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Free subgroups within the images of quantum representations

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Abstract. We prove that, except for a few explicit roots of unity, the quantum image of any Johnson subgroup of the mapping class group contains an explicit free non-abelian subgroup.

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1 Introduction and statements

The aim of this paper is to study the images of the mapping class groups by quantum representations. Some results in this direction are already known. We refer the reader to [30] and [19] for earlier treatments of quantum representations. In [8] we proved that the images are infinite and non-abelian (for all but finitely many explicit cases) using earlier results of Jones who proved in [17] that the same holds true for the braid group representations factorizing through the Temperley–Lieb algebra at roots of unity. Masbaum then found in [24] explicit elements of infinite order in the image. General arguments concerning Lie groups actually show that the image should contain a free non-abelian group. Furthermore, Larsen and Wang showed (see [21]) that the image of the quantum representations of the mapping class groups at roots of unity of the form $\exp(\frac{2\pi i}{4r})$, for prime $r \geq 5$, is dense in the projective unitary group.

In order to be precise we have to specify the quantum representations we are considering. Recall that in [2] the authors defined the TQFT functor $\mathcal{V}_p$, for every $p \geq 3$ and a primitive root of unity A of order $2p$. These TQFT should correspond

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to the so-called SU(2)-TQFT, for even p and to the SO(3)-TQFT, for odd p (see also [21] for another SO(3)-TQFT).

Definition 1.1. Let $p \in \mathbb{Z}_+$, $p \geq 3$, such that $p \not\equiv 2 \pmod 4$. The quantum representation ρ_p is the projective representation of the mapping class group associated to the TQFT $\mathcal{V}_{\frac{p}{2}}$ for even p and $\mathcal{V}_p$ for odd p, corresponding to the following choices of the root of unity:

$$A_p = \begin{cases} -\exp\left(\frac{2\pi i}{p}\right), & \text{if } p \equiv 0 \pmod 4, \\ -\exp\left(\frac{(p+1)\pi i}{p}\right), & \text{if } p \equiv 1 \pmod 2. \end{cases}$$

Notice that A_p is a primitive root of unity of order p when p is even and of order $2p$ otherwise.

Remark 1.2. The eigenvalues of a Dehn twist in the TQFT $\mathcal{V}_p$, i.e., the entries of the diagonal T-matrix, are of the form

$$\mu_l = (-A_p)^{l(l+2)},$$

where l belongs to the set of admissible colors (see [2, 4.11]). The set of admissible colors is $\{0, 1, 2, \ldots, \frac{p}{2} - 2\}$ for even p and is $\{0, 2, 4, \ldots, p-3\}$ for odd p. Therefore the order of the image of a Dehn twist by ρ_p is p.

We will now consider the Johnson filtration by the subgroups $I_g(k)$ of the mapping class group M_g of the closed orientable surface of genus g, consisting of those elements having a trivial outer action on the k-th nilpotent quotient of the fundamental group of the surface, for some $k \in \mathbb{Z}_+$. As is well known the Johnson filtration shows up within the framework of finite type invariants of 3-manifolds (see e.g. [11]).

Our next result shows that the image is large in the following sense (see also Propositions 3.2 and 3.5):

Theorem 1.3. *Assume that we have $g \geq 3$ and $p \notin \{3, 4, 8, 12, 16, 24\}$ or $g = 2$ and $p \notin \{3, 4, 8, 12, 16, 24, 40\}$. Then for any k, the image $\rho_p(I_g(k))$ of the k-th Johnson subgroup by the quantum representation ρ_p contains a free non-abelian group.*

The values of p which are excluded correspond to the TQFTs $\mathcal{V}_3, \mathcal{V}_2, \mathcal{V}_4, \mathcal{V}_6, \mathcal{V}_8$ and $\mathcal{V}_{12}$ and it is known (see [23, 32, 33], where the authors determined explicitly the images of quantum representations for $\mathcal{V}_8$ and $\mathcal{V}_{12}$) that the images of quantum representations are finite in these cases. When $g = 2$ and p is odd, $p \geq 7$, our proof is not effective and relies entirely on the result of [21] and Tits' theorem (see Section 3.1). Eventually, the exclusion of the case $g = 2$ and $p = 40$ seems to be an artefact of our method.

The idea of proof for this theorem is to embed a pure braid group within the mapping class group and to show that its image is large. Namely, a 4-holed sphere suitably embedded in the surface leads to an embedding of the pure braid group PB_3 in the mapping class group. The quantum representation contains a particular sub-representation which is the restriction of Burau's representation (see [8]) to a free subgroup of PB_3. One way to obtain elements of the Johnson filtration is to consider elements of the lower central series of PB_3 and extend them to all of the surface by identity. Therefore it suffices to find free non-abelian subgroups in the image of the lower central series of PB_3 by Burau's representation at roots of unity in order to prove Theorem 1.3.

The analysis of the contribution of mapping classes supported on small subsurfaces of a surface, which are usually holed spheres, to various subgroups of the mapping class groups was also used in an unpublished paper by T. Oda and J. Levine (see [22]) for obtaining lower bounds for the ranks of the graded quotients of the Johnson filtration.

Our construction also provides explicit free non-abelian subgroups (see Theorems 3.10 and 3.12 for precise statements).

2 Burau's representations of B_3 and triangle groups

In what follows, let B_n denote the braid group on n strands with the standard generators $g_1, g_2, \dots, g_{n-1}$. Squier was interested to compare the kernel of Burau's representation β_q at a k-th root of unity q with the normal subgroup $\mathrm{B}_n[k]$ of B_n generated by g_j^k, $1 \le j \le n-1$. Recall that:

Definition 2.1. The (reduced) Burau representation $\beta : \mathrm{B}_n \to \mathrm{GL}(n-1, \mathbb{Z}[q, q^{-1}])$ is defined on the standard generators

$$\beta_q(g_1) = \begin{pmatrix} -q & 1 \\ 0 & 1 \end{pmatrix} \oplus \mathbf{1}_{n-3},$$

$$\beta_q(g_j) = \mathbf{1}_{j-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ q & -q & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus \mathbf{1}_{n-j-2}, \quad \text{for } 2 \le j \le n-2,$$

$$\beta_q(g_{n-1}) = \mathbf{1}_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ q & -q \end{pmatrix}.$$

The paper [10] is devoted to the complete description of the image of Burau's representation of B_3 at roots of unity. Similar results were obtained in [20, 24, 26]. For the sake of completeness we review here the essential tools from [10] to be used later.

