is, loosely speaking[1], a functor from the category of cobordisms between surfaces into the category of finite dimensional vector spaces. Specifically, the objects of the first category are closed oriented surfaces endowed with colored banded points and morphisms between two objects are oriented cobordisms decorated by colored uni-trivalent ribbon graphs compatible with the banded points. A banded point on a surface is a point with a tangent vector at that point, or equivalently a germ of an oriented interval embedded in the surface. There is a corresponding surface with colored boundary obtained by deleting a small neighborhood of the banded points and letting the boundary circles inherit the colors of the respective points.

We will use the TQFT functor $\mathcal{V}_p$, for $p \geq 3$ and a primitive root of unity A of order $2p$, as defined in [5]. The vector space associated by the functor $\mathcal{V}_p$ to a surface is called the *space of conformal blocks*. Let Σ_g denote the genus g closed orientable surface, H_g be a genus g handlebody with $\partial H_g = \Sigma_g$. Assume given a finite set $\mathcal{Y}$ of banded points on Σ_g. Let G be a uni-trivalent ribbon graph embedded in H_g in such a way that H_g retracts onto G, its univalent vertices are the banded points $\mathcal{Y}$ and it has no other intersections with Σ_g.

The natural number $p \geq 3$ is called the *level* of the TQFT. We define the *set of colors* in level p to be $\mathcal{C}_p = \{0, 2, 4, \ldots, p-3\}$, if p is odd and $\mathcal{C}_p = \{0, 1, 2, \ldots, \frac{p-4}{2}\}$, if p is even, respectively.

An edge coloring of G is called *p-admissible* if the following conditions are satisfied:

[1]See [53] for a correct definition and a discussion of TQFT with anomalies.

(1) The *triangle inequality* is satisfied at each trivalent vertex of G. Namely, if i, j, k are the colors of the incoming edges, one has $|j-k| \leq i \leq j+k$.
(2) The sum of of the three colors around a vertex is even, this condition being empty for odd p.
(3) The sum of the three colors around a vertex is at most $q = 2p-4$, if p is odd and at most $q = p-4$, if p is even, respectively. We call it the *q-bound inequality.*

Fix a coloring of the banded points $\mathcal{Y}$. Then there exists a basis of the space of conformal blocks associated to the surface $(\Sigma_g, \mathcal{Y})$ with the colored banded points (or the corresponding surface with colored boundary) which is indexed by the set of all p-admissible colorings of G extending the boundary coloring. We denote by $W_{g,p,(i_1,i_2,\ldots,i_r)}$ (or, dropping p from the notation, by $W_{g,(i_1,i_2,\ldots,i_r)}$, if p is uniquely determined by the context) the vector space associated to the closed surface Σ_g with r banded points colored by $i_1, i_2, \ldots, i_r \in \mathcal{C}_p$. Note that banded points colored by 0 do not contribute.

Observe that an admissible p-coloring of G provides an element of the skein module [50] $S_A(H_g, \mathcal{Y}, (i_1, i_2, \ldots, i_r))$ [2] of the handlebody with the banded boundary points $\mathcal{Y}$ colored by $(i_1, i_2, \ldots, i_r)$, evaluated at the primitive $2p$-th root of unity A. This skein element is obtained by cabling the edges of G by the Jones-Wenzl idempotents prescribed by the coloring and having banded points colors fixed. Jones-Wenzl idempotents were discovered by V. Jones [32] and their inductive construction is due to Wenzl [55].

We suppose that H_g is embedded in a standard way into the 3-sphere S^3, so that the closure of its complement is also a genus g handlebody $\overline{H}_g$. There is then a sesquilinear form:

$$\langle\ ,\ \rangle = \langle\ ,\ \rangle_p : S_A(H_g, \mathcal{Y}, (i_1, i_2, \ldots, i_r)) \times S_A(\overline{H}_g, -\mathcal{Y}, (i_1, i_2, \ldots, i_r)) \to \mathbb{C}$$

defined by

$$\langle x, y\rangle = \langle x \sqcup y\rangle.$$

Here $x \sqcup y$ is the element of $S_A(S^3)$ is the union of x and y along $\mathcal{Y}$ in $H_g \cup \overline{H}_g = S^3$, and $\langle\ \rangle : S_A(S^3) \to \mathbb{C}$ is the Kauffman bracket invariant [33].

The space of conformal blocks $W_{g,(i_1,i_2,\ldots,i_r)}$ is the quotient $S_A(H_g)/\ker\langle\ ,\ \rangle$ by the left kernel of the sesquilinear form above. It follows that $W_{g,(i_1,i_2,\ldots,i_r)}$ is endowed with an induced *Hermitian form* H_A.

The projections of skein elements associated to the p-admissible colorings of a trivalent graph G as above form an orthogonal basis of $W_{g,(i_1,i_2,\ldots,i_r)}$ with respect to H_A. It is known ([5]) that H_A only depends on the p-th root of unity $\zeta_p = A^2$ and that in this orthogonal basis the diagonal entries belong to the totally real maximal subfield $\mathbb{Q}(\zeta_p + \overline{\zeta_p})$ (after rescaling). If σ is a p-admissible coloring of G extending the boundary coloring, then the diagonal term of H_A associated to this basis vector is

$$\eta^{g+1} \frac{\prod_{v \text{ vertex}} \langle v\rangle}{\prod_{e \text{ edge}} \langle e\rangle}, \tag{1}$$

where products are taken over the set of all trivalent vertices v and all edges e of the graph G, and

$$\eta = \frac{1}{2p}(A\kappa)^3(A^2 - A^{-2}) \sum_{j=1}^{2p} (-1)^j A^{j^2},$$

where

$$\kappa^6 = A^{-6-\frac{p(p+1)}{2}}$$

is a fixed parameter of the theory. Note that changing κ to $-\kappa$ will change the signature of H_A into its opposite. For an edge e labeled by the color a we put

$$\langle e\rangle = (-1)^a [a+1],$$

[2] If the context allows, we will use the shorthand notations $S_A(H_g, \mathcal{Y}, (i_1, i_2, \ldots, i_r)) = S_A(H_g, \mathcal{Y}) = S_A(H_g)$.

and for a trivalent vertex whose adjacent edges are labeled in counterclockwise order by the colors a, b, c we put

$$\langle v\rangle = (-1)^{i+j+k+1}\frac{[i+j+k+1]![i]![j]![k]!}{[a]![b]![c]!}, \tag{2}$$

where

$$i = \frac{a+b-c}{2},\ j = \frac{b+c-a}{2},\ k = \frac{c+a-b}{2}, \tag{3}$$

$$[m] = \frac{A^{2m}-A^{-2m}}{A^2-A^{-2}},\ [m]! = [1][2]\cdots[m-1][m]. \tag{4}$$

Let $G' \subset G$ be a uni-trivalent subgraph whose degree one vertices are colored, corresponding to a sub-surface Σ' of Σ_g with colored boundary. The projections in $W_{g,(i_1,i_2,\ldots,i_r)}$ of skein elements associated to the p-admissible colorings of G' form an orthogonal basis of the space of conformal blocks associated to the surface Σ' with colored boundary components.

There is a linear geometric action of the mapping class groups of the handlebodies with marked banded boundary $(H_g, \mathcal{Y})$ and $(\overline{H}_g, -\mathcal{Y})$ respectively on their skein modules and hence on the space of conformal blocks. Moreover, these actions extend to a projective action $\rho_{g,p,(i_1,\ldots,i_r),A}$ of $\mathrm{Mod}(\Sigma_g^r)$ on $W_{g,(i_1,i_2,\ldots,i_r)}$ respecting the Hermitian form $H_{\zeta_p} = H_A$. When referring to $\rho_{g,p,(i_1,\ldots,i_r),A}$ the subscript specifying the genus g will most often be dropped when its value will be clear from the context. Notice that the mapping class group of an essential (i.e. without annuli or disks complements) sub-surface $\Sigma' \subset \Sigma_g$ is a subgroup of $\mathrm{Mod}(\Sigma_g)$ which preserves the subspace of conformal blocs associated to Σ' with colored boundary. It is worthy to note that $\rho_{p,(i_1,\ldots,i_r),A}$ only depends on $\zeta_p = A^2$, so we can unambigously shift the notation for this representation to $\rho_{p,(i_1,\ldots,i_r),\zeta_p}$. We will often drop the reference to the root of unity ζ_p, as changing it amounts to use a Galois conjugacy of the real cyclotomic field.

There is a central extension $\widetilde{\mathrm{Mod}(\Sigma_g)}$ of $\mathrm{Mod}(\Sigma_g)$ by $\mathbb{Z}$ and a linear representation $\widetilde{\rho}_{p,\zeta_p}$ on W_g which resolves the projective ambiguity of ρ_{p,ζ_p}. The largest such central extension has class 12 times the Euler class (see [29, 45]), but the central extension considered in this paper is an index 12 subgroup of it, called $\widetilde{\Gamma}_1$ in [45].

We denote by $\Sigma_{g,n}^r$ the compact orientable surface of genus g with n boundary components and r marked points. Then $\mathrm{Mod}(\Sigma_{g,n}^r)$ denotes the pure mapping class group of $\Sigma_{g,n}^r$ which fixes pointwise boundary components and marked points.

We consider a subsurface $\Sigma_{g,r} \subset \Sigma_{g+r}$ whose complement consists of r copies of $\Sigma_{1,1}$. Let $\widetilde{\mathrm{Mod}(\Sigma_{g,r})}$ be the pull-back of the central extension $\widetilde{\mathrm{Mod}(\Sigma_g)}$ to the subgroup $\mathrm{Mod}(\Sigma_{g,r}) \subset \mathrm{Mod}(\Sigma_{g+r})$. Then $\widetilde{\mathrm{Mod}(\Sigma_{g,r})}$ is also a central extension, which we denote $\widetilde{\mathrm{Mod}(\Sigma_g^r)}$, of $\mathrm{Mod}(\Sigma_g^r)$ by $\mathbb{Z}^{r+1}$.

Definition 2.1. *Let $p \geq 4$ and ζ_p a primitive p-th root of unity. We denote by $\widetilde{\rho}_{p,\zeta_p,(i_1,i_2,\ldots,i_r)}$ the linear representation of the central extension $\widetilde{\mathrm{Mod}(\Sigma_g^r)}$ which acts on the vector space $W_{g,p,(i_1,i_2,\ldots,i_r)}$ associated by the TQFT to the surface with the corresponding colored banded points (see [29, 45]).*

The functor $\mathcal{V}_p$ associates to a handlebody H_g the projection of the skein element corresponding to the trivial coloring of the trivalent graph G by 0. The invariant associated to a closed 3-manifold is given by pairing the two vectors associated to handlebodies in a Heegaard decomposition of some genus g and taking into account the twisting by the gluing mapping class action on W_g.

One should notice that the skein TQFT $\mathcal{V}_p$ is unitary, in the sense that H_{ζ_p} is a positive definite Hermitian form when $\zeta_p = (-1)^p \exp\left(\frac{2\pi i}{p}\right)$, corresponding to $A_p = (-1)^{\frac{p-1}{2}} \exp\left(\frac{(p+1)\pi i}{2p}\right)$. For the sake of notational simplicity, from now we will drop the subscript p in ζ_p, when the order of the root of unity will be clear from the context and the precise choice of the root of given order won't matter. Note that for a general primitive p-th root of unity, the isometries of H_ζ form a pseudo-unitary group.

Now, the image $\rho_{p,\zeta}(T_\gamma)$ of a right hand Dehn twist T_γ in a convenient basis given by a trivalent graph is easy to describe. Assume that the simple curve γ is the boundary of a small disk intersecting once transversely an edge e of the graph G. Consider $v \in W_g$ a vector of the basis given by colorings of the graph G and assume that edge e is labeled by the color $c(e) \in \mathcal{C}_p$. Then the action of the (canonical) lift $\widetilde{T_\gamma}$ of the Dehn twist T_γ in $\widetilde{\mathrm{Mod}(\Sigma_g)}$ is given by (see [5], 5.8) :

$$\widetilde{\rho}_{p,\zeta}(\widetilde{T_\gamma})v = (-1)^{c(e)} A^{c(e)(c(e)+2)} v.$$

Actually the central extension $1 \to \mathbb{Z} \to \widetilde{\mathrm{Mod}(\Sigma_g)} \to \mathrm{Mod}(\Sigma_g) \to 1$ is trivial over $\mathrm{Mod}(H_g) < \mathrm{Mod}(\Sigma_g)$ which enables to define $\widetilde{\rho}_{p,\zeta}(\widetilde{\sigma})$ for every mapping class $\sigma \in \mathrm{Mod}(H_g)$ as the geometric action of σ on the reduced skein module of H_g. The condition on γ implies that $T_\gamma \in \mathrm{Mod}(H_g)$.

2.3. **Spaces of conformal blocks in genus zero.** In the rest of the article, p will be a positive integer, n a non negative integer, $i_1, \dots, i_n$ a choice of n colors in $\mathcal{C}_p$ and $\rho_{p,(i_1,\dots,i_n)}$ will denote the homomorphism of $\mathrm{Mod}(\Sigma_g^n)$ in the *projective* general linear group of the space of conformal blocks deduced from $\tilde{\rho}_p$. We use the convention that the color 0 be dropped from the boundary colors, as the corresponding representations are the same.

A simple closed curve on a surface is *essential* if it is neither null-homotopic nor homotopic to a boundary component. Essential simple closed curves are either called *non-separating*, when their complement is connected, or *separating*, otherwise. The *genus* of a separating simple closed curve is the minimum of the genera of the two connected subsurfaces which it bounds.

Definition 2.2. *Define $\delta_p(\mathbf{i})$, where $\mathbf{i} = (i_1, i_2, \dots, i_n)$, $i_1, i_2, \dots, i_n \in \mathcal{C}_p$, as the number of those $j \in \mathcal{C}_p$ such that $\dim W_{0,p,(i_1,i_2,\dots,i_n,j)} \neq 0$. The n-tuple $\mathbf{i} = (i_1, i_2, \dots, i_n)$ is said to be* generic *when $\delta_p(\mathbf{i}) \geq 3$.*

The aim of this section is to give explicit formulas for $\delta_p(i_1, i_2, \dots, i_n)$.

Lemma 2.3. *Let $p \geq 5$ be odd. Let $i_1, i_2, \dots, i_m$, $m \geq 3$, be colors from $\mathcal{C}_p$, not all equal to 0. Then the space of conformal blocks $W_{0,p,(i_1,i_2,\dots,i_m)}$ is non-zero if and only if the condition $P(k,m)$ below is satisfied:*

$$2\sum_{\alpha=1}^{2k+1} i_{s_\alpha} \leq 2k(p-2) + \sum_{t=1}^{m} i_t, \tag{5}$$

for all $0 \leq k \leq \frac{m-1}{2}$ and all subsets $\{s_1, s_2, \dots, s_{2k+1}\} \subset \{1, 2, \dots, m\}$.

Proof. These conditions for $k = 0$ (triangle inequalities) and $k = 1$ (the q-bound inequality) are equivalent to the p-admissibility, in the first non-trivial case $m = 3$.

Consider now a pants decomposition of the holed sphere, every pair of pants having one boundary curve from $\partial\Sigma_{0,m}$ except two of them each of which has precisely two boundary curves from $\partial\Sigma_{0,m}$. We choose a subset of k vertices of the dual graph such that their stars are disjoint. We sum the q-bound inequalities corresponding to these k vertices with the triangle inequalities of the remaining vertices, which concern the lower bounds for the internal edges, and obtain the inequality $P(k,m)$.

Conversely, we claim that the $P(k,m)$ conditions are sufficient to guarantee the existence of a p-admissible coloring of the dual graph. We proceed by induction on m. Suppose the claim is true for all spheres with at most m boundary components and consider $(i_1, i_2, \dots, i_{m+1})$ satisfying the conditions $P(k, m+1)$. We want to find $j_m \in \mathcal{C}_p$ such that the triple (j_m, i_m, i_{m+1}) satisfies the p-admissibility condition and $(i_1, i_2, \dots, i_{m-1}, j_m)$ satisfies the conditions $P(k,m)$. This amounts to the following two sets of inequalities:

$$|i_{m+1} - i_m| \leq j_m \leq \min(i_m + i_{m+1}, 2(p-2) - (i_m + i_{m+1}), p-3), \tag{6}$$

$$(7)\qquad 2\sum_{\alpha=1}^{2k+1} i_{s_\alpha} - 2k(p-2) - \sum_{t=1}^{m-1} i_t \le j_m \le 2k(p-2) + \sum_{t=1}^{m-1} i_t - 2\sum_{\alpha=1}^{2k} i_{s_\alpha}.$$

These inequalities have solutions $j_m \in \mathcal{C}_p$ if and only if the intervals defined above have non-empty intersection, namely if

$$(8)\qquad |i_{m+1} - i_m| \le 2k(p-2) + \sum_{t=1}^{m-1} i_t - 2\sum_{\alpha=1}^{2k} i_{s_\alpha}$$

and

$$(9)\qquad 2\sum_{\alpha=1}^{2k+1} i_{s_\alpha} - 2k(p-2) - \sum_{t=1}^{m-1} i_t \le \min(i_m + i_{m+1}, 2(p-2) - (i_m + i_{m+1}), p-3).$$

Inequality (8) is equivalent to $P(k, m+1)$. Inequality (9) consists of three inequalities, which are respectively equivalent to $P(k, m+1)$, $P(k+1, m+1)$ and

$$(10)\qquad 2\sum_{\alpha=1}^{2k+1} i_{s_\alpha} \le 2k(p-2) + (p-3) + \sum_{t=1}^{m-1} i_t.$$

This is a consequence of the two former inequalities and the fact that all i_j are even. $\square$

Lemma 2.4. *Let $p \ge 5$ be odd. Then for every $n \ge 2$, $\mathbf{i} = (i_1, i_2, \dots, i_n) \in \mathcal{C}_p^n$, such that $i_1 \le i_2 \le \dots \le i_n$ [3], we have:*

$$\delta_p(\mathbf{i}) = 1 + \frac{1}{2}\left(J_{p,\max}(\mathbf{i}) - J_{p,\min}(\mathbf{i})\right),$$

where

$$(11)\qquad J_{p,\max}(\mathbf{i}) = \min_{0\le \ell \le \frac{n}{2}} \min\left(p-3, \sum_{t=1}^{n-2\ell} i_t - \sum_{s=n-2\ell+1}^{n} i_s + 2\ell(p-2)\right),$$

$$(12)\qquad J_{p,\min}(\mathbf{i}) = \max_{0\le k\le \frac{n}{2}} \max\left(0, \sum_{s=n-2k}^{n} i_s - \sum_{t=1}^{n-2k-1} i_t - 2k(p-2)\right).$$

In particular, $\delta_p(\mathbf{i}) \ge 1$.

Proof. From Lemma 2.3 it suffices to analyze the solutions $j \in \mathcal{C}_p$ of the system of inequalities $P(k, n+1)$ and prove they are all even colors j such that $J_{p,\min} \le j \le J_{p,\max}$ observing that $J_{p,\min} \equiv 0 \pmod 2$, $J_{p,\max} \equiv 0 \pmod 2$.

This is equivalent to finding $j \in \mathcal{C}_p$ such that:

$$(13)\qquad \sum_{s=n-2k}^{n} i_s - \sum_{t=1}^{n-2k-1} i_t - 2k(p-2) \le j \le \sum_{t=1}^{n-2\ell} i_t - \sum_{s=n-2\ell+1}^{n} i_s + 2\ell(p-2),$$

for all $0 \le k \le \frac{n}{2}, 0 \le l \le \frac{n}{2}$. This system has solutions if and only if for every k, ℓ we have

$$(14)\qquad \max\left(0, \sum_{s=n-2k}^{n} i_s - \sum_{t=1}^{n-2k-1} i_t - 2k(p-2)\right) \le \min\left(p-3, \sum_{t=1}^{n-2\ell} i_t - \sum_{s=n-2\ell+1}^{n} i_s + 2\ell(p-2)\right).$$

Consider one of the inequalities involved in (14), say:

$$\sum_{s=n-2k}^{n} i_s - \sum_{t=1}^{n-2k-1} i_t - 2k(p-2) \le \sum_{t=1}^{n-2\ell} i_t - \sum_{s=n-2\ell+1}^{n} i_s + 2\ell(p-2),$$

[3] There is no loss of generality in assuming $i_1 \le i_2 \le \dots \le i_m$.

which reads:

$$\sum_{s=n-2k}^{n} i_s + \sum_{s=n-2\ell+1}^{n} i_s \leq \sum_{t=1}^{n-2\ell} i_t + \sum_{t=1}^{n-2k-1} i_t + 2(k+\ell)(p-2).$$

For $k+1 \leq \ell$ this is equivalent to:

$$2\sum_{s=n-2k}^{n} i_s \leq 2\sum_{t=1}^{n-2\ell} i_t + 2(k+\ell)(p-2).$$

This inequality follows from:

$$2\sum_{s=n-2k}^{n} i_s < 2(2k+1)(p-2) \leq 2\sum_{t=1}^{n-2\ell} i_t + 2(k+\ell)(p-2)$$

and equality cannot occur. Further, if $k \geq \ell$ this amounts to

$$2\sum_{s=n-2\ell+1}^{n} i_s \leq 2\sum_{t=1}^{n-2k-1} i_t + 2(k+\ell)(p-2),$$

which follows again from

$$2\sum_{s=n-2\ell+1}^{n} i_s \leq 4\ell(p-2) \leq 2\sum_{t=1}^{n-2k-1} i_t + 2(k+\ell)(p-2)$$

with equality only if $k=\ell=0$ and

$$i_1 = i_2 = \cdots i_{n-1} = 0.$$

Similar arguments show that all inequalities involved in (14) are valid for all $i_s \in \mathcal{C}_p$ and hence $\delta_p(i_1, i_2, \ldots, i_n) \geq 1$. ☐

We will need later also the following related function:

Definition 2.5. *We let* $\delta_p^{(1)}(\mathbf{i})$, $\mathbf{i} = (i_1, i_2, \ldots, i_n)$, $i_1, i_2, \ldots, i_n \in \mathcal{C}_p$, *be the number of those* $j \in \mathcal{C}_p$ *with the property that* $\dim W_{0,p,(i_1,i_2,\ldots,i_n,j,j)} \neq 0$.

