A REFINEMENT OF THE SIMPLE CONNECTIVITY AT INFINITY OF GROUPS

LOUIS FUNAR AND DANIELE ETTORE OTERA

ABSTRACT. We give another proof for a result of Brick ([2]) stating that the simple connectivity at infinity is a geometric property of finitely presented groups. This allows us to define the rate of vanishing of π_1^∞ for those groups which are simply connected at infinity. Further we show that this rate is linear for cocompact lattices in nilpotent and semi-simple Lie groups, and in particular for fundamental groups of geometric 3-manifolds.

Keywords: Simple connectivity at infinity, quasi-isometry, colored Rips complex, Lie groups, geometric 3-manifolds.

MSC Subject: 20 F 32, 57 M 50.

1. Introduction

The first aim of this note is to prove the quasi-isometry invariance of the simple connectivity at infinity for groups, in contrast with the case of spaces. We recall that:

Definition 1. *The metric spaces (X, d_X) and (Y, d_Y) are quasi-isometric if there are constants λ, C and maps $f : X \to Y$, $g : Y \to X$ (called (λ, C)-quasi-isometries) such that the following:*

$$d_Y(f(x_1), f(x_2)) \leqslant \lambda d_X(x_1, x_2) + C,$$

$$d_X(g(y_1), g(y_2)) \leqslant \lambda d_Y(y_1, y_2) + C,$$

$$d_X(fg(x), x) \leqslant C,$$

$$d_Y(gf(y), y) \leqslant C,$$

hold true for all $x, x_1, x_2 \in X, y, y_1, y_2 \in Y$.

Definition 2. *A connected, locally compact, topological space X with $\pi_1 X = 0$ is simply connected at infinity (abbreviated s.c.i. and one writes also $\pi_1^\infty X = 0$) if for each compact $k \subseteq X$ there exists a larger compact $k \subseteq K \subseteq X$ such that any closed loop in $X - K$ is null homotopic in $X - k$.*

Remark 1. *The simple connectivity at infinity is not a quasi-isometry invariant of spaces ([15]). In fact $(S^1 \times \mathbf{R}) \underset{S^1\times\mathbb{Z}}{\cup} D^2$ and $(S^1 \times \mathbf{R}) \underset{S^1\times\{0\}}{\cup} D^2$ are simply connected, quasi-isometric spaces although the first is simply connected at infinity while the second is not.*

This notion extends to a group-theoretical framework as follows (see [3], p.216):

Partially supported by GNSAGA.

Definition 3. *A group G is simply connected at infinity if for some (equivalently any) finite complex X such that $\pi_1 X = G$ one has $\pi_1^\infty \widetilde{X} = 0$, where $\widetilde{X}$ denotes the universal covering of X.*

The independence on the particular complex X is proved in [17] and [15]. Roughly speaking the simple connectivity at infinity depends only on the 2-skeleton and any finite 2-complex corresponds to a presentation of G. Since Tietze transformations act transitively on the set of group presentations it suffices to check the invariance under such moves. We refer to [17] for details.

All groups considered in the sequel will be finitely generated and a system of generators determines a word metric on the group. Although this depends on the chosen generating set the different word metrics are quasi-isometric. Therefore properties which are invariant under quasi-isometries are independent on the particular word metric and will be called *geometric properties*. It is well-known that being finitely presented, word hyperbolic or virtually free are geometric properties, while being virtually solvable or virtually torsion-free are not geometric (see [5]).

Our main result is :

Theorem 1 ([2])**.** *The simple connectivity at infinity of groups is a geometric property.*

This was originally proved by Brick in [2]. We provide a simpler and more conceptual proof, by analyzing the colored Rips complex.

