Ptolemy groupoids actions on Teichmüller spaces

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The aim of this paper is to present some ingredients necessary for the quantization of Teichmüller spaces (see [5, 7, 10]), from an elementary perspective. We refer to ([4, 1, 6, 12]) for the most recent advances concerning this subject.

1 Thurston-Bonahon-Penner-Fock coordinates on the Teichmüller spaces

1.1 Preliminaries on fatgraphs

Let Γ be a finite graph. We denote by V_Γ and E_Γ the set of its vertices and edges respectively.

Definition 1.1 *An* orientation *at a vertex v is a cyclic ordering of the (half-) edges incident at v. A* fatgraph *(sometimes called* ribbon graph*) is a graph endowed with an orientation at each vertex of Γ. A* left-hand-turn path *in Γ is a directed closed path in Γ such that if e_1, e_2 are successive edges in the path meeting at v, then e_2, e_1 are successive edges with respect to the orientation at v. The ordered pair e_1, e_2 is called a left-turn. We sometimes call* faces *of Γ the left-hand-turn paths and denote them by F_Γ.*

A fatgraph is usually represented in the plane, by assuming that the orientation at each vertex is the counter-clockwise orientation induced by the plane, while the intersections of the edges at points other than the vertices are ignored. There is a natural surface, which we denote by Γ^t obtained by thickening the fatgraph. We usually call Γ^t the ribbon graph associated to Γ. We replace the half-edges around a vertex by thin strips joined at the vertex, whose boundary arcs have natural orientations. For each edge of the graph we connect the thin strips corresponding to the vertices by a ribbon which follows the orientation of their boundaries. We obtain an oriented surface with boundary. The boundary circles are in one-to-one correspondence with the left-hand-turn paths. If one caps each left-hand-turn path by a 2-disk we find a closed surface Γ^c, and this explains why we called these paths faces. The centres of the 2-disks will be called punctures of Γ^c and $\Gamma^o = int(\Gamma^t)$ is homeomorphic to the punctured surface.

There is a canonical embedding $\Gamma \subset \Gamma^t$, and one can associate to each edge e of Γ a properly embedded orthogonal arc $e^\perp$ which joins the two boundary components of the thin strip lying over e. The dual arcs $e^\perp$ divide the ribbon Γ^t into hexagons. When we consider the completion Γ^c, we join the boundary points of these dual arcs to the punctures within each 2-disk face and obtain a set of arcs connecting the punctures, denoted by the same symbols. Then the dual arcs divide Γ^c into triangles. We set $\Delta(\Gamma)$ for the triangulation obtained this way. The vertices of $\Delta(\Gamma)$ are the punctures of Γ^c. Remark that $\Delta(\Gamma)$ is well-defined up to isotopy. Now the fatgraph $\Gamma \subset \Gamma^t$ can be recovered from $\Delta(\Gamma)$ as follows. Mark a point in the interior of each triangle, and connect points corresponding to adjacent triangles. This procedure works for any given triangulation Δ of an oriented surface and produces a fatgraph $\Gamma = \Gamma(\Delta)$ with the property that $\Delta(\Gamma) = \Delta$. The orientation of Γ comes from the surface.

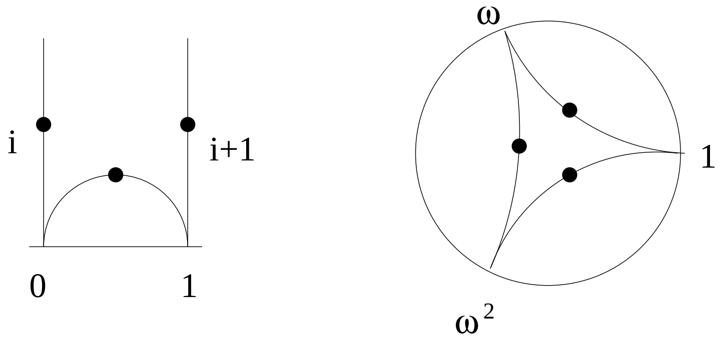
If Γ^o is the surface Σ_g^s of genus g with s punctures then by Euler characteristic reasons we have: $\sharp V_\Gamma = 4g-4+2s$, $\sharp E_\Gamma = 6g-6+3s$, $\sharp F_\Gamma = s$.

1.2 Coordinates on Teichmüller spaces

Marked ideal triangles Let us denote by $\mathbb{D}$ the unit disk, equipped with the hyperbolic metric. Recall that any two ideal triangles are isometric, since we may find a Möbius transformation, which takes one onto the other. Choose a point on each edge of the ideal triangle. The chosen points will be called *tick-marks.*

Definition 1.2 *A* marked ideal triangle *is an ideal triangle with a tick-mark on each one of its three sides. An isomorphism between two marked ideal triangles is an isomorphism between the ideal triangles which preserves the tick-marks.* A standard marked ideal triangle *is one which is isometric to the marked ideal triangle whose vertices in the disk model are given by* $v_1 = 1,\ v_2 = \omega,\ v_3 = \omega^2$ *and whose tick-marks are* $t_1 = -(2-\sqrt{3}),\ t_2 = -(2-\sqrt{3})\omega,\ t_3 = -(2-\sqrt{3})\omega^2$, *where* $\omega = e^{2\pi i/3}$.

The ideal triangle and its tick-marks are pictured below in both the half-plane model and the disk model; they correspond each other by the map $z \mapsto \frac{z-(\omega+1)}{z-(\bar{\omega}+1)}$.



Coordinates on the Teichmüller space of punctured surfaces Set $\mathcal{T}_g^s$ for the Teichmüller space of surfaces of genus g with s punctures. Let Γ be a fatgraph with the property that Γ^c is a surface of genus g with s punctures and let S denote the surface Γ^c endowed with a hyperbolic structure of finite volume, having the cusps at the punctures.

