Contemporary Mathematics

.

Cubulations mod bubble moves

Louis Funar

Abstract. The aim of this paper is to consider the set of cubical decompositions of a compact manifold mod out by some combinatorial moves (bubble) analogous to the bistellar moves earlier considered by Pachner. We prove that, in general there are obstructions for two cubulations of the same PL-manifold to be related by bubble moves, answering negatively a question of Habegger.

1. Introduction

In the twenties Alexander proved that any two triangulations of a polyhedron (or equivalently of two PL-homeomorphic manifolds) are related by a set of combinatorial moves, called stellar moves. After seventy years Alexander's moves were refined to a set of finite local moves on the triangulations of manifolds which were used to prove that certain state-sums associated to a triangulation provide topological invariants of 3-manifolds, the so-called Turaev-Viro invariants. The new moves are the bistellar moves and Pachner ([**27**]) has proved that they relate any two triangulations of a polyhedron, settling a long standing conjecture in combinatorial topology. Basically such a move in dimension n corresponds to replace a ball B by another ball B', where B and B' are complementary balls, unions of simplexes in the boundary of the standard $(n+1)$-simplex. For a nice exposition of Pachner's result and various extensions, see [**22**].

On the other hand Alexander's theorem becomes trivial in the context of some more general cell decompositions ("cellulation régulière") considered by Siebenmann [**28**] where the analogous moves are called "bisections". Here the cells are convex subsets in some Euclidean space, with an arbitrary number of vertices.

The Turaev-Viro invariants carry less information than the Reshetikhin-Turaev invariants, which are defined using Dehn surgery presentations of the manifolds instead of triangulations. Actually the latter have a strong 4-dimensional flavor, as explained by the theory of shadows developed by Turaev (see [**31**]). This motivates the study of state-sums based on cubulations, as an alternative way to get intrinsic invariants possibly containing more information (e.g. the phase factor). In order to apply the state-sum machinery to these decompositions we need an analogue of the Alexander's or Pachner's theorems. Specifically, N.Habegger asked (see problem 5.13 from R.Kirby's list ([**18**])) the following:

Problem 1. *Suppose M and N are PL-homeomorphic cubulated n-manifolds. Are they related by the following set of moves: excise B and replace it by B', where B and B' are complementary balls (union of n-cubes) in the boundary of the standard $(n+1)$-cube?*

1991 *Mathematics Subject Classification.* Primary 57 Q 15, 57 R 42, Secondary 51 M 20, 57 Q 35.

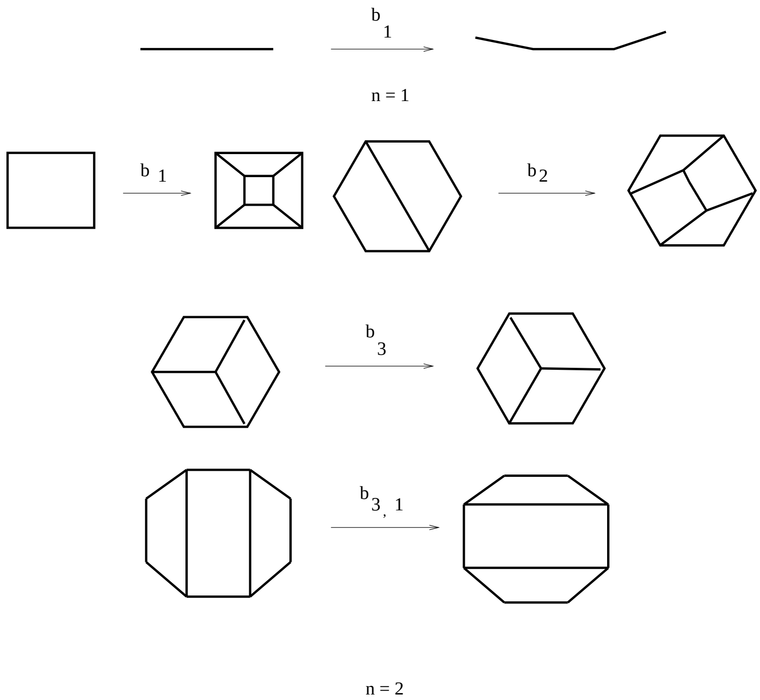

FIGURE 1. Bubble moves for $n = 1$ and $n = 2$

These moves will be called *bubble* moves in the sequel. Those bubble moves for which at least one of B and B' does not contain parallel (when viewed in the $n + 1$-cube) faces are called *np-bubble* moves. There are $n + 1$ distinct np-bubble moves b_k, $k = 1, 2, ..., n + 1$ and their inverses, b_k replacing B which is the union of exactly k cubes by its complementary. For $n = 2$ there is one bubble moves which is not a np-bubble (see picture 1).

Set $C(M)$ for the the set of cubulations of a closed manifold M (without boundary), $CBB(M)$ for the equivalence classes of cubulations mod np-bubble moves and $CB(M)$ for the equivalence classes of cubulations mod bubble moves.

For $n = 1$ the move b_1 divides an edge into three smaller edges and it follows that: $CB(\bigsqcup_n S^1) = CBB(\bigsqcup_n S^1) = (\mathbf{Z}/2\mathbf{Z})^n$, where n is the number of components. Thus there are non-trivial obstructions for two cubulations be np-bubble (bubble) equivalent. Assume from now on, that the manifolds considered are connected unless the contrary will be specified.

In the following section we will describe similar obstructions in higher dimensions. The set of all transforms of a given f-vector $\mathbf{f}$ under np-bubble/bubble moves is $\mathbf{f} + \Lambda(n)$, where $\Lambda(n) \subset \mathbf{Z}^{n+1}$ is a sublattice, depending only on the dimension. Therefore the class of $\mathbf{f} \in \mathbf{Z}^{n+1}/\Lambda(n)$ is an invariant taking values in a finite Abelian group. Our first result states that

THEOREM 2. *The obstruction in* $\mathbf{Z}^{n+1}/\Lambda(n)$ *is not trivial.*

We cannot expect the obstruction map $f : CB(M) \longrightarrow \mathbf{Z}^{n+1}/\Lambda(n)$ be injective in high dimensions.

Acknowledgements: Part of this work was done during the author's visit at Columbia University, whose support and hospitality are gratefully acknowledged. I'm thankful to E.Babson, J.Birman, C.Blanchet, R.Casali, C.Chan, N.Habegger, L.Guillou, T.Kashiwabara, A.Marin, S.Matveev, V.Poénaru for helpful discussions, suggestions and improvements.