Remark 2. *It seems that the fundamental group at infinity, whenever it is well-defined (see* [9] *for an extensive discussion on this topic), should also be a quasi-isometry invariant of the group.*

Definition 4. *Let X be a simply connected non-compact metric space with $\pi_1^\infty X = 0$. The rate of vanishing of π_1^∞, denoted $V_X(r)$, is the infimal $N(r)$ with the property that any loop which sits outside the ball $B(N(r))$ of radius $N(r)$ bounds a 2-disk outside $B(r)$.*

Remark 3. *It is easy to see that V_X can be an arbitrary large function.*

It is customary to introduce the following equivalence relation on functions: $f \sim g$ if there exists constants c_i, C_j (with $c_1, c_2 > 0$) such that

$$c_1 f(c_2 R) + c_3 \leq g(R) \leq C_1 f(C_2 R) + C_3.$$

It is an easy consequence of the proof of theorem 1 that the equivalence class of $V_X(r)$ is a quasi-isometry invariant. In particular $V_G = V_{\widetilde{X_G}}$ is a quasi-isometry invariant of the group G, where $\widetilde{X_G}$ is the universal covering space of a compact simplicial complex X_G, with $\pi_1(X_G) = G$ and $\pi_1^\infty(G) = 0$.

Remark 4. *For most groups G coming from geometry V_G is trivial, i.e. linear. Obviously if M has an Euclidean structure then $V_{\pi_1(M)}$ is linear. Since metric balls in the hyperbolic space are diffeomorphic to standard balls in $\mathbf{R}^n$ one derives that $V_{\pi_1(M)}$ is linear for any compact hyperbolic manifold M.*

Remark 5. *Notice that there exists (see* [4]*) word hyperbolic groups G (necessary of dimension $n \geq 4$ by* [1]*) which are not simply connected at infinity and hence V_G is not defined. Moreover if G is a word hyperbolic torsion-free group with $\pi_1^\infty(G) = 0$ then it seems that V_G is linear.*

Theorem 2. *V_G is linear for uniform lattices in:*

1. *semi-simple Lie groups.*
2. *nilpotent groups.*
3. *solvable stabilizers of horospheres in product of symmetric spaces of rank at least two, or generic horospheres in products of rank one symmetric spaces.*

Interesting examples of groups for which $\pi_1^\infty(G) = 0$ are the (infinite) fundamental groups of geometric 3-manifolds (and conjecturally of all 3-manifolds). We can show that:

Corollary 1. *The fundamental groups of geometric 3-manifolds have linear V_G.*

Remark 6. *The existence of groups G acting freely and cocompactly on $\mathbf{R}^n$, which have super-linear V_G seems most likely. The examples described in ([11], section 4), which have large acyclicity radius, strongly support this claim. The first point is that the rate of vanishing of π_1^∞ is rather related to higher (i.e. dimension $n-2$) connectivity radii, which are less understood. The second difficulty is that these groups are not s.c.i. The simplest way to overcome it is to consider group extensions. For instance $\pi_1^\infty(G \times \mathbf{Z}^2) = 0$, for any finitely presented group G; alternatively $\pi_1^\infty(V^n \times \mathbf{R}) = 0$ for any contractible manifold V^n ($n \geq 2$). However this idea does not work because $V_{G\times\mathbf{Z}^2}$ is always linear.*

Remark 7. *One needs some extra arguments in order to extend the proof to all solvable Lie groups. However, it seems very likely that cocompact lattices in all connected Lie groups have linear rate of vanishing of π_1^∞.*

Acknowledgements : The authors are indebted to C. Druţu, P. Pansu and F. Paulin for helpful conversations and to the referee for the useful comments improving the exposition.

2. Proof of Theorem 1

For positive d, set $P_d(G)$ for the simplicial complex defined as follows:

- its vertices are the elements of G,
- the elements $x_1......x_n$ of G span an n-simplex, if $d(x_i, x_j) \leq d$ for all i, j (where $d(.,.)$ is the word metric).

Remark 8. *If G is δ-hyperbolic then $P_d(G)$ is contractible as soon as $d > 4\delta + 1$ (see [8]).*

Although $P_d(G)$ is not contractible in general, one can prove that it is simply connected under a mild restriction. Let $G =< x_1, ..., x_n | R_1, ..., R_p >$ be a presentation of G and r denotes the maximum length among the relators R_i.