As already explained above we have a triangulation $\Delta(\Gamma)$ associated to Γ. One deforms the arcs of $\Delta(\Gamma)$ within their isotopy class in order to make them geodesic. We shall associate a real number $t_e \in \mathbb{R}$ to each edge of $\Delta(\Gamma)$ (equivalently, to each edge of Γ). Set Δ_v and Δ_w for the two triangles sharing the edge $e^\perp$. We consider next two adjacent lifts of these triangles (which we denote by the same symbols) to the hyperbolic space $\mathbb{H}^2$. Then both Δ_v and Δ_w are isometric to the standard ideal triangle of vertices v_1, v_2 and v_3. These two isometries define (by pull-back) canonical tick-marks t_v and respectively t_w on the geodesic edge shared by Δ_v and Δ_w. Set t_e for the (real) length of the translation along this geodesic needed to shift t_v to t_w. Notice that this geodesic inherits an orientation as the boundary of the ideal triangle Δ_v in $\mathbb{H}^2$ which gives t_e a sign. If we change the role of v and w the number t_e is preserved.

An equivalent way to encode the translation parameters is to use the cross-ratios of the four vertices of the glued quadrilateral $\Delta_v \cup \Delta_w$, which are considered as points of $\mathbb{R}P^1$. It is convenient for us to consider $\mathbb{R}P^1$ as the boundary of the upper half-plane model of $\mathbb{H}^2$, and hence the ideal points have real (or infinite) coordinates. Let assume that Δ_v is the ideal triangle determined by $[p_0 p_{-1} p_\infty]$ and Δ_w is $[p_0 p_\infty p]$. We consider then the following cross-ratios:

$$z_e = [p_{-1}, p_\infty, p, p_0] = [p, p_0, p_{-1}, p_\infty] = \log - \frac{(p_0 - p)(p_{-1} - p_\infty)}{(p_\infty - p)(p_{-1} - p_0)}.$$

This cross-ratio reflects both the quadrilateral geometry and the decomposition into two triangles. In fact the other possible decomposition into two triangle of the same quadrilateral leads to the value z_e.

The relation between the two translation parameters t_e and z_e is immediate. Consider the ideal quadrilateral of vertices $-1, 0, e^z$ and ∞, whose cross-ratio is $z_e = z$, where $e = [0\infty]$. The left triangle tick-mark is located at i, while the right one is located at ie^{-z}, after the homothety sending the triangle into the standard triangle. Taking in account that the orientation of the edge e is up-side one derives that t_e is the signed hyperbolic distance between i and $e^{-z_e} i$, which is z_e.

Proposition 1.1 *The map* $\mathbf{t}_\Gamma : \mathcal{T}_g^s \to \mathbb{R}^{E_\Gamma}$ *given by* $t_\Gamma(S) = (t_e)_{e\in E_\Gamma}$ *is a homeomorphism onto the linear subspace* $\mathbb{R}^{E_\Gamma/F_\Gamma} \subset \mathbb{R}^{E_\Gamma}$ *given by equations:*

$$t_\gamma := \sum_{k=1}^{n} t_{e_k} = 0,$$

for all left-hand-turn closed paths $\gamma \in F_\Gamma$*, which is expressed as a cyclic chain of edges* $e_1, ..., e_n$*.*

Remark 1.1 *Notice that there are exactly s left-hand-turn closed paths, which lead to s independent equations hence the subspace* $\mathbb{R}^{E_\Gamma/F_\Gamma}$ *from above is of dimension* $6g-6+2s$*.*

Proof. The map $\mathbf{t}_\Gamma$ is continuous, and it suffices to define an explicit inverse for it. Let Γ be a trivalent fatgraph whose edges are labelled by real numbers $\mathbf{r} = (r_e)_{e\in E_\Gamma}$. We want to paste one copy Δ_v of the standard marked ideal triangle on each vertex v of Γ and glue together by isometries these triangles according to the edges connections. Since the edges of an ideal triangle are of infinite length we have the freedom to use arbitrary translations along these geodesics when gluing together adjacent sides. If $e = [vw]$ is an edge of Γ then one can associate a real number $t_e \in \mathbb{R}$ as follows. There are two tick-marks, namely t_v and t_w on the common side of Δ_v and Δ_w. We denote by t_e the amount needed for translating t_v into t_w according to the orientation inherited as a boundary of Δ_v. Given now the collection of real numbers $\mathbf{r}$ we can construct unambiguously our Riemann surface $S(\Gamma, \mathbf{r})$, which moreover has the property that $t_\Gamma(S(\Gamma, \mathbf{r})) = \mathbf{r}$. Furthermore it is sufficient now to check whenever this constructions yields a complete Riemann surfaces. The completeness at the puncture determined by the left-hand-turn path γ is equivalent to the condition $t_\gamma = 0$, and hence the claim. The cusps of $S(\Gamma)$ are in bijection with the left-hand-turn paths in Γ, and the triangulation of $S(\Gamma)$ obtained by our construction corresponds to Γ. □

Remark 1.2 *W.Thurston associated to an ideal triangulation a system of shearing coordinates for the Teichmüller space in mid eighties (see [20]). However, the systematic study of such coordinates appeared only later in the papers of F.Bonahon [3] and from a slightly different perspective in Penner's treatment of the decorated Teichmüller spaces ([17]). V.Fock unravelled the elementary aspects of this theory which lead him further to the quantification of the Teichmüller space.*

The Fuchsian group associated to Γ and $\mathbf{r}$ The surface $S(\Gamma, \mathbf{r})$ is uniformized by a Fuchsian group $G = G(\Gamma, \mathbf{r}) \subset \mathrm{PSL}(2, \mathbb{R})$, i.e. $S(\Gamma, \mathbf{r}) = \mathbb{H}^2/G(\Gamma, \mathbf{r})$. We can explicitly determine the generators of the Fuchsian group, as follows.