2. Other statements and comments

2.1. Elementary obstructions. We want to identify obstructions, similar to that for the 1-dimensional case, in higher dimensions: set for $x \in C(M)$, and M of dimension n, $f_i(x) = card\{i\text{-dimensional cubes in } x\}$. We obtain a map $f : C(M) \longrightarrow \mathbf{Z}^{n+1}$ whose components are f_i, and it is usually called the f-vector in the theory of polytopes. Notice that once we start transforming a cubication of f-vector $\mathbf{f}$ by using all np-bubble/bubble moves we obtain a set of possible f-vectors having the form $\mathbf{f} + \Lambda(n) \subset \mathbf{Z}^{n+1}$, where $\Lambda(n)$ is a lattice. Therefore the first obstruction we encounter is the class of $\mathbf{f} \in \mathbf{Z}^{n+1}/\Lambda(n)$. The latter is a finite Abelian group and we will see that it is non-trivial.

PROPOSITION 2.1. *There exist some natural numbers $a_i(n) \in \mathbf{Z}_+$ such that:*

1. *All the $a_i(n)$ are non-trivial, and divisible by 2.*
2. *The map f induces a well-defined map $fb : CBB(M) \longrightarrow \prod_{i=0}^{n} \mathbf{Z}/a_i(n)\mathbf{Z}$ by*
$$fb(x) = (f_i(x)(mod\ a_i(n))_{i=0,1,\ldots,n}.$$
3. *The greatest such numbers $a_i(n)$ verify*

$$a_n(n) = 2, a_{n-1}(n) = 2n, a_{n-2}(n) = 2, a_0(n) = 2, a_1(n) = 3 + (-1)^n, (n > 2)$$

We computed the vector $\mathbf{a}(n)$ whose components are $a_i(n)$ for small dimensions: $\mathbf{a}(2) = (2, 4, 2)$, and $\mathbf{a}(3) = (2, 2, 6, 2)$. After some messy computations we obtain $\mathbf{a}(4) = (2, 4, 2, 8, 2)$, $\mathbf{a}(5) = (2, 2, 4, 2, 10, 2)$, and $a_2(6) = 6$. We denote by $fb^{(2)}$ the values of fb reduced mod $(2, 2, 2, ..., 2, 2n, 2)$.

Next we can extend this result on obstructions to bubble moves:

PROPOSITION 2.2. *There exist some natural numbers $\tilde{a}_i(n) \in \mathbf{Z}_+$ such that:*

1. *All the $\tilde{a}_i(n)$ are non-trivial, and divisible by 2.*
2. *The map f induces a well-defined map $fb : CB(M) \longrightarrow \prod_{i=0}^{n} \mathbf{Z}/\tilde{a}_i(n)\mathbf{Z}$ by*
$$fb(x) = (f_i(x)(mod\ \tilde{a}_i(n))_{i=0,1,\ldots,n}.$$
3. *The greatest such numbers $\tilde{a}_i(n)$ verify*

$$\tilde{a}_n(n) = 2, \tilde{a}_{n-1}(n) = 2n, \tilde{a}_{n-2}(n) = 2, \tilde{a}_0(n) = 2, \tilde{a}_1(n) = 3 + (-1)^n, (n > 2)$$

The two sequences a_j and $\tilde{a}_j$ are not identical since we have:

$$\tilde{a}_2(6) = 2 \neq a_2(6),\ \tilde{a}_2(4) = 2 \neq a_2(4).$$

Notice that $f_{n-1}(x) = nf_n(x)$ for all x. This means that the image of fb_{n-1} is $\mathbf{Z}/2\mathbf{Z} \cong \{0, n\} \subset \mathbf{Z}/2n\mathbf{Z}$. By the way the component fb_{n-1} is determined by fb_n. Notice that for $n = 2$ the image of fb is determined by f_0, and for $n = 3$ by f_0 and f_1.

A natural problem now is to know the images $fb(CB(M))$, $fbb(CBB(M))$, or at least their mod 2 reduction. This will give a hint about how much these invariants are powerful. We are far from having a complete answer now, and this problem is more difficult than it seems at first glance. There are some partial results for the case of the mod 2 reductions $fb^{(2)}(CB(M))$ and $fbb^{(2)}(CBB(M))$. Actually this is equivalent to characterize those f-vectors mod 2 which can be realized by cubulations of the manifold M. Obviously there are constraints for the existence of a simplicial polyhedra with a given f-vector and fixed topological type. For convex simplicial polytopes we have for instance the McMullen conditions (conjectured by McMullen in [**23**] and proved in [**4, 5, 29**]; the reader may consult also other proofs and results in [**23, 3, 24, 25**]). The complete characterization of the f-vectors of simplicial polytopes (and PL-spheres) was obtained by Stanley in [**30**]. The analogous problem of the realization of f-vectors by cubical polytopes has also been addressed in some recent papers, for example [**7, 2, 16, 17**] and references therein. The Dehn-Sommerville equations have a counterpart for cubical polytopes as in [**15**]. The lower bound conjecture and the upper bound conjecture have counterparts in the cubical case. The new feature is that, unlike in the simplicial case, there are parity restrictions on the f-vectors. This was firstly observed in [**7**]. Remark that it is exactly these restrictions in which we are interested. We have, as a simple application of the Dehn-Sommerville equation, a first constraint on the range of the mod 2 image:

PROPOSITION 2.3. *The rank of the affine module $fb^{(2)}(CB(M))$ is at most $\left[\frac{n+1}{2}\right]$.*

The relationship between cubical PL n-spheres and the immersions was described in the following beautiful result of Babson and Chan (see [**2**]):

PROPOSITION 2.4. *Let $\varphi : M \longrightarrow S^n$ a codimension 1 normal crossing immersion. Then there exists a PL cubical n-sphere K, such that*

$$f_i(K) = \chi(X_i(M, \varphi))(mod\ 2),$$

where χ denotes the Euler characteristics, and $X_i(M, \varphi) = \{x \in S^n;\ card\ \varphi^{-1}(x) = i\}$.

As a consequence, there exists a PL cubical n-sphere K with given $f_i(mod\ 2)$ if and only if there exists a codimension 1 immersion (M, φ) in S^n, for which the Euler characteristic of the multiple point loci $X_i(M, \varphi)$ of degree i equals f_i mod 2.

Remark that this result extends immediately to other varieties than the spheres. We have only to consider immersions $\varphi : M \longrightarrow N$ such that the image $\varphi(M)$ is a spine of N, which means that $N - \varphi(M)$ is an union of balls.