Lemma 2.6. *Let* $p \geq 5$ *be odd. Then for every* $n \geq 1$, $\mathbf{i} = (i_1, i_2, \ldots, i_n)$, $i_s \in \mathcal{C}_p$ *such that* $i_1 \leq i_2 \leq \cdots \leq i_n$, *we have*

$$\delta_p^{(1)}(\mathbf{i}) = 1 + \left\lfloor \frac{J_{p,\max}^{(1)}(\mathbf{i})}{4} \right\rfloor - \left\lceil \frac{J_{p,\min}(\mathbf{i})}{4} \right\rceil,$$

where

$$J_{p,\max}^{(1)}(\mathbf{i}) = \min_{1\leq \ell \leq \frac{n+1}{2}} \min\left(2(p-3), \sum_{t=1}^{n-2\ell+1} i_t - \sum_{s=n-2\ell+2}^{n} i_s + 2\ell(p-2)\right), \tag{15}$$

$$J_{p,\min}^{(1)}(\mathbf{i}) = \max_{0\leq k \leq \frac{n-1}{2}} \max\left(0, \sum_{s=n-2k}^{n} i_s - \sum_{t=1}^{n-2k-1} i_t - 2k(p-2)\right). \tag{16}$$

Proof. The claim follows by direct computation from Lemma 2.4, reducing us to count the number of even solutions $j \in \mathcal{C}_p$ of:

$$J_{p,\min}^{(1)}(\mathbf{i}) \leq 2j \leq J_{p,\max}^{(1)}(\mathbf{i}).$$

☐

Remark 2.7. *Observe we always have* $\delta_p^{(1)}(\mathbf{i}) \geq 1$ *if* $\mathbf{i} \neq (0, \ldots, 0)$. *This follows from* $\delta_p(i_1, \ldots, i_n, i_n) \geq 1$. *It is not completely obvious from the formulas given above that*

$$\left\lfloor \frac{J^{(1)}_{p,\max}(\mathbf{i})}{4} \right\rfloor \geq \left\lceil \frac{J_{p,\min}(\mathbf{i})}{4} \right\rceil .$$

Lemma 2.8. *Let* $p \geq 6$ *be even. The space of conformal blocks* $W_{0,p,(i_1,i_2,\ldots,i_m)}$, $m \geq 3$, *the colors* $i_1, i_2, \ldots, i_m$ *not all equal to* 0, *is non-zero if and only if, first the following parity obstruction*

$$\sum_{t=1}^{m} i_t \equiv 0 \pmod 2 \tag{17}$$

holds and second the condition $P(k, m)$ *below is satisfied:*

$$2 \sum_{\alpha=1}^{2k+1} i_{s_\alpha} \leq k(p-4) + \sum_{t=1}^{m} i_t \tag{18}$$

for all $0 \leq k \leq \frac{m-1}{2}$ *and all subsets* $\{s_1, s_2, \ldots, s_{2k+1}\} \subset \{1, 2, \ldots, m\}$.

Proof. The proof follows the pattern of the corresponding result for odd p. The parity obstruction, and the $P(k, m)$ condition for $k = 0$ (triangle inequalities) and $k = 1$ (the q-bound inequality) are equivalent to the p-admissibility, in the first non-trivial case $m = 3$.

Consider now a pants decomposition of the holed sphere, every pair of pants having one boundary curve from $\partial\Sigma_{0,m}$ except two of them each of which has precisely two boundary curves from $\partial\Sigma_{0,m}$. Summing up the parity obstruction for vertices we obtain the parity obstruction for arbitrary m. The $P(k, m)$ condition is proved as above.

Conversely, we claim that the $P(k, m)$ conditions are sufficient to guarantee the existence of a p-admissible coloring of the dual graph. We proceed by induction on m. Suppose the claim is true for all spheres with at most m boundary components and consider $(i_1, i_2, \ldots, i_{m+1})$ satisfying the conditions $P(k, m+1)$. We want to find $j_m \in \mathcal{C}_p$ such that the triple (j_m, i_m, i_{m+1}) satisfies the p-admissibility condition and $(i_1, i_2, \ldots, i_{m-1}, j_m)$ satisfies the conditions $P(k, m)$. This amounts to one parity obstruction

$$i_m + i_{m+1} + j_m \equiv 0 \pmod 2, \tag{19}$$

along with the following two sets of inequalities:

$$|i_{m+1} - i_m| \leq j_m \leq \min\left(i_m + i_{m+1}, p - 4 - (i_m + i_{m+1}), \frac{p-4}{2}\right), \tag{20}$$

$$2 \sum_{\alpha=1}^{2k+1} i_{s_\alpha} - k(p-4) - \sum_{t=1}^{m-1} i_t \leq j_m \leq k(p-4) + \sum_{t=1}^{m-1} i_t - 2 \sum_{\alpha=1}^{2k} i_{s_\alpha}. \tag{21}$$

These inequalities have solutions $j_m \in \mathcal{C}_p$ if and only if

$$|i_{m+1} - i_m| \leq k(p-4) + \sum_{t=1}^{m-1} i_t - 2 \sum_{\alpha=1}^{2k} i_{s_\alpha} \tag{22}$$

and

$$2 \sum_{\alpha=1}^{2k+1} i_{s_\alpha} - k(p-4) - \sum_{t=1}^{m-1} i_t \leq \min\left(i_m + i_{m+1}, p - 4 - (i_m + i_{m+1}), \frac{p-4}{2}\right). \tag{23}$$

Inequality (22) is equivalent to $P(k, m+1)$. Inequality (23) consists of three inequalities, which respectively equivalent to $P(k, m+1)$, $P(k+1, m+1)$ and

$$2 \sum_{\alpha=1}^{2k+1} i_{s_\alpha} \leq \left(k + \frac{1}{2}\right)(p-4) + \sum_{t=1}^{m-1} i_t. \tag{24}$$

This is a consequence of the former two inequalities.

It remains to observe that there exist solutions satisfying the parity condition above. In fact both endpoints of the interval specified by inequality (20) are compatible with the parity obstruction (19), while the endpoints of the interval given by (21) are both congruent to $i_m + i_{m+1} \pmod 2$. It follows that there exists a solution j_m satisfying the parity obstruction. $\square$

Lemma 2.9. *Let $p \geq 6$ be even. Then for every $n \geq 2$, $\mathbf{i} = (i_1, i_2, \dots, i_n)$, $i_1, i_2, \dots, i_n \in \mathcal{C}_p$ such that $i_1 \leq i_2 \leq \dots \leq i_n$, we have*

$$\delta_p(\mathbf{i}) = 1 + \frac{1}{2}\left(J_{p,\max}(\mathbf{i}) - J_{p,\min}(\mathbf{i})\right),$$

where

$$J_{p,\max}(\mathbf{i}) = \min_{0\leq \ell \leq \frac{n}{2}} \min\left(\frac{p-4}{2} - \varepsilon_{\mathbf{i}}(p), \sum_{t=1}^{n-2\ell} i_t - \sum_{s=n-2\ell+1}^{n} i_s + \ell(p-4)\right), \tag{25}$$

$$J_{p,\min}(\mathbf{i}) = \max_{0\leq k \leq \frac{n}{2}} \max\left(\varepsilon_{\mathbf{i}}, \sum_{s=n-2k}^{m} i_s - \sum_{t=1}^{n-2k-1} i_t - k(p-4)\right), \tag{26}$$

$$\varepsilon_{\mathbf{i}} \in \{0,1\} \text{ such that } \varepsilon_{\mathbf{i}} \equiv \sum_{s=1}^{n} i_s \pmod 2, \tag{27}$$

$$\varepsilon_{\mathbf{i}}(p) = \begin{cases} \varepsilon_{\mathbf{i}}, & \text{if } p \equiv 0 \pmod 4; \\ 1-\varepsilon_{\mathbf{i}}, & \text{if } p \equiv 2 \pmod 4. \end{cases} \tag{28}$$

In particular, $\delta_p(\mathbf{i}) \geq 1$.

Proof. From Lemma 2.8 it suffices to analyze the solutions $j \in \mathcal{C}_p$ of the system of inequalities $P(k, n+1)$ with the additional parity constraint (17). There is no loss of generality in assuming that $i_1 \leq i_2 \leq \dots \leq i_n$. Therefore our system is equivalent to finding $j \in \mathcal{C}_p$ such that:

$$\sum_{s=m-2k}^{n} i_s - \sum_{t=1}^{n-2k-1} i_t - k(p-4) \leq j \leq \sum_{t=1}^{n-2\ell} i_t - \sum_{s=n-2\ell+1}^{m} i_s + \ell(p-4). \tag{29}$$

This system has solutions if and only if for every k, ℓ we have

$$\max\left(0, \sum_{s=n-2k}^{m} i_s - \sum_{t=1}^{n-2k-1} i_t - k(p-4)\right) \leq \min\left(\frac{p-4}{2}, \sum_{t=1}^{n-2\ell} i_t - \sum_{s=n-2\ell+1}^{n} i_s + \ell(p-4)\right). \tag{30}$$

Moreover, we have solutions j which also satisfy the parity condition (17) if and only if

$$J_{p,\min}(\mathbf{i}) \leq J_{p,\max}(\mathbf{i})$$

and then $\delta_p(i_1, i_2, \dots, i_n)$ is the number of integers in this interval congruent to $\varepsilon_{\mathbf{i}}$ modulo 2, as claimed.

Consider one of the inequalities involved in (30), say:

$$\sum_{s=n-2k}^{n} i_s - \sum_{t=1}^{n-2k-1} i_t - k(p-4) \leq \sum_{t=1}^{n-2\ell} i_t - \sum_{s=n-2\ell+1}^{n} i_s + \ell(p-4),$$

which reads:

$$\sum_{s=n-2k}^{m} i_s + \sum_{s=n-2\ell+1}^{n} i_s \leq \sum_{t=1}^{n-2\ell} i_t + \sum_{t=1}^{n-2k-1} i_t + (k+\ell)(p-4).$$

For $k+1 \leq \ell$ this is equivalent to:

$$2\sum_{s=n-2k}^{n} i_s \leq 2\sum_{t=1}^{n-2\ell} i_t + (k+\ell)(p-4).$$

This inequality follows from:

$$2\sum_{s=n-2k}^{n} i_s \leq (2k+1)(p-4) \leq 2\sum_{t=1}^{n-2\ell} i_t + 2(k+\ell)(p-4).$$

Further, if $k \geq \ell$ this amounts to

$$2\sum_{s=n-2\ell+1}^{n} i_s \leq 2\sum_{t=1}^{n-2k-1} i_t + (k+\ell)(p-4),$$

which follows again from

$$2\sum_{s=n-2\ell+1}^{n} i_s \leq 2\ell(p-4) \leq 2\sum_{t=1}^{n-2k-1} i_t + (k+\ell)(p-4)$$

with equality only if $k=\ell=0$ and

$$i_1 = i_2 = \cdots i_{n-1} = 0.$$

Similar arguments show that all inequalities involved in (30) are valid for all $i_s \in \mathcal{C}_p$ and hence $\delta_p(\mathbf{i}) \geq 1$. □

Lemma 2.10. *Let $p \geq 6$ be even. Then for every $n \geq 1$, $\mathbf{i} = (i_1, i_2, \ldots, i_n)$, $i_1, i_2, \ldots, i_n \in \mathcal{C}_p$ such that $i_1 \leq i_2 \leq \cdots \leq i_n$, we have:*

$$\delta_p^{(1)}(\mathbf{i}) = \begin{cases} 0, & \text{if } \varepsilon_{(i_1,i_2,\ldots,i_n)} = 1; \\ 1 + \frac{J^{(1)}_{p,\max}(\mathbf{i}) - J^{(1)}_{p,\min}(\mathbf{i})}{2}, & \text{if } \varepsilon_{(i_1,i_2,\ldots,i_n)} = 0, \end{cases} \tag{31}$$

where

$$J^{(1)}_{p,\max}(\mathbf{i}) = \min_{1\leq \ell \leq \frac{n+1}{2}} \min\left(p-4, \sum_{t=1}^{n-2\ell} i_t - \sum_{s=n-2\ell+1}^{n} i_s + \ell(p-4)\right), \tag{32}$$

$$J^{(1)}_{p,\min}(\mathbf{i}) = \max_{0\leq k \leq \frac{n+1}{2}} \max\left(0, \sum_{s=n-2k}^{n} i_s - \sum_{t=1}^{n-2k-1} i_t - k(p-4)\right). \tag{33}$$

Proof. Observe first from Lemma 2.8 that $\varepsilon_{(i_1,i_2,\ldots,i_n)} = 0$ is a necessary condition for the existence of such j. Further, from the proof of Lemma 2.9 we have to count the number of solutions $j \in \mathcal{C}_p$ of the double inequalities:

$$J^{(1)}_{p,\min}(\mathbf{i}) \leq 2j \leq J^{(1)}_{p,\max}(\mathbf{i})$$

The claim follows by direct computation. □

2.4. **The order of Dehn twists in the projective quantum representations.** The explicit description from the previous section leads to the following computation:

Lemma 2.11. *Let $p \geq 5$ be odd, c be an essential simple closed curve on $\Sigma_{g,n}$, $g \geq 1$ and $i_1, i_2, \ldots, i_n$ be non-zero colors from $\mathcal{C}_p$. Then $\rho_{p,(i_1,\ldots,i_n)}(T_c)^p = 1$. Furthermore,*

(1) Assume that one of the following conditions is satisfied:
  - *(a) c is separating of strictly positive genus;*
  - *(b) c is non-separating and either $g \geq 2$ or $g=1$ and $n=0$;*
  - *(c) c bounds a holed sphere with other boundary curves colored by the non-zero colors $i_1, i_2, \ldots, i_m$, for some $2 \leq m \leq n$, where*
    - *(i) $\delta_p(i_1, i_2, \ldots, i_m) \geq 3$, or*

(ii) $\delta_p(i_1, i_2, \ldots, i_m) = 2$ *and* $J_{p,\min}(i_1, i_2, \ldots, i_m) = 0$.

(d) c *is non-separating,* $g = 1$ *and*

(i) $n \geq 1$, $\delta_p^{(1)}(i_1, i_2, \ldots, i_n) \geq 3$, *or*

(ii) $n \geq 1$, $\delta_p^{(1)}(i_1, i_2, \ldots, i_n) = 2$ *and* $J_{p,\min}(i_1, i_2, \ldots, i_n) = 0$.

Then $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ *has order* p.

(2) If c *bounds a holed sphere with other boundary curves colored by the non-zero colors* $i_1, i_2, \ldots, i_m$, *for some* $2 \leq m \leq n$, *where* $\delta_p(i_1, i_2, \ldots, i_m) = 2$. *Then* $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ *has order*

$$p/\text{g.c.d.}\left(J_{p,\max}(i_1, i_2, \ldots, i_m), p\right).$$

(3) c *is non-separating,* $g = 1$ *and* $n \geq 1$, $\delta_p^{(1)}(i_1, i_2, \ldots, i_n) = 2$ *and* $J_{p,min}(i_1, i_2, \ldots, i_n) > 0$. *Then* $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ *has order*

$$p/\text{g.c.d.}\left(\left\lfloor \frac{J^{(1)}_{p,\max}(i_1, i_2, \ldots, i_n)}{4} \right\rfloor, p\right).$$

(4) Suppose that:

(a) either c *bounds a holed sphere with other boundary curves colored by the non-zero colors* $i_1, i_2, \ldots, i_m$, *for some* $2 \leq m \leq n$, *where* $\delta_p(i_1, i_2, \ldots, i_m) = 1$, *or*

(b) c *is non-separating and* $g = 1$ $n \geq 1$ *and* $\delta_p^{(1)}(i_1, i_2, \ldots, i_n) = 1$.

Then $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ *has order* 1.

Proof: Choose a pants decomposition of the surface $\Sigma_{g,n}$ obtained by deleting disjoint disks centered at the n punctures, which contains the simple closed curve c. Note that boundary circles have to be part of the decomposition and are already colored as $(i_1, i_2, \ldots, i_r)$. If we cut open $\Sigma_{g,n}$ along c we either obtain $\Sigma_{g-1,n+2}$ – when c is non-separating – or else a disjoint union $\sigma_{h,s} \sqcup \Sigma_{g-h,n+1-s}$. We color the two new boundary circles of the resulting possibly disconnected surface by some color j and let $W(\Sigma_{g,n} \setminus \{c\}; j)$ denote the space of conformal blocks associated to it. Observe also that its identification with a subsurface of $\Sigma_{g,n}$ induces an injection of $W(\Sigma_{g,n} \setminus \{c\}; j)$ into $W_{g,(i_1,i_2,\ldots,i_n)}$, and we have the following decomposition:

$$W_{g,(i_1,i_2,\ldots,i_n)} = \oplus_{j \in \mathcal{C}_p} W(\Sigma_{g,n} \setminus \{c\}; j)$$

Now $\rho_{p,(i_1,i_2,\ldots,i_n)}(T_c)$ acts as a scalar on every subspace $W(\Sigma_{g,n} \setminus \{c\}; j)$ and with respect to the direct sum decomposition above we have:

$$\rho_{p,(i_1,\ldots,i_n)}(T_c) = \oplus_{j \in \mathcal{C}_p} (-1)^j A^{j(j+2)} \mathbf{1}_{W(\Sigma_{g,n} \setminus \{c\}; j)}$$

where $\mathbf{1}_W$ is the identity operator on the subspace W.

On one hand eigenvalues of $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ are powers of A^8 and hence

$$\rho_{p,(i_1,\ldots,i_n)}(T_c)^p = 1$$

In order to compute the order of $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ it suffices to find which colors j appear *effectively* in the direct sum above, namely for which $j \in \mathcal{C}_p$ the vector space $W(\Sigma_{g,n} \setminus \{c\}; j)$ is non-zero.

According to ([26], Lemma 3.3) we have

$$\dim W_{1,p,(i,j)} = \frac{(p - 1 - \max(i,j))(\min(i,j) + 1)}{2} \neq 0$$

Thus, if c bounds subsurfaces of positive genera on both sides, then all possible colors could be realized effectively.

If c is non-separating and $g \geq 2$, the colors $j = 0$ and $j = 2$ do appear effectively and, in particular, $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ has among its eigenvalues 1 and A^8. Since the order of A^8 is p, the order of $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ is actually p. This proves the first two items of our Lemma.

If c is like in (1.c.(i)), then there at least three colors j which appear effectively. The proof of Lemma 2.4 shows that these colors can be chosen to be consecutive even numbers $2k, 2k + 2, 2k + 4$. Since $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ is the image of a diagonal matrix with $1, A^{8(k+1)}, A^{8(2k+3)}$ as coefficients, A^8 is primitive $p-th$ root of 1, and g.c.d.$(p,k+1,2k+3)=1$, the order of $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ is p.

If c is like in (1.c.(ii)), then there are two colors which appear effectively, hence they are 0 and 2, which implies that the order $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ is the order of A^8, namely p.

When $g=1$ and c is separating, then two copies of c bound a holed sphere. Observe that $j,j,i_1,\ldots,i_m$ satisfies the conditions $P(k,m+2)$ for $j\in\{0,2\}$, if $i_1,\ldots,i_m$ satisfies the conditions $P(k,m)$. Then Lemma 2.3 permits to conclude the proof of the last item (1.d).

If c bounds a holed sphere and $\delta_p(i_1,i_2,\ldots,i_m)=2$, then those j which appear effectively are the consecutive even integers $J_{p,\min}(i_1,i_2,\ldots,i_m)$ and $J_{p,\max}(i_1,i_2,\ldots,i_m)$. Then the order of $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ is the order of $A^{4J_{p,\max}(i_1,i_2,\ldots,i_m)}$, which is $p/$g.c.d.$(J_{p,\max}(i_1,i_2,\ldots,i_m),p)$.

If c is non-separating, $g=1$ and $\delta_p^{(1)}(i_1,i_2,\ldots,i_m)=2$, then the j which appear effectively are the consecutive even integers $2\left\lceil\frac{J_{p,\min}(i_1,i_2,\ldots,i_m)}{4}\right\rceil$ and $2\left\lfloor\frac{J^{(1)}_{p,\max}(i_1,i_2,\ldots,i_m)}{4}\right\rfloor$. Then the order of $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ is the order of $A^{8\left\lfloor\frac{J^{(1)}_{p,\max}(i_1,i_2,\ldots,i_m)}{4}\right\rfloor}$, which is $p/$g.c.d. $\left(2\left\lfloor\frac{J^{(1)}_{p,\max}(i_1,i_2,\ldots,i_m)}{4}\right\rfloor,p\right)$.