We have natural isomorphisms between the fundamental group $\pi_1(S(\Gamma, \mathbf{r}) \cong \pi_1(\Gamma^t) \cong \pi_1(\Gamma)$. Any path γ in Γ is a cyclic sequence of adjacent directed edges $e_1, e_2, e_3, ..., e_n$, where e_i and e_{i+1} have the vertex v_i in common. We insert between e_i and e_{i+1} the symbol lt if e_i, e_{i+1} is a left-hand-turn, the symbol rt if it is a right-hand-turn and no symbol otherwise (i.e. when e_{i+1} is e_i with the opposite orientation). Assume now that we have a Riemann surface whose coordinates are $\mathbf{t}_\Gamma(S) = \mathbf{r}$. We define then a representation $\rho_\mathbf{r} : \Pi_1(\Gamma) \to \mathrm{PSL}(2, \mathbb{R})$ of the path groupoid $\Pi_1(\Gamma)$ by the formulas:

$$\rho_\mathbf{r}(e) = \begin{pmatrix} 0 & e^{\frac{r_e}{2}} \\ -e^{-\frac{r_e}{2}} & 0 \end{pmatrix}, \text{ and } \rho_\mathbf{r}(lt) = \rho_\mathbf{r}(rt)^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

This is indeed well-defined since $\rho_\mathbf{r}(e)^2 = -1 = 1 \in \mathrm{PSL}(2, \mathbb{R})$, and hence the orientation of the edge does not matter, and $\rho_\mathbf{r}(lt)^3 = \rho_\mathbf{r}(rt)^3 = 1$. Furthermore the fundamental group $\pi_1(\Gamma)$ is a subgroup of $\Pi_1(\Gamma)$.

Proposition 1.2 *The Fuchsian group* $G(\Gamma, \mathbf{r})$ *is* $\rho_\mathbf{r}(\pi_1(\Gamma)) \subset \mathrm{PSL}(2, \mathbb{R})$*.*

Proof. We can begin doing the pasting without leaving the hyperbolic plane, until we get a polygon P, together with a side pairing. We may think of each triangle as having a white face and a black face, and build the polygon P such that all the triangles have white face up. We attach to each side pairing (s_i, s_j) an orientation preserving isometry A_{ij}, such that $A_{ij}(s_i) = s_j$, A_{ij} sends tick-marks into the tick-marks shifted by r_e, and $P \cap A_{ij}(P) = \emptyset$. Denote by G the subgroup of $\mathrm{ISO}^+(\mathbb{D})$ generated by all the side-pairing transformations. In order to apply the Poincaré Theorem all the vertex-cycle transformations must be parabolic. This amounts to ask that for every left-hand-turn closed path γ we have $t_\gamma = 0$. Then by the Poincaré theorem G is a discrete group of isometries with P as its fundamental domain and $\mathbb{H}^2/G$ is the complete hyperbolic Riemann surface $S(\Gamma, \mathbf{r})$.

We need now the explicit form of the matrices A_{ij}. We obtain them by composing the isometries sending a marked triangle into the adjacent one, in a suitable chain of triangles, where consecutive ones have a common edge. If e is such an edge we remark that $\rho_{\mathbf{r}}(e)$ do the job we want, because it sends the triangle $[-1, 0, \infty]$ into $[e^{r_e}, \infty, 0]$. Moreover the quadrilateral $[-1, 0, e^{r_e}, \infty]$, with this decomposition into two triangles, has associated the cross-ratio r_e. We need next to use $\rho_{\mathbf{r}}(lt)$ which permutes counter-clockwise the tick-marks and the vertices $-1, 0$ and ∞ of the ideal triangle. Then one identifies the matrices A_{ij} with the images of the closed paths by $\rho_{\mathbf{r}}$. □

Remark 1.3 *We observe that the left-hand-turn paths are preserved under an isomorphism of graphs which preserves the cyclic orientation at each vertex. Thus any automorphism of the fatgraph Γ induces an automorphism of $S(\Gamma)$.*

Belyi Surfaces Let assume now that S is endowed with the structure of Riemann surface. It is well known that there exists a non-constant meromorphic function on S, $\varphi : S \to \mathbb{CP}^1$. The Riemann surface S is called a *Belyi surface* if there exists a ramified covering $\varphi : S \to \mathbb{CP}^1$, which is branched only over 0, 1 and ∞. A surprising theorem of Belyi ([2]) states that S is a Belyi surface if and only if it is defined over $\overline{\mathbb{Q}}$ i.e. as an algebraic curve in $\mathbb{CP}^2$ its minimal polynomial lies over some number field.

Following [17, 15] we can characterise Belyi surfaces in terms of fat graphs as follows: A Riemann surface S can be constructed as $S(\Gamma) = S(\Gamma, \mathbf{0})$ for some trivalent fatgraph Γ if and only if S is a Belyi surface.

Coordinates on the Teichmüller space of surfaces with geodesic boundary Set $\mathcal{T}_{g,s;or}$ for the Teichmüller space of surfaces of genus g with s *oriented* boundary components. Here *or* denotes the choice of one orientation for each of the boundary components. Since the surface has a canonical orientation, we can set unambiguously $or : \{1, 2, ..., s\} \to \mathbb{Z}/2\mathbb{Z}$ by assigning $or(j) = +1$ if the orientation of the j-th component agrees with that of the surface and $or(j) = -1$, otherwise. We suppose that each boundary component is a geodesic in the hyperbolic metric, and possibly a cusp (hence in some sense this space is slightly completed). Let Γ be a fatgraph with the property that Γ^t is a surface of genus g with s boundary components and let S denote the surface Γ^t endowed with a hyperbolic structure, for which the boundary is geodesic. Assume that, in this metric, the boundary geodesics b_j have length l_j.