Let us denote by $A = \beta_{-q}(g_1^2)$ and $B = \beta_{-q}(g_2^2)$ and $C = \beta_{-q}((g_1g_2)^3)$. As is well known PB_3 is isomorphic to the direct product $\mathbb{F}_2 \times \mathbb{Z}$, where $\mathbb{F}_2$ is freely generated by g_1^2 and g_2^2 and the factor $\mathbb{Z}$ is the center of B_3 generated by $(g_1g_2)^3$.

It is simple to check that

$$A = \begin{pmatrix} q^2 & 1+q \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -q-q^2 & q^2 \end{pmatrix}, \quad C = \begin{pmatrix} -q^3 & 0 \\ 0 & -q^3 \end{pmatrix}.$$

Recall that $PSL(2, \mathbb{Z})$ is the quotient of B_3 by its center. Since C is a scalar matrix, it follows that the homomorphism $\beta_{-q} : B_3 \to GL(2, \mathbb{C})$ factors to a homomorphism $PSL(2, \mathbb{Z}) \to PGL(2, \mathbb{C})$.

We will be concerned below with the subgroup Γ_{-q} of $PGL(2, \mathbb{C})$ generated by the images of A and B in $PGL(2, \mathbb{C})$. When β_{-q} is unitarizable, the group Γ_{-q} can be viewed as a subgroup of the complex-unitary group $PU(1, 1)$.

Before we proceed we make a short digression on triangle groups. Let Δ be a geodesic triangle in the hyperbolic plane of angles $\frac{\pi}{m}, \frac{\pi}{n}, \frac{\pi}{p}$, so that $\frac{1}{m} + \frac{1}{n} + \frac{1}{p} < 1$. The extended triangle group $\Delta^*(m, n, p)$ is the group of isometries of the hyperbolic plane generated by the three reflections R_1, R_2, R_3 with respect to the edges of Δ. It is well known that a presentation of $\Delta^*(m, n, p)$ is given by

$$\Delta^*(m, n, p) = \langle R_1, R_2, R_3 \; ; R_1^2 = R_2^2 = R_3^2 = 1, \\ (R_1R_2)^m = (R_2R_3)^n = (R_3R_1)^p = 1 \rangle.$$

The second type of relations have a simple geometric meaning. In fact, the product of the reflections with respect to two adjacent edges is a rotation by the angle which is twice the angle between those edges. The subgroup $\Delta(m, n, p)$ generated by the rotations $a = R_1R_2$, $b = R_2R_3$, $c = R_3R_1$ is a normal subgroup of index 2, which coincides with the subgroup of isometries preserving the orientation. One calls $\Delta(m, n, p)$ the triangle (also called triangular, or von Dyck) group associated to Δ. Moreover, the triangle group has the presentation

$$\Delta(m, n, p) = \langle a, b, c \; ; \; a^m = b^n = c^p = 1, \; abc = 1 \rangle.$$

Observe that $\Delta(m, n, p)$ also makes sense when m, n or p are negative integers, by interpreting the associated generators as clockwise rotations. The triangle Δ is a fundamental domain for the action of $\Delta^*(m, n, p)$ on the hyperbolic plane. Thus a fundamental domain for $\Delta(m, n, p)$ consists of the union Δ^* of Δ with the reflection of Δ in one of its edges.

Proposition 2.2 ([10]). *Let $m < k$ be such that $\gcd(m,k) = 1$ where $k \geq 4$. Then the group $\Gamma_{-\exp(\frac{\pm 2m\pi i}{2k})}$ is a triangle group with the presentation*

$$\Gamma_{-\exp(\frac{\pm 2m\pi i}{2k})} = \langle A, B \; ; \; A^k = B^k = (AB)^k = 1 \rangle.$$

If n is odd $n = 2k+1$, then the group Γ_{-q} is a quotient of the triangle group associated to Δ, which embeds into the group associated to some sub-triangle Δ' of Δ.

Proposition 2.3 ([10]). *Let $0 < m < 2k+1$ be such that $\gcd(m, 2k+1) = 1$ and $k \geq 3$. Then $\Gamma_{-\exp(\frac{\pm 2m\pi i}{2k+1})}$ is isomorphic to the triangle group $\Delta(2,3,2k+1)$ and has the following presentation (in terms of our generators A, B):*

$$\Gamma_{-\exp(\frac{\pm 2m\pi i}{2k+1})} = \langle A, B \; ; A^{2k+1} = B^{2k+1} = (AB)^{2k+1} = 1, \\ (A^{-1}B^k)^2 = 1, \; (B^k A^{k-1})^3 = 1 \rangle.$$

Proof. Here is a sketch of the proof. Deraux proved in [5, Theorem 7.1] that the group $\Delta(\frac{2k+1}{2}, \frac{2k+1}{2}, \frac{2k+1}{2})$, which is generated by the rotations a, b, c around the vertices of the triangle Δ, embeds into the triangle group associated to a smaller triangle Δ'. One constructs Δ' by considering all geodesics of Δ joining a vertex and the midpoint of its opposite side. The three median geodesics pass through the barycenter of Δ and subdivide Δ into six equal triangles. We can take for Δ' any one of the six triangles of the subdivision. It is immediate that Δ' has angles $\frac{\pi}{2k+1}$, $\frac{\pi}{2}$ and $\frac{\pi}{3}$ so that the associated triangle group is $\Delta(2,3,2k+1)$. This group has the presentation

$$\Delta(2,3,2k+1) = \langle \alpha, u, v \; ; \; \alpha^{2k+1} = u^3 = v^2 = \alpha u v = 1 \rangle,$$

where the generators are the rotations of double angle around the vertices of the triangle Δ'.