Eventually, if $\delta_p(i_1,i_2,\ldots,i_m)=1$, or $\delta_p^{(1)}(i_1,i_2,\ldots,i_n)=1$ then there is only one color j which could appear effectively as label of the curve c and hence $\rho_{p,(i_1,i_2,\ldots,i_n)}(T_c)$ is a scalar, namely a trivial element in the projective unitary group. □

Lemma 2.12. *Let $p\geq 6$ be even, c an essential simple closed curve on $\Sigma_{g,n}$, $g\geq 1$ and $(i_1,i_2,\ldots,i_n)$ be non-zero colors from $\mathcal{C}_p$. We assume that*

$$\sum_{s=1}^{n} i_s\equiv 0 \pmod 2, \tag{34}$$

as otherwise $W_{p,g,(i_1,i_2,\ldots,i_n)}$ is a null vector space. Then $\rho_{p,(i_1,\ldots,i_n)}(T_c)^{2p}=1$. Furthermore:

1. *If c is separating of strictly positive genus, assume that the other boundary components of one connected complementary subsurface are colored $i_1,i_2,\ldots,i_m$, with $0\leq m\leq n$.*
   - *(a) If*
   $$\sum_{s=1}^{m} i_s\equiv 0 \pmod 2$$
   then $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ has order
   $$\begin{cases}1, & \text{if } p=6;\\ \frac{p}{4}, & \text{if } p\equiv 0 \pmod 4, p\geq 8;\\ \frac{p}{2}, & \text{if } p\equiv 2 \pmod 4, p\geq 10.\end{cases}$$
   - *(b) If*
   $$\sum_{s=1}^{m} i_s\equiv 1 \pmod 2$$
   then $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ has order
   $$\begin{cases}1, & \text{if } p\in\{6,8\};\\ 5, & \text{if } p=10;\\ 2, & \text{if } p=12;\\ \frac{p}{2}, & \text{if } p\geq 14.\end{cases}$$
2. *If c is non-separating and $g\geq 2$, or $g=1$ and $n=0$, then $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ has order*
   $$\begin{cases}4, & \text{if } p=6;\\ 2p, & \text{if } p\geq 8.\end{cases}$$
3. *Suppose that c bounds a holed sphere with other boundary curves colored by the non-zero colors $i_1,i_2,\ldots,i_m$, for some $2\leq m\leq n$, where $\delta_p(i_1,i_2,\ldots,i_m)\geq 3$.*

(a) *If*
$$\sum_{s=1}^{m} i_s \equiv 0 \pmod 2,$$
then $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ *has order*
$$\begin{cases} 1, & \text{if } p = 6; \\ \frac{p}{4}, & \text{if } p \equiv 0 \pmod 4, p \geq 8 \\ \frac{p}{2}, & \text{if } p \equiv 2 \pmod 4, p \geq 8 \end{cases}$$
(b) *If*
$$\sum_{s=1}^{m} i_s \equiv 1 \pmod 2,$$
then $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ *has order*
$$\begin{cases} 1, & \text{if } p \in \{6, 8\}; \\ 5, & \text{if } p = 10; \\ 2, & \text{if } p = 12; \\ \frac{p}{2}, & \text{if } p \geq 14. \end{cases}$$

(4) *If c is non-separating, $g = 1$ and $\delta_p^{(1)}(i_1, i_2, \ldots, i_n) \geq 3$, so that $p \geq 8$, then $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ has order $2p$.*

(5) *Let c bound a holed sphere with other boundary curves colored by the non-zero colors $i_1, i_2, \ldots, i_m$, for some $2 \leq m \leq n$, where $\delta_p(i_1, i_2, \ldots, i_m) = 2$. Then $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ has order*
$$p/2 \,\text{g.c.d.}\left(J_{p,\max}(i_1, i_2, \ldots, i_m), p/2\right).$$

(6) *Let c be non-separating, $g = 1$, $n \geq 1$, $\delta_p^{(1)}(i_1, i_2, \ldots, i_n) = 2$. Then $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ has order*
$$2p/\text{g.c.d.}\left(2 + J^{(1)}_{p,\max}(i_1, i_2, \ldots, i_n), p\right).$$

(7) *Suppose that:*
  (a) *either c bounds a holed sphere with other boundary curves colored by the non-zero colors $i_1, i_2, \ldots, i_m$, for some $2 \leq m \leq n$, where $\delta_p(i_1, i_2, \ldots, i_m) = 1$, or*
  (b) *c is non-separating and $g = 1$, $n \geq 1$ and $\delta_p^{(1)}(i_1, i_2, \ldots, i_n) = 1$.*

  Then $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ has order 1.

Proof: First note that the sum of the parity conditions over all vertices yields the global parity condition (34). Further, we follow the arguments from Lemma 2.11 above and use the same notation. We then have:
$$\widetilde{\rho}_{p,(i_1,\ldots,i_n)}(\widetilde{T}_c) = \oplus_{j \in \mathcal{C}_p} (-1)^j A^{j(j+2)} \mathbf{1}_{W(\Sigma_{g,n} \setminus \{c\}; j)}$$
where $\mathbf{1}_W$ is the identity operator on the subspace W. The eigenvalues of $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ are powers of A and hence
$$\rho_{p,(i_1,\ldots,i_n)}(T_c)^{2p} = 1$$
In order to compute the order of $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ it suffices to find which colors j appear effectively in the direct sum above, namely for which $j \in \mathcal{C}_p$ the vector space $W(\Sigma_{g,n} \setminus \{c\}; j)$ is non-zero. Let us first give some information about spaces of conformal blocks in positive genus:

Lemma 2.13.
$$\dim W_{1,p,(i,j)} = \begin{cases} \left(\frac{p-2}{2} - \max(i,j)\right)(1 + \min(i,j)), & \text{if } i \equiv j \pmod 2; \\ 0, & \text{if } i \not\equiv j \pmod 2. \end{cases} \tag{35}$$
Hence $\dim W_{1,p,(i,j)} = 0$ *if only if* $i \not\equiv j \pmod 2$.

Proof: Direct computation. □

Corollary 2.14. *If $g \geq 2$, then* $\dim W_{g,p,(i,j)} = 0$ *if only if* $i \not\equiv j \pmod 2$.

Proof: Decompose a 2-holed surface of genus g as $\Sigma_{g,2} = \Sigma_{1,2} \cup_{S^1} \Sigma_{1,2} \cup_{S^1} \ldots \cup_{S^1} \Sigma_{1,2}$ and apply the splitting principle coloring each glued boundary circle with the color i. □

Corollary 2.15. *If $g \geq 1$, $m \geq 2$* $\dim W_{g,p,(i_1,\ldots i_m)} \neq 0$ *if only if*

$$\sum_{t=1}^{m} i_t \equiv 0 \pmod 2.$$

Proof: The case $m = 2$ follows from the previous statements. So we may assume $m \geq 3$. Decompose a 2-holed surface of genus g as $\Sigma_{g,m} = \Sigma_{1,2} \cup_{S^1} \Sigma_{1,2} \cup_{S^1} \ldots \cup_{S^1} \Sigma_{1,2} \cup_{S^1} \Sigma_{0,m}$ assuming i_1 colors the left boundary curve of the first $\Sigma_{1,2}$ and apply the splitting principle coloring each glued boundary circle with a color j_1 such that $W_{0,p,(j_1,i_2,\ldots,i_m)} \neq 0$ which exists by Lemma 2.9 and has the same parity as i_1. □

Suppose now that c bounds two subsurfaces of positive genera on both sides. As above, the colors j that could be realized effectively on c are precisely those satisfying the parity condition:

$$j \equiv \sum_{s=1}^{m} i_s \pmod 2$$

and we call the splitting of the surface odd/even according to the parity of the sum of colors on either side.

Let c provide an even splitting. If $p = 6$ there is only one even color, namely 0 and hence

$$\rho_{g,6,(i_1,i_2,\ldots,i_n)}(T_c) = 1$$

For even $p \geq 8$, there exist at least two even colors 0 and 2 which appear effectively thanks to Corollary 2.15. All other even labels yield eigenvalues which are powers of A^8. Since A^8 has order $\frac{p}{4}$, when $p \equiv 0 \pmod 4$ and $\frac{p}{2}$, otherwise, the same holds for $\rho_{g,p,(i_1,i_2,\ldots,i_n)}(T_c)$.

Let c provide an odd splitting. If $p = 6$ or $p = 8$ there is only one odd color in $\mathcal{C}_p$, namely 1. The matrix so obtained is then scalar and hence its image in the projective unitary group is trivial:

$$\rho_{g,6,(i_1,i_2,\ldots,i_n)}(T_c) = 1, \ \rho_{g,8,(i_1,i_2,\ldots,i_n)}(T_c) = 1$$

If $p = 10$ or $p = 12$, there are exactly two odd colors 1 and 3. They effectively appear by Corollary 2.15. The associated eigenvalues are $-A^3$ and $-A^{15}$. The smallest exponent for which the powers of these two eigenvalues coincide are 2, for $p = 12$ and 5 for $p = 10$. For $p \geq 14$, there are at least three odd colors $1, 3$ and 5, with associated eigenvalues $-A^3, -A^{15}$ and $-A^{35}$. The smallest k for which k-th powers are equal should verify $A^{12k} = A^{20k} = 1$ and hence $A^{4k} = 1$, so that $k = \frac{p}{2}$. Higher odd colors lead to eigenvalues of the form powers of A^4 times $-A^3$. Thus the order of $\rho_{g,p,(i_1,i_2,\ldots,i_n)}(T_c)$ is $\frac{p}{2}$. This proves the first item of our Lemma.

Assume that c is nonseparating and $g \geq 2$. Choose a pants decomposition such that c is a meridian of a one holed torus bounded by the simple closed curve d. The parity obstruction for the complementary subsurface shows that d must be even in any p-admissible coloring. By Corollary 2.15, any even color for d appears effectively. Therefore any color for c appears effectively, in particular the colors $j = 0, 1$ and also $j = 2$, when $p \geq 8$. Now $-A^3$ had order 4, for $p = 6$. If $p \geq 8$, the l.c.m. of orders of $-A^3$ and A^8 is $2p$ and hence $\rho_{p,(i_1,\ldots,i_n)}(T_c)$ has order $2p$. The proof is similar for $g = 1$, $n = 0$. This proves the second item of the Lemma.

Assume c is separating a holed sphere, providing an even splitting. If $p = 6$, there is only one even color 0, and hence $\rho_{6,(i_1,i_2,\ldots,i_n)} = 1$. If $\delta_p(i_1, i_2, \ldots, i_m) \geq 3$, there are at least three consecutive even colors j which appear effectively for c. Then the order of $\rho_{p,(i_1,i_2,\ldots,i_n)}$ is the order of A^8, which is as stated, by the same argument as in the odd case. If $\delta_p(i_1, i_2, \ldots, i_m) = 2$ and $J_{p,\min}(i_1, i_2, \ldots, i_m) = 0$, then the colors 0 and 2 appear effectively and hence the order of $\rho_{p,(i_1,i_2,\ldots,i_n)}$ is the order of A^8 again.

Assume c is separating a holed sphere, providing an odd splitting. If $p = 6$ or $p = 8$, there is only one odd color 1, and hence $\rho_{p,(i_1,i_2,\ldots,i_n)} = 1$. If $\delta_p(i_1, i_2, \ldots, i_m) \geq 3$, there are at least three consecutive odd colors j which appear effectively for c and the order $\rho_{p,(i_1,i_2,\ldots,i_n)}$ is the order of A^4, which is as stated, except for small values of $p \leq 12$. This concludes the proof of the third item.

When $g = 1$ and c is non-separating and $\delta_p^{(1)}(i_1, i_2, \ldots, i_m) \geq 3$ then there are at least three consecutive colors $j, j+1, j+2$ which appear effectively. The corresponding eigenvalues have a common N-th power iff $(-A)^{(2j+3)N} = (-A)^{(2j+5)N} = 1$, and hence $A^N = 1$, so that N is divisible by $2p$. This proves the fourth item.

If c bounds a holed sphere and $\delta_p(i_1, i_2, \ldots, i_m) = 2$, then j must be even and there are exactly two even colors j and $j + 2$ which appear effectively and the order of $\rho_{p,(i_1,i_2,\ldots,i_n)}$ is the order of $A^{4(j+2)}$, which is as claimed.

If c is non-separating, $g = 1$, $\delta_p^{(1)}(i_1, i_2, \ldots, i_m) = 2$, then there are exactly two colors j and $j + 1$ which appear effectively and the order of $\rho_{p,(i_1,i_2,\ldots,i_n)}$ is the order of A^{2j+3} which is as claimed.

Eventually, recall that if $\delta_p(i_1, i_2, \ldots, i_m) = 1$ or $\delta_p^{(1)}(i_1, i_2, \ldots, i_m) = 1$, then $\rho_{p,(i_1,i_2,\ldots,i_n)}$ is a scalar and hence of order 1. □

Remark 2.16. *Assume now that all i_s are equal to $\frac{p-4}{2}$. We claim that the space $W_{0,p,(i_1,\ldots,i_n)}$ could only be non-trivial when $n \geq 4$ is even, in which case it is of dimension 1. This follows either by using Lemma 2.8, or else directly by analyzing the dual tree of the pants decomposition used in its proof. The internal edges form a chain; their labels are forced by the p-admissibility condition to be alternatively $0, \frac{p-4}{2}, 0, \frac{p-4}{2}, \ldots$. This implies that the only color which could appear effectively on c is 0, if m is odd and $\frac{p-4}{2}$, if m is even. In both cases $\rho_{p,(i_1,i_2,\ldots,i_n)}$ is a scalar and hence it is of order 1.*

2.5. **Infiniteness results.** Recall from the introduction that $\mathrm{Mod}(\Sigma_g^n[\mathbf{k}])$ denotes the normal subgroup generated by k_0-th powers of Dehn twist along non-separating simple closed curves and k_j-th powers of Dehn twists along simple closed curves of type j, where $\mathbf{k} = (k_0; k_1, k_2, \ldots, k_{N_{g,n}})$. As a shortcut we use $\mathrm{Mod}(\Sigma_g^n[k; m])$ for $\mathbf{k} = (k, m, m, m \ldots, m)$ and $\mathrm{Mod}(\Sigma_g^n[k; -])$ for $\mathbf{k} = (k;)$, where k_i are absent for $i > 0$.

Proposition 2.17. *Assume $g \geq 1$, $n \geq 0$. Let $(i_1, \ldots, i_n) \in \mathcal{C}_p^n$ such that*

$$\sum_{t=1}^{n} i_t \equiv 0 \pmod 2.$$

(1) *For odd $p \geq 5$, $\rho_{p,(i_1,i_2,\ldots,i_n)}$ factors through $\mathrm{Mod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[p]$.*
(2) *For even $p \geq 10$, $\rho_{p,(i_1,i_2,\ldots,i_n)}$ factors through $\mathrm{Mod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[2p; p/\mathrm{g.c.d.}(4,p)]$, if all colors are even, and through out $\mathrm{Mod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[2p; p/2]$, in general.*

Proof. This is immediate from Lemmas 2.11 and 2.12. □

Proposition 2.18. *Assume that $g \geq 0$, $n \geq 0$, $2g - 2 + n > 0$, $(g, n) \neq (0, 3)$ and that $p \geq 5$, if p is odd, or $p \geq 10$ and $(g, p) \neq (2, 12)$, if p is even. When $(g, n) = (1, 1)$ we assume that either $p \geq 7$ is odd or $p \geq 12$ is even. Then $\rho_p(\mathrm{Mod}(\Sigma_{g,n}))$ is infinite.*

Proof. For all $g \geq 2$, $p \geq 5$ except $p \in \{6, 8, 12\}$, $(g, p) = (2, 20)$ this is proved in [24]; Masbaum found explicit elements of infinite order in [44]. The case $p = 20$ is settled in ([17], Appendix A). The proofs use in fact explicit computations in the case of $\Sigma_{0,4}$ and $\Sigma_{0,5}$ and these provide immediately the desired result also in the case when $g \in 0, 1$.

It remains to prove the claim for $g \geq 3$ and $p = 12$. Note that Wright proved that the image is finite when $p = 12$ and $g = 2$ (see [58]). We follow the approach developed by Coxeter (see [14, 15]) who used it to prove that quotients of braid groups by powers of braid generators is infinite. This was used in [39] to study the case of punctured torus.

Lemma 2.19 ([14],p.116, or [15],p.121)**.** *If a group acts irreducibly on a finite dimensional complex vector space keeping invariant a non-degenerate indefinite Hermitian form, then this group must be infinite.*

We need also the following:

Lemma 2.20 ([38])**.** *Let p be even. Then the action of $\widetilde{\rho}_{p,(1,i_2,\ldots,i_n)}(\widetilde{\mathrm{Mod}(\Sigma_{g,n})})$ on the space $W_{g,p,(1,i_2,\ldots,i_n)}$ is irreducible for all $g \geq 1$ and all colors $(1, i_2, \ldots, i_n)$ for which this vector space is non-zero.*

Consider now a torus $\Sigma_{1,2}$ with two boundary components colored $(1,1)$. We need the following result whose proof will be postponed a few lines later:

Lemma 2.21. *The Hermitian form on the space $W_{1,p,(1,1)}$ is indefinite, if $p \geq 10$ is even or $p \geq 5$ is odd.*

The three lemmas above imply that the image of $\rho_{12,(1,1)}(\mathrm{Mod}(\Sigma_{1,2}))$ is infinite. Further observe that for every $g \geq 3$ the embedding $\Sigma_{1,2} \subset \Sigma_g$ obtained by gluing $\Sigma_{g-2,2}$ to $\Sigma_{1,2}$ along the boundary components induces an injection at the mapping class group level: $\mathrm{Mod}(\Sigma_{1,2}) \to \mathrm{Mod}(\Sigma_g)$. It follows that $\rho_{12}(\mathrm{Mod}(\Sigma_g))$ contains the subgroup $\rho_{12,(1,1)}(\mathrm{Mod}(\Sigma_{1,2}))$ and it is therefore infinite. This proves Proposition 2.18. $\square$

Proof of Lemma 2.21. The graph G consists of a loop with two pending edges, labeled 1. Let p be even. The two remaining edges, whose union form the loop, are colored by (i, j); then by the admissibility conditions at level p we need that $i + j$ be odd, smaller than or equal to $p - 5$ and such that $(1, i, j)$ satisfy the triangle inequalities, so that $|i - j| = 1$. Thus the admissible colorings are of the form $(k, k + 1)$ and $(k + 1, k)$, with $k = \left\{0, 1, 2, \frac{p-4}{2}\right\}$, so that $\dim W_{1,p,(1,1)} = p - 4$. Moreover, by the formulas above the diagonal term of the Hermitian form corresponding to the coloring (i, j) is $\eta^2 \frac{1}{[i]![j]!}$. The quotient of two diagonal terms associated to the colorings $(k, k + 1)$ and $(k + 1, k + 2)$ is therefore given by:

$$[k + 1][k + 2].$$

Since A was a primitive $2p$-th root of unity, $A = \exp\left(\frac{i\pi\ell}{p}\right)$, with g.c.d.$(\ell, 2p) = 1$. Now $[k] = \frac{\sin\left(\frac{2i\pi\ell k}{p}\right)}{\sin\left(\frac{2i\pi\ell}{p}\right)}$. If $p \geq 10$, there exists always some ℓ such that $[1][2] < 0$, namely the two diagonal terms have opposite sign and thus the Hermitian form is indefinite. The case p odd is similar and we skip the details. $\square$

The remaining cases, not covered by the Proposition 2.18 will be discussed in the next section.

Choose a basepoint on the first boundary circle of $\Sigma_{g,n}$. If we adjoin a disk capping off the first hole the embedding $\Sigma_{g,n} \to \Sigma_{g,n-1}$ induces an exact sequence:

$$1 \to \pi_1(U\Sigma_{g,n-1}) \to \mathrm{Mod}(\Sigma_{g,n}) \to \mathrm{Mod}(\Sigma_{g,n-1}) \to 1.$$

Here $\pi_1(U\Sigma_{g,n-1})$ is the fundamental group of the unit tangent bundle at $\Sigma_{g,n-1}$, namely a central extension of $\pi_1(\Sigma_{g,n-1})$. Then the projective representation $\rho_{p,(i_1,i_2,\ldots,i_n)}$ of $\mathrm{Mod}(\Sigma_{g,n})$ induces a representation for which we keep the same notation:

$$\rho_{p,(i_1,i_2,\ldots,i_n)} : \pi_1(\Sigma_{g,n-1}) \to PU(W_{g,p,(i_1,i_2,\ldots,i_n)}).$$

Recall that a subsurface $\Sigma_{g',n'}$ of $\Sigma_{g,n}$ is *essential* if the morphism $\pi_1(\Sigma_{g',n'}) \to \pi_1(\Sigma_{g,n})$ induced by the inclusion is injective. It is well-known that a subsurface is essential if its complement does not contain disk components.

Proposition 2.22. *Assume $n \geq 1$. Let $\Sigma \cong \Sigma_{0,3}$ or $\Sigma_{1,1}$ be an essential subsurface of Σ_g^{n-1} equipped with an orientation preserving homeomorphism onto a general fiber of the map $f_n^1 : \mathcal{M}_g^n \to \mathcal{M}_g^{n-1}$ which forgets the first puncture.*

(1) *Assume that $g \geq 2$ and $p \geq 5$ is odd. Then $\rho_{p,(i_1,i_2,\ldots,i_n)}(\pi_1(\Sigma))$ contains a free nonabelian group, if $i_1 \neq 0$.*
(2) *Assume that $g \geq 2$ and $p \geq 10$ is even, $(g,p) \neq (2,12)$, $n \geq 2$ and $1 + i_2 + \cdots + i_n \equiv 0 \pmod 2$. Then $\rho_{p,(1,i_2,\ldots,i_n)}(\pi_1(\Sigma))$ contains a free nonabelian group.*
(3) *Assume that $g \geq 2$ and $p \geq 10$ is even, $(g,p) \neq (2,12)$, $n \geq 1$ and $i_2 + \cdots + i_n \equiv 0 \pmod 2$. Then $\rho_{p,(2,i_2,\ldots,i_n)}(\pi_1(\Sigma))$ contains a free nonabelian group.*

Proof. When $i_1 = 2$ this is the main result of ([37], Thm.4.1) noting that their proof works for all $p \geq 5$ not only for large enough p. For other values of $i_1 \neq 0$ this is contained in the proof of ([26], Prop. 3.2, see also [25]).

Let c be a curve separating the subsurface $\Sigma_{1,2}$ of $\Sigma_{g,n}$ with boundary labeled 1. As $g \geq 2$, the color $j = 1$ appears effectively for the curve c. Then Lemma 2.21 implies that the Hermitian form on the subspace $W_{1,p,(1,1)}$ is indefinite, and hence the Hermitian form on $W_{g,p,(1,i_2,\ldots,i_n)}$ is indefinite (when the space is nonzero). Then the claim follows from Lemmas 2.19 and 2.20.

The last item follows by observing that the proof given in [37] for $i_1 = 2$, can be adapted without any change to even p (see [26]). □

2.6. **Finiteness for small values of p.** We start by recalling the following well-known results of Humphries ([31]):

Lemma 2.23 ([31]). *(1) The group $\mathrm{Mod}(\Sigma_{g,n})/\mathrm{Mod}(\Sigma_{g,n})[2;-]$ is finite for every $g \geq 1$, $n \leq 1$, and, in particular, $\mathrm{Mod}(\Sigma_{g,n})/\mathrm{Mod}(\Sigma_{g,n})[2]$ is finite. Moreover,*

$$\mathrm{Mod}(\Sigma_{g,n})[2;-] = \mathrm{Mod}(\Sigma_{g,n})[2;1] = \ker(\mathrm{Mod}(\Sigma_{g,n}) \to Sp(2g, \mathbb{Z}/2\mathbb{Z})).$$

(2) The group $\mathrm{Mod}(\Sigma_{g,n})/\mathrm{Mod}(\Sigma_{g,n})[3;-]$ is finite for $(g,n) \in \{(2,0),(2,1),(3,0)\}$, and, in particular, $\mathrm{Mod}(\Sigma_{g,n})/\mathrm{Mod}(\Sigma_{g,n})[3]$ is finite. Moreover,

$$\mathrm{Mod}(\Sigma_2)[3;-] = \mathrm{Mod}(\Sigma_2)[3;1] \supset \ker(\mathrm{Mod}(\Sigma_2) \to Sp(2g, \mathbb{Z})),$$
$$\mathrm{Mod}(\Sigma_3)[3;-] = \mathrm{Mod}(\Sigma_3)[3;1].$$

(3) For $n \geq 0$, $p \geq 4$, the group $\mathrm{Mod}(\Sigma_{2,n})/\mathrm{Mod}(\Sigma_{2,n})[p;-]$ is infinite.