There is a wide literature on immersions, and especially on the following function θ_n, considered first by Freedman ([**13**]), where $\theta_n(\varphi)$ is the number of multiple n-points mod 2. The beginning of this theory was the result of Banchoff [**1**] saying that the number of normally triple points of a closed surface immersed in $\mathbf{R}^3$ is congruent mod 2 with its Euler characteristic. The function θ_n is easily seen to be well-defined as a function on the Abelian group B_n of bordism classes of immersions of $(n-1)$-manifolds in S^n. We have therefore an induced homomorphism:

$$\theta_n : B_n \longrightarrow \mathbf{Z}/2\mathbf{Z}.$$

Remark that the question on whether θ_n is surjective (i.e. nontrivial) is equivalent to find the image of $fb^{(2)}_{n-1}(S^n)$. From the results concerning the function θ_n obtained in [**13, 10, 11, 12, 20, 21, 8, 9**] we deduce that the f-vectors of a n-sphere have the following properties (see also [**2**]):

(1) For $n = 2$ we have $f_0 = f_2(mod\ 2)$ and $f_1 = 0(mod\ 2)$ and thus $fb^{(2)}(CB(S^2)) = fb^{(2)}(CBB(S^2)) = \mathbf{Z}/2\mathbf{Z}$.
(2) For $n = 3$, $f_0 = f_1 = 0(mod\ 2)$, $f_2 = f_3(mod\ 2)$. From the existence of Boy's immersion $j : \mathbf{R}P^2 \longrightarrow S^2$, with a single degree 3 intersection we find that there exists a PL 3-sphere with an odd number of facets. Therefore $fb^{(2)}(CB(S^3)) = fb^{(2)}(CBB(S^3)) = \mathbf{Z}/2\mathbf{Z}$.
(3) The problem of characterizing the image $fb^{(2)}_{n-1}(S^n)$ is reduced to a homotopy problem. Namely, the image is $\mathbf{Z}/2\mathbf{Z}$ if and only if
   (a) either n is $1, 3, 4$ or 7.
   (b) or else $n = 2^a - 2$, with $a \in \mathbf{Z}_+$, and there exists a framed n-manifold with Kervaire invariant 1. The latter is known to be true for $n = 2, 6, 14, 30, 62$.
(4) If we consider only the class of edge orientable cubulations in the sense of [**16**] the problem of characterizing the image $fb^{(2)}_{n-1}(S^n)$ is also reduced to a homotopy problem, which is completely solved. In fact the condition of edge orientability is equivalent to ask that the associated manifold M immersed in S^n be orientable. Thus we have to consider only the restriction of the map θ_n at the subgroup of oriented bordism classes of immersions, as originally considered by Freedman [**13**]. Away from the trivial cases $n = 1, 2$ the only case when the restriction of θ_n remains surjective in the orientable context is $n = 4$. Thus $f_{n-1} = 0(mod\ 2)$ if $n \neq 1, 2, 4$.

Thus, at this time only a finite number of n is known, for which the last component of $fb^{(2)}$ is nontrivial. Anyway the remarks from above show that the invariants fb and fbb are interesting and nontrivial. Notice that fb does not determine the class of the f-vector in $\mathbf{Z}^{n+1}/\Lambda(n)$, since the lattice $\Lambda(n)$ is not a product, in general. We have for instance:

Proposition 2.5. *The complete f-obstruction for 3-manifolds is specified by $(f_0, f_1) \in \mathbf{Z}/2\mathbf{Z}$ and the additional $f_0 + f_1 \in \mathbf{Z}/4\mathbf{Z}$.*

2.2. The 2-dimensional case. In the 2-dimensional case the image of the immersion associated to a cubulation C is an union of immersed circles K_i.

Proposition 2.6. *The collection of homotopy classes of the circles K_i is np-bubble invariant.*

Proof. The local pictures of the moves b_1, b_2, b_3, on the boundary of the 3-ball are relative homotopy equivalences. Since the moves have their supports in small disks (part of the cubulated surface) we are done. □

Notice that we have a collection, and not a set, since some elements can be multiple. It follows that $CBB(M)$ is infinite provided that the genus of M is at least 1. This generalizes also in higher dimensions for manifolds with nontrivial topology. This makes the np-bubble equivalence too rigid: for manifolds with nontrivial topology the homothetic cubulations M and λM will be np-bubble nonequivalent. Thus the np-bubble equivalence is interesting only for spheres. We say that a two dimensional cubulation is *simple* if the associated circles K_i are individually embedded in the respective surface. A cubulation is called now *semi-simple* if each image circle $\varphi(K_i)$ has an even number of double points, which form cancelling pairs, i.e. pairs of double points connected by two distinct and disjoint arcs bounding a disk. We have the following characterization of $CB(S^2)$:

Proposition 2.7. *The np-bubble moves act transitively on the set of simple cubulations of S^2. The orbit of the cubulation ∂C^{n+1} is the set of semi-simple cubulations. The map $fb^{(2)}$ is an isomorphism between $CB(S^2)$ and $\mathbf{Z}/2\mathbf{Z}$.*

Proof. The moves b_2, b_3 act like the second and the third Reidemester moves in plane, and b_1 creates (its inverse destroys) a circle. There is a complication due to the presence of some other arcs in the diagrams, which remain untouched by the moves. A combinatorial analysis shows that all immersed circles can be reduced to some standard circles similar to the framed versions of the unknot, with exactly one curl for each component. The move $b_{3,1}$ can add several components and then creating new cancelling pairs which can be annihilated by np-bubbles. The details are given in [**14**]. □

2.3. Around bordisms. Let now consider the set $\mathcal{I}(M)$ (respectively $\mathcal{I}^+(M)$ in the orientable case) of bordisms of codimension 1 nc (i.e. normal crossings) immersions into a differentiable manifold M. Two nc-immersions $f_i : N_i \longrightarrow M$ of the $n-1$-dimensional manifolds N_i are bordant if there exists a proper nc-immersion $f : N \longrightarrow M \times [0,1]$ from some cobordism N between N_1 and N_2 such that the restriction of f to N_i is isotopic to f_i. Notice that the transversality allows us to get rid of the nc-assumption.