Consider the restriction of the hyperbolic metric to $int(\Gamma^t) = \Gamma^o$. Then Γ^o is canonically homeomorphic to the punctured surface $\Gamma^c - \{p_1, ..., p_s\}$. In particular there is a canonically induced hyperbolic metric on $\Gamma^c - \{p_1, ..., p_s\}$, which we denote by S^*. Moreover this metric is not complete at the punctures p_j. Suppose that the punctures p_j corresponds to the left-hand-turn closed paths γ_j, or equivalently the boundary components geodesics b_j, of length l_j. Assume that we have an ideal triangulation of S^* by geodesic simplices, whose ideal vertices are the punctures p_j. Then the holonomy of the hyperbolic structure around the vertex p_j is a non-trivial, and it can be calculated in the following way (see [19], Prop.3.4.18, p.148). Consider a geodesic edge α entering the puncture and a point $p \in \alpha$. Then the geodesic spinning around p_j in the positive direction (according to the orientation of the boundary circle) is intersecting again α a first time in the point $h_{p_j}(p)$. The hyperbolic distance between the points p and $h_{p_j}(p)$ is the length l_j of the boundary circle in the first metric. Moreover the point $h_{p_j}(p)$ lies in the ray determined by p and the puncture p_j. Notice that if we had chose the loop encircling the puncture to go in opposite direction then the iterations $h_{p_j}(p)$ would have gone faraway from the puncture, and the length would have been given the negative sign. Set therefore l_j^o* for the signed length.

We construct as above the geodesic ideal triangulation $\Delta(\Gamma)$ of the non-complete hyperbolic punctured surface S^*. We can therefore compute the holonomy map using the thick-marks on some edge abutting to the puncture p_j. It is immediately that the the holonomy displacement on this edge is given by t_{γ_j}, where γ_j is the left-hand-turn closed path corresponding to this puncture. In particular we derive that:

$$|t_{\gamma_j}| = l_j, \text{ for all } j \in \{1, 2, ..., s\}.$$

Using the method from the previous section we know how to associate to any edge e of Γ a real number $t_e = t_e(S^*)$ measuring the shift between two ideal triangles in the geodesic triangulation of the surface S^*.

Proposition 1.3 *The map $\mathbf{t}_\Gamma : \mathcal{T}_{g,s;or} \to \mathbb{R}^{E_\Gamma}$ given by $t_\Gamma(S) = (t_e)_{e \in E_\Gamma}$ is a homeomorphism.*

Proof. The construction of an inverse map proceeds as above. Given $\mathbf{r} \in \mathbb{R}^{E_\Gamma}$ we construct a non-complete hyperbolic surface S^* with s punctures with the given parameters, by means of gluing ideal triangles. As shown in ([19], Prop. 3.4.21, p.150) we can complete this hyperbolic structure to a surface with geodesic boundary S, such that $int(S) = S^*$. Further if $t_{\gamma_j} > 0$, then we assign the orientation of γ_j for the boundary component b_j, otherwise we assign the reverse orientation. When $t_{\gamma_j} = 0$ it means that we have a cusp at p_j. □

Remark 1.4 *The two points of $\mathcal{T}_{g,s;or}$ given by the same hyperbolic structure on the surface $\Sigma_{g,s}$ but with distinct orientations of some boundary components lie in the same connected component. Nevertheless the previous formulas shows that a path connecting them must pass through the points of $\mathcal{T}_{g,s;or}$ corresponding to surfaces having a cusp at the respective puncture.*

Set $\mathcal{T}_{g,s}$ for the Teichmüller space of surfaces of genus g with s *non-oriented* boundary components, i.e. hyperbolic metrics for which the boundary components are geodesic. There is a simple way to recover coordinates on $\mathcal{T}_{g,s}$ from its oriented version. Let $\psi : \mathbb{R}^{E_\Gamma} \to \mathbb{R}^{F_\Gamma}$ be the map $\psi(\mathbf{t}) = (t_{\gamma_i})_{\gamma_i \in F_\Gamma}$. Choose a projector $\psi^* : \mathbb{R}^{E_\Gamma} \to \ker\psi = \mathbb{R}^{E_\Gamma/F_\Gamma}$, and set $\iota_{|.|} : \mathbb{R}^{F_\Gamma} \to \mathbb{R}^{F_\Gamma}$ for the map given on coordinates by $\iota_{|.|}(y_j)_{j=1,\sharp F_\Gamma} = (|y_j|)_{j=1,\sharp F_\Gamma}$. Then $\mathcal{T}_{g,s}$ is the quotient by the $(\mathbb{Z}/2\mathbb{Z})^{F_\Gamma}$-action on $\mathcal{T}_{g,s;or}$ which changes the orientation of the boundary components.

Proposition 1.4 *We have a homeomorphism $\mathbf{t}_\Gamma : \mathcal{T}_{g,s} \to \mathbb{R}^{6g-6+2s} \oplus \mathbb{R}^s$, which is induced from the second line of the following commutative diagram:*

$$\begin{array}{ccc}
\mathcal{T}_{g,s;or} & \xrightarrow{(\psi^*\oplus\psi)\circ\mathbf{t}_\Gamma} & \mathbb{R}^{E_\Gamma/F_\Gamma} \oplus \mathbb{R}^{F_\Gamma} \\
\downarrow & & \downarrow id \oplus \iota_{|.|} \\
\mathcal{T}_{g,s} & \longrightarrow & \mathbb{R}^{E_\Gamma/F_\Gamma} \oplus \mathbb{R}_+^{F_\Gamma} \\
\uparrow & & \uparrow id \oplus 0 \\
\mathcal{T}_g^s & \longrightarrow & \mathbb{R}^{E_\Gamma/F_\Gamma}
\end{array}$$