Further ρ_6 factors through $\mathrm{Mod}(\Sigma_g)/\mathrm{Mod}(\Sigma_g)[4;1]$, which is finite.

Wright proved in [57] and [58] the following finiteness results:

Lemma 2.24 ([57]). *The image $\rho_8(\mathrm{Mod}(\Sigma_g))$ is finite for every $g \geq 1$.*

Lemma 2.25 ([58]). *The image $\rho_{12}(\mathrm{Mod}(\Sigma_2))$ is finite for $g = 2$.*

Note that $\rho_{12}(\mathrm{Mod}(\Sigma_g))$ is a quotient of $\mathrm{Mod}(\Sigma_g)/\mathrm{Mod}(\Sigma_g)[24;3]$.

This has to be compared with the finiteness question about $\mathrm{Mod}(\Sigma_g)/\mathrm{Mod}(\Sigma_g)[3]$, which was also answered affirmatively only for $g \in \{2,3\}$, by Humphries (see [31]).

3. The Kähler Orbifold $\overline{\mathcal{M}_g^{n\ an}}[p]$ and its fundamental group

3.1. **Conventions.** Throughout this paper we use the language of stacks. Although most constructions could be performed by working with orbifolds in the topological category, there are inherent complications when we want to promote the construction of root stacks of a normal crossing divisor which is not simple to the category of algebraic spaces or schemes. We then follow [28] building on Vistoli's work [54].

As advocated by [42], we follow the conventions of [20] and view an orbifold as a smooth Deligne-Mumford stack with trivial generic isotropy groups relative to the category of complex analytic spaces with Hausdorff topology. There is an analytification 2-functor from algebraic Deligne-Mumford stacks over $\mathbb{C}$ to complex analytic Deligne-Mumford stacks and an underlying topological stack functor from complex analytic Deligne-Mumford stacks to topological Deligne-Mumford stacks [46, 47, 3]. We will only consider separated quasi-compact algebraic stacks locally of finite type and separated analytical/topological stacks with proper diagonal.

On a complex or topological stack we use the Hausdorff topology, see [52] for the relation with fancier topologies in the algebraic case. In the complex analytic and topological case, we can always refine a covering of a Deligne-Mumford stack by a covering by open substacks that are quotients of a proper action of a finite group. If $\mathcal{X}$ is an algebraic Deligne-Mumford stack $\mathcal{X}$ there is a structural sheaf $\mathcal{O}_{\mathcal{X}}$, an abelian category of coherent algebraic sheaves and similarly in the complex analytic case, see [40, 13]. When we represent a stack by an étale groupoid a sheaf comes by descent from an equivariant sheaf on the groupoid.

An effective Cartier divisor of an algebraic/analytic stack is a closed substack which is defined by an invertible ideal sheaf, locally generated by a nonzero divisor. It is actually better to slightly enlarge the notion and in our terminology a Cartier divisor on $\mathcal{X}$ will be a pair $(s, \mathcal{L})$ of an invertible sheaf $\mathcal{L}$ and a section $s : \mathcal{O}_{\mathcal{X}} \to \mathcal{L}$ up to equivalence (multiplication by an invertible function). The resulting groupoid of Cartier divisors is naturally equivalent to the groupoid $\mathrm{Hom}(\mathcal{X}, [\mathbb{A}^1/\mathbb{G}_m])$ (see [48], p.778) where $[\mathbb{A}^1/\mathbb{G}_m])$ is the quotient stack of the usual action of $\mathbb{G}_m$ on $\mathbb{A}^1$, and is an algebraic stack which is not Deligne-Mumford.

The multiplication map $m : [\mathbb{A}^1/\mathbb{G}_m] \times [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ enables one to define the sum of two divisors by the composition $\mathcal{X} \to [\mathbb{A}^1/\mathbb{G}_m] \times [\mathbb{A}^1/\mathbb{G}_m] \xrightarrow{m} [\mathbb{A}^1/\mathbb{G}_m]$ and the pull back $f^* = \circ f : \mathrm{Hom}(\mathcal{X}, [\mathbb{A}^1/\mathbb{G}_m]) \to \mathrm{Hom}(\mathcal{Y}, [\mathbb{A}^1/\mathbb{G}_m])$, for every $f \in Ob(\mathrm{Hom}(\mathcal{Y}, \mathcal{X}))$.

3.2. Recall (see [47, 42]) that an *orbifold* is a smooth Deligne-Mumford stack with trivial generic isotropy groups relative to the category of complex analytic spaces with the classical topology. An orbifold $\mathcal{X}$ is called *developable* if its universal covering stack is a smooth manifold, namely if every local inertia morphism $\pi_1(\mathcal{X}, x)_{loc} \to \pi_1(\mathcal{X}, x)$ is injective, for x an orbifold point of $\mathcal{X}$. The orbifold $\mathcal{X}$ is *uniformizable* whenever the profinite completion of a local inertia morphism, namely $\pi_1(\mathcal{X}, x)_{loc} \to \widehat{\pi_1(\mathcal{X}, x)}$ is still injective. In particular developable orbifolds with residually finite fundamental groups are uniformizable.

3.3. For $2g-2+n > 0$ the stack $\overline{\mathcal{M}_g^n}$ of n-punctured genus g stable curves is a smooth proper Deligne-Mumford stack, whose analytification $\overline{\mathcal{M}_g^{n\ an}}$ is a smooth proper complex analytic Deligne-Mumford stack too (see [35], Thm. 2.7), actually an orbifold if $(g, n) \neq (2, 0)$. The coarse moduli space $\overline{M_g^{n\ an}}$ of the stack $\overline{\mathcal{M}_g^{n,\ an}}$ is the moduli space of n-punctured genus g stable curves (see [16]). Note that there is a natural map of stacks $\overline{\mathcal{M}_g^{n\ an}} \to \overline{M_g^{n\ an}}$ such that the fiber over a point of $M_g^{n\ an}$ can be identified with the classifying space of its $\mathrm{PMod}(\Sigma_g^n)$-stabilizer. Moreover, $\overline{\mathcal{M}_g^{n\ an}}$ carries a Kähler metric in the sense of [22], from Knudsen-Mumford's Theorem that the moduli space $\overline{M_g^n}$ is projective (see [36], Thm.6.1). Observe that the forgetful map $f_1 : \overline{\mathcal{M}_g^1} \to \overline{\mathcal{M}_g}$ is proper and representable ([16]), more precisely it is schematic (we use the convention of [40] for representability).

3.4. The stack of smooth curves is realized as an open substack $j : \mathcal{M}_g \hookrightarrow \overline{\mathcal{M}_g}$ and $\overline{\mathcal{M}_g} \setminus \mathcal{M}_g$ is a union of irreducible Cartier divisors $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_{\lfloor \frac{g}{2} \rfloor}$.

The generic point of $\mathcal{D}_0$ is a genus $g-1$ curve with one node and the generic point of $\mathcal{D}_k$ has two irreducible smooth curves as components which are meeting at a node and whose respective genera are k and $g-k$. The total divisor

$$\mathcal{D} = \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} \mathcal{D}_i,$$

is a normal crossing divisor (see [16]).

3.5. The map $j^* f_1$ is the forgetful map $\mathcal{M}_g^1 \to \mathcal{M}_g$ and is representable, smooth and proper. Hence $\mathcal{D}^1 := \overline{\mathcal{M}_g^1} \setminus \mathcal{M}_g^1$ is a closed substack with a representable proper map $f_1|_{\mathcal{D}^1} : \mathcal{D}^1 \to \mathcal{D}$.

Now,

$$\mathcal{D}^1 = \sum_{i=0}^{g-1} \mathcal{D}^1_i$$

where the generic point of $\mathcal{D}^1_i$ for $i > 0$ represents a nodal curve with a component of genus i carrying the one marked point and a component of genus $g - i$, both meeting at a node which is distinct from the marked point. On the other hand the generic point of $\mathcal{D}^1_0$ represents an irreducible nodal curve of genus $g-1$ with one marked point. It follows that $f_1^*(\mathcal{D}_i) = \mathcal{D}^1_i + \mathcal{D}^1_{g-i}$, is a divisor whose two components meet transversally in codimension 1, when $0 < i < \frac{g}{2}$ whereas $f_1^*(\mathcal{D}_0) = \mathcal{D}^1_0$. If $g \equiv 0 \pmod 2$, then $f_1^*(\mathcal{D}_{\frac{g}{2}}) = \mathcal{D}^1_{\frac{g}{2}}$. The divisors $\mathcal{D}^1_i$ are irreducible, the total divisor $\mathcal{D}^1$ is a normal crossing divisor (see [35], Thm.2.7) but not a simple normal crossing one. For instance $\mathcal{D}^1_0$ is singular at a singular point of the generic fiber of $\mathcal{D}^1_0 \to \mathcal{D}_0$.

More generally, the stack $\mathcal{M}^n_g$ of smooth n-pointed smooth curves of genus g is an open substack $j : \mathcal{M}^n_g \to \overline{\mathcal{M}^n_g}$ of the Deligne-Mumford stack of stable n-pointed curves of genus g, when $2g - 2 + n > 0$. Then $\mathcal{D}^n = \overline{\mathcal{M}^n_g} \setminus \mathcal{M}^n_g$ is a union of irreducible Cartier divisors $\mathcal{D}_i$, $i = 0, \dots, N_{g,n} - 1$. Knudsen proved in [35] that $\overline{\mathcal{M}^{n+1}_g}$ is equivalent to the stack of n-pointed stable curves endowed with an extra section, namely the universal n-pointed stable curve. Moreover, $\mathcal{D}^n = \overline{\mathcal{M}^n_g} \setminus \mathcal{M}^n_g$ is a union of irreducible Cartier divisors $\mathcal{D}_i$, $i = 0, \dots, N_{g,n} - 1$ and a normal crossings divisor. More specifically the projection map $f_{n+1} : \overline{\mathcal{M}^{n+1}_g} \to \overline{\mathcal{M}^n_g}$ has the property that

$$\mathcal{D}^{n+1} = f_{n+1}^{-1}(\mathcal{D}^n) + \sum_{i=1}^{n} S^{i,n+1},$$

where a point of $S^{i,n+1}$ corresponds to a stable curve with a rational component having two marked points labeled i and $n+1$, namely $S^{i,n+1}$ is the image of the canonical i-th section $\overline{\mathcal{M}^n_g} \to \overline{\mathcal{M}^{n+1}_g}$.

Lemma 3.1. *We have $f_{n+1}^*\mathcal{D}^n = \mathcal{D}^{n+1} - \sum_{i=1}^n S^{i,n+1}$ and in particular, $f_1^*\mathcal{D} = \mathcal{D}^1$.*

Proof. See ([35], Proof of Thm.2.7). □

3.6. **The canonical stack of the root stack of a normal crossing divisor that may not be simple.** The first ingredient of our construction is the *root stack* $\mathcal{S}\left[\sqrt[p]{\mathcal{E}}\right]$ associated to a Cartier divisor $\mathcal{E}$ on a stack $\mathcal{S}$, which was introduced independently in [1, 10] and used in [19]. Specifically, it is defined as:

$$\mathcal{S}\left[\sqrt[p]{\mathcal{E}}\right] := \mathcal{S} \times_{\phi_{\mathcal{E}}, [\mathbb{A}^1/\mathbb{G}_m], \_^p} \left[\mathbb{A}^1/\mathbb{G}_m\right],$$

where $\phi_{\mathcal{E}} : \mathcal{S} \to \left[\mathbb{A}^1/\mathbb{G}_m\right]$ is the natural map and $\_^p$ is the p-th power map, $p \in \mathbb{N}^*$, the case $p = 1$ being trivial. We use 2-fibered products above as one should (see [40]).

Note that the morphism $\mathcal{S}\left[\sqrt[p]{\mathcal{E}}\right] \to \mathcal{S}$ is universal among morphisms for which $\mathcal{E}$ pulls back to p times a Cartier divisor.

The same construction can be performed with respect to the topological and the analytic categories after application of $\_^{an}$, $\_^{top}$ since these lax 2-functors preserve finite homotopy limits.

The second ingredient is the construction of the *canonical stack* due to Vistoli (see [54]). Recall that a Deligne-Mumford stack $\mathcal{X}$ over a field of characteristic zero has quotient singularities if it has a (locally étale in the algebraic case) cover which is a scheme with quotient singularities, namely locally the quotient of a smooth variety by a finite group. If $\mathcal{X}$ is a separated analytic Deligne-Mumford stack of finite type with generic trivial isotropy having only quotient singularities then there is a smooth canonical stack $\mathcal{X}^{can} \to \mathcal{X}$ which is an equivalence in codimension 1 and which is unique up to equivalence.

Let $\mathcal{S}^{an}$ be an analytic smooth Deligne-Mumford stack and $\mathcal{E}$ be a normal crossings divisor. Set Δ^n for the open disk around the origin in $\mathbb{C}^n$ and $E_j = \{z_j = 0\} \subset \Delta^n$ be the coordinate hyperplanes associated to a system of complex coordinates (z_j). An étale map $\eta : \Delta^n \to \mathcal{S}^{an}$ is said to be a chart adapted to $\mathcal{E}$ if

$$\eta^*\mathcal{E} = \left\{ (z_i)_{i=1}^n ;\ \prod_{c=1}^k z_{i_c} = 0 \right\} = \sum_{c=1}^k E^\eta_{i_c},$$

where the map $c \mapsto i_c$ is injective. In order to be precise, we shall at some point display the dependence of k in η and use the notation $k = k(\eta)$.

Proposition 3.2. *If $\mathcal{S}^{an}$ is an analytic smooth Deligne-Mumford stack with trivial generic isotropy and $\mathcal{E}$ is a normal crossing divisor on $\mathcal{S}^{an}$, then there exists a smooth Deligne-Mumford stack $e_p : \mathcal{S}^{an}[p, \mathcal{E}] \to \mathcal{S}^{an}$ unique up to equivalence with the same moduli space as $\mathcal{S}^{an}$ such that for every adapted chart η we have an equivalence:*

$$\Delta^n \times_{\mathcal{S}^{an}, \eta, e_p} \mathcal{S}^{an}[p, \mathcal{E}] \simeq \Delta^n \left[\sqrt[p]{E^\eta_{i_1}} \right] \times_{\Delta^n} \times \ldots \times_{\Delta^n} \Delta^n \left[\sqrt[p]{E^\eta_{i_k}} \right].$$

Proof. We first claim that $\mathcal{S}^{an}\left[\sqrt[p]{\mathcal{E}_1}\right] \times_{\mathcal{S}} \mathcal{S}\left[\sqrt[p]{\mathcal{E}_2}\right] \times_{\mathcal{S}} \cdots \times_{\mathcal{S}} \mathcal{S}\left[\sqrt[p]{\mathcal{E}_k}\right]$ has quotient singularities.

Indeed, if one removes a codimension ≥ 2 closed substack $\mathcal{Z}$ from the *smooth* stack

$$\mathcal{V} = \Delta^n \left[\sqrt[p]{E^\eta_{i_1}} \right] \times_{\Delta^n} \cdots \times_{\Delta^n} \Delta^n \left[\sqrt[p]{E^\eta_{i_k}} \right]$$

the fundamental group will not change. Such a $\mathcal{Z}$ is actually the substack over $Z \subset \Delta^n$ where Z is a closed analytic subspace. Then the universal covering stack of $\mathcal{V}$ is Δ^n and $\mathcal{V} = [\mu_p^k \backslash \Delta^n]$. Here μ_p is the group of p-th roots of unity and the s-th factor acts by multiplication of the s-th coordinate.

Consider now the stack $\mathcal{U} = \Delta^n \left[\sqrt[p]{E_{I_1}}\right] \times_{\Delta^n} \cdots \times_{\Delta^n} \Delta^n \left[\sqrt[p]{E_{I_m}}\right]$, where we denote $E_I = \sum_{j \in I} E^\eta_j$, and I_j, for $1 \leq l \leq m$ form a partition of the set $\{i_1, \ldots, i_k\}$. The inclusion of the residual gerbe at the origin in $\mathcal{U}$ is a deformation retract. Then the local uniformizability of analytic Deligne-Mumford stacks implies that $\mathcal{U} \simeq [G\backslash U]$, where U is an analytic space, G has a fixed point over the origin and the natural map $m_{\mathcal{U}} : U \to \Delta^n$ is G-equivariant. Then $\mathcal{U} \setminus m^{-1}(Z)$ is a connected uniformizable stack étale over $\Delta^n \setminus Z$, in particular $[\mathcal{U} \setminus m^{-1}(Z)] \simeq [G' \backslash (\Delta^n \setminus m_{\mathcal{V}}^{-1}(Z))]$, for some finite group G'. As the root stack construction behaves well under pull back, there is an isomorphism $U = G'\backslash\Delta^n$, where $G' = \ker(\mu_p^k \to \mu_p^m)$ is the kernel of the homomorphism $(\zeta_l) \mapsto (\prod_{l \in I_1} \zeta_l, \ldots, \prod_{l \in I_m} \zeta_l)$. This proves the claim.

Furthermore, we can define, using Vistoli's canonical stack which applies to a Deligne-Mumford stack $\mathcal{X}$ with quotient singularities and yields a smooth Deligne-Mumford stack together with a map $\psi : \mathcal{X}^{can} \to \mathcal{X}$ which is an equivalence in codimension one, see [28]:

$$\mathcal{S}^{an}[p, \mathcal{E}] = \left(\mathcal{S}^{an}\left[\sqrt[p]{\mathcal{E}_1}\right] \times_{\mathcal{S}} \mathcal{S}\left[\sqrt[p]{\mathcal{E}_2}\right] \times_{\mathcal{S}} \cdots \times_{\mathcal{S}} \mathcal{S}\left[\sqrt[p]{\mathcal{E}_k}\right] \right)^{can}.$$

The uniqueness statement is the main theorem in [28] at least in the algebraic case. The proof given there can be adapted to the (easier) analytic case.

□

Obviously, the construction does not require the ramification indices to be equal and the proof above shows that:

Proposition 3.3. *Let $\mathcal{E}_1, \ldots, \mathcal{E}_r$, $1 \leq r \leq \infty$, be the irreducible components of $\mathcal{E}$ and fix $\mathbf{p} := (p_1, \ldots, p_r) \in \mathbb{N}^{*r}$. There is a unique (up to equivalence) smooth Deligne-Mumford stack $\mathcal{S}^{an}[\mathbf{p}, \mathcal{E}] \to \mathcal{S}^{an}$ which is an equivalence outside $\mathcal{E}$ and ramifies with index p_i on $\mathcal{E}_i$.*

This is probably valid in the algebraic category over an algebraically closed field assuming tame ramification.

Remark 3.4. *(See [22] for a proof.) If $\mathcal{S}^{an}$ carries a Kähler metric, so does $\mathcal{S}^{an}[\mathbf{p},\mathcal{E}]$.*

3.7. The analytic stacks $\overline{\mathcal{M}_g^{an}}[\mathbf{p}]$.

Definition 3.5. *Given $\mathbf{p} \in (\mathbb{N}^*)^{N_{g,n}}$ we denote by $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{p}]$ the smooth proper Deligne Mumford stack $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{p},\mathcal{D}]$. If $p,q \geq 2$ we shall use the specific notation $\overline{\mathcal{M}_g^{n\ an}}[p,q]$ for the smooth proper Deligne Mumford stack $\overline{\mathcal{M}_g^{n\ an}}[(p,q,q,\dots,q),\mathcal{D}]$ and will drop q from the notation, when $q = p$.*

Theorem 3.6. *Let $g,n \in \mathbb{N}$ such that $2g-2+n > 0$, $(g,n) \notin \{(1,1),(2,0)\}$. The quotient* $\mathrm{PMod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[\mathbf{p}]$ *is the fundamental group of the compact Kähler orbifold $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{p}]$.*

Proof of Theorem 3.6. There is an isomorphism $\pi_1(\mathcal{M}_g^{n\ an}) \simeq \mathrm{Mod}(\Sigma_g^n)$ which carries the conjugate of the meridian loops around the various components of $\mathcal{D}^n$ to the Dehn twists along simple closed curves. By Proposition 3.2 the analytic stack $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{p}]$ is a proper smooth Deligne Mumford orbifold endowed with a map $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{p}] \to \overline{\mathcal{M}_g^{n\ an}}$ inducing an isomorphism on the moduli space. As $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{p}]$ is smooth, we can delete the codimension 2 singular substack of $\mathcal{D}^n$ without changing the orbifold fundamental group. The stack-theoretic Van Kampen theorem from [59] implies that the fundamental group of the orbifold $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{p}]$ is $\mathrm{PMod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[\mathbf{p}]$.

Eventually, the orbifold $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{p}]$ has projective moduli space and hence is a Kähler orbifold by [22]. □

For the easy description of the isotropy groups of these orbifolds, see section 4.2.

Example 3.7. *The group* $\mathrm{PMod}(\Sigma_0^5)/\mathrm{Mod}(\Sigma_0^5)[5]$ *is a uniform complex hyperbolic group constructed by Hirzebruch: it corresponds to the quotient of the surface of degree 5^5 over $\mathbb{P}^2$ which ramifies with multiplicity 5 on $CEVA(2)$ by the (abelian) group of this Galois covering. For general p see [19].*

A similar statement works for $\mathrm{Mod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[p]$ taking the quotient of $\overline{\mathcal{M}_g^{n\ an}}[p]$ by the group of permutations of n punctures, see [51] for group actions on stacks.