We define now a *marked* cubulation as a cubulation C of the manifold M, endowed with a PL-homeomorphism of the subjacent space $|C|$, $|C| \longrightarrow M$ up to an isotopy. We ask for simplicity the compatibility of the PL and DIFF structures on the image, and M to be a DIFF manifold. The set of marked combinatorial cubulations is denoted by $\widetilde{CB}(M)$. We can associate to each marked cubulation C of the DIFF manifold M (compatible with the DIFF structure) a codimension 1 nc-immersion $\varphi_C : N_C \longrightarrow M$. We associate to each cube the set of n section hyperplanes, each of them splitting it into two equal halves. The union of all these section hyperplanes form the image of a normal crossings codimension 1-immersion after a suitable smoothing. If the cubulation C is edge-orientable (following [**16**]), and M is oriented then N_C is an oriented manifold. We denote therefore by $\widetilde{CB}^+(M)$ and $CB^+(M)$ the associated objects in the oriented case. We have then the following result:

Proposition 2.8. *There are applications $I : \widetilde{CB}(M) \longrightarrow \mathcal{I}(M)$, and in the oriented case $\widetilde{CB}^+(M) \longrightarrow \mathcal{I}^+(M)$, given by $C \to \varphi_C$.*

The map I is always surjective (see [**14**]) and injective for the 2-sphere (see above).

Proof. We consider the local picture of a bubble move, viewed in the boundary of the $n+1$-cube. The set of sections which make the immersion on the boundary are intersections of the section hyperplanes of the $n+1$-cube with the faces. The n-balls B and B' (interchanged by the bubble move) form the boundary of the sphere S^n. We infer between B and B' a short cylinder which identifies the $n+1$-ball with a cobordism between B and B'. When adding to this picture the hyperplane sections (trivially extended over the cylinder) we find a bordism (with normal crossings) between the immersions φ_B and $\varphi_{B'}$ into the n-ball. □

A partial result about the converse direction was obtained in [**2**]. We think that $\widetilde{CB}(M)$ depends only on the homotopy type of M and the functor $\widetilde{CB}$ which associates to M the set $\widetilde{CB}(M)$ is (homotopically) representable, i.e. there exists a space $\mathcal{CB}$ such that $\widetilde{CB}(M) = [M, \mathcal{CB}]$.

Notice that $CB(M) = \widetilde{CB}(M)/\mathcal{M}(M)$, where $\mathcal{M}(M)$ is the mapping class group of M, i.e. the group of diffeomorphisms of M up to an isotopy. However the description of $\mathcal{I}(M)$ is not a simple task. The classical Pontryagin-Thom construction (see e.g. [**32**]) implies that we have a homotopical description as follows: $\mathcal{I}(M) = [M_c, \Omega^\infty S^\infty \mathbf{RP}^\infty]$, and $\mathcal{I}^+(M) = [M_c, \Omega^\infty S^\infty S^1]$, where M_c is the one point compactification, Ω is the loop space, S the reduced suspension and the brackets denote the homotopy classes of maps. Moreover $\mathcal{I}^+(M) = \pi^1(M_c)$ can be identified then with the first cohomotopy group $\pi^1(M_c)$, which is however hard to compute. There are some cases when the cohomotopy groups could in principle be computed, for instance in the case of spheres. This gives $\mathcal{I}(S^n) = \pi_n^s(\mathbf{RP}^\infty)$, where $\pi_n^s(\mathbf{RP}^\infty)$ is the n-th stable homotopy group, and $\mathcal{I}^+(S^n) = \pi_n^s(S^1) = \pi_{n-1}^s$. For instance we can use $\pi_1^s(\mathbf{RP}^\infty) = \pi_2^s(\mathbf{RP}^\infty) = \mathbf{Z}/2\mathbf{Z}$, and $\pi_3^s(\mathbf{RP}^\infty) = \mathbf{Z}/8\mathbf{Z}$, and few values of the stable stems are tabulated below:

n	0	1	2	3	4	5	6	7	8
π_n^s	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/24$	0	0	$\mathbf{Z}/2$	$\mathbf{Z}/240$	$\mathbf{Z}/2 \oplus \mathbf{Z}/2$

Eventually let us introduce the bordism set of cubulations $\mathcal{C}(M)$ of the manifold M: two cubulations C_1 and C_2 are bordant if there exists a cubulation C of $M \times [0,1]$ whose restrictions on the boundaries are C_i. Notice that this definition has a strong combinatorial flavor, since the topology of the cobordant cubulation is fixed. Observe that the identity induces a map $CB(M) \longrightarrow \mathcal{C}(M)$,and we would like to know whether there is an inverse arrow. This has some similitudes with Wall's theorem about the existence of formal deformations between simple homotopy equivalent CW complexes of dimension n, by passing throughout $n+1$-dimensional cells ($n \neq 2$). Remark that any two cubulations become bordant after suitable subdivisions. Alternatively let us look at cubulations of the sphere S^n which are bubble equivalent to the standard one. We can view the bubble moves as the result of gluing and deleting $n+1$ cubes (after a suitable thickening) to the given cubulation. It follows that any such cubulations bounds, i.e. it is the boundary of a cubulation of the $n+1$-ball. For instance if $n = 1$ we should have a polygon with an even number of vertices. It is possible that for $n = 2, 3$ the converse is also true: the boundary of a ball cubulation is bubble equivalent to a trivial one. We don't expect the same phenomenon for $n \geq 4$. let put it another way: it is known that there exist non-shellable triangulations of the ball for $n \geq 3$. We define a *shuffling* of a cubulation (triangulation) as being a sequence of moves where alternate shellings (adding iteratively cells which intersect the previous stage union into balls) and inverse shellings (deleting iteratively cells which intersect the previous stage boundary into balls). Then Pachner theorem says that all triangulations can be shuffled. We saw that an obstruction for shuffling is that the cubulation bounds. However that cubulations of the $n+1$-ball which cannot be shuffled actually exist. For instance consider the connected sum $x \sharp x$ for a cubulation x whose image $I(x)$ is the generator of the third stable stem. It is proved in [**14**] that the connected sum of cubulations makes $CB(S^n)$ a monoid. The connected sum is obtained by removing one cell from each cubulation and then joining them by a standardly cubulated cylinder as a piping tube. It is clear that $x \sharp x$ is a cubulated sphere which bounds a cubulated ball, namely $(x-$ one cell $)\times[0,1]$. If $x\sharp x$ could be shuffled then x must have order 2 in $CB(S^n)$. But the map I is functorial (see [**14**]) and the bordism group $I^+(S^4)=\mathbf{Z}/24\mathbf{Z}$ hence x cannot have order 2.

3. Proofs

3.1. The proof of Proposition 2.1. At the beginning we make some notations: D_k^+ is the union of k cubes (of dimension n) which are the faces of a $(n+1)$-dimensional cube, and no two of them are parallel faces, for $k=1,2,...,n+1$. The complementary union of $2(n+1)-k$ cubes is denoted D_k^-. So a np-bubble move b_k replaces D_k^+ by D_k^-. Now $f_p(D_k^+)$ is the number of p-dimensional cubes in D_k^+. The number of interior p-cubes is $f_p(int(D_k^+))=f_p(D_k^+)-f_p(\partial D_k^+)$. Notice that this is a notational convention because it does not count the number of p-cubes in the interior of D_k^+ but the number of open such p-cubes sitting in the interior of D_k^+.