Lemma 1. *If $2d > r$, then $\pi_1(P_d(G)) = 0$.*

Proof. Let $l = [1, \gamma_1, \gamma_2, ..., \gamma_n, 1]$ be a (simplicial) loop in $P_d(G)$ based at the identity. Two successive vertices of l are at distance at most d. One can interpolate between two consecutive γ_j's a sequence of elements of G (of length at most d), consecutive ones being adjacent when viewed as elements of the Cayley graph (hence at distance one). The product of elements corresponding to these edges of length one of l is trivial in G. Therefore it is a product of conjugates of relators: $\prod g_i^{-1} R_i g_i$. The diameter of each R_i is at most $r/2$, and the assumption $2d > r$, implies that each loop $g_i^{-1} R_i g_i$ is contractible in $P_d(G)$. This ends the proof. □

The natural group action of G on itself by left translations gives rise to an action on $P_d(G)$. In particular, if G has no torsion then it acts freely on $P_d(G)$ and $P_d(G)/G = X$ is a compact simplicial complex with $\pi_1(P_d(G)/G) = G$.

Proposition 1. *The vanishing of π_1^∞ is a geometric property of torsion-free groups.*

Proof. One has to show that if the group H is quasi-isometric to G then $\pi_1^\infty P_d(G) = 0$ implies that $\pi_1^\infty P_a(H) = 0$ for large enough a.

Let $f : H \to G$ and $g : G \to H$ be (k, C)-quasi-isometries between G and H. Fix $x_0 \in P_a(H)$ and $f(x_0) \in P_d(G)$ as base points.

Lemma 2. *If $\pi_1^\infty P_d(G) = 0$ then $\pi_1^\infty P_D(G) = 0$ for $D \geq d$.*

Proof. An edge in $P_D(G)$ corresponds to a path (of length uniformly bounded by $\frac{D}{d} + 1$) in $P_d(G)$. Thus a loop l in $P_D(G)$ at distance at least R from a given point corresponds to a loop l' in $P_d(G)$ at distance at least $R - \frac{D}{d} - 1$ from the same point. By assumption l' will bound a 2-disk D^2 far away in $P_d(G)$.

Now the union of an edge $[x_1 x_n]$ in $P_D(G)$ and its corresponding path $[x_1, x_2, ..., x_n]$ in $P_d(G) \subset P_D(G)$ form the boundary of a 2-disk in $P_D(G)$, which is triangulated by using the triangles $[x_1, x_j, x_n]$. Consider one such triangulated 2-disk for each edge of l and glue to the previously obtained D^2 to get a 2-disk in $P_D(G)$ bounding l and far away. □

By hypothesis for each r there exists $N(r) > 0$ such that every loop l in $P_d(G)$ satisfying $d(l, f(x_0)) > N(r)$ bounds a disk outside $B(f(x_0), r)$. This means that there exists a simplicial map $\varphi : D^2 \to P_d(G) - B(f(x_0), r)$ such that $\varphi(\partial D^2)$ is the given loop l, when D^2 is suitably triangulated. A loop $l = [x_1, x_2, ..., x_n, x_1]$ in $P_d(G)$, based at x_1, is the one-dimensional simplicial sub-complex with vertices x_j and edges $[x_i x_{i+1}]$, $i = 1, n$ (with the convention $n + 1 = 1$).

Set $M(R) = kN(kR + kC + 3C) + 3C$. We claim that:

Lemma 3. *Any loop l in $P_a(H)$ sitting outside the ball $B(x_0, M(R))$ bounds a 2-disk not intersecting $B(x_0, R)$.*

Proof. Set $l = [x_1, ..., x_n]$. Using the previous lemma one can assume that d is large enough such that $\frac{d-C}{k} > 1$. As in lemma 1 one can add extra vertices between the consecutive ones such that $d(x_i, x_{i+1}) \leq \varepsilon$ holds, where $k\varepsilon + C = d$.