Remark 3.8. *If we delete the singular locus $\mathcal{D}^{n,sing}$ of $\mathcal{D}$, the divisor of the pair*

$$(\overline{\mathcal{M}_g^{n\ an}} \setminus \mathcal{D}^{n,sing}, \mathcal{D}^{n,reg})$$

is actually an effective smooth divisor. The root stack

$$\overline{\mathcal{M}_g^{n\ an\ 0}}[\mathbf{p}] := (\overline{\mathcal{M}_g^{n\ an}} \setminus \mathcal{D}^{n,sing})[\sqrt[\mathbf{p}]{\mathcal{D}^{n,reg}}]$$

is the complement of a closed substack of codimension at least 2 in $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{p}]$ hence the natural map $\overline{\mathcal{M}_g^{n\ an\ 0}}[\mathbf{p}] \to \overline{\mathcal{M}_g^{n\ an}}[\mathbf{p}]$ induces an isomorphism of the fundamental groups.

Uniformizability of $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{p}]$ can then be formulated only in terms of the moduli map

$$\chi : \overline{\mathcal{M}_g^{n\ an\ 0}}[\mathbf{p}] \to \overline{M_g^{n\ an}}.$$

Indeed the isotropy morphism at (a lift of) a point $x \in \overline{M_g^{n\ an}}$ identifies with the morphism $\pi_1(\chi^{-1}(U)) \to \overline{\mathcal{M}_g^{n\ an\ 0}}[\mathbf{p}]$ where $U \subset \overline{M_g^{n\ an}}$ is a small neighborhood of x. If it holds we can construct an étale Galois covering scheme $Z \to \overline{\mathcal{M}_g^{n\ 0}}[\mathbf{p}]$, smooth and quasiprojective, such that, if G is the Galois group, $[Z/G] = \overline{\mathcal{M}_g^{n\ 0}}[\mathbf{p}]$, the normalization $\overline{Z}$ of $\overline{M_g^n}$ in the field of rational functions of Z is smooth, the G action extends to $\overline{Z}$ and $[\overline{Z}/G] \cong \overline{\mathcal{M}_g^n}[\mathbf{p}]$

3.8. Orbifold compactifications and Teichmüller level structures.

A *Teichmüller level structure* λ is a finite index normal subgroup $H_\lambda \triangleleft \mathrm{PMod}(\Sigma_g^n)$ (see [9]). The natural finite étale $G_\lambda := H_\lambda \backslash \mathrm{PMod}(\Sigma_g^n)$-covering $\mathcal{M}_g^{n,\lambda} \to \mathcal{M}_g^n$ can be compactified uniquely to a finite representable G_λ-ramified covering $\overline{\mathcal{M}_g^{n,\lambda}} \to \overline{\mathcal{M}_g^n}$ (see [7]).

Denote by $\overline{\mathcal{M}_g^n}[\lambda]$ the stack quotient $[G_\lambda \backslash \overline{\mathcal{M}_g^{n,\lambda}}]$ (in the sense of [51]). Then $\overline{\mathcal{M}_g^n}[\lambda]$ is a Deligne-Mumford stack compactifying $\mathcal{M}_g^n$ and dominating $\overline{\mathcal{M}_g^n}$. In particular, its moduli space is $\overline{M_g^n}$.

The scheme $\overline{\mathcal{M}_g^{n,\lambda}}$ or equivalently the stack $\overline{\mathcal{M}_g^n}[\lambda]$ may or may not be smooth. The abelian level structure correspond to root stacks of $\mathcal{D}_0$ that are not smooth at the double locus of $\mathcal{D}_0$. For some level structures λ, $\overline{\mathcal{M}_g^n}[\lambda]$ is however smooth (which is equivalent to $\overline{\mathcal{M}_g^{n,\lambda}}$ being smooth) and uniformizable (see [43] for $n = 0$, and [6] in general), hence $\overline{M_g^n}$ is a quotient of a smooth variety by a finite group.

Let $\mathcal{D}_g^n := \overline{\mathcal{M}_g^n} \setminus \mathcal{M}_g^n$. Then according to Knudsen (see [35]) this is a divisor with normal crossings with an irreducible decomposition

$$\mathcal{D}_g^n = \sum_{i=1}^{N_{g,n}} \mathcal{D}_{g,i}^n.$$

Given a Teichmüller level structure λ, we define $k_i(\lambda) \in \mathbb{N}^*$ to be the order of the image of a Dehn twist corresponding to a loop encircling once $D_{g,i}^n$ in G_λ and denote by $\mathbf{k}(\lambda)$ the vector $(k_i)_i = (k_i(\lambda))_i \in \mathbb{N}_{\geq 1}^{N_{g,n}}$.

Proposition 3.9. *The Deligne-Mumford stack $\overline{\mathcal{M}_g^n}[\lambda]$ lies in a diagram of Deligne-Mumford stacks whose moduli space is $\overline{M_g^n}$ in which all maps are étale in codimension 1:*

$$\overline{\mathcal{M}_g^n}[\mathbf{k}(\lambda), \mathcal{D}_g^n] \to \overline{\mathcal{M}_g^n}[\lambda] \to \overline{\mathcal{M}_g^n}\left[\sqrt[k_1]{\mathcal{D}_{g,1}^n}\right] \times_{\overline{\mathcal{M}_g^n}} \overline{\mathcal{M}_g^n}\left[\sqrt[k_2]{\mathcal{D}_{g,2}^n}\right] \times_{\overline{\mathcal{M}_g^n}} \cdots$$

Furthermore, $\overline{\mathcal{M}_g^n}[\lambda]$ is smooth if and only if the first map is an equivalence and $\overline{\mathcal{M}_g^n}[\lambda] \simeq \overline{\mathcal{M}_g^n}[\mathbf{k}(\lambda), \mathcal{D}_g^n]$

Proof. The first statement is clear, the second one is a consequence of the purity of the branch locus. □

It seems to be a rather delicate problem to understand the $\mathbf{k}(\lambda)$ that occur and give rise to a smooth uniformizable compactification.

4. TQFT representations on isotropy groups of stable curves

4.1. **Stabilizers of pants decompositions.** The aim of this section is to describe the image of the stabilizer of a pants decomposition by quantum representations. If P is a pants decomposition of some surface $\Sigma_{g,n}$ we keep the same notation for the isotopy class of the multicurve consisting of all loops from P. Note that the order of the curves is irrelevant. Set further $\mathrm{Mod}(\Sigma_{g,n}, P)$ for the pure mapping classes of orientation-preserving homeomorphisms which preserve the isotopy class of the multicurve P and preserve pointwise the boundary components.

Let G be a uni-trivalent graph embedded in the handlebody H_g in such a way that the endpoints of G sit on the boundary and H_g retracts onto G. Let $\Sigma_{g,n}$ be the result of drilling small holes around the endpoints of G within the boundary surface. Then $\Sigma_{g,n}$ inherits a pants decomposition $P = P(G)$ which consists of the set of simple loops γ_e, where γ_e bounds a small disk intersecting once e, for every edge e of G.

Let $\mathrm{Aut}(G)$ denote the group of automorphisms of G which preserve pointwise the leaves.

The graph G has two types of internal trivalent vertices, as follows. A *generic* vertex has 3 distinct incoming edges and provides an essential $\Sigma_{0,3} \subset \Sigma_{g,n}$ and the *tadpole* has only 2 distinct incoming edges and gives raise to an essential $\Sigma_{1,1} \subset \Sigma_{g,n}$. Fix an orientation of the tadpole loops and define $\mathrm{Aut}^+(G) \subset \mathrm{Aut}(G)$ to be the subgroup preserving the orientation of the tadpoles.

If $P \subset \Sigma_{g,n}$ is a pants decomposition of $\Sigma_{g,n}$ we denote by $T(P)$ the free abelian group generated by the Dehn twists along the non-peripheral curves in P and the n boundary components. Thus $T(P)$ is isomorphic to $\mathbb{Z}^{3g-3+n} \times \mathbb{Z}^n$.

Lemma 4.1. *The group* $\mathrm{Mod}(\Sigma_{g,n}, P)$ *is an extension*

$$1 \to T(P) \to \mathrm{Mod}(\Sigma_{g,n}, P) \to \mathrm{Aut}(G) \to 1. \tag{36}$$

Proof. Any mapping class preserving P induces an automorphism of G. On tadpole loops, the orientation is preserved if the mapping class preserves the two connected components of a deleted annulus neighborhood of the simple curve in the one holed torus corresponding to the loop and reversed if these two components are exchanged. If this automorphism is trivial then each pair of pants in the decomposition P is fixed. It is well-known that the mapping class group $\mathrm{Mod}(\Sigma_{0,3})$ of a pair of pants is the abelian group generated by the boundary Dehn twists.

This establishes the exact sequence above, except for the surjectivity. First of all, observe $\mathrm{Aut}(G) = (\mathbb{Z}/2\mathbb{Z})^t \rtimes \mathrm{Aut}^+(G)$ where t is the number of tadpoles. There is a mapping class $h \in \mathrm{Mod}(\Sigma_{1,1})$ which lifts the elliptic involution $-\mathbf{1} \in SL_2(\mathbb{Z}) = \mathrm{Mod}(\Sigma_1^1)$. Its square is the Dehn twist along the boundary curve. Given $(\epsilon_i)_{1\le i\le t} \in (\mathbb{Z}/2\mathbb{Z})^t$, we can glue h^{ϵ_k} on the k-th tadpole and the identity on the rest of the surface to get a mapping class that projects to $(\epsilon_i)_{1\le i\le t}$ in the first factor.

If $\phi \in \mathrm{Aut}^+(G)$ and Π is the pair of pants corresponding to a generic vertex v, we construct a homeomorphism from Π to Π' the pair of pants corresponding to $\phi(v)$ mapping in an orientation preserving fashion the three boundary curves represented by the incident edges $\{e, e', e''\}$ to the three boundary curves represented by $\{\phi(e), \phi(e'), \phi(e'')\}$. These homeomorphisms glue into a homeomorphism of $\Sigma_{g-t,n+t} = \Sigma_{g,n} \setminus \cup_{1\le k\le t}\Sigma_{1,1}$ preserving $P \cap \Sigma_{g-t,n+t}$. Then, we can extend h to a homeomorphism $\bar{h}$ of $\Sigma_{g,n}$ preserving P and the orientation on tadpoles. Obviously $\bar{h}$ maps to ϕ.

□

The following lemma will not be used here, but we include it nevertheless.

Lemma 4.2. *The exact sequence (36) splits over* $\mathrm{Aut}^+(G)$.

Proof. Each pair of pants decomposes into the union of two hexagons glued along three disjoint segments in their boundary. On each pair of pants we color one hexagon in black, the other one in white. This expresses $\Sigma_{g,n}$ as an union of $6g-6+2n$ hexagons, $3g-3+n$ of them being black and $3g-3+n$ being white.

At each boundary component of a pant we may adjust locally the hexagons so that the parts of their boundaries that are internal to the pant match and glue into a union of simple closed curves and that the black hexagons of two adjacent pants (including self-adjacent ones) match at the boundary of the pant.

This induces a partition of $\Sigma_{g,n}$ into two (non necessarily connected) subsurfaces, one being white, the other one black. We consider the group A of mapping classes of orientation preserving homeomorphisms of $\Sigma_{g,n}$ that permute the maximal set of $3g-3+n$ non-peripheral simple curves and preserve the black/white partition.

There is natural morphism $A \to \mathrm{Aut}^+(G)$, where $\mathrm{Aut}^+(G)$ fixes an orientation of the tadpole loops. Now an element of $\mathrm{Aut}^+(G)$ defines a permutation of the set of pairs of black and white hexagons in the same pant respecting boundary hence lifts to an element of A. The kernel of this morphism is trivial since the mapping class group of an hexagon (or a disk) modulo its boundary is trivial, by Alexander's lemma. □

Let $\mathbf{k}$ be a integral vector indexed by the set of $\mathrm{Aut}(G)$-orbits of non-peripheral simple closed curves and a component 1 attached to the boundary components on $(\Sigma_{g,n}, P)$. We denote by $T(P)[\mathbf{k}]$ the free abelian subgroup of $T(P)$ generated by k_j-th powers of Dehn twists along curves in P of type j. Then $T(P)[\mathbf{k}]$ is a normal subgroup of $\mathrm{Mod}(\Sigma_{g,n}, P)$ and we have an extension

$$1 \to \frac{T(P)}{T(P)[\mathbf{k}]} \to \frac{\mathrm{Mod}(\Sigma_{g,n}, P)}{T(P)[\mathbf{k}]} \to \mathrm{Aut}(G) \to 1. \tag{37}$$

We use the shorthand notation $[q, r]$ to denote the integral vector whose components are q on the $\mathrm{Aut}(G)$)-orbits of non-separating curves and r on the orbits of the separating ones.

Our main result here is then the following:

Proposition 4.3. *Let* $\mathbf{i} = (2, 2, \ldots, 2)$, $g \geq 0$, $n \geq 0$ *and* $2g - 2 + n > 0$ *and* $(g, n) \neq (1, 0), (1, 1), (2, 0)$.

(1) If $p \geq 5$ *is odd, then* $\rho_{p,(\mathbf{i})}(\mathrm{Mod}(\Sigma_{g,n}, P))$ *is isomorphic to* $\frac{\mathrm{Mod}(\Sigma_{g,n},P)}{T(P)[p]}$.

(2) If $p \geq 8$ *is even then* $\rho_{p,(\mathbf{i})}(\mathrm{Mod}(\Sigma_{g,n}, P))$ *is isomorphic to* $\frac{\mathrm{Mod}(\Sigma_{g,n},P)}{T(P)[2p,p/\mathrm{g.c.d.}(p,4)]}$.

Proof. Let us first investigate $\ker \rho_{p,(\mathbf{i})} \cap T(P)$. The description for ρ_p shows that $\rho_p|_{T(P)}$ maps into a maximal torus of the projective group. In the standard basis attached to P, all matrices associated to the Dehn twists along curves in P are simultaneously diagonal.

We did not look at the order of Dehn twists in genus 0. However, for this particular vector of colors $\mathbf{i} = (2, \ldots, 2)$, in any genus, we can always color the internal edges except one with the color 2 and the remaining one with the colors 0 or 2. Hence we are reduced to the same calculation as in the proof of Lemma 2.11 .(1).(c) resp. Lemma 2.12 .(3).(a) for separating curves. So the order is always p if p is odd and is $p/\mathrm{g.c.d.}(p, 4)$ if p even. In higher genus, nothing changes for separating curves. Since the non-separating edges can be colored with the colors $0, 2$ if p is odd, with the colors $0, 1, 2$ if p is even, the non-separating order is p when p is odd, $2p$ when p is even.

It follows that $T(P)[p] \subseteq \ker \rho_{p,(\mathbf{i})} \cap T(P)$, if p is odd and $T(P)[2p, p/\mathrm{g.c.d.}(p, 4)] \subseteq \ker \rho_{p,(\mathbf{i})} \cap T(P)$, if p is even, respectively.

We prove by induction the genus that these inclusions are equalities.

First if $g = 0$ the case $n = 3$ is trivial, the cases $n = 4, 5$ follow by inspection. Assume now that the induction hypothesis holds for all $k \leq n$ and consider $(\Sigma_{0,n+1}, P)$ a pants decomposition. Let us fix an enumeration $(e_i)_{1\leq i\leq 3g-3+n+1}$ of the internal edges of G which is a tree. Suppose that we have a relation of the form

$$\prod_{i=1}^{3g-3+n+1} \rho_{p,(\mathbf{i})}(T_{c_i})^{m_i} = 1.$$

We view this as an identity in $PGL(W(\Sigma_{0,n+1})_{(\mathbf{i})})$. Let us cut along the internal edge e_1 and get two holed spheres with m, resp. m' number of holes $3 \leq m, m' \leq n$. Color the respective boundary components corresponding to e_1 with the color 2. This gives an inclusion of vector spaces

$$W((\Sigma_{0,m})_{(\mathbf{i})}) \oplus W((\Sigma_{0,m'})_{(\mathbf{i})}) \subset W((\Sigma_{0,n+1})_{(\mathbf{i})}).$$

Moreover $\widetilde{\rho}_{p,(\mathbf{i})}$ preserves the two factors and restrict to their $\widetilde{\rho}_{p,(\mathbf{i})}$ on their $T(P)$. It follows that the separating order divides e_i for $i \geq 2$. Hence the relation reduces to $\rho_{p,(\mathbf{i})}(T_{c_1})^{m_1} = 1$. And the separating order divides m_1 as desired.

The case $g = 0$ being settled, the higher genus case follows by induction on g. One uses the same argument as above cutting along non-separating edges. One gets only a subspace

$$W((\Sigma_{g,n})_{(\mathbf{i})}) \subset W((\Sigma_{g+1,n})_{(\mathbf{i})}).$$

the important point is the functoriality of the restriction of $\widetilde{\rho}_{p,(\mathbf{i})}$ to this subspace.

The kernel of $\rho_{p,(\mathbf{i})} : \frac{\mathrm{Mod}(\Sigma_{g,n},P)}{T(P)[p]} \longrightarrow PGL(W((\Sigma_{g,n})_{(\mathbf{i})})$ is thus a subgroup K which maps isomorphically onto its image in $\bar{K} < \mathrm{Aut}(G)$. The action of $\mathrm{Aut}(G)$ on the admissible colorings of G is then an invariant subset of its action by $\rho_{p,(\mathbf{i})}$ on the projective space of conformal blocks $\mathbb{P}(W((\Sigma_{g,n})_{(\mathbf{i})}))$. It follows that $\bar{K}$ is contained in the subgroup of $\mathrm{Aut}(G)$

which fixes all admissible colorings of G. Since one can always put the color 2 on all edges except one, it follows that $\bar{K}$ fixes all edges of G.

Assume $k \in \bar{K}$ does not fix a generic vertex v. Then the three edges emanating from v arrive at $k.v$. It follows that G is a theta graph. Hence $(g,n) = (2,0)$. Assume k fixes every generic vertex and does not fix a tadpole vertex v. It follows that k permutes 2 tadpole vertices joined by an edge and $(g,n) = (2,0)$ too. Actually we reach a contradiction in this case since it permutes the two tadpole loops.

In all other cases k fixes all vertices and edges. Hence, $\bar{K}$ is a subgroup of the $(\mathbb{Z}/2\mathbb{Z})^t$ corresponding to the tadpoles hence to essential subsurfaces $\Sigma_{1,1} \subset \Sigma_{g,n}$. If we take the $h \in Mod(\Sigma_{1,1})$ be the lift of the elliptic involution and push it as a mapping class of $\Sigma_{g,n}$ we have $h^2 = T_c$ where c is the boundary of Σ_1^1. Hence if this boundary is non peripheral we deduce from $\rho_{p,1,(2)}(T_c) \neq 1$ that $\rho_{p,1,(2)}(h) \neq 1$. We conclude as above by induction on g. The excluded cases are exceptions, the corresponding elements being central in the corresponding mapping class groups. □

4.2. **Isotropy groups of general stable curves.** If C is a stable curve of genus g with n marked punctures, its automorphism group acts naturally on the dual graph Γ_C of C, the unoriented marked graph whose vertices are the irreducible components of C marked by their genus and the marking of the punctures they contain and whose edges are the double points connecting them, loops being allowed. A topological model of C is obtained from a closed oriented surface S of genus g with n marked points by pinching every loop of the multicurve $\underline{\alpha}$ to a point. Then Γ_C is isomorphic to the weighted dual graph $\Gamma(\underline{\alpha})$ of the multicurve $\underline{\alpha}$. Denote by $\widetilde{C}$ the normalization of C along with the node branches data.

Proposition 4.4. *Let $x : \{pt\} \to \overline{\mathcal{M}_g^{n\ an}}$ be a point representing an n-pointed stable curve C_x and $B\mathrm{Aut}(C_x) \to \overline{\mathcal{M}_g^{n\ an}}$ its residual gerbe. Let U be a neighborhood of x such that $B\mathrm{Aut}(C_x) \to U$ is a deformation retract. Then there is a short exact sequence:*

$$1 \to \mathbb{Z}^{\underline{\alpha}} \to \pi_1(U \cap \mathcal{M}_g^{n\ an}, *) \to \mathrm{Aut}(C_x) \to 1,$$

where $\mathrm{Aut}(C_x)$ acts through its natural action on the free abelian group over the vertex set of the dual graph Γ_x of C_x. Moreover, if C_x is maximally degenerate, then the exact sequence above is isomorphic to the sequence in Lemma 4.1 for an appropriate pants decomposition.

Let $\mathbf{p} = (p_0, p_1, \dots, p_r)$ be a ramification multi-index. If C is a stable curve whose topological type is obtained from the multicurve $\underline{\alpha}$ let us define $\mu_{\mathbf{p}}(C)$ be the product $\mu_{p_i}^{k_i}$, where k_i is the number of loops in α of type i.

Proposition 4.5. *Consider a point $x[\mathbf{p}]$ of $\overline{\mathcal{M}_g^{n\ an}}[\mathbf{p}]$ mapping to a point x of $\overline{\mathcal{M}_g^{n\ an}}$ which corresponds to the stable curve C_x. Then its isotropy group is given by the exact sequence:*

$$1 \to \mu_{\mathbf{p}}(C_x) \to \pi_1(\overline{\mathcal{M}_g^{n\ an}}[\mathbf{p}], x[\mathbf{p}])_{loc} \to \mathrm{Aut}(C_x) \to 1.$$

4.3. **Faithfully representing isotropy groups.** The purpose of this section is to find the image of the isotropy group under suitable finite dimensional representations. Let C be a stable curve of genus g with n marked points with a topological model obtained from a closed oriented surface S with n marked points by pinching every loop of a multicurve $\underline{\alpha}$ to a point. Let $\Gamma = \Gamma(\underline{\alpha})$ be the weighted dual graph of $\underline{\alpha}$. Denote by $\widetilde{C}$ the normalization of C along with the node branches data. We then have an exact sequence:

$$1 \to \widetilde{\mathrm{Aut}}(\widetilde{C}) \to \mathrm{Aut}(C) \xrightarrow{B} G\mathrm{Aut}(\Gamma(\underline{\alpha})) \to 1,$$

where $\widetilde{\mathrm{Aut}}(\widetilde{C})$ is the group of automorphisms of $\widetilde{C}$ preserving each node branch and $G\mathrm{Aut}(\Gamma(\underline{\alpha}))$ is the subgroup of the group $\mathrm{Aut}(\Gamma(\underline{\alpha}))$ of automorphisms of the graph which can be lifted to C.