In order to find the corresponding $a_i(n)$ we have to compute the numbers $f_p(b_k(x))-f_p(x)=f_p(D_k^+)-f_p(D_k^-)=a(k,n,p)$; then $a_i(n)=gcd\{f_p(b_k(x))-f_p(x),k=1,2,...,n+1\}$.

We use the method of generating functions: set $F_X(T)=\sum_{p=0}^n f_p(X)T$. It is well-known that $f_p(C_n)=\left(\begin{array}{c} n \\ p \end{array}\right)2^{n-p}$ for the n-cube C_n so that $F_{C_n}(T)=(2+T)^n$.

Let e_i, $i=1,..n+1$ be the vectors spanning C_{n+1} and L_i be the n-cube spanned by e_1, e_2,..., e_{i-1}, e_{i+1},..., e_{n+1}, where e_i is omitted. Then a combinatorial model for D_k^+ is $\bigsqcup_{i=1}^k L_i$. The inclusion-exclusion principle states that:

$$\begin{aligned} f_p(\bigsqcup_{i=1}^k L_i) &= \sum_{i=1}^k f_p(L_i)-\sum_{i<j}^k f_p(L_i\cap L_j)+... \\ & +(-1)^{l+1}\sum_{i_1<i_2<...<i_l}^k f_p(L_{i_1}\cap L_{i_2}\cap...\cap L_{i_l})+... \\ & +(-1)^{k+1}f_p(L_1\cap L_2\cap ...\cap L_k). \end{aligned}$$

We observe that $L_{i_1}\cap L_{i_2}\cap...\cap L_{i_l}$ is combinatorially the cube C_{n+1-l} and we derive

$$f_p(\bigsqcup_{i=1}^k L_i)=\sum_{i=1}^k(-1)^{i+1}\left(\begin{array}{c} n-i+1 \\ p \end{array}\right)\left(\begin{array}{c} k \\ i \end{array}\right)2^{n+1-i-p}.$$

It follows that, at the level of generating functions, we have:

$$\begin{aligned} F_{D_k^+}(T) &= \sum_{i=1}^k(-1)^{i+1}\left(\begin{array}{c} k \\ i \end{array}\right)F_{C_{n+1-i}}(T)=\sum_{i=1}^k(-1)^{i+1}\left(\begin{array}{c} k \\ i \end{array}\right)(2+T)^{n+1-i}= \\ & (2+T)^{n+1-i}\left((2+T)^k-\sum_{i=1}^k(-1)^i\left(\begin{array}{c} k \\ i \end{array}\right)(2+T)^{k-i}\right)= \\ & (2+T)^{n+1}-(2+T)^{n+1-k}(1+T)^k. \end{aligned}$$

Therefore the generating function counting the interior cubes in D_k^- is simply

$$F_{int(D_k^-)}(T)=(2+T)^{n+1-k}(1+T)^k-T^{n+1}-kT^n.$$

In fact the total number of p-cubes in D_k^+ and D_k^- is $f_p(C_{n+1})$, but there are no $(n+1)$-dimensional faces and also the n-dimensional cubes are not interior in D_k^-, so k of them have to be removed from the total.

It remains to compute the number of interior p-cubes in D_k^+: all of them come as intersections $L_{i_1} \cap L_{i_2} \cap ... \cap L_{i_{n-p+1}}$ with the additional condition that $n-p+1 \geq 2$. It follows that

$$f_p(int(D_k^+)) = \begin{cases} \binom{k}{n-p+1} & \text{if} \quad n-1 \geq p \geq n-k+1 \\ 0 & \text{elsewhere} \end{cases}$$

The generating function is therefore

$$F_{int(D_k^+)}(T) = \sum_{p=n-k+1}^{n-1} \binom{k}{n-p+1} T^p = T^{n-k+1}[(1+T)^k - kT^{k-1} - T^k].$$

We find that the series associated to the jumps of the f-vector is

$$-F_{int(D_k^+)}(T) + F_{int(D_k^-)}(T) = \left((2+T)^{n+1-k} - T^{n+1-k}\right)(T+1)^k.$$

so that

$$\sum_{p=0}^{n} a(k,n,p)T^p = \left((2+T)^{n+1-k} - T^{n+1-k}\right)(T+1)^k.$$

As an immediate corollary we derive that all $a(k,n,p)$ are divisible by 2 since $(2+T)^{n+1-k} - T^{n+1-k}$ has even coefficients. This prove the first two claims. Developing the terms we obtain by a simple computation $a(k,n,n) = 2(n+1-k)$, $a(k,n,n-1) = 2n(n+1-k)$, and

$$a(k,n,n-2) = \frac{n+1-k}{3}\left(3n(n-1) + (n-k)(n-1-k)\right).$$

Also

$$a(k,n,0) = \begin{cases} 2^{n-k+1}, & if \quad k < n+1 \\ 0 & elsewhere \end{cases}$$

$$a(k,n,1) = \begin{cases} 2^{n-k}(n+1+k), & if \quad k \leq n-1 \\ 2n, & if \quad k = n \\ 0 & elsewhere \end{cases}$$

$$a(k,n,2) = \begin{cases} 2^{n-k-2}\left((n+1-k)(n+k) + 4k(k-1)\right), & if \quad k < n-1 \\ 2n(n-1), & if \quad k = n-1 \\ n(n-1), & if \quad k = n \\ 0 & elsewhere \end{cases}$$

A tedious verification ends the proof.

3.2. The proof of Proposition 2.2. Set more generally $D_{k,r}^+$ for the union of k cubes (of dimension n) which are the faces of a $(n+1)$-dimensional cube, and exactly $2r$ of them arise in pairs of parallel faces, for $k = 1, 2, ..., n+1$, and $2r \leq k$. The complementary set is denoted by $D_{k,r}^-$. Denote by $b_{k,r}$ the bubble move which replaces $D_{k,r}^+$ by $D_{k,r}^-$.