The image $f(l) = [f(x_1), ..., f(x_n), f(x_1)]$ of the loop l has the property that $d(f(x_i), f(x_{i+1})) \leq k\varepsilon + C$. Using $d(x, gf(x)) \leq C$ one obtains that $d(gf(x), gf(y)) \geq d(x, y) - 2C$, which implies $d(x, y) \leq kd(f(x), f(y)) + 3C$ and thus:

$$d(f(x), f(y)) \geq \frac{d(x, y) - 3C}{k}, \text{ for all } x, y \in P_a(H).$$

From this inequality one derives that:

$$d(f(x_i), f(x_0) \geq (M(R) - 3C)/k = N(kR + kC + 3C)$$

and thus the loop $f(l)$ sits outside the ball $B(f(x_0), N(kR + kC + 3C))$, and hence by assumption $f(l)$ bounds a disk which does not intersect $B(f(x_0), kR + kC + 3C)$.

Let $y_1, ..., y_t$ be the vertices of the simplicial complex $\varphi(D^2)$ bounded by the loop $f(l)$. The vertices $f(x_1), ..., f(x_n)$ are contained among the y_j's. One can suppose that any triangle $[y_i, y_j, y_m]$ of $\varphi(D^2)$ has edge length at most d. Therefore we have:

$$d(g(y_j), x_i) \leq d(g(y_j), gf(x_i)) + C \leq kd(y_j, f(x_i)) + 2C \leq k^2\varepsilon + (k + 2)C.$$

This proves that $x_i, x_j, g(y_m)$ span a simplex of $P_a(H)$ (for all i, j, m) whenever we choose a larger than $k^2\varepsilon + (k+2)C$. Moreover:

$$d(x_0, g(y_i)) \geq d(gf(x_0), g(y_i)) - C \geq \frac{d(f(x_0), y_i) - 3C}{k} - C \geq R.$$

Further there is a simplicial map $\psi : \varphi(D^2) \to P_a(H)$ which sends $f(x_j)$ into x_j and all other vertices y_k into the corresponding $g(y_k)$. It is immediate now that $\psi\varphi(D^2)$ is a simplicial sub-complex bounded by l, which has the required properties. □

This proves proposition 1. □

When G has torsion, one can construct a highly connected polyhedron with a free and cocompact G-action as follows (see [1]):

Definition 5. *The colored Rips complex $P(d, m, G)$ (for natural m) is the sub-complex of the m-fold join $G * G ..... * G$ consisting of those simplexes whose vertices are at distance at most d in G.*

Lemma 4. *For $m \geq 3$ and d large enough, G acts freely on the 2-skeleton of $P(d, m, G)$ and $\pi_1(P(d, m, G)) = 0$.*

Proof. Clearly G acts freely on the vertices of $P_d(G)$, hence any non-trivial $g \in G$ fixing a simplex has to permute its vertices. Adding $m \geq 3$ colors prevents therefore the action from having fixed simplexes of dimension less than 3. Now using the proof of lemma 1 one obtains also the simple connectivity. □

The theorem 1 follows now from the proof of proposition 1, suitably adapted to the 2-skeleton of $P(d, m, G)$.

Remark 9. *The same technique shows that the higher connectivity at infinity is also a quasi-isometry invariant of groups.*

We have then:

Corollary 2. *The equivalence class of $V_X(r)$ is a quasi-isometry invariant of X.*

Proof. The result is implied by theorem 1 and lemma 3. □

3. Uniform lattices in Lie groups

Proposition 2. *Uniform lattices in (non-compact) semi-simple Lie groups have linear rate of vanishing of π_1^∞.*

Proof. We will denote below by d_X the distance function and by B_X the respective metric balls for the space X.

Let K be the maximal compact subgroup of the simple Lie group G and G/K the associated symmetric space. It is well-known that the Killing metric on G/K is non-positively curved, and hence the metric balls are diffeomorphic to standard balls, by the Hadamard theorem.