Further let $\widetilde{I}(C) \subseteq \mathrm{PMod}(S) = \mathrm{PMod}(\Sigma_g^n)$ denote the lift of the automorphism group of the stable curve C to the mapping class group $\mathrm{PMod}(S)$, which fits into the exact sequence:

$$1 \to \mathbb{Z}^{\underline{\alpha}} \to \widetilde{I}(C) \stackrel{Q}{\to} \mathrm{Aut}(C) \to 1.$$

Let $\theta_{p,S} : \mathrm{Mod}(S) \to \mathrm{Aut}(H_1(S; \mathbb{Z}/p\mathbb{Z}))$ be the homology representation for the surface S on the homology with $\mathbb{Z}/p\mathbb{Z}$ coefficients.

Lemma 4.6. *For every stable curve C with associated multicurve $\underline{\alpha}$ and $p \geq 3$ we have*

$$\ker(B \circ Q) \cap \ker(\theta_{p,S}) \subset \mathbb{Z}^{\underline{\alpha}}.$$

Proof. Let $\widetilde{C}$ have the connected components C_i which are of genus g_i, k_i marked points coming from the branched nodes and r_i additional marked points inherited from the marked points of C. Note that our initial surface S is obtained from the union of surfaces $\Sigma_{g_i,k_i}^{r_i}$ by identifying boundary circles corresponding to the pinched curves from $\underline{\alpha}$.

As the homological representation $\theta_{p,S}$ is functorial, the inclusion induces a homomorphism $H_1(\Sigma_{g_i,k_i}^{r_i}; \mathbb{Z}/p\mathbb{Z}) \to H_1(S; \mathbb{Z}/p\mathbb{Z})$ which is equivariant with respect to the natural map:

$$\Phi_i : \mathrm{Mod}(\Sigma_{g_i,k_i}^{r_i}) \longrightarrow \mathrm{Mod}(S, \underline{\alpha}).$$

Set further:

$$\Phi = \prod_i \Phi_i : \prod_i \mathrm{Mod}(\Sigma_{g_i,k_i}^{r_i}) \to \mathrm{Mod}(S, \underline{\alpha}).$$

Let $\Lambda \subset H_1(S, \mathbb{Z}/p\mathbb{Z})$ and $\Lambda_i \subset H_1(\Sigma_{g_i,k_i}^{r_i}; \mathbb{Z}/p\mathbb{Z})$ be the subgroups generated by the homology classes of the components of the multicurves $\underline{\alpha}$ and $\underline{\alpha}_i = \underline{\alpha} \cap \Sigma_{g_i,k_i}^{r_i}$, respectively. These are isotropic subspaces for the intersection form which are invariant by the image of Φ and $\mathrm{Mod}(\Sigma_{g_i,k_i}^{r_i})$, respectively. Then, there is an injective homomorphism

$$I : \bigoplus_i H_1(\Sigma_{g_i,k_i}^{r_i}; \mathbb{Z}/p\mathbb{Z})/\Lambda_i \longrightarrow H_1(S; \mathbb{Z}/p\mathbb{Z})/\Lambda,$$

which is equivariant under the group homomorphism Φ, where the groups act via the homology representation.

Note that the action of the subgroup $\mathbb{Z}^{\underline{\alpha}_i}$ on $H_1(\Sigma_{g_i,k_i}^{r_i}; \mathbb{Z}/p\mathbb{Z})/\Lambda_i$ is trivial. Therefore the action of $\mathrm{Mod}(\Sigma_{g_i,k_i}^{r_i})$ on $H_1(\Sigma_{g_i,k_i}^{r_i}; \mathbb{Z}/p\mathbb{Z})$ induces an action of $\mathrm{Mod}(\Sigma_{g_i}^{k_i+r_i})$ on the quotient $H_1(\Sigma_{g_i,k_i}^{r_i}; \mathbb{Z}/p\mathbb{Z})/\Lambda_i$. The latter action coincides, under the isomorphism induced by the inclusion $H_1(\Sigma_{g_i,k_i}^{r_i}; \mathbb{Z}/p\mathbb{Z}) \to H_1(\Sigma_{g_i}^{k_i+r_i}; \mathbb{Z}/p\mathbb{Z})$, with the reduced homology representation of $\mathrm{Mod}(\Sigma_{g_i}^{k_i+r_i})$ on $H_1(\Sigma_{g_i}^{k_i+r_i}; \mathbb{Z}/p\mathbb{Z})/\Lambda_i$. Note that the action of the pure mapping class group $\mathrm{PMod}(\Sigma_{g_i}^{k_i+r_i})$ on Λ_i is trivial.

Although the elements of $\widetilde{I}(C)$ are not in the image of Φ, they do preserve Λ. Observe that the action of the subgroup $\mathbb{Z}^{\underline{\alpha}} \subset \widetilde{I}(C)$ on $H_1(S; \mathbb{Z}/p\mathbb{Z})/\Lambda$ is trivial, as well. By what precedes, the group $\widetilde{I}(C)$ acts on $\bigoplus_i H_1(\Sigma_{g_i,k_i}^{r_i}; \mathbb{Z}/p\mathbb{Z})/\Lambda_i$ via its quotient $\mathrm{Aut}(C) \subset \mathrm{Aut}(\widetilde{C})$. Observe that the map I is $\widetilde{I}(C)$-equivariant.

Furthermore there is a natural map $\ker(B \circ Q) \to \prod_i \mathrm{Aut}(C_i)$. For $\phi \in \ker(B \circ Q)$ we denote by ϕ_i the corresponding automorphism of C_i viewed as a mapping class in $\mathrm{PMod}(\Sigma_{g_i}^{k_i+r_i})$.

The well-known Serre lemma states that the action of a nontrivial finite order element of $\mathrm{Mod}(\Sigma_g^n)$ on $H_1(\Sigma_g^n)$ is nontrivial, provided Σ_g^n is hyperbolic, i.e. $2g-2+n > 0$. Since the curve C is stable, each component C_i should be hyperbolic. If $\theta_{p,S}(\phi) = 1$, then ϕ_i acts trivially on $H_1(\Sigma_{g_i}^{k_i+r_i}; \mathbb{Z}/p\mathbb{Z})/\Lambda_i$. As ϕ_i is a pure mapping class, its homological representation on $H_1(\Sigma_{g_i}^{k_i+r_i}; \mathbb{Z}/p\mathbb{Z})$ is trivial as well. Therefore, Serre's lemma enables to conclude that $\phi_i = 1$. $\square$

Remark 4.7. *The map I extends naturally to an injective homomorphism:*

$$I : \bigoplus_i H_1(\Sigma_{g_i,k_i}^{r_i}, \partial\Sigma_{g_i,k_i}^{r_i}; \mathbb{Z}/p\mathbb{Z}) \longrightarrow H_1(S, \underline{\alpha}; \mathbb{Z}/p\mathbb{Z})$$

which is equivariant under the group homomorphism Φ and fits into the exact sequence:

$$1 \to \bigoplus_i H_1(\Sigma^{r_i}_{g_i,k_i}, \partial\Sigma^{r_i}_{g_i,k_i}; \mathbb{Z}/p\mathbb{Z}) \xrightarrow{I} H_1(S, \underline{\alpha}; \mathbb{Z}/p\mathbb{Z}) \to H_1(\Gamma(\underline{\alpha}); \mathbb{Z}/p\mathbb{Z}) \to 1.$$

Lemma 4.8. *Let C be a stable curve with n marked points with associated multicurve $\underline{\alpha}$, $x \in \widetilde{I}(C)$ and $p \geq 3$. If $B \circ Q(x) \neq 1 \in \mathrm{Aut}(\Gamma(\underline{\alpha}))$, then $\rho_{p,\mathbf{i}}(x) \neq 1$, where $\mathbf{i} = (i_1, \ldots, i_n)$ is any coloring of the marked points such that there exists a coloring $(j_1, \ldots, j_q)$ of the nodes of C so that for all i $W((\Sigma_{g_i,r_i+k_i})_{(\mathbf{i}_i,\mathbf{j}_i)}) \neq 0$ where $(\mathbf{i})_i$ is the restriction of $(\mathbf{i})$ to the r_i marked points in Σ_{g_i,r_i+k_i} and $(\mathbf{j}_i)$ is the restriction of $(\mathbf{j})$ to the k_i nodes in Σ_{g_i,r_i+k_i}.*

Proof. We can choose a basis of the space of conformal blocks associated to S corresponding to a pants decomposition containing the multicurve α. Then the action of $\rho_p(x)$ on the conformal blocks permutes the conformal blocks associated to the connected components of C minus the nodes according to the nontrivial action of $B \circ Q(x)$. □

Proposition 4.9. *Let $\mathbf{i} = (2, 2, \ldots, 2)$, $g \geq 0, n \geq 0$, $(g, n) \neq (2, 0)$, $2g - 2 + n > 0$.*

(1) If $p \geq 5$ is odd, then $\rho_{p,(\mathbf{i})} \oplus \theta_{p,\Sigma^n_g}$ factors through $\mathrm{Mod}(\Sigma^n_g)/\mathrm{Mod}(\Sigma^n_g)[p]$ and sends $\pi_1(\overline{\mathcal{M}^{n\ an}_g}[p], x[p])_{loc}$ isomorphically onto its image.

(2) If $p \geq 10$ is even and $(g, n, p) \neq (2, 0, 12)$ then the representation $\rho_{p,(\mathbf{i})} \oplus \theta_{2p,\Sigma^n_g}$ factors through $\mathrm{Mod}(\Sigma^n_g)/\mathrm{Mod}(\Sigma^n_g)[2p, p/\mathrm{g.c.d.}(p, 4)]$ and sends the isotropy group $\pi_1(\overline{\mathcal{M}^{n\ an}_g}[2p, p/\mathrm{g.c.d.}(p, 4)], x[2p, p/\mathrm{g.c.d.}(p, 4)])_{loc}$ isomorphically onto its image.

Proof. It is well-known that $\ker \theta_p$ contains both T^p_γ, for nonseparating γ and T_γ, when γ is separating. Then Proposition 2.17 implies the claims concerning the factorization of the given representations, when $g \geq 1$, while a direct verification also settles the case when $g = 0$. From Lemmas 4.6 and 4.8 the kernel of these representations is contained within $\mathbb{Z}^{\underline{\alpha}}$. Then the faithfulness is a consequence of Proposition 4.3 which describes the kernel on $\mathbb{Z}^{\underline{\alpha}}$ and Proposition 4.5 which describes the isotropy group. □

4.4. $\overline{\mathcal{M}^{n\ an}_g}[p]$ **is uniformizable.** The last step in the proof of Theorem 1.1 is to show that the orbifolds we consider are uniformizable.

Proposition 4.10. *Assume $g \geq 0, n \geq 0$ and $2g - 2 + n > 0$.*

If $p \geq 5$ is odd, then $\overline{\mathcal{M}^{n\ an}_g}[p]$ is uniformizable. Further, for even $p \geq 10$, and $(g, n, p) \neq (2, 0, 12)$, the orbifold $\overline{\mathcal{M}^{n\ an}_g}[2p, p/\mathrm{g.c.d.}(p, 4)]$ is uniformizable.

Proof. Let p be odd. Proposition 4.9 provided a projective representation of $\pi_1(\overline{\mathcal{M}^{n\ an}_g}[p])$ which is faithful on the isotropy subgroups. This can be easily converted into a linear representation by first composing it with the adjoint representation of the projective unitary group, as the later is centerfree. By Proposition 2.18 the image of the linear representation is infinite and by the Selberg lemma it has a finite index torsionfree subgroup J. By taking the quotient of the image by J we obtain a homomorphism from $\pi_1(\overline{\mathcal{M}^{n\ an}_g}[p])$ into a finite group G in which all isotropy groups should inject. Therefore $\overline{\mathcal{M}^{n\ an}_g}[p]$ is uniformizable, as claimed. When p is even, the proof is similar. □

Remark 4.11. *This gives an alternate and perhaps simpler version of the existence of smooth finite Galois coverings of the moduli space of stable curves due to Looijenga, Pikaart and Boggi. Letting $p \to \infty$, we also recover the universal ramification of the Teichmüller tower ([9], corrected in [43]).*

4.5. **End of proof of Theorem 1.1.** Proposition 4.10 and an argument from [18] implies that $\pi_1(\overline{\mathcal{M}^{n\ an}_g}[p])$ is a Kähler group. In fact, consider a Galois étale covering $M \to \overline{\mathcal{M}^{n\ an}_g}[p]$ with deck group G. Then M can be chosen to be a smooth projective variety. On the other hand there exists a projective surface Y such that $\pi_1(Y) = G$, as G is finite. If $\widetilde{Y}$ denotes its universal covering, then $\pi_1(\overline{\mathcal{M}^{n\ an}_g}[p])$ acts diagonally on $\widetilde{M} \times Y$ freely and properly discontinuously. Then the quotient is a smooth projective variety having fundamental group $\pi_1(\overline{\mathcal{M}_g^{n\ an}}[p])$. The case of even p is similar.

This indeed shows that the groups $\mathrm{Mod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[p]$, for odd $p \geq 5$, and respectively $\mathrm{Mod}(\Sigma_g^n)/\mathrm{Mod}(\Sigma_g^n)[2p, p/\mathrm{g.c.d.}(p,4)]$, for even $p \geq 10$ are Kähler and ends the proof of Theorem 1.1.

5. Bogomolov-Katzarkov Surfaces

5.1. Let $\pi : S \to C$ be a non isotrivial fibration of a smooth complex projective algebraic surface S onto a smooth curve C such that:

(1) π is a projective morphism with connected fibres,
(2) for generic $c \in C$, the fiber $\pi^{-1}(c)$ is a genus $g \geq 2$ curve
(3) the singular fibres of π are reduced stable curves, in particular the singular set consists of ordinary double points.

In a transcendental neighborhood of $\sigma \in S$ a node of a singular fiber there are complex coordinates (x, y) on S and a complex coordinate z on C near $\pi(\sigma)$ such that $z(\pi(x, y)) = xy$.

By the definition of the stack $\overline{\mathcal{M}_g}$ of genus g stable curves, we have a map $\psi : C \to \overline{\mathcal{M}_g}$, which induces a holomorphic map $C^{an} \to \overline{\mathcal{M}_g^{an}}$. By construction this map is covered by a map of stacks $S \to \overline{\mathcal{M}_g^1}$ and we have a 2-commutative diagram:

$$\begin{tikzcd} S \arrow[r] \arrow[d, "\pi"'] & \overline{\mathcal{M}_g^1} \arrow[d, "f_1"] \arrow[dl, phantom, "\swarrow"] \\ C \arrow[r, "\psi"] & \overline{\mathcal{M}_g} \end{tikzcd}$$

The square is cartesian, i.e.: $S \simeq C \times_{\psi, f_1} \overline{\mathcal{M}_g^1}$, π being equivalent to $\psi^* f_1$.

Fix an integer $N \geq 2$. If $N = 2$ and $g(C) = 0$ assume π has at least 4 singular fibers, in general there at least 3 singular fibers. We will mostly be interested in the case N is large enough, meaning $N \geq 5$ if N is odd.

5.2. **The stack $\mathcal{S}_\pi[\mathbf{k}]$.** For every multi-index $\mathbf{k}$ we define the following stack:

$$\mathcal{S}_\pi[\mathbf{k}] := S^{an} \times_{\overline{\mathcal{M}_g^{1\ an}}} \overline{\mathcal{M}_g^{1\ an}}[\mathbf{k}].$$

According to the convention from the introduction we write $\mathcal{S}_\pi[N]$ for $\mathcal{S}_\pi[N, N, N, \ldots, N]$ and $\mathcal{S}_\pi[N, L]$ for $\mathcal{S}_\pi[N, L, L, \ldots, L]$.

Proposition 5.1. *The stack $\mathcal{S}_\pi[N]$ is a smooth compact Kähler orbifold with moduli space S. There is a proper map $\mathcal{S}_\pi[N] \to \mathcal{C}[N]$ where $\mathcal{C}[N]$ is a non-elliptic orbicurve whose general fiber is a smooth curve of genus g giving rise to a short exact sequence :*

$$1 \to I_N(\mathbf{C}_\pi) \to \pi_1(\mathcal{S}_\pi[N]) \to \pi_1(\mathcal{C}[N]) \to 1.$$

where $I_N(\mathbf{C}_\pi)$ is the image of the fundamental group of a generic fiber into $\pi_1(\mathcal{S}_\pi[N])$). Furthermore:

$$I_N(\mathbf{C}_\pi) \cong \frac{\pi_1(\Sigma_g)}{\langle\langle c^N,\ c \in \mathbf{C}_\pi \rangle\rangle}$$

where $\mathbf{C}_\pi$ is a monodromy invariant family of conjugacy classes in $\pi_1(\Sigma_g)$ representing free homotopy classes of simple closed oriented curves.

Proof. This is in effect proved in [8]. For the convenience of the reader we will rephrase their paper in our terminology.

The smoothness of S translates into the statement that given an uniformizing parameter z for C at $q \in C$ and η a chart adapted to $\mathcal{D}$ centered at $\psi(q)$, $\eta^{-1}\psi$ assumes the form $z \mapsto (z_1, \ldots, z_{3g-3})$ with $\frac{dz_i}{dz}(q) \neq 0$ for $1 \leq i \leq k(\eta) = \#Sing(\pi^{-1}(p))$. Furthermore, near the image Q of a singular point q' of $\pi^{-1}(q)$ we can find an étale chart of $\overline{\mathcal{M}_g^{1\ an}}$ at Q such that the map $S \to \overline{\mathcal{M}_g^1}^{an}$ takes the form $(x,y) \mapsto (x,y,z_2(x,y),\dots,z_{3g-3}(x,y))$ in local coordinates (x,y) of S near q', whereas the map to C is given by $t = xy$. Using the étale charts in [16] in which f_1 takes the form $(\xi_1,\eta_1,z_2,\dots,z_{3g-3})$ we also can arrange to have $z_k(x,y) = z_k(xy)$ since $\pi \simeq \psi^* f_1$.

So the map from S to $\overline{\mathcal{M}_g^1}^{an}$ is transverse to $\mathcal{D}_1$ and the pull back of $\mathcal{D}_1$ is just the sum $\Sigma := \sum_{q\in B} S_q$ of the (reduced) singular fibres of π, B being the set of critical values of π. Certainly the formation of the smooth root stack of a normal crossing divisor commutes with transversal mappings and we get:

$$\mathcal{S}_\pi[N] \simeq S[N,\Sigma].$$

There is a natural map $\mathcal{S}_\pi[N] \to \mathcal{C}[N]$ where $\mathcal{C}[N] = C^{an}[\sqrt[N]{B}]$ according to ([10], Rem.2.2.2). Actually $S^{an} \times_{C^{an}} \mathcal{C}[N] \simeq S[\sqrt[N]{\Sigma}]$ and $\mathcal{S}_\pi[N] \to S[\sqrt[N]{\Sigma}]$ is the canonical stack. Indeed the germ $S[\sqrt[N]{\Sigma}]_{q'}$ is equivalent to the germ at the origin of the stack with a cyclic quotient singularity of type $\frac{1}{N}(1,-1)$:

$$[\mu_N \backslash A_N] = [\mu_N \backslash \{t^N = xy\}].$$

Now $\mathcal{C}[N]$ is an uniformizable orbicurve since otherwise C were rational and π would have ≤ 2 singular fibers, but this situation is excluded by the big Picard theorem and the fact that the period domain for ppav is a bounded domain. Let C_N be a complex projective curve with an action of a finite group G such that $[G\backslash C_N] = \mathcal{C}[N]$, namely the Galois cover $C_N \to C$ ramifies precisely over B with index N.

Consider now the following *singular surface* $S_N := S \times_C C_N$. Then S_N is a normal surface with A_{N-1} singularities and the canonical stack $(S \times_C C_N)^{can}$ is an étale covering of $\mathcal{S}_\pi[N]$. Actually since $(S\times_C C_N)^{can}$ is smooth its fundamental group is isomorphic to the fundamental group of the open substack obtained by deleting the finite substack of its orbifold points, which is then precisely the open surface considered in [8]. There C_N is denoted by R, $S \times_C C_N$ by S and the singular set of $S\times_C C_N$ by Q which is in a natural bijection with the sets of nodes.

Let D be a small coordinate disk in C_N centered at a singular fiber of $S_N \to C_N$ which is a degree N covering of a small coordinate disk D' of C centered at a critical point of π, such that D only ramifies over D' at that critical point. Let T' be a non singular fiber of $\pi : S_N \times_{C_N} D \to D$ identified with a non singular fiber of $\pi^{-1}(D')$.

Then there is a collection $c_1',\dots,c_q'$ of disjoint simple curves in T' which appear as geometric vanishing cycles in a Clemens contraction for the family of stable curves $S\times_C D' \to D'$. Denote by $\bar c_1,\dots,\bar c_q$ their conjugacy classes in $\pi_1(T')$. Introduce base points denoted by $*'$ which are preserved by the maps under consideration.

Lemma 5.2. *The inclusion map $(T',*') \hookrightarrow ((S_N\times_{C_N} D)^{can},*')$ induces at fundamental group level the quotient group homomorphism onto:*

$$\pi_1((S_N \times_{C_N} D)^{can},*') = \pi_1(T',*')/\langle\langle \bar c_1^N,\dots,\bar c_q^N\rangle\rangle,$$

where $\langle\langle\_\rangle\rangle$ means the normal subgroup generated by the respective elements.

For the sake of simplicity we kept the notation $\pi_1(T',*')$ instead of the more appropriate $\pi_1(T',*_{T'})$.

Proof. This is just a reformulation of ([8], Thm.2.1), using Van Kampen for stacks. □

One has $g(C_N) \ge 1$ and we denote the universal covering $\widetilde{C_N^{univ}} \cong \mathbb{C}$ or Δ by $\mathbb{E}$ to emphasize it is homeomorphic to an euclidian plane. Consider the canonical stack

$$\mathcal{T} := (S_N \times_{C_N} \mathbb{E})^{can} \simeq S_N^{can} \times_{C_N} \mathbb{E}$$

and introduce a coherent choice of base points $*$. We can assume that $*_{\mathcal{T}}$ does not lie on a singular fiber of the projection to $\mathbb{E}$ and has consequently a trivial inertia group. Let T be the fiber of $\pi_{\mathbb{E}} := \pi \times_C 1_{\mathbb{E}} : \mathcal{T} \to \mathbb{E}$ at $*_{\mathbb{E}}$.