Remark 1.5 *Observe that the embedding $\mathcal{T}_g^s \hookrightarrow \mathcal{T}_{g,s}$ given in terms of coordinates by adding on the right a string of zeroes lifts to an embedding $\mathcal{T}_g^s \hookrightarrow \mathcal{T}_{g,s;or}$.*

Putting together the results of the last two sections we derive that:

Proposition 1.5 *The map $\mathbf{t}_\Gamma : \mathcal{T}^s_{g,n;or} \to \mathbb{R}^{E_\Gamma}$ given by $t_\Gamma(S) = (t_e)_{e\in E_\Gamma}$ is a homeomorphism of the Teichmüller space of surfaces of genus g with n oriented boundary components and s punctures onto the linear subspace $\mathbb{R}^{E_\Gamma/F^*\Gamma}$ of dimension $6g - 6 + 3n + 2s$ given by the equations: $t_{\gamma_j} = 0$, for those left-hand-turn closed paths γ_j corresponding to the punctures, $\gamma_j \in F^*_\Gamma \subset F_\Gamma$.*

2 The mapping class group action

2.1 General facts about mapping class groups

Consider Σ a surface, possibly with boundary and punctures or marked points, compact and orientable. We denote by $\mathrm{Homeo}^+(\Sigma)$ the group of homeomorphisms of Σ preserving the orientation, endowed with the compact-open topology.

Definition 2.1 *The mapping class group of Σ is $\mathrm{Mod}(\Sigma) = \mathrm{Homeo}^+(\Sigma)/\simeq$, where $f, g : \Sigma \to \Sigma$ are equivalent if they are homotopic. This is equivalent to consider the quotient $\mathrm{Homeo}^+(\Sigma)/\mathrm{Homeo}_0(\Sigma)$, where $\mathrm{Homeo}_0(\Sigma)$ is the connected component of identity. If Σ has boundary or marked points then one requires that the homeomorphisms and the homotopies we are concerned of to fix this boundary/marking data (pointwise or setwise).*

It is known since the work of M.Dehn and J.Nielsen that there is a natural isomorphism

$$\mathrm{Mod}(\Sigma_g) \to \mathrm{Out}^+(\pi_1(\Sigma_g)) = \mathrm{Aut}(\pi_1(\Sigma_g))/\mathrm{Inn}(\pi_1\Sigma_g)$$

which sends the class of the homeomorphism φ into the class of the map $\varphi_* : \pi_1\Sigma_g \to \pi_1\Sigma_g$ induced by φ in homotopy. Here $\mathrm{Inn}(\Gamma)$ is the set of inner automorphisms (acting by conjugacy), which corresponds to the freedom in choosing the base point for the fundamental group. The result can be stated in the case when Σ_g has boundary but one needs to add extra conditions on the automorphisms in the right hand side, by asking them to preserve the conjugacy classes of boundary loops.

There is a close relation between mapping class groups and Teichmüller spaces. By using the identification between $\mathrm{Mod}(\Sigma)$ and $\mathrm{Out}^+(\pi, \Sigma)$, the mapping class group acts properly discontinuously by left composition on the space $\mathcal{T}(\Sigma)$. In fact $\mathcal{T}(\Sigma)$ is also the subspace of group representations:

$$\mathcal{T}(\Sigma) = \mathrm{Hom}^+_{f,d}\,(\pi_1\Sigma, \mathrm{PSL}(2,\mathbb{R}))\,/\,\mathrm{PSL}(2,\mathbb{R})$$

where the subscripts f, d mean that we restrict to those representations that are faithful and discrete. Now, we have the action

$$\mathrm{Out}^+(\pi,\Sigma) \times \mathrm{Hom}^+_{f,d}\,(\pi_1\Sigma, \mathrm{PSL}(2,\mathbb{R}))\,/\,\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{Hom}^+_{f,d}\,(\pi_1\Sigma, \mathrm{PSL}(2,\mathbb{R}))\,/\,\mathrm{PSL}(2,\mathbb{R})$$

given by

$$(\varphi, [\rho]) \longrightarrow [\rho \circ \varphi^{-1}]$$

Moreover, $\mathrm{Mod}(\Sigma)$ acts by real analytic homeomorphisms.

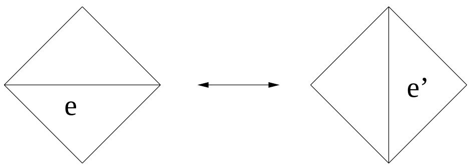
2.2 The Ptolemy modular groupoid

The modular groupoid was considered by Mosher in his thesis and further as a key ingredient in [13, 14], it is implicit in Harer's paper on the arc complex (see [8]) and then studied by Penner (see [17, 18]; notice that the correct definition is that from [18]) who introduced also the terminology.

Recall that a groupoid is a category whose morphisms are invertible, such that between any two objects there is at least one morphism. The morphisms from an object to itself form a group (the group associated to the groupoid).

Remark 2.1 *Suppose that we have an action of a group G on a set M. We associate a groupoid $\mathcal{G}(G, M)$ as follows: its objects are the G-orbits on M, and the morphisms are the G-orbits of the diagonal action on $M \times M$. If the initial action was free then G embeds in $\mathcal{G}(G, M)$ as the automorphisms group of any object.*

Assume that we have an ideal triangulation $\Delta(\Gamma)$ of a surface Σ_g^s. If e is an edge shared by the triangles Δ_v and Δ_w of the triangulation then we change the triangulation by excising the edge e and replacing it by the other diagonal of the quadrilateral $\Delta_v \cup \Delta_w$, as in figure below. This operation $F[e]$ was called flip in [7] or elementary by Mosher and Penner.