Lemma 2.4. *The natural embedding of $\Delta(\frac{2k+1}{2}, \frac{2k+1}{2}, \frac{2k+1}{2})$ into $\Delta(2,3,2k+1)$ is an isomorphism.*

Proof. A simple geometric computation shows that

$$a = \alpha^2, \quad b = v\alpha^2 v = u^2\alpha^2 u, \quad c = u\alpha^2 u^2.$$

Therefore

$$\alpha = a^{k+1} \in \Delta\left(\frac{2k+1}{2}, \frac{2k+1}{2}, \frac{2k+1}{2}\right).$$

From the relation $\alpha u v = 1$ we derive $a^{k+1}uv = 1$, and thus $u = a^k v$. The relation $u^3 = 1$ reads now $a^k(va^k v)a^k v = 1$ and replacing b^k by $va^k v$ we find that $v = a^k b^k a^k \in \Delta(\frac{2k+1}{2}, \frac{2k+1}{2}, \frac{2k+1}{2})$.

Further

$$u = a^k v = a^{-1} b^k a^k \in \Delta\left(\frac{2k+1}{2}, \frac{2k+1}{2}, \frac{2k+1}{2}\right).$$

This means that $\Delta(\frac{2k+1}{2}, \frac{2k+1}{2}, \frac{2k+1}{2})$ is actually $\Delta(2, 3, 2k+1)$, as claimed. □

It suffices now to find a presentation of $\Delta(2, 3, 2k+1)$ that uses the generators $A = a$, $B = b$. It is not difficult to show that the group with the presentation of the statement is isomorphic to $\Delta(2, 3, 2k+1)$, the inverse homomorphism sending α into A^{k+1}, u into $A^{-1}B^kA^k$ and v into $A^kB^kA^k$. □

A direct consequence of Propositions 2.2 and 2.3 is the following abstract description of the image of Burau's representation:

Corollary 2.5. *If q is not a primitive root of unity of order in $\{1, 2, 3, 4, 6, 10\}$, then Γ_q is an infinite triangle group.*

Alternatively, we obtain a set of normal generators for the kernel of Burau's representation, as follows:

Corollary 2.6. *Let $n \notin \{1, 6\}$ and let q be a primitive root of unity of order n. We denote by $N(G)$ the normal closure of a subgroup G of $\langle g_1^2, g_2^2\rangle$. Then the kernel* $\ker \beta_{-q} : \langle g_1^2, g_2^2\rangle \to \mathrm{PGL}(2, \mathbb{C})$ *of the restriction of Burau's representation is given by*

$$\begin{cases} N(\langle g_1^{2k}, g_2^{2k}, (g_1^2g_2^2)^k\rangle), & n = 2k, \\ N(\langle g_1^{2(2k+1)}, g_2^{2(2k+1)}, (g_1^2g_2^2)^{2k+1}, (g_1^{-2}g_2^{2k})^2, (g_2^{2k}g_1^{2(k-1)})^3\rangle), & n = 2k+1. \end{cases}$$

3 Johnson subgroups and proof of Theorem 1.3

3.1 Proof of Theorem 1.3

For a group G we denote by $G_{(k)}$ the lower central series defined by

$$G_{(1)} = G, \quad G_{(k+1)} = [G, G_{(k)}],\ k \geq 1.$$

An interesting family of subgroups of the mapping class group is the set of higher Johnson subgroups defined as follows.

Definition 3.1. The k-th Johnson subgroup $I_g(k)$ is the group of mapping classes of homeomorphisms of the closed orientable surface Σ_g whose action by outer automorphisms on $\pi/\pi_{(k+1)}$ is trivial, where $\pi = \pi_1(\Sigma_g)$.

Thus $I_g(0) = M_g$, $I_g(1)$ is the Torelli group commonly denoted by T_g, while $I_g(2)$ is the group generated by the Dehn twists along separating simple closed curves and considered by Johnson and Morita (see e.g. [16, 27]), which is often denoted by K_g.

The proof of Theorem 1.3 follows from the same argument as in [8], where we proved that the image of the quantum representation ρ_p is infinite for all p in the given range.

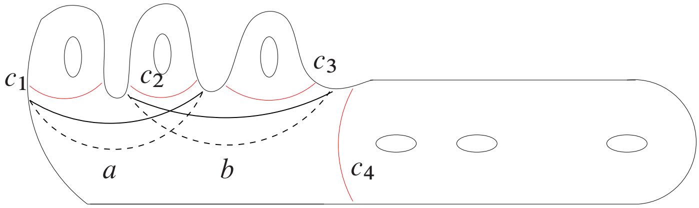
Before we proceed, we have to make the cautionary remark that ρ_p is only a projective representation. Here and henceforth when speaking about Burau's representation we will mean the representation $\beta_q : \mathrm{B}_3 \to \mathrm{PGL}(2, \mathbb{C})$ taking values in matrices modulo scalars.

We will first consider the generic case where the genus is large and the 10-th roots of unity are discarded. This will prove Theorem 1.3 in most cases. Specifically we will prove first:

Proposition 3.2. *Assume that $g \geq 4$. Then the image $\rho_p\big((\langle g_1^2, g_2^2\rangle)_{(k)}\big)$ contains a free non-abelian group for every k and $p \notin \{3, 4, 8, 12, 16, 24, 40\}$.*

Proof. The first step of the proof provides us with enough elements of $I_g(k)$ having their support contained in a small subsurface of Σ_g.

Specifically we embed $\Sigma_{0,4}$ into Σ_g by means of curves c_1, c_2, c_3, c_4 as in the figure below. Then the curves a and b which are surrounding two of the holes of $\Sigma_{0,4}$ are separating.



The pure braid group PB_3 embeds into $M_{0,4}$ using a non-canonical splitting of the surjection $M_{0,4} \to \mathrm{PB}_3$. Furthermore, $M_{0,4}$ embeds into M_g when $g \geq 4$, by using the homomorphism induced by the inclusion of $\Sigma_{0,4}$ into Σ_g as in the figure. Then the group generated by the Dehn twists a and b is identified with the free subgroup generated by g_1^2 and g_2^2 into PB_3. Moreover, PB_3 has a natural action on a subspace of the space of conformal blocks associated to Σ_g as in [8], which is isomorphic to the restriction of Burau's representation at some root of unity depending on p. Notice that the two Dehn twists above are elements of K_g.