The set of branch points $B(\mathbb{E})$ of $\pi_{\mathbb{E}}$ is the preimage of branch points of π and thus it is discrete in $\mathbb{E}$. Hence there is a countable discrete subset $J \subset]0, +\infty[$ such that $\partial \mathbb{B}_{\mathbb{E}}(*, R)$ does not intersect $B(\mathbb{E})$ when $0 < R \notin J$. For each such R we can construct a tree $\mathcal{A}_R$ rooted at $*$ in $\mathbb{B}_{\mathbb{E}}(*, R)$ and ending at the various points of $B(\mathbb{E}) \cap \mathbb{B}_{\mathbb{E}}(*, R)$. Obviously we can arrange so that $R' \geq R \Rightarrow \mathcal{A}_R \subset \mathcal{A}_{R'}$ with equality between two consecutive values of J. We may also assume that the lifts of the various $*'$ lie on $\mathcal{A}_R$. Using Ehresmann lemma (and an horizontal normal bundle to $\pi_{\mathbb{E}}|_{\mathbb{E}\setminus B(\mathbb{E})}$ along the geodesic rays of $\mathcal{A}_R$ emanating from $*_{\mathbb{E}}$ we can transport the geometric vanishing cycles at each $*'$ to a system of disjoint simple curves in T. Doing this for all $*'$ yields a family $\{c_\alpha\}_{\alpha \in A(R)}$ of closed simple curves in T. This yields a family $\{\bar{c}_\alpha\}_{\alpha \in A(R)}$ of conjugacy classes in $\pi_1(T, *)$ and $R \mapsto A(R)$ is an increasing function. Taking the union for all R we get a family $\mathbf{C}_\pi = \{\bar{c}_\alpha\}_{\alpha \in A}$ of conjugacy classes in $\pi_1(T, *)$.

Lemma 5.3. *We have*

$$\pi_1(\pi_{\mathbb{E}}^{-1}(\mathbb{B}_{\mathbb{E}}(*, R)), *) = \pi_1(T, *)/\langle\langle \bar{c}_\alpha^N,\ \alpha \in A(R)\rangle\rangle.$$

Moreover, the inclusion map $(T, *) \hookrightarrow (\pi_{\mathbb{E}}^{-1}(\mathbb{B}_{\mathbb{E}}(*, R)), *)$ induces the quotient group homomorphism. In particular, when $R = \infty$, we obtain

$$\pi_1(\mathcal{T}, *) = \pi_1(T, *)/\langle\langle \bar{c}_\alpha^N,\ \alpha \in A\rangle\rangle.$$

Proof. Van Kampen for stacks. □

It follows from the covering theory for topological stacks ([47]) that there is an exact sequence:

$$1 \to \pi_1(\mathcal{T}, *) \to \pi_1(S_N^{can}, *) \to \pi_1(C_N, *) \to 1,$$

the normal subgroup being the image of $\pi_1(T, *)$ and similarly

$$1 \to \pi_1(\mathcal{T}, *) \to \pi_1(\mathcal{S}[N], *) \to \pi_1(\mathcal{C}[N], *) \to 1.$$

We set therefore $I_N(\mathbf{C}_\pi) = \pi_1(\mathcal{T}, *)$. This completes the proof of Proposition 5.1. □

5.3. **The fiber group $I_N(\mathbf{C}_\pi)$.** As argued in [8], the fiber group is a very interesting object. Consider the fiber exact sequence of the proper smooth map obtained from π by deleting the singular fibers:

$$1 \to \pi_1(\Sigma_g)) \to \pi_1(S \setminus \pi^{-1}(B)) \to \pi_1(C \setminus B) \to 1$$

Consider $\rho : \pi_1(C \setminus B) \to \mathrm{Out}^+(\pi_1(\Sigma_g))$ and $M_\pi = Im(\rho)$ the monodromy group. The group M_π acts on conjugacy classes of elements of $\pi_1(\Sigma_g)$. It is a non trivial invariant of the intriguing combinatorial object $M_\pi \curvearrowright \mathbf{C}_\pi$.

Lemma 5.4. *$\mathbf{C}_\pi$ is a finite union of M_π-orbits.*

Proof. This is a reformulation of the discussion before ([8], Def. 3.1). □

Apart from that, few things seem to be known on the the actions $M_\pi \curvearrowright \mathbf{C}_\pi$ which can occur in algebraic families of curves, except the restrictions coming from Deligne's semisimplicity theorem and also his theorem on the fixed part when one uses the natural map $M_\pi \curvearrowright \mathbf{C}_\pi \to M_{\pi,H_1} \curvearrowright H_1(\Sigma_g, \mathbb{Z})$.

However there is a surjective morphism

$$\pi_1(\Sigma_g) \to B_N(\Sigma_g) = \pi_1(\Sigma_g)/\langle\langle x^N, x \in \pi_1(\Sigma_g)\rangle\rangle$$

onto its N-th verbal quotient, also called the N-th Burnside group of the surface. The Restricted Burnside Problem, solved affirmatively by Zelmanov, states that the profinite completion of the Burnside group (of a free group) is finite. For large enough N, $B_N(\Sigma_g)$ is infinite having the N-th Burnside group of the free group on g generators as a quotient and hence it is not residually finite. This won't imply that $\pi_1(\mathcal{S}[N])$ is not residually finite but it implies that the fiber group is infinite. This would be a very desirable situation since the only 'obvious' representations of $\pi_1(\mathcal{S}[N])$ come from $\pi_1(\mathcal{C}[N])$.

Passing to $B_N(\Sigma_g)$ is an extreme step. It is better to look at the Burnside type quotient by the N-th powers of the primitive elements. Indeed, this quotient occurs as a fiber group if the monodromy group of π is the mapping class group and every topological type of simple closed curve occurs in $\mathbf{C}_\pi$, a case that can be realized. The following proposition makes precise some infiniteness statements in [8].

Proposition 5.5. *The fiber group $I_N(\mathbf{C}_\pi)$ is infinite when $g = 2$ and $N \geq 4$ or $g \geq 3$ and $N \notin \{2, 3, 4, 6, 8, 12\}$.*

Proof. This is a corollary of the proof of [2, Cor. 4], the main ingredient being [31] when $g = 2$ and the Koberda-Santharoubane theorem in the version of [26, Proof of Proposition 3.2]. $\square$

5.4. **The surface $S(N)$.**

5.4.1. *N odd.*

Proposition 5.6. *If N is odd, the stack $\mathcal{S}_\pi[N]$, is a uniformizable compact Kähler orbifold with a projective moduli space.*

Proof. This is [8, Lemma 2.7], which uses a very specific Teichmüller level. First, they check that $(S_N \times_{C_N} D)^{can}$ is uniformizable.

Let us describe their argument as [8] skips some details. Set $\Pi = \pi_1(T')$, for the fundamental group of the generic fiber, which is isomorphic to $\pi_1(\Sigma_g)$, $\Pi_{(3)} = [[\Pi, \Pi], \Pi]$ for the third term of the lower central series and $\Pi^N = \langle\langle x^N, x \in \Pi\rangle\rangle$ be the normal subgroup generated by the N-th powers. Consider next the quotient $U_N(\Pi) = \Pi/\Pi_{(3)} \cdot \Pi^N$.

The Lie algebra of $\Pi/\Pi_{(3)}$ is the quotient of $\mathbb{Z}[A_i, B_i, E_{l,k}, F_{m,n}, G_{m,n}\}]_{1\leq i\leq g, 1\leq l\leq k\leq g, 1\leq m<n\leq g}$ by the relations $[A_l, B_k] = E_{l,k}$ $[A_m, A_n] = F_{m,n}$ $[B_m, B_n] = G_{m,n}$ where $E_{l,k}$ $F_{m,n}$ $G_{m,n}$ are central, and $\sum_{k=1}^{g} E_{k,k} = 0$. Moreover, it defines by means of the Baker-Campbell-Hausdorff formula an unipotent group scheme U which is smooth over $\mathbb{Z}[1/2]$, so that $U(\mathbb{Z}) = \Pi/\Pi_{(3)}$. There is a map $\Pi \to U_N(\Pi) \to U(\mathbb{Z}/N\mathbb{Z})$ sending the classes $\bar{c}_i$ to elements of order N in $U(\mathbb{Z}/N\mathbb{Z})$. In fact nontrivial elements of the nilpotent group $U(\mathbb{Z}/N\mathbb{Z})$ have order N and thus $U(\mathbb{Z}/N\mathbb{Z})$ is finite. Then Lemma 5.2 implies that $(S_N \times_{C_N} D)^{can}$ is uniformizable.

Observe that the subgroup $H_N = \Pi_{(3)} \cdot \Pi^N$ is characteristic, i.e. invariant by all the automorphisms of Π and of finite index. Hence the same quotient of $\pi_1(\Sigma_g)$ works for all singular fibers.

Since $B \neq \emptyset$, the group $\pi_1(C \setminus B)$ is free and so ρ lifts to $\mathrm{Aut}^+(\pi_1(\Sigma_g))$. Moreover, the fundamental group of $S \setminus \pi^{-1}(B)$ is a semidirect product $\pi_1(\Sigma_g) \rtimes \pi_1(C \setminus B)$. Since the automorphism group of a finite group is finite, there is a normal subgroup of finite index $H \lhd \pi_1(C \setminus B)$ which acts trivially on $U_N(\pi_1(\Sigma_g))$. Then, $H_N \rtimes H$ is a finite index subgroup of $\pi_1(S \setminus \pi^{-1}(B))$. Moreover, $\pi_1(S \setminus \pi^{-1}(B))$ surjects onto the finite group $U_N(\pi_1(\Sigma_g)) \rtimes (\pi_1(C \setminus B)/H)$. Note that this surjective homomorphism maps the elements of $\mathbf{C}_\pi$ to elements of order N. In particular, $K_N(\mathbf{C}_\pi) := \ker(\pi_1(\Sigma_g) \to I_N(\mathbf{C}_\pi)) \subseteq H_N$.

Since H_N is a characteristic subgroup of $\pi_1(\Sigma_g)$ it is also a normal subgroup of $\pi_1(S \setminus \pi^{-1}(B))$ contained in $\pi_1(\Sigma_g)$. We then obtain the following diagram, keeping in mind the natural surjective morphism $\pi_1(S_N \setminus \pi^{-1}(B_N)) \to \pi_1(S_N^{can})$:

$$
\begin{tikzcd}
1 \arrow[r] & \pi_1(\Sigma_g) \arrow[r] \arrow[d, equal] & \pi_1(S \setminus \pi^{-1}(B)) \arrow[r] & \pi_1(C \setminus B) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(\Sigma_g) \arrow[r] \arrow[d] & \pi_1(S_N \setminus \pi^{-1}(B_N)) \arrow[r] \arrow[u] \arrow[d] & \pi_1(C_N \setminus B_N) \arrow[r] \arrow[u] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(\Sigma_g)/H_N \arrow[r] \arrow[d, equal] & \pi_1(S_N \setminus \pi^{-1}(B_N))/H_N \arrow[r] \arrow[d] & \pi_1(C_N \setminus B_N) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(\Sigma_g)/H_N \arrow[r] & \pi_1(S_N^{can})/\mathrm{Im}(H_N) \arrow[r] & \pi_1(C_N) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(\Sigma_g)/K_N(\mathbf{C}_\pi) \arrow[r] \arrow[u] & \pi_1(S_N^{can}) \arrow[r] \arrow[u] & \pi_1(C_N) \arrow[r] & 1.
\end{tikzcd}
$$

Consider the intersection H' of H with $\pi_1(C_N \setminus B)$. Then $H_N \rtimes H'$ is a finite index normal subgroup of $\pi_1(S_N \setminus \pi^{-1}(B_N))$. Projecting it down to $\pi_1(S_N \setminus \pi^{-1}(B_N))/H_N$ we get a normal subgroup of finite index $H'' < \pi_1(S_N \setminus \pi^{-1}(B_N))/H_N$ intersecting $\pi_1(\Sigma_g)/H_N$ trivially. Projecting it down to $\pi_1(S_N^{can})/\mathrm{Im}(H_N)$ we claim that we get a normal subgroup of finite index $H''' < \pi_1(S_N^{can})/\mathrm{Im}(H_N)$ intersecting $\pi_1(\Sigma_g)/H_N$ trivially. To this purpose we need the following elementary lemma:

Lemma 5.7. *Let A, B be groups and $B \to \mathrm{Aut}(A)$ a group homomorphism. Let $C \triangleleft B$ be a normal subgroup acting trivially on A. Then $A \rtimes B/C$ is a quotient of $A \rtimes B$.*

Our claim is then the consequence of Lemma 5.7 applied to $C = \ker(H' \to \pi_1(C_N))$, $B = H'$, $A = \pi_1(\Sigma_g)/H_N$ setting $H''' = \{1\} \times B/C$ and observing $A \times B/C$ is the image of $A \times H' \triangleleft \pi_1(S_N \setminus \pi^{-1}(B_N))/H_N$ a finite index normal subgroup of $\pi_1(S_N^{can})/\mathrm{Im}(H_N)$.

Eventually we obtain a homomorphism $\pi_1(S_N^{can}) \to G$ where G is finite such that $\pi_1(\Sigma_g)/I_N(\mathbf{C}_\pi)$ is sent to a subgroup isomorphic to $U_N(\pi_1(\Sigma_g))$ by a surjective morphism mapping the elements in $\mathbf{C}_\pi$ to elements of order N.

This concludes the proof of Proposition 5.6. $\square$

Remark 5.8. *If $\overline{\mathcal{M}_g^1}^{an}[N]$ is uniformizable then $\mathcal{S}_\pi[N]$ is uniformizable too.*

We then denote by $S(N) = S_\pi(N)$, the π being dropped if the context allows, the smooth projective surface uniformizing the stack $\mathcal{S}_\pi[N]$, namely such that $[G\backslash S(N)] = S_\pi[N]$. Actually this does not define a smooth projective surface but a commensurability class[4] of such surfaces.

5.4.2. *N even.* The result from [8] should be restated as:

Proposition 5.9. *If $N \equiv 0 \pmod 4$, the stack $\mathcal{S}_\pi[N, N/2]$ is a uniformizable compact Kähler orbifold with a projective moduli space.*

Proof. This is the result of [8, Lemma 2.7] slightly modified in the even case. In fact, the order of a separating primitive element in $\frac{\pi_1(\Sigma_g)}{\pi_1(\Sigma_g)_{(3)} \cdot \pi_1(\Sigma_g)^N}$ is actually $\frac{N}{2}$ (see [49, Lemma 6.3]) and not N, correcting [8, Lemma 2.5]. All arguments used above in the odd case are then valid. Alternatively, we can derive it from the uniformisability of $\overline{\mathcal{M}_g^1}^{an}[N, N/2]$, which follow the same way as the uniformisability of $\overline{\mathcal{M}_g^{an}}[N, N/2]$ from [49, Thm.3.1.1]. $\square$

[4]Two projective varieties are *commensurable* if and only if they have a common finite étale covering.

If $\mathcal{X}$ is a compact complex orbifold (or an analytification of a Deligne-Mumford stack globally of finite type, say with a quasiprojective moduli space) we can do the following construction. Consider $H \triangleleft \pi_1(\mathcal{X})$ be a finite index subgroup such that the kernels of the isotropy morphisms $\pi_1^{loc}(\mathcal{X}, x) \to \pi_1(\mathcal{X})/H$ coincide with the kernels of their universal counterpart $\pi_1^{loc}(\mathcal{X}, x) \to \widehat{\pi_1(\mathcal{X})}$. Then the covering stack of $\mathcal{X}$ corresponding to H is a complex orbifold $\mathcal{X}'$ with a finite étale covering $\mathcal{X}' \to \mathcal{X}$. Consider the moduli space X' of $\mathcal{X}'$. It is a normal variety with quotient singularities at worst which carries an action of $G = \pi_1(\mathcal{X})/H$. This gives a map of stacks with identical moduli spaces $\mathcal{X} \to [X'/G]$ with the property that any finite dimensional (or finite image) representation of $\pi_1(\mathcal{X})$ factors through $\pi_1([X'/G])$.

Definition 5.10. *We define $S(N) = S_\pi(N)$ for every $N \in \mathbb{N}$ by applying this construction to $\mathcal{S}_\pi[N]$, $S(N) = S'_\pi[N]$ being a normal complex projective surface with isolated quotient singularities.*

Remark 5.11. *The minimal resolution of singularities $\widehat{S(N)} \to S(N)$ induces an isomorphism on π_1.*

Remark 5.12. *It is proposed in [8] to investigate the surfaces $S(N)$ to find counterexamples to Shafarevich conjecture on holomorphic convexity among them.*

6. Using TQFT representations for the Shafarevich Conjecture

6.1. **The Birman exact sequence.** All the above fiber sequences come from the fiber sequence of f_1, which by means of the isomorphisms $\pi_1(\mathcal{M}_g^n) \cong \mathrm{PMod}(\Sigma_g^n)$ can be identified with the Birman exact sequence:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\Sigma_g) \arrow[r, "i_*"] \arrow[d, equal] & \pi_1(\mathcal{M}_g^1) \arrow[r, "f_{1*}"] \arrow[d, "\cong"] & \pi_1(\mathcal{M}_g) \arrow[r] \arrow[d, "\cong"] & 1 \\
1 \arrow[r] & \pi_1(\Sigma_g) \arrow[r, "pp"] & \mathrm{Mod}(\Sigma_g^1) \arrow[r] & \mathrm{Mod}(\Sigma_g) \arrow[r] & 1,
\end{tikzcd}$$

where pp is Birman's point pushing homomorphism.

The point pushing of a simple closed loop is a product of two Dehn twists along disjoint simple curves in $\mathrm{Mod}(\Sigma_g^1)$. Let $S(\Sigma_g)$ be the set of conjugacy classes of of $\pi_1(\Sigma_g)$ which can be represented by simple closed curves on Σ_g. Also Dehn twists are preserved by point forgetting maps.

Lemma 6.1. *The map f_1 lifts to a map of stacks $f_1[p] : \overline{\mathcal{M}_g^{1\ an}}[p] \to \overline{\mathcal{M}_g^{an}}[p]$ compactifying f_1 inducing at the level of fundamental groups the canonical map $\mathrm{Mod}(\Sigma_g^1)/\mathrm{Mod}(\Sigma_g^1)[p] \to \mathrm{Mod}(\Sigma_g)/\mathrm{Mod}(\Sigma_g)[p]$.*

Proof. This follows from Lemma 3.1, the fact that the root construction behaves well under pullback (see [10]) and the universal property of canonical stacks [23, Theorem 4.6]. □

It follows from Theorem 3.6 that the Birman exact sequence gives rise to a level p analogue, as proved in [2]:

$$1 \longrightarrow BT(\pi_1(\Sigma_g), p) \longrightarrow \pi_1(\overline{\mathcal{M}_g^{1\ an}}[p]) \xrightarrow{f_1[p]_*} \pi_1(\overline{\mathcal{M}_g^{an}}[p]) \longrightarrow 1$$

where

$$BT(\pi_1(\Sigma_g), p) = \pi_1(\Sigma_g)/\langle\langle x^p, x \in S(\Sigma_g)\rangle\rangle,$$

This is of course the fiber sequence for the map of stacks $f_1[p] : \overline{\mathcal{M}_g^{1\ an}}[p] \to \overline{\mathcal{M}_g^{an}}[p]$. The group $BT(\pi_1(\Sigma_g), p)$ is actually the fiber group $I_p(\mathbf{C}_{f_1})$. Here we abuse notation, such a fiber group arises from a family π such that ρ is surjective.

By construction, the map $C \to \overline{\mathcal{M}_g}$ lifts to a map of stacks $\mathcal{C}[p] \to \overline{\mathcal{M}_g}[p]$. Comparing with Proposition 5.1, we find a diagram:

$$\begin{tikzcd}
1 \arrow[r] & I_p(\mathbf{C}_\pi) \arrow[r] \arrow[d] & \pi_1(\mathcal{S}_\pi[p]) \arrow[r] \arrow[d] & \pi_1(\mathcal{C}[p]) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & B(\pi_1(\Sigma_g), p) \arrow[r] & \pi_1(\overline{\mathcal{M}_g^{1\ an}})[p] \arrow[r] & \pi_1(\overline{\mathcal{M}_g^{an}})[p] \arrow[r] & 1
\end{tikzcd}$$

with the leftmost vertical map being given by the natural quotient map attached to $\mathbf{C}_\pi \subset \mathbf{C}_{f_1} = S(\Sigma_g)$.

6.2. **The Stein property for the universal covering spaces of most Bogomolov-Katzarkov surfaces.** The starting point of this project was the realization that [37] implies the following result for large odd numbers N:

Theorem 6.2. *If $g \geq 2$, $N \geq 5$ is odd, N even and $N/2 \geq 5$ is odd, or $N \equiv 0 (\mathrm{mod}\ 4)$ and $N \geq 20$ and $(g, N) \neq (2, 24)$ then there exists a complex finite dimensional linear representations ρ of $\pi_1(\mathcal{S}_\pi[N])$ such that $\rho \circ \iota_*$ has infinite image containing a free group on two generators, where ι is the inclusion of the general fiber into $\mathcal{S}_\pi[N]$.*

Proof. If N is odd, define $p = N$. By construction there is map of stacks $i : \mathcal{S}_\pi[p] \to \overline{\mathcal{M}_g^{1\ an}}[p]$ and we can define $\rho = \rho_{p,(2)} \circ \iota_*$. The result is then an immediate consequence of Propositions 2.17 and 2.22.

If N is even, define $p = \frac{N}{2}$. By construction there is map of stacks $i : \mathcal{S}_\pi[N] \to \overline{\mathcal{M}_g^{1\ an}}[2p, p/g.c.d(p, 4)]$ and we can define $\rho = \rho_{p,(2)} \circ \iota_*$. The result is then an immediate consequence of Propositions 2.17 and 2.22. □

Corollary 6.3. *Under the hypotheses of Theorem 6.2, $S(N)$ has a holomorphically convex infinite Galois covering space with the property that the codomain of its Cartan-Remmert reduction is 2-dimensional.*

Proof. Recall that the Cartan-Remmert reduction of a holomorphically convex normal complex space T is the unique proper holomorphic map $T \to U$ such that U is a normal Stein space.