LEMMA 3.1. *The set of all combinatorially distinct unions of k cubes, $k \leq n+1$, of dimension n, which are faces of the $n+1$-dimensional cube, and are topologically disks, as well as their complementary sets, is exactly the set of all $D_{k,r}^+$ with $2r \leq k$. Therefore the bubble moves are the $b_{k,r}$ and their inverses.*

PROOF. Let γ_n be the adjacency graph of the cube ∂C^{n+1}. Then γ_{n+1} is the suspension $S(\gamma_n)$: this means that there are two new vertices which are connected with all of γ_n vertices but there is no edge between them. In fact it suffices to see that after excising two opposite faces of ∂C^{n+1} we obtain $\partial C^n \times [0,1]$, and the latter has the adjacency graph γ_n. Alternatively, this shows that γ_n is the complementary of the pairing graph P_n as subgraph of the complete graph $K_{2(n+1)}$. The pairing graph P_n has $2(n+1)$ vertices which are connected by $n+1$ edges, in pairs. Now we can use a recurrence argument. If $k \leq n$ and Γ is the adjacency graph of the union D of k cubes, then there exists a pair of opposite faces such that none of them is contained in D. Thus D is contained in $\partial C^n \times [0,1]$. We can write then $D = \tilde{D} \times [0,1]$, where $\tilde{D} \subset \partial C^n$ is the union of k cubes one dimension lower. First $\tilde{D}$ is a ball because it has trivial homotopy. The complementary of D inside $\partial C^n \times [0,1]$ should be also a ball, because it is the same as $\partial C^{n+1} - D$ without two cells touching the boundary (which are shelled of). Thus the complementary of $\tilde{D}$ in ∂C^n must be a ball and we can apply the recurrence hypothesis, in order to find the shape of D.

If $k = n+1$ there are two cases: either there is again one pair of opposite faces outside D (and then $D \subset \partial C^n \times [0,1]$, and the argument from above works), or else no such pair exists. Therefore D contains from each pair of opposite faces exactly one. In this case there is only one such D, up to the cubical symmetries, namely the $n+1$ coordinate hyperplanes intersection with the standard cubes. □

We have the following:

LEMMA 3.2. *For all k, r, p we have the identity:*

$$f_p(D^+_{k,r}) - f_p(D^-_{k,r}) = f_p(D^+_k) - f_p(D^-_k) (\ mod\ 2).$$

PROOF. Set $k = 2r + q$. We denote the faces of C_{n+1} by L_i^{ε}, where $L_i^0 = L_i$ and L_i^1 is the opposite face. Then $D_{k,r}$ is combinatorially equivalent to some

$$D_{k,r} \cong \bigsqcup_{i=1}^{r} L_i^0 \bigsqcup_{i=1}^{r} L_i^1 \bigsqcup_{i=r+1}^{r+q} L_i^{\varepsilon_i},$$

for some ε_i. But now the choice of the ε_i is irrelevant because there is an isometry of C_{n+1} which transforms an ε_j into $1 - \varepsilon_j$ and preserve the others. In terms of coordinates x_i this is defined by $x_i \longrightarrow x_i$ for $i \neq j$, and $x_j \longrightarrow x_j$. Therefore we can choose a combinatorial model for $D_{k,r}$ with all $\varepsilon_i = 0$.

The only difference with the previous case is that

$$\bigcap_{s=1}^{m} L_{i_s}^{\varepsilon_s} = \begin{cases} C_{n+1-m} & \text{if there are no } s_1, s_2 \text{ such that } i_{s_1} = i_{s_2}, \varepsilon_{s_1} \neq \varepsilon_{s_2} \\ \emptyset & \text{otherwise} \end{cases}$$

Therefore, with respect to the computation we made previously we have to take into account that some of intersections are void. The void intersections correspond to the combinations of h-tuples $L_i^0, ..., L_i^1, ...$ (the remaining $h-2$ faces being arbitrary) which has been counted as cubes C_{n+1-h} previously. In the calculation of $f_p(\bigsqcup_{i=1}^{r} L_i^0 \bigsqcup_{i=1}^{r} L_i^1 \bigsqcup_{i=r+1}^{r+q} L_i^{\varepsilon_i})$ we have to see how each term arising in the inclusion-exclusion principle, namely $X_h = \sum_{L_{i_s}^{\varepsilon_s} \in A} f_p(\cap_{s=1}^{h} L_{i_s}^{\varepsilon_s})$, has changed. Here A is the set of faces of $D_{k,r}$.

For $h = 1$ there are no intersections so that this factor is conserved. For $h = 2$ all the factors $L_i^0 \cap L_i^1$ have been counted before, but now their contribution is zero. There are r such factors which implies that:

$$X_2 = \binom{n-1}{p} 2^{n-p-1} \left(\binom{k}{2} - r \right)$$

For $h = 3$ the now vanishing combinations are $r(k-2)$ since a couple can be chosen among the r pairs and the third can be chosen in $k-2$ ways. It follows that :

$$X_3 = \binom{n-2}{p} 2^{n-p-2} \left(\binom{k}{3} - r(k-2) \right)$$

We continue with $h = 4$: we have $r\binom{k-2}{2}$ possibilities to get at least one couple of parallel faces but the couples $L_i^0, L_i^1, L_j^1, L_j^0$ are counted two times. Applying again the inclusion-exclusion principle we derive that

$$X_4 = \binom{n-3}{p} 2^{n-p-3} \left(\binom{k}{4} - r\binom{k-2}{2} + \binom{r}{2} \right)$$

This generalizes easily by induction to:

$$X_s = \binom{n+1-s}{p} 2^{n+1-p-s} \left(\sum_{j=1}^{2j \le s} (-1)^j \binom{k-2j}{s-2j} \binom{r}{j} \right)$$

We are able now to write:

$$f_p(D_{k,r}^+) = f_p(D_k^+) + \sum_{s=2}^{n-p+1} (-1)^s \binom{n+1-s}{p} 2^{n+1-p-s} \left(\sum_{j=1}^{2j \le s} (-1)^j \binom{k-2j}{s-2j} \binom{r}{j} \right)$$

Now we need to know how the number of interior p-cubes $f_p(int(D_{k,r}^+))$ has been changed. Of course these interior p-cubes are always coming as intersections

$$\bigcap_{s=1; L_{i_s}^{\varepsilon_s} \in A}^{n-p+1} L_{i_s}^{\varepsilon_s}$$

Again some of these intersections are void because parallel faces are allowed to be in A. But the inclusion-exclusion principle gives (in the non-trivial case $n - 1 \ge p \ge n - k + 1$) :

$$f_p(int(D_{k,r}^+)) = f_p(int(D_{k,0}^+)) \sum_{j=1}^{2j \le n-p+1} (-1)^j \binom{k-2j}{n-p+1-2j} \binom{r}{j}$$

It suffices now to see that the difference $f_p(D_{k,r}^+) - f_p(int(D_{k,r}^+))$ has the same parity for all r. In the previous formula for $f_p(D_{k,r}^+)$ only the term for $s = n-p+1$ has not a coefficient divisible by 2. But for $s = n - p + 1$ the contributing term in $f_p(D_{k,r}^+)$ is exactly the same as the total contributing term in $f_p(int(D_{k,r}^+))$ and they cancel each other.