Let $\mathcal{IT}(\Sigma_g^s))$ denote the set of isotopy classes of ideal triangulations of Σ_g^s. The reduced *Ptolemy groupoid* $\overline{P_g^s}$ is the groupoid generated by the flips action on $\mathcal{IT}(\Sigma_g^s))$. Specifically its elements are classes of sequences $\Delta_0, \Delta_1, ..., \Delta_m$, where Δ_{j+1} is obtained from Δ_j by using a flip. Two sequences $\Delta_0, ..., \Delta_m$ and $\Delta'_0, ..., \Delta'_n$ are equivalent if their initial and final terms coincide i.e. there exists a homeomorphism φ preserving the punctures such that $\varphi(\Delta_0) \cong \Delta'_0$ and $\varphi(\Delta_m) \cong \Delta'_n$, where $\cong$ denotes the isotopy equivalence. Notice that any two (isotopy classes of) ideal triangulations are connected by a chain of flips (see [9] for an elementary proof), and hence $\overline{P_g^s}$ is indeed an groupoid. Moreover $\overline{P_g^s}$ is the groupoid $\mathcal{G}(\mathcal{M}_g^s, \mathcal{IT}(\Sigma_g^s))$ associated to the obvious action of the mapping class group $\mathcal{M}_g^s$ on the set of isotopy classes of ideal triangulations $\mathcal{IT}(\Sigma_g^s))$. One problem in considering $\overline{P_g^s}$ is that the action of $\mathcal{M}_g^s$ on $\mathcal{IT}(\Sigma_g^s))$ is *not free* but there is a simple way to remedy it. For instance in [13, 14] one adds the extra structure coming from fixing an *oriented arc* of the ideal triangulation. A second problem is that we want that the mapping class group action on the Teichmüller space extends to a groupoid action.

Consider now an ideal triangulation $\Delta = \Delta(\Gamma)$, where Γ is its dual fatgraph. A labelling of Δ is a numerotation of its edges $\sigma_\Gamma : E_\Gamma \to \{1, 2, ..., \sharp E_\Gamma\}$. Set now $\mathcal{LIT}(\Sigma_g^s))$ for the set of labelled ideal triangulations. The

Ptolemy groupoid P_g^s of the punctured surface Σ_g^s is the groupoid generated by flips on $\mathcal{LIT}(\Sigma_g^s)$). The flip $F[e]$ associated to the edge $e \in E_\Gamma$ acts on the labellings in the obvious way:

$$\sigma_{F[e](\Gamma)}(f) = \begin{cases} \sigma_\Gamma(f), & \text{if } f \neq e' = F[e](e) \\ \sigma_\Gamma(e), & \text{if } f = e', \end{cases}$$

According to ([18] Lemma 1.2.b), if $2g-2+s \geq 2$ then any two labelled ideal triangulations are connected by a chain of flips, and thus P_g^s is indeed a groupoid. Moreover, this allows us to identify P_g^s with $\mathcal{G}(\mathcal{M}_g^s, \mathcal{LIT}(\Sigma_g^s))$.

Remark 2.2 *In the remaining cases, namely Σ_0^3 and Σ_1^1, the flips are not acting transitively on the set of labelled ideal triangulations. In this situation an appropriate labelling consist in an oriented arc, as in [13]. The Ptolemy groupoid associate to this labelling has the right properties, and it acts on the Teichmüller space.*

Proposition 2.1 *We have an exact sequence*

$$1 \to \mathcal{S}_{6g-6+3s} \to P_g^s \to \overline{P_g^s} \to 1,$$

where $\mathcal{S}_n$ denotes the symmetric group on n letters. Notice that $P_1^1 = \overline{P_1}^1$. If $(g,s) \neq (1,1)$ then $\mathcal{M}_g^s$ naturally embeds in P_g^s as the group associated to the groupoid.

Proof. The first part is obvious. The following result is due to Penner ([18],Thm.1.3):

Lemma 2.1 *If $(g,s) \neq (1,1)$ then $\mathcal{M}_g^s$ acts freely on $\mathcal{LIT}(\Sigma_g^s)$.*

Proof. A homeomorphism keeping invariant a labelled ideal triangulation either preserves the orientation of each arc or else it reverses the orientation of all arcs. In fact once the orientation of an arc lying in some triangle is preserved, the orientation of the other boundary arcs of the triangle must also be preserved. Further in the first situation either the surface is Σ_0^3 (when $\mathcal{M}_0^3 = 1$) or else each triangle is determined by its 1-skeleton, and the Alexander trick shows that the homeomorphism is isotopic to identity. In the second case we have to prove that $(g,s) = (1,1)$. Since the arcs cannot have distinct endpoints we have $s = 1$. Let Δ_1 be an oriented triangle and $D \subset \Delta_1$ be a 2-disk which is a slight retraction of Δ_1 into its interior. The image D' of D cannot lie within Δ_1 because the homeomorphism is globally orientation preserving while the orientation of the boundary of D' is opposite to that of $\partial\Delta$. Thus D' lies outside Δ_1 and the region between $\partial D'$ and $\partial\Delta_1$ is an annulus, so the complementary of Δ_1 consists of one triangle. Therefore $g = 1$. □

Remark 2.3 *The punctured torus Σ_1^1 has an automorphism which reverse the orientation of each of the three ideal arcs.*

□

The case of the punctured torus is settled by the following:

Proposition 2.2 *Let $\Delta_{st} = \{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_1 = (1,0), \alpha_2 = (1,1), \alpha_3 = (0,1)$ be the standard labelled ideal triangulation of the punctured torus $\Sigma_1^1 = \mathbb{R}^2/\mathbb{Z}^2 - \{0\}$.*