The main theorem in [21] states that, given a complex projective manifold X and $M \in \mathbb{N}$, there is a Cartan-Remmert reduction:

$$\widetilde{s_M} : H_M \backslash \widetilde{X^{univ}} \to \widetilde{U(M)} \quad \text{with } H_M = \bigcap_{\rho : \pi_1(X) \to GL_M(\mathbb{C})} \ker(\rho).$$

Since $\Gamma_M = \pi_1(X)/H_M$ acts in a properly discontinuous way, this map descends to a proper holomorphic map, the GL_M-th Shafarevich morphism of X:

$$Sh_M : X \to Sh_M(X) = \Gamma_M \backslash \widetilde{U(M)}.$$

The fibers of $\widetilde{s_M}$ are the connected components of the lifts of the fibers of Sh_M which are characterized as the maximal connected subvarieties $Z \subset X$ such that $\mathrm{Im}(\pi_1(Z)) \to \Gamma_M)$ is finite. In particular, when all positive dimension irreducible subvarieties Z satisfy:

$$\mathrm{Im}(\pi_1(Z^{norm}) \to \Gamma_M) \text{ is infinite,}$$

where $Z^{norm} \to Z$ is the normalization, then $\widetilde{s_M} = \mathrm{Id}$ and $\widetilde{U(M)}$ is Stein, and when all positive dimensional irreducible subvarieties Z through the general point satisfy this property, then $\dim \widetilde{U(M)} = \dim(X)$.

In the situation of the corollary, $\pi_1(S(N))$ has a finite degree linear representation with infinite image that comes from the base of the fibration π whose image is a non parabolic curve. Assume that its degree is M_0. Therefore, whenever Z is a connected closed subspace such that all linear representations of $\pi_1(S(N))$ have finite images in Γ_M for $M \geq M_0$ upon restriction to $\pi_1(Z)$, then Z is contained in the fibers of π. Then Theorem 6.2 gives a finite degree representation, say of degree M_1, which has infinite image on the general fiber of π. Hence, if $\pi_1(Z) \to \Gamma_M$ has finite image for $M \geq \max(M_0, M_1)$, then Z lies in a special fiber of π. This implies the corollary.

□

Theorem 6.4. *Assume that $g \geq 2$ and $N \notin \{1, 2, 3, 4, 6, 8, 12, 16, 20\}$ and $(g, N) \neq (2, 24)$. Then the universal covering space of $S(N)$ is Stein.*

Proof. We will assume $N = p$ is odd in order to simplify the notation. We claim that there is a complex representation ρ of $\pi_1(\mathcal{S}_\pi[p])$ such that the image of $\pi_1(Z)$ is infinite, for every irreducible component Z of the special fiber of the natural morphism to a curve $\psi : S(p) \to C'_p$ obtained as the Stein factorization of the composition of $S(p) \to S \xrightarrow{\pi} C$. We will show that the representation $\rho_{p,(2)}$ is convenient.

Since the fibres of π are stable, the complement of the singular set of each such component Z contains as a Zariski dense open subset a finite étale covering of the complement of the punctures in a stable curve with $n > 0$ marked points. In particular it contains an essential subsurface Σ with boundary which is homeomorphic to a pair of pants $\Sigma_{0,3}$ or to a $\Sigma_{1,1}$. This subsurface Σ can be deformed to a subsurface of the general fiber of π since π is smooth near Σ. In particular it can be identified with a subsurface of a fiber of f_1. Then we can compute $\rho(\pi_1(Z))$ by using the restriction of $\rho_{p,(i_1)}$ to the fundamental group $\pi_1(\Sigma_g)$ of the general fiber.

Since Z contains a finite étale cover of the Σ, the representation ρ has infinite image on the fundamental group of this subsurface and hence on the fundamental group of Z, by Propositions 2.17 and 2.22.

Hence the Shafarevich morphism of $S(p)$ used in the proof of Corollary 6.3 cannot contract any component of the preimage of Z in $S(p)$. Since the universal covering space of a Stein manifold is Stein, this concludes the proof.

To do the proof in the even case, one has to use the fact that if X, Y are normal spaces, Y being Stein, $X \to Y$ is a holomorphic map has the property that the preimage of a small open subset $\Omega \subset Y$ is a disjoint union of proper finite ramified covering spaces of Ω, X is Stein ([41]).

In fact [21] is not needed here, the earlier result [34] allows to conclude from the semisimplicity of $\rho_{p,(i_1)}$.

□

6.3. **The stack $\mathcal{S}_\pi[2p, \frac{p}{\text{g.c.d.}(4,p)}]$.** In the proofs above we can use the quantum representations ρ_p, for even $p \geq 10$, to show that $\mathcal{S}_\pi[2p, \frac{p}{\text{g.c.d.}(4,p)}]$ has an holomorphically convex universal covering which is Stein.

6.4. **Unstable families of curves and end of proof of Theorem 1.2.** If $\pi : S \to C$ is not supposed to be stable but has only nodal singularities on its fibers as in [8] then any maximal chain Γ of unstable components is either a (-1)-rational curve or a chain of n_Γ (-2)-rational curves for some $n_\Gamma \in \mathbb{N}^*$. It can be blown down and one gets a fibered surface $\pi : S^* \to C$ with stable singular fibers. Each singular point $s \in S^{*,sing}$ is an $A_{n(s)}$-singularity for some $n(s) \in \mathbb{N}^*$ and a node of singular fiber it belongs to.

One can do the same construction as in [8] for S. Let D be the sum of the singular fibers, each component being counted with multiplicity one, so that $\pi^* B = D$ where $B \subset C$ is the set of singular values of π.

However, if there is a (-1)-rational curve, $\mathcal{S}_\pi[N]$ need not be uniformizable. This situation occurs when we blow up some smooth points of the singular fiber of a smooth stable fibration $\pi : S^* \to C$. The orbisurface $S(N)$ is then the canonical stack of a weighted blow up of $S^*(N)$ of weights $(1, N)$ hence has a A_{N-1} singularity. Thus, without loss of generality we can assume S to be minimal which rules out these (-1)-rational curves.

Then, we have a commutative diagram:

$$\begin{tikzcd}
S \arrow[r] \arrow[dr] & S^* \arrow[r] \arrow[d, "\pi"'] & \overline{\mathcal{M}_g^1} \arrow[d, "f_1"] \arrow[dl, Rightarrow] \\
& C \arrow[r, "\psi"] & \overline{\mathcal{M}_g}
\end{tikzcd}$$

The square is cartesian, i.e.: $S^* \simeq C \times_{\psi, f_1} \overline{\mathcal{M}_g^1}$, π being equivalent to $\psi^* f_1$. Denote by D^* the sum of the singular fibers of $\pi : S^* \to C$. The stack $S^*(\sqrt[N]{D^*})$ has a quotient singularity of type $pn(s)$ over $s \in S^{*,sing}$ and we may set $\mathcal{S}^*_\pi[N] := S(\sqrt[N]{D^*})^{can}$. It is not clear whether $\mathcal{S}^*_\pi[N]$ is uniformizable. The root stack being a natural construction, we have a map of stacks $(S^* \setminus S^{*,sing}) \sqrt[N]{D} \to \overline{\mathcal{M}_g^{1\ 0}}[N]$ with the notation as in Remark 3.8. It induces a morphism on the fundamental groups and since $\pi_1((S^* \setminus S^{*,sing})(\sqrt[N]{D^*})) \simeq \pi_1(\mathcal{S}^*_\pi[N])$ and $\pi_1(\overline{\mathcal{M}_g^{1\ 0}}[N]) \simeq \pi_1(\overline{\mathcal{M}_g^1}[N])$ a group morphism $\pi_1(\mathcal{S}^*_\pi[N]) \to \pi_1(\overline{\mathcal{M}_g^{1\ an}}[N])$ factorizing the natural morphism $\pi_1(S^* \setminus D^*) \to \pi_1(\overline{\mathcal{M}_g^{1\ an}}[N])$. Consider now the normal surface $S^*_\pi[N]'$ constructed at the end of section 5. The quantum representations give rise to representations of $\pi_1(S^*_\pi[N]')$ (and of $\pi_1(\widehat{S^*_\pi[N]'})$) and the proof of Theorem 6.4 applies to the effect that $S^*_\pi[N]'$ has a Stein universal covering space and $\widehat{S^*_\pi[N]'}$ has a holomorphically convex universal covering space which resolves the singularities of $\widetilde{S^*_\pi[N]'}^{univ}$.

The same construction gives actually a group morphism $\pi_1(\mathcal{S}_\pi[N]) \to \pi_1(\overline{\mathcal{M}_g^{1\ an}}[N])$ factorizing the natural morphism $\pi_1(S \setminus D) \to \pi_1(\overline{\mathcal{M}_g^{1\ an}}[N])$. It descends to a map of fundamental groups $\pi_1(\widehat{S(N)}) = \pi_1(S(N)) \to \pi_1(\overline{\mathcal{M}_g^{1\ an}}[N])$. Then, we can construct the GL_N Shafarevich morphism, which contracts exactly the connected components of the preimage of the chain of (-2) rational curves by the natural map to S, which have finite fundamental groups. This is enough to imply the holomorphic convexity of the univeral covering spaces of $\widehat{S(N)}$ and $S(N)$.

6.5. **Concluding remarks.** If $\pi : S \to C$ is an irrational pencil of curves (resp. a rational pencil) on a smooth projective surface and has more complicated singularities, such as non reduced components of the singular fibers, we can use the semistable reduction theorem [16], to find $C' \to C$ a ramified covering, which ramifies at the unstable singular fibers (resp. one may have to introduce a further ramification point on the smooth locus) to get a normal projective surface S' birational to $S \times_C C'$ with an irrational pencil (resp. a general pencil of curves) with only stable curves as scheme theoretic singular fibers and one is reduced to the previous situation.

So in order to find a counterexample to the Shafarevich conjecture along the lines of [8], the only remaining possibilities with a base change having a common ramification index, require a ramification index $N \in \{1, 2, 3, 4, 6, 8, 12, 16, 20, 24\}$ ($N = 1$ corresponds to no base change at all) or allowing a fibration with non-reduced fibers.

Another possibility is to perform a fiber product with $\overline{\mathcal{M}_g^{1\ an}}[\mathbf{k}]$ with general weight vector $\mathbf{k}$. The fundamental groups of the $\overline{\mathcal{M}_g^{1\ an}}[\mathbf{k}]$ with general weight vector $\mathbf{k}$ do not seem to be easy to study. Let Π be the fundamental group of a hyperbolic surface (possibly punctured). Let $\{\gamma_i\}$ be a set of isotopy classes of simple closed curves whose orbits under the action of the mapping class group are pairwise disjoint. Then, Wise and Bajpai proved in [4, 56] that there is some $d \geq 1$ such that for every positive integral vector $\mathbf{k} = (k_i)$ there exists some finite quotient $\Pi \to \Pi/K$ in which images of the classes γ_i have orders dk_i, respectively. We don't know whether the kernel K might be chosen to be invariant with respect to the mapping class group action. If it could and additionally K satisfies Boggi's condition from [6, Thm. 3.9] (for example, if $K = [L, L]L^\ell$, for some $\ell \geq 3$, where L is invariant subgroup of Π) then the compactified Deligne-Mumford stack of curves endowed with Looijenga's levels associated to K and $m \geq 2$ would be smooth and representable. This further implies that $\overline{\mathcal{M}_g^{an}}[md\mathbf{k}]$ is uniformisable.

In a similar vein, one may as in [8] start with a stable fibration $\pi : S \to C$ with reduced fibers and try and study the fundamental groups of the canonical stacks obtained by assigning different multiplicities to the fibers. We don't known how to decide whether the fiber group is finite or infinite when the multiplicities are coprime.

References

[1] D. Abramovich, M. Olsson, A. Vistoli, *Tame stacks in positive characteristic*, Ann. Institut Fourier 58 (2008), 1057–1091. 2, 21

[2] J. Aramayona, L. Funar, *Quotients of the mapping class group by power subgroups*, Bull. London Math. Soc. **51** (2019), 385–398. 2, 33, 35

[3] K. Behrend and B. Noohi, *Uniformization of Deligne-Mumford curves*, J. Reine Angew. Math. **599** (2006), 111–153. 19

[4] J. Bajpai, *Omnipotence of surface groups*, Master Thesis, McGill Univ. 2007. 38

[5] C. Blanchet, N. Habegger, G. Masbaum, P. Vogel, *Topological quantum field theories derived from the Kauffman bracket*, Topology **34** (1995), 883–927. 4, 5, 7

[6] M. Boggi *Galois coverings of moduli spaces of curves and loci of curves with symmetry*, Geometriae Dedicata **168** (2014), 113–142. 2, 24, 38

[7] M. Boggi, M. Pikaart, *Galois coverings of the moduli of curves*, Compositio Math. **120** (2000) 171–191. 1, 2, 23

[8] F. Bogomolov, L. Katzarkov, *Complex projective surfaces and infinite groups*, Geom. Funct. Analysis **8** (1998), 243–272. 2, 3, 30, 31, 32, 33, 34, 35, 37, 38, 39

[9] J. L. Brylinski, *Propriétés de ramification à l'infini du groupe modulaire de Teichmüller*, Ann. Sci. Ec. Norm. Sup. Série 4, **12** (1979), 295–333. 1, 23, 29

[10] C. Cadman, *Using stacks to impose tangency conditions on curves*, Amer. J. Math. 129 (2007), 405–427. 21, 31, 35

[11] F. Campana, B. Claudon, P. Eyssidieux, *Représentations linéaires des groupes kählériens : Factorisations et conjecture de Shafarevich linéaire*, Compositio Math. **151** (2015), 351–376.

[12] A. Chiodo, *Stable twisted curves and their r-spin structures*, Ann. Institut Fourier 58 (2008), 1635–1689. 2

[13] N. Chriss, V. Guinzburg *Representation Theory and Complex Geometry*, Modern Birkhäuser Classics, Birkhäuser, 2010. 20

[14] H.S.M. Coxeter, *On factors of braid groups*, Proc. 4-th Canadian Math. Congress, Univ. Toronto Press, p.95–122, 1959. 17, 18

[15] H.S.M. Coxeter, W.O.J. Moser, Generators and relations for discrete groups, *Ergebnisse Math.* Springer-Verlag, 1980. 17, 18

[16] P. Deligne, D. Mumford, *The irreducibility of the space of curves of a given genus*, Publ. Math. I.H.E.S. **36** (1969), 75–109. 20, 31, 38

[17] J.K. Egsgaard, S. Fuglede Jørgensen, *The homological content of the Jones representation at $q = -1$*, J. Knot Theory Its Ramif. **25** (2016), No. 11, Article ID 1650062, 25p. 17

[18] P. Eyssidieux, *On the Uniformisation of Compact Kähler orbifolds*, Vietnam J. Math. **41** (2013), 399–407. 2, 29

[19] P. Eyssidieux, *Orbifold Kähler Groups and the Shafarevich Conjecture for Hirzebruch's covering surfaces with equal weights*, Asian J. Math **22** (2018), special volume in the honor of N. Mok, 315–330. 1, 21, 23

[20] P. Eyssidieux, *Orbifold Kähler Groups related to arithmetic complex hyperbolic lattices*, arxiv:1805.00767. 1, 19

[21] P. Eyssidieux, L. Katzarkov, T. Pantev, and M. Ramachandran, *Linear Shafarevich Conjecture*, Annals of Math. **176** (2012), 1545–1581. 3, 4, 36, 37

[22] P. Eyssidieux, F. Sala, *Instantons and framed sheaves on Kähler Deligne-Mumford stacks*, arxiv 1404.3504, Ann. Fac. Sci. Toulouse, Math. (6) 27 (2018), 599-628. 20, 23

[23] B. Fantecchi, E. Mann, F. Nironi, *Toric Deligne Mumfords stacks* J. Reine Angewand. Math. **648** (2010), 201-244. 35

[24] L. Funar, *On the TQFT representations of the mapping class groups*, Pacific J. Math. **188** (2)(1999), 251–274. 2, 3, 17

[25] L. Funar, *Zariski density and finite quotients of mapping class groups*, Int. Math. Res. Not., no.**9**, 2013, 2078-2096. 19

[26] L. Funar, P. Lochak *Profinite Completion of Burnside type quotients of surface groups* Comm. Math. Phys. **360** (2018), 1061–1082. 2, 13, 19, 33

[27] L. Funar, W. Pitsch, *Images of quantum representations of mapping class groups and Dupont-Guichardet-Wigner quasi-homomorphisms*, J. Inst. Math. Jussieu **17** (2018), 277–304. 2

[28] A. Geraschenko, M. Satriano, *A "bottom-up" characterization of smooth Deligne-Mumford stacks*, Int. Math. Res. Not., issue 21 (2017), 6469–6483. 2, 19, 22

[29] S. Gervais, *Presentation and central extensions of mapping class groups*, Trans. Amer. Math. Soc. **348** (1996), 3097–3132. 6

[30] P. Gilmer, G. Masbaum, *Integral lattices in TQFT*, Ann. Sci. Ecole Norm. Sup. Série (4) **40** (2007), 815–844.

[31] S.P. Humphries, *Normal closures of powers of Dehn twists in mapping class groups*, Glasgow J. Math. **34** (1992), 313–317. 2, 19, 33

[32] V. Jones, *Index for subfactors* Inventiones Math. **72** (1983), 1–25. 5

[33] L. H. Kauffman, *State models and the Jones polynomial*, Topology **26** (1987) 395–407. 5

[34] L. Katzarkov, M. Ramachandran, *On the universal coverings of algebraic surfaces*, Ann. Sci. Ecole Norm. Sup. Série 4, **31** (1998), 525–535. 4, 37

[35] F. Knudsen, *The projectivity of the moduli space of stable curves, II: The stacks $M_{g,n}$*, Math. Scand. **52** (1983), 161–199. 20, 21, 24

[36] F. Knudsen, *The projectivity of the moduli space of stable curves, III: The line bundles on $M_{g,n}$ and a proof of the projectivity of $\overline{M_{g,n}}$ in characteristic* 0, Math. Scand. **52** (1983), 200–212. 20

[37] T. Koberda, R. Santharoubane, *Quotients of surface groups and homology of finite covers via quantum representations*, Inventiones Math. **206** (2016), 269–292. 2, 3, 19, 36

[38] T. Koberda, R. Santharoubane, *Irreducibility of quantum representations of mapping class groups with boundary*, Quantum Topology **9** (2018), 633–641. 18

[39] J. Korinman, *On the (in)finiteness of the image of Reshetikhin-Turaev representations*, Archiv Math. 111 (2018), 247–256. 17

[40] G. Laumon, L. Moret-Bailly, Champs Algébriques, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3. Folge **39**, Springer, 2000. 20, 21

[41] P. Le Barz, *A propos des revêtements ramifiés d'espaces de Stein*, Math. Ann. **222** (1976), 63–69. 37

[42] E. Lerman, *Orbifolds as stacks?*, Enseign. Math. (2), 56 (2010), 315–363. 19, 20

[43] E. Looijenga, *Smooth Deligne-Mumford compactifications by means of Prym level structures*, J. Alg. Geom. **3** (1994) 283–293. 1, 24, 29

[44] G. Masbaum, *An Element of Infinite Order in TQFT-Representations of Mapping Class Groups*, Low-Dimensional Topology (Funchal, 1998), 137–39. Contemporary Mathematics 233. Providence, RI: American Mathematical Society, 1999. 17

[45] G. Masbaum, J. Roberts, *On central extensions of mapping class groups*, Math. Ann. **302** (1995), 131–150. 6

[46] B. Noohi, *Fundamental groups of algebraic stacks*, J. Inst. Math. Jussieu **3** (2004), 69–103. 19

[47] B. Noohi, *Fundations of topological stacks*, arXiv math/0503247 (2005). 19, 20, 32

[48] M. Olsson, *Logarithmic geometry and algebraic stacks*, Ann. Sci. Ecole Norm. Sup. Série 4, **36** (2003), 747–791 20

[49] M. Pikaart, A. J. de Jong, *Moduli of curves with non-Abelian level structure*, In: R.Dijkgraaf, C. Faber and G. van der Geer (eds), The Moduli Space of Curves, Progress in Mathematics **129**, Birkhäuser, 1995, pp. 483–510

[50] J. H. Przytycki, *Skein modules of 3-manifolds*, Bull. Acad. Pol.: Math. **39** (1991), 91-100. 2, 34 5

[51] M. Romagny, *Group actions on stacks and applications*, Michigan Math. J. **53** (2005), 209-236. 23, 24

[52] The Stacks Project Authors, *Stacks Project* http://stacks.math.columbia.edu (2018). 20

[53] V. Turaev *Quantum Invariants of Knots and 3-Manifolds* de Gruyter Series in Mathematics **18**, de Gruyter (1994). 4

[54] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Inventiones Math. 97 (1989), 613–670. 2, 19, 21

[55] H. Wenzl, *On sequences of projections*, C. R. Math. Rep. Acad. Sci. Canada **9** (1987), 5–9. 5

[56] D. T. Wise, *Subgroup separability of graphs of free groups with cyclic edge groups*, 38

[57] G. Wright, *The Reshetikhin-Turaev representation of the mapping class groups*, J. Knot Theory Its Ramif. **3** (1994), 547–574. 19

[58] G. Wright, *The Reshetikhin-Turaev representation of the mapping class groups at sixth roots of unity*, J. Knot Theory Its Ramif. **5** (1996), 721–741. 17, 19

[59] V. Zoonekynd, *Théorème de Van Kampen pour les champs algébriques*, Ann. Math. Blaise Pascal **9** (2002), 101–145. 23

Univ. Grenoble Alpes, CNRS, Institut Fourier, 38000 Grenoble, France
Email address: philippe.eyssidieux@univ-grenoble-alpes.fr
URL: http://www-fourier.univ-grenoble-alpes.fr/∼eyssi/

Univ. Grenoble Alpes, CNRS, Institut Fourier, 38000 Grenoble, France
Email address: louis.funar@univ-grenoble-alpes.fr
URL: http://www-fourier.univ-grenoble-alpes.fr/∼funar/