We will need the following proposition whose proof will be given in Section 3.1.

Proposition 3.3. *The above embedding of* PB_3 *into* M_g *sends* $(\mathrm{PB}_3)_{(k)}$ *into* $I_g(k)$.

Recall now that $\langle g_1^2, g_2^2 \rangle$ is a normal free subgroup of PB_3. The second ingredient needed in the proof of Proposition 3.2 is the following proposition which will be proved in Section 3.1:

Proposition 3.4. *Assume that $g \geq 4$. Then the image $\rho_p\big((\langle g_1^2, g_2^2 \rangle)_{(k)}\big)$ contains a free non-abelian group for every k and $p \notin \{3, 4, 8, 12, 16, 24, 40\}$.*

Thus the group $\rho_p\big((\mathrm{PB}_3)_{(k)}\big)$ contains $\rho_p\big((\langle g_1^2, g_2^2 \rangle)_{(k)}\big)$ and so it also contains a free non-abelian group. Therefore, Proposition 3.3 above implies that $\rho_p(I_g(k))$ contains a free non-abelian subgroup, which will complete the proof of Proposition 3.2. □

We further consider the remaining cases and briefly outline in Section 3.2 the modifications needed to make the same strategy work also for small genus surfaces and for those values of the parameter p which were excluded above, namely:

Proposition 3.5. *Assume that g and p verify one of the following conditions:*

(i) *$g = 2$, p is even and $p \notin \{4, 8, 12, 16, 24, 40\}$,*

(ii) *$g = 3$ and $p \notin \{3, 4, 8, 12, 16, 24, 40\}$,*

(iii) *$g \geq 4$ and $p = 40$.*

Then $\rho_p(M_g)$ contains a free non-abelian group.

Then Propositions 3.2 and 3.5 above will prove Theorem 1.3.

Proof of Proposition 3.3

Choose the base point $*$ for the fundamental group $\pi_1(\Sigma_g)$ on the circle c_4 that separates the sub-surfaces $\Sigma_{3,1}$ and $\Sigma_{g-3,1}$. Let φ be a homeomorphism of $\Sigma_{0,4}$ that is the identity on the boundary and whose mapping class b belongs to the group $\mathrm{PB}_3 \subset M_{0,4}$. Consider its extension $\widetilde{\varphi}$ to Σ_g by identity outside $\Sigma_{0,4}$. Its mapping class B in M_g is the image of b in M_g.

In order to understand the action of B on $\pi_1(\Sigma_g)$ we introduce three kinds of loops based at $*$:

(i) Loops of type I are those included in $\Sigma_{g-3,1}$.

(ii) Loops of type II are those contained in $\Sigma_{0,4}$.

(iii) Begin by fixing three simple arcs $\lambda_1, \lambda_2, \lambda_3$ embedded in $\Sigma_{0,4}$ joining $*$ to the three other boundary components c_1, c_2 and c_3, respectively. Loops of type III are of the form $\lambda_i^{-1} x \lambda_i$, where x is some loop based at the endpoint of λ_i and contained in the 1-holed torus bounded by c_i. Thus loops of type III generate $\pi_1(\Sigma_{3,1}, *)$.

Now, the action of B on the homotopy classes of loops of type I is trivial. The action of B on the homotopy classes of loops of type II is completely described by the action of $b \in \mathrm{PB}_3$ on $\pi_1(\Sigma_{0,4}, *)$. Specifically, let $A : \mathrm{B}_3 \to \mathrm{Aut}(\mathbb{F}_3)$ be the Artin representation (see [1]). Here $\mathbb{F}_3$ is the free group on three generators x_1, x_2, x_3 which is identified with the fundamental group of the 3-holed disk $\Sigma_{0,4}$.

Lemma 3.6. *If $b \in (\mathrm{PB}_3)_{(k)}$, then*

$$A(b)(x_i) = l_i(b)^{-1} x_i l_i(b), \quad \text{where } l_i(b) \in (\mathbb{F}_3)_{(k)}.$$

Proof. This statement is folklore. Moreover, it is valid for any number n of strands instead of 3. Here is a short proof avoiding heavy computations. It is known that the set $\mathrm{PB}_{n,k}$ of those pure braids b for which the length m Milnor invariants of their Artin closures vanish for all $m \leq k$ is a normal subgroup $\mathrm{PB}_{n,k}$ of B_n. Furthermore, the central series of subgroups $\mathrm{PB}_{n,k}$ verifies (see e.g. [29])

$$[\mathrm{PB}_{n,k}, \mathrm{PB}_{n,m}] \subset \mathrm{PB}_{n,k+m}, \quad \text{for all } n, k, m,$$

and hence, we have $(\mathrm{PB}_n)_{(k)} \subset \mathrm{PB}_{n,k}$.

Now, if b is a pure braid, then $A(b)(x_i) = l_i(b)^{-1} x_i l_i(b)$, where $l_i(b)$ is the so-called *longitude* of the i-th strand. Next we can interpret Milnor invariants as coefficients of the Magnus expansion of the longitudes. In particular, this correspondence shows that $b \in \mathrm{PB}_{n,k}$ if and only if $l_i(b) \in (\mathbb{F}_n)_{(k)}$. This proves the claim. □

The action of B on the homotopy classes of loops of type III can be described in a similar way. Let a homotopy class a of this kind be represented by a loop $\lambda_i^{-1} x \lambda_i$. Then $\lambda_i^{-1} \varphi(\lambda_i)$ is a loop contained in the surface $\Sigma_{0,4}$, whose homotopy class $\eta_i = \eta_i(b)$ depends only on b and λ_i. Then it is easy to see that

$$B(a) = \eta_i^{-1} a \eta_i.$$

Let now y_i and z_i be standard homotopy classes of loops based at a point of c_i which generate the fundamental group of the holed torus bounded by c_i, so that $\{y_1, z_1, y_2, z_2, y_3, z_3\}$ is a generator system for $\pi_1(\Sigma_{3,1}, *)$, which is the free group $\mathbb{F}_6$ of rank 6.