If G is not $SL(2, \mathbf{R})$ then K is different from S^1 and therefore it has finite fundamental group. In particular the universal covering $\widetilde{K}$ is compact. The Iwasawa decomposition $G = KAN$ yields a canonical diffeomorphism $G \to K \times G/K$. Furthermore we have a an induced canonical quasi-isometry $\widetilde{G} \to \widetilde{K} \times G/K$. Large balls in $\widetilde{G}$ can be therefore compared with products of the (compact) $\widetilde{K}$ and metric balls in G/K, as follows:

$$\widetilde{K} \times B_{G/K}(r-a) \subset B_{\widetilde{G}}(r) \subset \widetilde{K} \times B_{G/K}(a+r) \subset B_{\widetilde{G}}(2a+r), \text{ for } r \text{ large enough,}$$

which implies our claim.

The case of $V_{\widetilde{SL(2,\mathbf{R})}}$ is quite similar, and a consequence of the (well-known) fact that $\widetilde{SL(2,\mathbf{R})}$ and $H^2 \times \mathbf{R}$ are canonically quasi-isometric. We will sketch a proof below. Let us outline the construction of the $\widetilde{SL(2,\mathbf{R})}$ geometry. A Riemannian metric on a manifold allows us to construct canonically a Riemannian metric on its tangent bundle, usually called the Sasaki metric. In particular one considers the restriction of the Sasaki metric to the unit tangent bundle UH^2 of the hyperbolic plane. Further there exists a natural diffeomorphism between UH^2 and $PSL(2,\mathbf{R})$, which gives $PSL(2,\mathbf{R})$ a Riemannian metric, and hence induces a metric on its universal covering, namely $\widetilde{SL(2,\mathbf{R})}$. This is the Riemannian structure describing the geometry of $\widetilde{SL(2,\mathbf{R})}$. Observe that $\widetilde{SL(2,\mathbf{R})}$ and $H^2 \times \mathbf{R}$ are two metric structures on the same manifold, and both are Riemannian fibrations over H^2 (the former being metrically non-trivial while the later is trivial).

It is clear now that the identity map between the manifolds UH^2 and $H^2 \times S^1$ is a quasi-isometry, lifting to a quasi-isometry between $\widetilde{SL(2,\mathbf{R})}$ and $H^2 \times \mathbf{R}$. This implies that there are two constants $a > 0, b$ such that:

$$\frac{1}{a} d_{H^2\times\mathbf{R}}(x,y) - b \leq d_{\widetilde{SL(2,\mathbf{R})}}(x,y) \leq a d_{H^2\times\mathbf{R}}(x,y) + b, \text{ for all } x,y \in H^2 \times \mathbf{R},$$

holds true. In particular we have the following inclusions between the respective metric balls:

$$B_{H^2\times\mathbf{R}}\left(\frac{r}{c}\right) \subset B_{\widetilde{SL(2,\mathbf{R})}}(r) \subset B_{H^2\times\mathbf{R}}(cr) \subset B_{\widetilde{SL(2,\mathbf{R})}}(c^2 r),$$

for r large enough and $c > 0$. The claim follows. □

Remark 10. *The same argument shows that the acyclicity radius for semisimple Lie groups is linear (see* [11]*, section 4).*

The way to prove the claim for nilpotent and solvable groups consists in the large scale comparison with some other metrics, whose balls are known to be diffeomorphic to standard balls. While locally the Riemannian geometry of a nilpotent Lie group is Euclidean, globally it is similar to the Carnot-Caratheodory non-isotropic geometry.