1. *If $\Delta = \{\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}\}$ is flip equivalent to Δ_{st} then σ is the identity.*
2. *A mapping class which leaves invariant Δ_{st} is either identity or $-id \in SL(2,\mathbb{Z}) = \mathcal{M}_1^1$.*
3. *Let $\Delta = \{\gamma_1, \gamma_2, \gamma_3\}$ be an arbitrary ideal triangulation. Then there exists an unique $\sigma(\Delta) \in \mathcal{S}_3$ such that Δ is flip equivalent with the labelled diagram $\{\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}\}$.*
4. *In particular if $\Delta = \varphi(\Delta_{st})$ then we obtain a group homomorphism $\sigma : SL(2,\mathbb{Z}) \to \mathcal{S}_3$, given by $\sigma(\varphi) = \sigma(\varphi(\Delta_{st}))$, whose values can be computed from:*

$$\sigma\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (23), \quad \sigma\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = (12), \quad \sigma\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (13).$$

We need therefore another labelling for Σ_1^1, which amounts to fix a *distinguished oriented edge (d.o.e.)* of the triangulation. The objects acted upon flips are therefore pairs (Δ, e), where e is the d.o.e. of Δ. A flip acts on the set of labelled ideal triangulations with d.o.e. as follows. If the flip leaves e invariant then the new d.o.e. is the old one. Otherwise the flip under consideration is $F[e]$, and the new d.o.e. will be the image e' of e, oriented so that the frame (e, e') at their intersection point is positive with respect to the surface orientation. The groupoid Pt_g^s generated by flips on (labelled) ideal triangulations with d.o.e. of is called the *extended Ptolemy groupoid*. Since any edge permutation is a product of flips (when $(g, s) \neq 1$) it follows that any two labelled triangulations with d.o.e. can be connected by a chain of flips.

The case of the punctured torus is subjected to caution again: it is more convenient to define the groupoid Pt_1^1 as that generated by iterated compositions of flips on the standard (labelled or not) ideal triangulation Δ_{st} of Σ_1^1 with a fixed d.o.e., for instance α_1. In fact proposition 2.2 implies that there are three distinct orbits of the flips on triangulations with d.o.e., according to the the position of the d.o.e. within Δ_{st}.

Remark 2.4 *For all (g, s) we have an exact sequence:*

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathsf{Pt}_g^s \to P_g^s \to 1.$$

Moreover $\mathcal{M}_g^s \to P_g^s$ lifts to an embedding $\mathcal{M}_g^s \hookrightarrow \mathsf{Pt}_g^s$.

Remark 2.5 *We can define the groupoid $\overline{\mathsf{Pt}_g^s}$ by considering flips on ideal triangulations with d.o.e. without labellings.*

Remark 2.6 *The kernel of the map $\mathcal{M}_1^1 \to P_1^1$ is the group of order two generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Therefore any (faithful) representation of P_1^1 induces a (faithful) representation of* $\mathrm{PSL}(2, \mathbb{Z})$.

Remark 2.7 *One reason to consider P_g^s instead of $\overline{P_g^s}$ is that P_g^s acts on the Teichmüller space while $\overline{P_g^s}$ does not. The other reason is that $\mathcal{M}_g^s$ injects into P_g^s (if $(g, s) \neq (1, 1)$). The kernel of $\mathcal{M}_g^s \to \overline{P_g^s}$ is the image of the automorphism group $Aut(\Gamma)$ in $\mathcal{M}_g^s$.*

Proof. An automorphism of Γ is a combinatorial automorphism which preserves the cyclic orientation at each vertex. Notice that an element of $Aut(\Gamma)$ induces a homeomorphism of Γ^t and hence an element of $\mathcal{M}_g^s$. Now, if φ is in the kernel then φ is described by a permutation of the edges i.e. an element of $\varphi_* \in \mathcal{S}_{\sharp E_\Gamma}$. One can assume that the orientations of all arcs are preserved by φ when $(g, s) \neq (1, 1)$. Then φ_* completely determines φ, by the Alexander trick. Further φ induces an element of $Aut(\Gamma)$ whose image in $\mathcal{S}_{\sharp E_\Gamma}$ is precisely φ_*. This establishes the claim. Notice that the map $Aut(\Gamma) \to \mathcal{S}_{\sharp E_\Gamma}$ is injective for most but not for all fatgraphs Γ. The fatgraphs Γ for which the map $Aut(\Gamma) \to \mathcal{S}_{\sharp E_\Gamma}$ fails to be injective are described in [15]. □

We can state now a presentation for Pt_g^s which is basically due to Penner ([18]):

Proposition 2.3 *Pt_g^s is generated by the flips $F[e]$ on the edges. The relations are:*

1. *Set J for the change of orientation of the d.o.e. Then*

$$F[F[e]e]F[e] = \begin{cases} 1, & \text{if } e \text{ is not the d.o.e.} \\ J, & \text{if } e \text{ is the d.o.e.} \end{cases}$$

2. $J^2 = 1$.

3. *Consider the pentagon from picture below, and $F[e_j]$ be the flips on the dotted edges. Let $\tau_{(12)}$ denote the transposition interchanging the labels of the two edges e_1 and f_1 from the initial triangulation. Then we have:*

$$F[e_1]F[e_2]F[e_3]F[e_4]F[e_5] = \begin{cases} J\tau_{(12)}, & \text{if } e_1 \text{ is not the d.o.e.} \\ \tau_{(12)}, & \text{if } e_1 \text{ is the d.o.e.} \end{cases}$$

The action of $\tau_{(12)}$ on triangulations with d.o.e. is at follows: if none of the permuted edges e, f is the d.o.e. then $\tau_{(12)}$ leaves the d.o.e. unchanged. If the d.o.e. is one of the permuted edges, say e, then the new d.o.e. is f oriented such that e (with the former d.o.e. orientation) and f with the given d.o.e. orientation form a positive frame on the surface. Notice that $[F[e_1]F[e_2]F[e_3]F[e_4]F[e_5] = \tau_{(12)}$ even if f_1 is the d.o.e.