Lemma 3.7. *If $b \in (\mathrm{PB}_3)_{(k)}$, then $\eta_i(b) \in (\mathbb{F}_6)_{(2k)}$.*

Proof. It suffices to observe that $\eta_i(b)$ is actually the i-th longitude $l_i(b)$ of the braid b, expressed now in the generators y_i and z_i instead of the generators x_i. We also know that $x_i = [y_i, z_i]$. Let then $\eta : \mathbb{F}_3 \to \mathbb{F}_6$ be the group homomorphism given on the generators by $\eta(x_i) = [y_i, z_i]$. Then $\eta_i(b) = \eta(l_i(b))$. Eventually, if $l_i(b) \in (\mathbb{F}_3)_{(k)}$, then $\eta(l_i(b)) \in (\mathbb{F}_6)_{(2k)}$ and the claim follows. □

Therefore the class B belongs to $I_g(k)$, since its action on every generator of $\pi_1(\Sigma_g, *)$ is a conjugation by an element of $\pi_1(\Sigma_g, *)_{(k)}$.

Proof of Proposition 3.4

First we want to identify some sub-representation of the restriction of ρ_p to the group $\mathrm{PB}_3 \subset M_g$. Specifically we have:

Lemma 3.8. *Let* $p \geq 5$. *The restriction of the quantum representation* ρ_p *to the group* $\mathrm{PB}_3 \subset M_{0,4}$ *has an invariant* 2*-dimensional subspace such that the corresponding sub-representation is equivalent to the Burau representation* β_{-q_p}, *where the root of unity* q_p *is given by*

$$q_p = \begin{cases} -A_p^{-4} = -\exp\left(-\frac{8\pi i}{p}\right), & \text{if } p \equiv 0 \pmod 4, \\ -A_5^{-4} = -\exp\left(-\frac{4\pi i}{5}\right), & \text{if } p = 5, \\ -A_p^{-8} = -\exp\left(-\frac{8(p+1)\pi i}{p}\right), & \text{if } p \equiv 1 \pmod 2,\ p \geq 7. \end{cases}$$

Proof. For even p this is the content of [8, Proposition 3.2]. We recall that in this case the invariant 2-dimensional subspace is the space of conformal blocks associated to the surface $\Sigma_{0,4}$ with all boundary components being labeled by the color 1. The odd case is similar. The invariant subspace is the space of conformal blocks associated to the surface $\Sigma_{0,4}$ with boundary labels $(2, 2, 2, 2)$, when $p = 5$ and $(4, 2, 2, 2)$, when $p \geq 7$ respectively. The eigenvalues of the half-twist can be computed as in [8]. □

Thus the image $\rho_p(\mathrm{PB}_3)$ of the quantum representation projects onto the image of the Burau representation $\beta_{-q_p}(\mathrm{PB}_3)$.

Up to a Galois conjugacy we can assume that β_{-q_p} is unitarizable and after rescaling, it takes values in U(2). Consider the projection of $\beta_{-q_p}\big((\mathrm{PB}_3)_{(k)}\big)$ into $\mathrm{U}(2)/\mathrm{U}(1) = \mathrm{SO}(3)$.

A finitely generated subgroup of SO(3) is either finite or abelian or else dense in SO(3). If the group is dense in SO(3), then it contains a free non-abelian subgroup. Moreover, solvable subgroups of SU(2) (and hence of SO(3)) are abelian. The finite subgroups of SO(3) are well known. They are the following: cyclic groups, dihedral groups, tetrahedral group (automorphisms of the regular tetrahedron), the octahedral group (the group of automorphisms of the regular octahedron) and the icosahedral group (the group of automorphisms of the regular icosahedron or dodecahedron). All but the last one are actually solvable groups. The icosahedral group is isomorphic to the alternating group A_5 and it is well known that it is simple (and thus non-solvable). As a side remark this group appeared in relation with the non-solvability of the quintic equation in Felix Klein's monograph [18].

Lemma 3.9. *If q is not a primitive root of unity of order in the set $\{1, 2, 3, 4, 6, 10\}$, then $(\Gamma_q)_{(k)}$ is non-solvable and thus non-abelian for any k. Moreover, $(\Gamma_q)_{(k)}$ cannot be A_5, for any k.*

Proof. If $(\Gamma_q)_{(k)}$ were solvable, then Γ_q would be solvable. But one knows that Γ_q is not solvable. In fact if q is as above, then Γ_q is an infinite triangle group by Corollary 2.5.

Now any infinite triangle group has a finite index subgroup which is a surface group of genus at least 2. Therefore, each term of the lower central series of that surface group embeds into the corresponding term of the lower central series of Γ_q, so that the latter is non-trivial. Since the lower central series of a surface group of genus at least 2 consists only of infinite groups, it follows that no term can be isomorphic to the finite group A_5 either. □

Lemma 3.9 shows that whenever p is as in the statement of Proposition 3.4, the group $\beta_{-q_p}\left((\langle g_1^2, g_2^2\rangle)_{(k)}\right)$ is neither finite nor abelian, so that it is dense in SO(3) and hence it contains a free non-abelian group. This proves Proposition 3.4.

Explicit free subgroups

The main interest of the elementary arguments in the proof presented above is that the free non-abelian subgroups in the image are abundant and explicit. For instance we have:

Theorem 3.10. *Assume that $g \geq 4$, $p \notin \{3, 4, 12, 16\}$ and $p \not\equiv 8 \pmod{16}$. Set $x = \rho_p([g_1^2, g_2^2])$ and $y = \rho_p([g_1^4, g_2^2])$. Then the group generated by the iterated commutators*

$$[x, [x, [x, \ldots, [x, y]] \ldots] \quad \textit{and} \quad [y, [x, [x, \ldots, [x, y]] \ldots]$$

of length $k \geq 3$ is a free non-abelian subgroup of $\rho_p(I_g(k))$.

It is well known that the order of the matrix $\beta_{-q}(g_i)$, $i \in \{1, 2\}$, in $PGL(2, \mathbb{C})$ is the order of the root of unity q, namely the smallest positive n such that q is a primitive root of unity of order n.