Proposition 3. *If G is a torsion-free nilpotent group then V_G is linear.*

Proof. It is known (see [13]) that G is a cocompact lattice in a real simply connected nilpotent Lie group $G_{\mathbf{R}}$, called the Malcev completion of G. We have also a nice characterization of the metric balls in real, nilpotent Lie groups given by Karidi (see [12]), as follows. Since $G_{\mathbf{R}}$ is diffeomorphic to $\mathbf{R}^n$ it makes sense to talk about parallelepipeds with respect to the usual Euclidean structure on $\mathbf{R}^n$. Next, there exists some constant $a > 0$ (depending on the group $G_{\mathbf{R}}$ and on the left invariant Riemannian structure chosen, but not on the radius r) such that the radius r-balls $B_{G_{\mathbf{R}}}(r)$ are sandwiched between two parallelepipeds, which are homothetic at ratio a, so:

$$P_r \subset B_{G_{\mathbf{R}}}(r) \subset aP_r \subset B_{G_{\mathbf{R}}}(ar), \text{ for any } r \geq 1.$$

This implies that we can take $V_{G_{\mathbf{R}}}(r) \sim ar$, and hence V_G is linear. □

Remark 11. *Metric balls in solvable Lie groups are quasi-isometric with those of discrete solvgroups, and so they are highly concave (see [7]): there exist pairs of points at distance c, sitting on the sphere of radius r, which cannot be connected by a path within the ball of radius r, unless its length is at least $r^{0.9}$. Further it can be shown that there are arbitrarily large metric balls which are not simply connected. Nevertheless we will prove that metric balls contain large slices of hyperbolic balls.*

Proposition 4. *Cocompact lattices in solvable stabilizers of horospheres in product of symmetric spaces of rank at least two, or generic horospheres in products of rank one symmetric spaces have linear V_G.*

Proof. We give the proof for our favorite solvable group, namely the 3-dimensional group Sol. It is well-known that Sol is isometric to a generic horosphere $\mathcal{H}$ in the product $H^2 \times H^2$ of two hyperbolic planes. Generic means here that the horosphere is associated to a geodesic ray which is neither vertical nor horizontal. The argument in ([11] 3.D.), or its generalization from [6], shows that such horospheres $\mathcal{H}$ are undistorted in the ambient space i.e. there exists $a \geq 1$, such that

$$\frac{1}{a} d_{H^2 \times H^2}(x, y) \leq d_{\mathcal{H}}(x, y) \leq a d_{H^2 \times H^2}(x, y), \text{ for all } x, y \in \mathcal{H},$$

holds true. Here $d_{H^2 \times H^2}$ and $d_{\mathcal{H}}$ denote the distance functions in $H^2 \times H^2$ and $\mathcal{H}$, respectively. In particular we have the following inclusions between the respective metric balls:

$$\mathcal{H} \cap B_{H^2 \times H^2}\left(\frac{r}{a}\right) \subset B_{\mathsf{Sol}}(r) \subset \mathcal{H} \cap B_{H^2 \times H^2}(ar) \subset B_{\mathsf{Sol}}(a^2 r).$$

Since the horoballs in $H^2 \times H^2$ are convex it follows that the intersections $\mathcal{H} \cap B_{H^2 \times H^2}(r)$ are diffeomorphic to standard balls. This proves that V_{Sol} is linear.

The linearity result extends without modifications to lattices in solvable stabilizers of generic horospheres in symmetric spaces of rank at least 2 (see [6]). □

Remark 12. *It is known that finitely presented solvable groups are either simply connected at infinity or are of a very special form, as described in* [14]. *On the other hand it is a classical result that any simply connected solvable Lie group is diffeomorphic to the Euclidean space. It would be interesting to know whether a simply connected solvable Lie group can be isometrically embedded as a horosphere in a symmetric space.*

Remark 13. *Notice that horospheres in hyperbolic spaces (and hence non-generic horospheres in products of hyperbolic spaces) have exponential distortion, namely $d_{H^n}(x, y) \sim \log d_{\mathcal{H}^{n-1}}(x, y)$, for $x, y \in \mathcal{H}^{n-1}$. This highly contrast with the higher rank and/or generic case.*

This ends the proof of theorem 2.