Remark that in fact the greater common divisors of $f_p(D_{k,r}^+) - f_p(D_{k,r}^-)$ are the analogous $\tilde{a}_i(n)$ of $a_i(n)$. It is clear than $\tilde{a}_i(n)$ are divisors of $a_i(n)$. However the explicit computations are more difficult. We omit the annoying details for checking that $\tilde{a}_0(n), \tilde{a}_1(n), \tilde{a}_{n-1}(n)$ are exactly those claimed. This ends the proof of the Proposition 2.2. □

3.3. The proof of Proposition 2.3. Let P_c^{n+1} denotes the family of convex cubical polytopes. Some of the cubulations of the sphere corresponds to the boundaries ∂P_c^{n+1} of elements from P_c^{n+1}. The last component of the f-vector of elements in P_c^{n+1} is trivial. Consider the affine $\mathbf{Z}$-submodule (or $\mathbf{Z}$-submodule coset) A_c^{n+1} generated by all the f-vectors of elements of P_c^{n+1} , viewed in $\mathbf{Z}^{n+1}$.

LEMMA 3.3. *(Dehn-Sommerville Equations) The affine space* $A_c^{n+1} \otimes \mathbf{Q} \subset \mathbf{Q}^{n+1}$ *is of dimension* $\left[\frac{n+1}{2}\right]$. *A set of defining equation is obtained from the Euler-Poincaré equation*

$$\sum_{i=0}^{n} (-1)^i f_i(x) = 1 - (-1)^{d+1}$$

together with

$$\sum_{j=k}^{n} (-1)^j \binom{j}{k} 2^{j-k} f_j = (-1)^{d-1} f_k, k = 0, 1, 2, ..., n-1.$$

Equivalently we have the Euler-Poincaré equation and the set of independent equations

$$\sum_{j=k}^{n} (-1)^j \binom{j}{k} f_j = 0, k \equiv n+1 (mod\ 2), 1 \leq k \leq n-1.$$

PROOF. See [**15**], p. 156-159. □

In the general case of an arbitrary manifold M, not necessary S^n, we have to replace the Euler-Poincaré equation by the corresponding:

LEMMA 3.4. *(Dehn-Sommerville Equations for a manifold) The affine space* $A_c^{n+1}(M) \otimes \mathbf{Q} \subset \mathbf{Q}^{n+1}$ *is of dimension at most* $\left[\frac{n+1}{2}\right]$. *A set of equations which define a flat containing* $A_c^{n+1}(M) \otimes \mathbf{Q}$ *is obtained from the Euler-Poincaré equation*

$$\sum_{i=0}^{n} (-1)^i f_i(x) = \chi(M)$$

together with

$$\sum_{j=k}^{n} (-1)^j \binom{j}{k} 2^{j-k} f_j = (-1)^{d-1} f_k, k = 0, 1, 2, ..., n-1.$$

Equivalently, we have the Euler-Poincaré equation and the set of independent equations

$$\sum_{j=k}^{n} (-1)^j \binom{j}{k} f_j = 0, k \equiv n+1 (mod\ 2), 1 \leq k \leq n-1.$$

This was proved by Klee for simplicial complexes in [**19**], see also [**15**], p. 152. The case of cubulations is similar, and we omit the proof.

As a corollary we derive that the affine $\mathbf{Z}/2\mathbf{Z}$-submodule of $(\mathbf{Z}/2\mathbf{Z})^{n+1}$ generated by $fb^{(2)}(CB(M))$ has rank at most $\left[\frac{n+1}{2}\right]$. In fact, the system of independent equations written above has the determinant 1 mod 2.

In the case $M = S^n$ the affine $\mathbf{Z}/2\mathbf{Z}$-submodule is a $\mathbf{Z}/2\mathbf{Z}$-submodule because the only hyperplane which was not incident to the origin was the (Euler-Poincaré)-hyperplane, but $1 + (-1)^d \equiv 0 (mod\ 2)$, so that also this hyperplane pass through the origin, when tensorizing with $\mathbf{Z}/2\mathbf{Z}$.

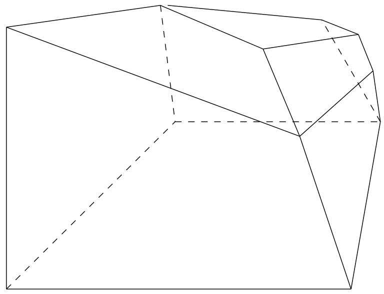

FIGURE 2. A cubulation with odd f_2

There exist cubulations of the sphere S^2 having an odd number of faces (or vertices), because immersed circles with an odd number of double points are easy to construct. An example of a convex cubical polytope with the f-vector (11,18,9) is drawn in picture 2. This proves that the image is in fact $\mathbf{Z}/2\mathbf{Z}$.

3.4. The complete f-obstruction. The complete f-obstruction is the class of the f-vector in the finite Abelian group $\mathbf{Z}^{n+1}/\Lambda(n)$. In general $\Lambda(n)$ is not a product in $\mathbf{Z}^{n+1}$ so that the projections fb do not contain all the information. In other words we saw that $\mathbf{f}(b_k(x)) = \mathbf{f}(x) + a(k,n)$, where $a(k,n) = (a(k,n,p)_{p=0,\dots,n})$. Therefore, starting from the f-vector α we will obtain β after some np-bubble moves if and only if the system of linear equations

$$\sum_{k=1}^{n+1} x_k a(k,n) = \beta - \alpha = b,$$

has integer solutions $x_k \in \mathbf{Z}$. Here x_k is the algebraic number of times the move b_k has been used (an inverse move b_k^{-1} counts as -1). Such a linear system has integer solutions only if $gcd(a(k,n,p), k = 1, n+1)$ divides b_p, for each $p = 0, 1, ..., n$. These obstructions on $\alpha - \beta$ are exactly those described in the first part, namely $f_i(mod\ a_i(n))$. . From 2.2 our the linear system is under-determined, of rank less than $\left[\frac{n+1}{2}\right]$. The complete obstruction is the set of compatibility relations making this system having solutions.

In the case of our system modeling the transformations of the f-vector by bubble/np-bubble moves, for $n = 1, 2$, we do not find any new obstructions. Now the result of proposition 2.6 can be deduced immediately:

PROOF. For $n = 3$ there is one new obstruction. There are at most two independent f_i, let choose for instance f_0 and f_1. Then the np-bubble moves b_i have the matrix $a(*, *)$:

$$\begin{pmatrix} 0 & 2 & 4 & 8 \\ 0 & 6 & 12 & 20 \end{pmatrix}.$$

This gives, away from the f_0, f_1 (mod 2) the that $(f_0 + f_1)(mod\ 4)$ is an invariant.