We considered in Lemma 3.8 the root of unity q_p with the property that β_{-q_p} is a sub-representation of the quantum representation ρ_p. We derive from Lemma 3.8 that the order of the root of unity q_p is $2o(p)$ where

$$o(p) = \begin{cases} \frac{p}{4}, & \text{if } p \equiv 4 \pmod 8, \\ \frac{p}{8}, & \text{if } p \equiv 0 \pmod{16}, \\ \frac{p}{16}, & \text{if } p \equiv 8 \pmod{16}, \\ p, & \text{if } p \equiv 1 \pmod 2,\ p \geq 5. \end{cases}$$

Therefore $\beta_{-q_p}(\langle g_1^2, g_2^2\rangle)$ is isomorphic to the triangle group $\Delta(o(p), o(p), o(p))$. Notice that in general $o(p) \in \frac{1}{2} + \mathbb{Z}$ and $o(p)$ is an integer if and only if p has the property that $p \not\equiv 8 \pmod{16}$, as we suppose from now on. Observe also that the order of $\beta_{-q_p}(g_1^2)$ is a proper divisor of the order p of a Dehn twist $\rho_p(g_1^2)$, when p is even.

In the proof of Theorem 3.10 we will need the following result concerning the structure of commutator subgroups of triangle groups:

Lemma 3.11. *The commutator subgroup $\Delta(r,r,r)_{(2)}$ of a triangle group $\Delta(r,r,r)$, $r \in \mathbb{Z} - \{0,1,2\}$, is a 1-relator group with generators $\widetilde{c_{ij}}$, for $1 \le i, j \le r-1$, and the relation*

$$\widetilde{c_{11}} \cdot \widetilde{c_{21}}^{-1} \cdot \widetilde{c_{22}} \cdot \widetilde{c_{32}}^{-1} \cdots \widetilde{c_{r\,r-1}}^{-1} \cdot \widetilde{c_{rr}} = 1.$$

Proof. The kernel K of the abelianization homomorphism

$$\mathbb{Z}/r\mathbb{Z} * \mathbb{Z}/r\mathbb{Z} \to \mathbb{Z}/r\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z}$$

is the free group generated by the commutators. Denote by $\widetilde{a}$ and $\widetilde{b}$ the generators of the two copies of the cyclic group $\mathbb{Z}/r\mathbb{Z}$. Then the kernel K is freely generated by $\widetilde{c_{ij}} = [\widetilde{a}^i, \widetilde{b}^j]$, where $1 \le i, j \le r-1$. The group $\Delta(r,r,r)$ is the quotient of $\mathbb{Z}/r\mathbb{Z} * \mathbb{Z}/r\mathbb{Z}$ by the normal subgroup generated by the element $(\widetilde{ab})^r \widetilde{a}^{-r} \widetilde{b}^{-r}$, which belongs to K. This shows that $\Delta(r,r,r)_{(2)}$ is a 1-relator group, namely the quotient of K by the normal subgroup generated by the element $(\widetilde{ab})^r \widetilde{a}^{-r} \widetilde{b}^{-r}$. In order to get the explicit form of the relation we have to express this element as a product of the generators of K, i.e., as a product of commutators of the form $[\widetilde{a}^i, \widetilde{b}^j]$. This can be done as follows:

$$(\widetilde{ab})^r \widetilde{a}^{-r} \widetilde{b}^{-r} = [\widetilde{a}, \widetilde{b}][\widetilde{b}, \widetilde{a}^2][\widetilde{a}^2, \widetilde{b}^2] \cdots [\widetilde{a}^{r-1}, \widetilde{b}^{r-1}][\widetilde{b}^{r-1}, \widetilde{a}^r][\widetilde{a}^r, \widetilde{b}^r].$$

Hence $\Delta(r,r,r)_{(2)}$ has a presentation with generators $\widetilde{c_{ij}}$, where $1 \le i \le j \le r$, and the relation in the statement of the lemma holds. □

Proof of Theorem 3.10. Recall the classical Magnus Freiheitsatz, which states that any subgroup of a 1-relator group which is generated by a proper subset of the set of generators involved in the cyclically reduced word relator is free.

Assume that $o(p) \in \mathbb{Z}$ and $o(p) \ge 4$. Then $\beta_{-q_p}([g_1^2, g_2^2])$ and $\beta_{-q_p}([g_1^4, g_2^2])$ are the elements $\widetilde{c_{11}}$ and $\widetilde{c_{21}}$ of $\Delta(o(p), o(p), o(p))_{(2)}$ respectively.

An easy application of the Freiheitsatz to the commutator subgroup of the infinite triangle group $\Delta(o(p), o(p), o(p))$ gives us that the subgroup generated by $\beta_{-q_p}([g_1^2, g_2^2])$ and $\beta_{-q_p}([g_1^4, g_2^2])$ is free. This implies that the subgroup generated by x and y is free.

Eventually the k-th term of the lower central series of the group generated by x and y is also a free subgroup which is contained in $\rho_p\big((\mathrm{PB}_3)_{(k)}\big) \subset \rho_p(I_g(k))$. This proves Theorem 3.10. $\square$

When $p \equiv 8 \pmod{16}$, $o(p)$ is a half-integer and $\beta_{-q_p}(\langle g_1^2, g_2^2\rangle)$ is isomorphic to the triangle group $\Delta(2,3,2o(p))$. If $\gcd(3, 2o(p)) = 1$, then

$$H_1(\Delta(2,3,2o(p))) = 0,$$

so the central series of this triangle group is trivial. Nevertheless $\Delta(2,3,2o(p))$ has many normal subgroups of finite index which are surface groups and thus contain free subgroups. In particular, any subgroup of infinite index of $\Delta(2,3,2o(p))$ is free. There is then an extension of the previous result in this case, as follows:

Theorem 3.12. *Assume that $g \geq 4$, $p \notin \{8, 24, 40\}$ and $p \equiv 8 \pmod{16}$ so that $p = 8n$, for odd $n = 2k+1 \geq 7$. Consider the following two elements of $\langle g_1^2, g_2^2\rangle$:*

$$s = g_1^{2k} g_2^{2k} g_1^{2(k-k^2)} g_2^{-2} g_1^{2k} g_2^{-2} g_1^{2(k-k^2)} g_2^{2k} g_1^{2k},$$

and

$$t = g_1^{2k} g_2^{2k} g_1^{2(k-k^2)} g_2^{-2} g_1^{2(k+1)} g_2^2 g_1^{2(k+k^2)} g_2^{2k} g_1^{10k}.$$