Remark 14. *One might notice a few similarities between V_G and the isodiametric function considered by Gersten (see* [11]*).*

Corollary 3. *The rate of vanishing of π_1^∞ is linear for the fundamental groups of geometric 3-manifolds.*

Proof. There are eight geometries in the Thurston classification (see [16]): the sphere S^3, $S^2 \times \mathbf{R}$, the Euclidean E^3, the hyperbolic 3-space H^3, $H^2 \times \mathbf{R}$, $\widetilde{SL(2, \mathbf{R})}$, Nil and Sol. Manifolds covered by S^3 have finite fundamental groups. Further the compact manifolds without boundary covered by $S^2 \times \mathbf{R}$ are the two S^2 bundles over S^1, $\mathbf{RP}^2 \times S^1$ or the connected sum $\mathbf{RP}^3 \sharp \mathbf{RP}^3$, and the claim can be checked easily. As we already observed, this is the also the case for the Euclidean and hyperbolic geometries. The same holds for the product $H^2 \times \mathbf{R}$, in which case metric balls are diffeomorphic to standard balls. The remaining cases are covered by theorem 2. □

References

[1] M.Bestvina and G.Mess, *The boundary of negatively curved groups*, J.A.M.S. **4**(1991), 469-481.

[2] S.Brick, *Quasi-isometries and ends of groups*, J.Pure Appl. Algebra **86**(1993), 23-33.

[3] S.Brick and M.Mihalik, *The QSF property for groups and spaces*, Math. Zeitschrift **220**(1995), 207-217.

[4] M.Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. **117**(1983), 293–324.

[5] A.Dyubina-Erschler, *Instability of the virtual solvability and the property of being virtually torsion-free for quasi-isometric groups*, I.M.R.N. 2000, no. 21, 1097–1101.

[6] C.Druţu, *Nondistorsion des horosphères dans des immeubles euclidiens et dans des espaces symétriques*, G.A.F.A. **7**(1997), 712–754.

[7] L.Funar, *Discrete cocompact solvgroups and Poénaru's condition*, Archiv Math.(Basel) **72**(1999), 81-85.

[8] E.Ghys and P.de la Harpe (Ed.), *Sur les groupes hyperboliques d'après M. Gromov*, Progress in Math., vol. **3**, Birkhauser, 1990.

[9] R.Geoghegan and M.Mihalik, *The fundamental group at infinity*, Topology **35**(1996), 655-669.

[10] M.Gromov, *Hyperbolic groups*, Essays in Group Theory (S. Gersten Ed.), MSRI publications, no. **8**, Springer-Verlag (1987).

[11] M.Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), 1–295, London Math. Soc. Lecture Note Ser., 182, (G.A.Niblo and M.A.Roller, Ed.), Cambridge Univ. Press, Cambridge, 1993.

[12] R.Karidi, *Geometry of balls in nilpotent Lie groups*, Duke Math. J. **74**(1994), 301–317.

[13] A.Malcev, *On a class of homogeneous spaces*, Izvestiya Akad. Nauk. SSSR. Ser. Mat. **13**(1949), 9–32, Amer. Math. Soc. Translation **39**(1951).

[14] M.Mihalik, *Solvable groups that are simply connected at* ∞, Math. Zeitschrift **195** (1987), 79–87.

[15] D.E.Otera, *On the simple connectivity of groups*, preprint 139(2001), Univ. Palermo, Bull.U.M.I. (to appear).

[16] P.Scott, *The geometries of 3-manifolds*, Bull.London Math. Soc., **15**(1983), 401-487.

[17] C.Tanasi, *Groups simply connected at infinity*, (Italian), Rend. Ist. Mat. Univ. Trieste **31** (1999), 61-78.

Institut Fourier BP 74, UMR 5582, Université Grenoble I, 38402 Saint-Martin-d'Hères Cedex, France

E-mail address: funar@fourier.ujf-grenoble.fr

Université Paris-Sud, Mathématiques, Bât 425, 91405 Orsay Cedex, France

E-mail address: Daniele.Otera@math.u-psud.fr