In the case we have to do with the bubble moves, there are two more moves to add to the previous list, namely $b_{3,1}$ and $b_{4,1}$, and the matrix is:

$$\begin{pmatrix} 0 & 2 & 4 & 8 & 0 & 0 \\ 0 & 6 & 12 & 20 & 0 & 4 \end{pmatrix}.$$

Thus $(f_0 + f_1)(mod\ 4)$ is also an invariant for bubble moves. □

Let observe that the standard cubulation of S^3 (the boundary of the 4-cube) has trivial $(f_0+f_1)(mod\, 4)$. On the other hand, $CB(S^3)$ is a group whose elements are of order two, hence $2(f_0+f_1)=0(mod\, 4)$ for any cubulation of the sphere. This implies that $f_0=f_1(mod\, 2)$, which agrees with the results of [**2**] for the polytopes. Remark however that the obstructions $f_0(mod\, 2)$, $f_1(mod\, 2)$ and $(f_0+f_1)(mod\, 4)$. are defined on $CB(M)$ for all 3-manifolds M, not only for the sphere.

References

[1] T.Banchoff, *Triple points and surgery of immersed surfaces* Proc.Amer.Math.Soc., **46**(1974), 407–413.

[2] E.K.Babson and C.S.Chan, *Counting faces of cubical spheres modulo two*, Preprint, 1997.

[3] D.W. Barnette, *The minimum number of vertices of a simple polytope*, Israel J.Math., **10**(1971), 121-125.

[4] D.W. Barnette, *A proof of the lower bound conjecture for convex polytopes* Pacific J.Math., **46**(1973), 349-354.

[5] L.J.Billera and C.W.Lee, *A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes* J.Combinatorial Theory Ser.A , **31**(1981), 237-255.

[6] R.H.Bing, *Some aspects of the topology of 3-manifolds related to the Poincaré conjecture*, Lectures on Modern Mathematics, (T.L.Saaty, Editor), 93-128, John Wiley and Sons, New York, 1964.

[7] G. Blind and R. Blind, *Gaps in the number of faces of cubical polytopes*, *Discrete and Comp. Geometry*, **11** (1994), 351–356.

[8] J. Scott Carter, *A further generalization of Boy's surface*, *Houston J.Math.*, **12**(1986), 11–31.

[9] J. Scott Carter, *Surgery on codimension one immersions*, Trans.Amer.Math.Soc., **298**(1986), 83–101.

[10] P.J. Eccles, *Codimension 1 immersions and the Kervaire invariant one problem* Math.Proc.Cambridge Phil.Soc.,**90**(1981), 483–493.

[11] P.J. Eccles, *Multiple points of codimension 1 immersions*, in Topology Symposium, Siegen 1979, U.Koschorke and W.Neumann Eds., Lecture Notes in Math. no.788, 23–38, 1980.

[12] P.J. Eccles *Multiple points of codimension one immersions of oriented manifolds*, Math.Proc.Cambridge Phil.Soc., **87**(1980), 213–220.

[13] M. Freedman, *Quadruple points of 3-manifolds in S^4*, Comment.Math.Helvetici, **53**(1978), 385–394.

[14] L.Funar, *Cubulations, immersions, mappability and a problem of Habegger*, math.GT/9804046, (available also at `http://www-fourier.ujf-grenoble.fr/ ~ funar`)

[15] R. Grünbaum, *Convex polytopes.* Pure and Appl. Math. , Interscience Publ. , John Wiley and Sons, London-New York - Sydney, vol. 16, 1967.

[16] G. Hetyei, *On the Stanley ring of a cubical complex*, Discrete and Comp. Geometry, **14**(1995), 305–330.

[17] W. Jockusch *The lower and upper bound problems for cubical polytopes*, Discrete and Comp. Geometry,**9**(1993), 159–163.

[18] R. Kirby, *Problems in low-dimensional topology*, in "Geometric Topology", Georgia International Topology Conference, (W.H.Kazez, Editor), AMS-IP Studies in Advanced Math., **2**, part 2, 35-473, 1995.

[19] V. Klee, *A combinatorial analogue of Poincare's duality theorem*, Canadian J.Math., **16**(1964), 517-531.

[20] U. Koschorke and B.Sanderson, *Selfintersections and higher Hopf invariants,* Topology, **17**(1978), 283–290.

[21] J. Lannes, *Sur les immersions de Boy*, in Algebraic Topology, Aarhus, 1982, Lecture Notes in Math., no.1051, 263–270, 1984.

[22] R. Lickorish, *Piecewise linear manifolds and Cerf theory*, in "Geometric Topology", Georgia International Topology Conference, (W.H.Kazez, Editor), AMS-IP Studies in Advanced Math., **2**, part 1, 375-387, 1995.

[23] P. McMullen, *On the number of faces of a simplicial polytope* Israel J.Math.,**9**(1971), 559-570.

[24] P. McMullen, *The maximum number of faces of a convex polytope*, Mathematika, **17**(1970), 179-184.
[25] P. McMullen, *On simple polytopes*, Inventiones Math., **113**(1993), 419-444.
[26] U. Pachner, *Shellings of simplicial balls and p.l. manifolds with boundary*, Discrete Math., **81**(1990), 37-47.
[27] U. Pachner, *Homeomorphic manifolds are equivalent by elementary shellings*, European J.Combinatorics, **12**(1991), 129-145.
[28] L. Siebenmann, *Les bisections expliquent le théorème de Reidemester-Singer*, Preprint Orsay, 81T16, 1981.
[29] R. Stanley, *The number of faces of a simplicial convex polytope*, Adv. Math., **35**(1980), 236-238.
[30] R. Stanley, *The number of faces of a simplicial polytopes and spheres*, in Discrete Geometry and Convexity, J.E.Goodman et. all. Eds., Annals of the N.Y.Academy of Sci., 440(1985), 212-223.
[31] V. Turaev, *Quantum invariants of links and 3-valent graphs in 3-manifolds*, de Gruyter Studies in Mathematics, vol.18, 1994.
[32] P. Vogel, *Cobordismes d'immersions* Ann.Ecole Normale Sup. **7**(1974), 317-358.

Institut Fourier, BP 74, Univ.Grenoble I, Mathématiques, 38402 Saint-Martin-d'Hères cedex, France

E-mail address: `funar@fourier.ujf-grenoble.fr`