Let $N(s,t)$ be the normal subgroup generated by the elements s and t in $\langle g_1^2, g_2^2\rangle$. Then for any choice of $f(n)$ elements $x_1, x_2, \ldots, x_{f(n)}$ from the subgroup $N(s,t)$ the image $\rho_p(\langle x_1, x_2, \ldots, x_{f(n)}\rangle)$ is a free group. Here the function $f(n)$ is given by

$$f(n) = |\mathrm{PSL}(2, \mathbb{Z}/n\mathbb{Z})| \cdot \frac{n-6}{6n}$$

and, in particular, when n is prime, by

$$f(n) = \frac{(n+1)(n-1)(n-6)}{12}.$$

Then the group generated by the iterated commutators of length $k \geq 3$ is a free subgroup of $\rho_p(I_g(k))$.

Proof. Observe that the map $\mathrm{PSL}(2,\mathbb{Z}) \to \mathrm{PSL}(2, \mathbb{Z}/n\mathbb{Z})$ factors through the triangular group $\Delta(2,3,n)$, namely we have a homomorphism

$$\psi : \Delta(2,3,n) \to \mathrm{PSL}(2, \mathbb{Z}/n\mathbb{Z})$$

defined by

$$\psi(\alpha) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \psi(u) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \psi(v) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The matrices $\psi(\alpha)$, $\psi(u)$ and $\psi(v)$ are obviously elements of orders n, 3 and 2 in $\mathrm{PSL}(2, \mathbb{Z}/n\mathbb{Z})$ respectively. Then the normal subgroup $K(2,3,n) = \ker\psi$ is torsion free, because every torsion element in $\Delta(2,3,n)$ is conjugate to some power of one the generators α, u, v (see [13]). Therefore $K(2,3,n)$ is a surface group, namely the fundamental group of a closed orientable surface which finitely covers the fundamental domain of $\Delta(2,3,n)$. The Euler characteristic $\chi(K(2,3,n))$ of this Fuchsian group can easily be computed by means of the formula

$$\chi(K(2,3,n)) = |\mathrm{PSL}(2, \mathbb{Z}/n\mathbb{Z})| \cdot \chi(\Delta(2,3,n)),$$

where the (orbifold) Euler characteristic $\chi(\Delta(2,3,n))$ has the well-known expression

$$-\chi(\Delta(2,3,n)) = 1 - \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{n}\right) = \frac{n-6}{6n}.$$

It is also known that any $-\chi(G) + 1$ elements of a closed orientable surface group G generate a free subgroup of G. Thus, in order to establish Theorem 3.12, it will suffice to show that the images of the elements s and t under Burau's representation β_{-q_p} are normal generators of the group $K(2,3,n)$. This is equivalent to showing that these images correspond to the relations needed to impose in $\Delta(2,3,n)$ in order to obtain the quotient $\mathrm{PSL}(2, \mathbb{Z}/n\mathbb{Z})$. However, one already knows presentations for this group (see [4, Lemma 1] and [14]) as follows:

$$\mathrm{PSL}(2, \mathbb{Z}/n\mathbb{Z}) = \langle \alpha, v, u\ ;\ \alpha^n = u^3 = v^2 = 1,\ gvgv = g\alpha g^{-1}\alpha^{-4} = 1 \rangle,$$

for odd n, where $g = v\alpha^k v\alpha^{-2} v\alpha^k$. The first three relations above correspond to the presentation of $\Delta(2,3,n)$ and the elements $gvgv$ and $g\alpha g\alpha^{-4}$ correspond to the images of s and t in $\Delta(2,3,n)$, by using the fact that $\alpha = a^{k+1}$, $v\alpha^2 v = b$, $v = a^k b^k a^k$ (see the proof of Lemma 2.4). □

Second proof of Proposition 3.2

We outline here an alternative proof which does not rely on the description of the image of Burau's representation in Corollary 2.5. This proof is shorter but less effective since it does not produce explicit free subgroups and uses the result of [21] and the Tits alternative, which need more sophisticated tools from the theory of algebraic groups. On the other hand this proof also works for odd p and $g = 2$.

The image $\rho_p(M_g)$ in $\mathrm{PU}(N(p,g))$ is dense in $\mathrm{PSU}(N(p,g))$ if $p \geq 5$ is prime (see [21]), where $N(p,g)$ denotes the dimension of the space of conformal blocks in genus g for the TQFT $\mathcal{V}_p$. In particular, the image of the representation ρ_p is Zariski dense in $\mathrm{PU}(N(p,g))$. By the Tits alternative (see [31]) the image is either solvable or else it contains a free non-abelian subgroup. However, if the image were solvable, then its Zariski closure would be a solvable Lie group, which is a contradiction. This implies that $\rho_p(M_g)$ contains a free non-abelian subgroup.

If p is not prime but has a prime factor $r \geq 5$, then the claim for p follows from that for r. If p does not satisfy this condition, then we have again to use the result of Proposition 3.4 for $k = 1$. This result can be obtained directly from the computations in [17] proving that the image of the Jones representation of B_3 is neither finite nor abelian for the considered values of p. This settles the case $k = 1$ of Proposition 3.2.

Furthermore, the group $\rho_p(T_g)$ is of finite index in $\rho_p(M_g)$, and hence it also contains a free non-abelian subgroup. Results of Morita (see [28]) show that for any given $k \geq 2$ the group $I_g(k+1)$ is the kernel of the k-th Johnson homomorphism $I_g(k) \to A_k$, where A_k is a finitely generated abelian group. This implies that $[I_g(k), I_g(k)] \subset I_g(k+1)$, for every $k \geq 2$. In particular, the k-th term of the derived series of $\rho_p(T_g)$ is contained into $\rho_p(I_g(k+1))$. But every term of the derived series of $\rho_p(T_g)$ contains the corresponding term of the derived series of a free subgroup and hence a free non-abelian group. This proves Proposition 3.2.

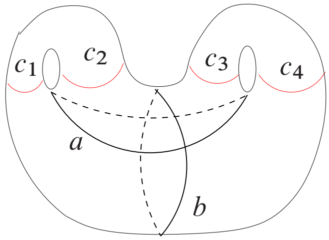
Remark 3.13. Using the strong version of Tits' theorem due to Breuillard and Gelander (see [3]) there exists a free non-abelian subgroup of $M_g/M_g[p]$ whose image in $\mathrm{PSU}(N(p,g))$ is dense. Here $M_g[p]$ denotes the (normal) subgroup generated by the p-th powers of Dehn twists.

3.2 Proof of Proposition 3.5

If the genus $g \in \{2, 3\}$, then the construction used in the proof of Proposition 3.2 should be modified. This is equally valid when we want to get rid of the values $p = 5$ and $p = 40$.

The proof follows along the same lines as Proposition 3.4, but the embeddings $\Sigma_{0,4} \subset \Sigma_g$ are now different. In all cases considered below the analogue of Proposition 3.3 will still be true, i.e., the image of the subgroup $(\langle g_1^2, g_2^2 \rangle)_{(k)}$ by the homomorphisms $M_{0,4} \to M_g$ will be contained within the Johnson subgroup $I_g(k)$.

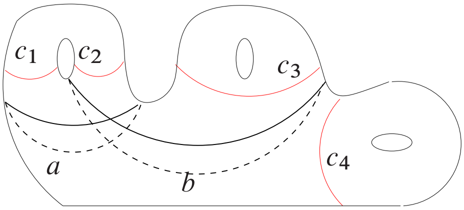
If $g = 2$, we use the following embedding $\Sigma_{0,4} \subset \Sigma_2$.



Although the homomorphism $M_{0,4} \to M_2$ induced by this embedding is not anymore injective, it sends the free subgroup $\langle g_1^2, g_2^2 \rangle \subset \mathrm{PB}_3 \subset M_{0,4}$ isomorphically onto the subgroup of M_2 generated by the Dehn twists along the curves a and b in the figure above.

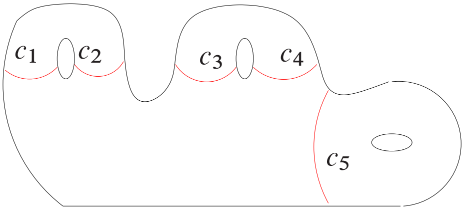
Consider for even p the space of conformal blocks associated to $\Sigma_{0,4}$ with boundary labels $(1,1,1,1)$. This 2-dimensional subspace is $\rho_p(\langle g_1^2, g_2^2 \rangle)$-invariant and the restriction of ρ_p to this subspace is still equivalent to Burau's representation β_{-q_p} (see [8]). Therefore Proposition 3.4 shows that $\rho_p\big((\langle g_1^2, g_2^2 \rangle)_{(k)}\big)$, and hence also $\rho_p(I_2(k))$, contains a free non-abelian group.

If $g = 3$ and $p \geq 7$ is odd, we consider the following embedding of $\Sigma_{0,4} \subset M_3$.



The homomorphism $M_{0,4} \to M_3$ induced by this embedding is not injective, but it also sends the free subgroup $\langle g_1^2, g_2^2 \rangle \subset \mathrm{PB}_3 \subset M_{0,4}$ isomorphically onto the subgroup of M_3 generated by the Dehn twists along the curves a and b in the figure above. The space of conformal blocks associated to $\Sigma_{0,4}$ with boundary labels $(2,2,2,4)$ is a 2-dimensional subspace invariant by $\rho_p(\langle g_1^2, g_2^2 \rangle)$ and the restriction of ρ_p to this subspace is equivalent to Burau's representation β_{-q_p}. Applying again Proposition 3.4, we find that $\rho_p\big((\langle g_1^2, g_2^2 \rangle)_{(k)}\big)$, and hence $\rho_p(I_3(k))$, contains a free non-abelian group. This also gives the desired results for any $g \geq 3$, and p as in the statement.

Eventually we have to settle the case $p = 40$ when $\beta_{-q_p}(\mathrm{B}_3)$ is known to have finite image (see [17]). We will consider instead the representation $\rho_p(i(\mathrm{PB}_4))$, where PB_4 embeds non-canonically into $M_{0,5}$ and $M_{0,5}$ maps into M_3 by the homomorphism $i : M_{0,5} \to M_3$ induced by the inclusion $\Sigma_{0,5} \subset \Sigma_g$ drawn below.



We consider the 3-dimensional space of conformal blocks associated to $\Sigma_{0,5}$ with the boundary labels $(1,1,1,1,2)$ when $p = 40$ and the labels $(2,2,2,2,2)$

when $p = 5$ respectively. This space of conformal blocks is $\rho_p(i(\mathrm{PB}_4))$-invariant. The restriction of $\rho_p|_{\mathrm{PB}_4}$ to this invariant subspace is known (see again [8]) to be equivalent to the Jones representation of B_4 at the corresponding root of unity.

Now we have to use a result of Freedman, Larsen and Wang (see[7]) subsequently reproved and extended by Kuperberg in [20, Theorem 1] saying that the Jones representation of B_4 at a 10-th root of unity on the two 3-dimensional conformal blocks we chose is Zariski dense in the group $\mathrm{SL}(3,\mathbb{C})$. A particular case of the Tits alternative says that any finitely generated subgroup of $\mathrm{SL}(3,\mathbb{C})$ is either solvable or else contains a free non-abelian group. A solvable subgroup has also a solvable Zariski closure. The denseness result from above implies then that $\rho_p(\mathrm{PB}_4)$, and hence also $\rho_p\big((\mathrm{PB}_4)_{(k)}\big)$, contains a free non-abelian group. The arguments in the proof of Proposition 3.4 carry on to this setting and this proves Proposition 3.5.

Corollary 3.14. *For any k the quotient group $I_g(k)/M_g[p] \cap I_g(k)$, and in particular, $K_g/K_g[p]$, for $g \geq 3$ and $p \notin \{3,4,8,12,16,24\}$, contains a free non-abelian subgroup. Here $K_g[p]$ denotes the normal subgroup of K_g generated by the p-th powers of the Dehn twist along separating simple closed curves.*

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