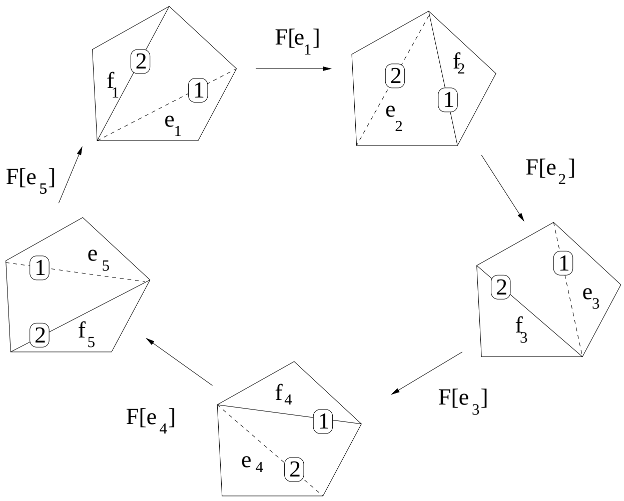
4. *If e and f are disjoint edges then $F[e]F[f] = F[f]F[e]$.*

5. *The relations in a $\mathbb{Z}/2\mathbb{Z}$ extension of the symmetric group, expressed in terms of flips. To be more specific, les us assume that the edges are labelled and the d.o.e. is labelled 0. Then we have:*

$$\tau_{(0i)}^2 = J,\ \tau_{(ij)}^2 = 1,\ \text{if } i,j \neq 0,\ \tau_{(st)}\tau_{(mn)} = \tau_{(mn)}\tau_{(st)}\ \text{if } \{m,n\} \cap \{s,t\} = \emptyset,$$

$$\tau_{(st)}\tau_{(tv)}\tau_{(st)} = \tau_{(tv)}\tau_{(st)}\tau_{(tv)},\ \text{if } s,t,v \text{ are distinct.}$$

6. *$F[\tau(e)]\tau F[e] = \tau$, for any label transposition τ (expressed as a product of flips as above), which says that the symmetric group is a normal subgroupoid of P_g^s.*



Proof. We analyse first the case where labellings are absent:

Lemma 2.2 *$\overline{P_g^s}$ is generated by the flips on edges $F[e]$. The relations are:*

1. *$F[e]^2 = 1$, which is a fancy way to write that the composition of the flip on $F[e](e)$ with the flip on e is trivial.*

2. *$F[e_1]F[e_2]F[e_3]F[e_4]F[e_5] = 1$, where $F[e_i]$ are the flips considered in the picture 2.2.*

3. *Flips on two disjoint edges commute each other.*

Proof. This result is due to Harer (see [8]). It was further exploited by Penner ([17, 18]). □
The complete presentation is now a consequence of the two exact sequences from proposition 2.1 and remark 2.4, relating $\overline{P_g^s}$, P_g^s and P_g^s. □

Remark 2.8 *By setting $J = 1$ above we find the presentation of P_g^s, with which we will be mostly concerned in the sequel.*

2.3 The mapping class group action on the Teichmüller spaces

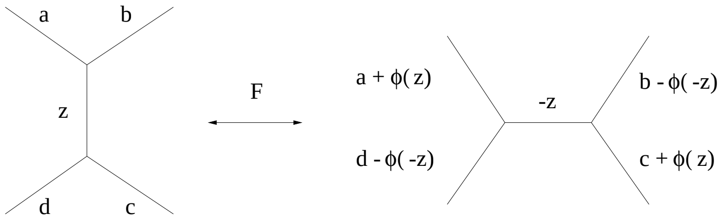
In order to understand the action on $\mathcal{T}_g^s$ we to consider also $\mathcal{T}_{g,s;or}$.

The action of $\mathcal{M}_g^s$ on the Teichmüller space extends to an action of P_g^s to $\mathcal{T}_g^s$. Geometrically we can see it as follows. An element of $\mathcal{T}_g^s$ is a marked hyperbolic surface S. The marking comes from an ideal triangulation. If we change the triangulation by a flip, and keep the hyperbolic metric we obtain another element of $\mathcal{T}_g^s$.

In the same way the $\mathcal{M}_{g,s}$ action on the Teichmüller space $\mathcal{T}_{g,s;or}$ extends to an action of the Ptolemy groupoid $P_{g,s}$. This action is very easy to understand in terms of coordinates. In more specific terms a flip between the graphs Γ and Γ' induces an analytic isomorphism $\mathbb{R}^{E_\Gamma} \to \mathbb{R}^{E_{\Gamma'}}$ by intertwining the coordinate systems t_Γ and $t_{\Gamma'}$. It is more convenient to identify $\mathbb{R}^{E_\Gamma}$ with a fixed Euclidean space, which is done by choosing a labelling $\sigma : E_\Gamma \to \{1, 2, ...\sharp E_\Gamma\}$ of its edges. Thus we have homeomorphism $\mathbf{t}_{\Gamma,\sigma} : \mathcal{T}_{g,s;or} \to \mathbb{R}^{\sharp E_\Gamma}$ given by $(\mathbf{t}_{\Gamma,\sigma}(S))_k = (\mathbf{t}_\Gamma(S))_{\sigma^{-1}(k)\in E_\Gamma}$. Further we can compare the coordinates $\mathbf{t}_{\Gamma,\sigma}$ and $\mathbf{t}_{F(\Gamma,\sigma)}$, for two labelled fatgraphs which are related by a flip.

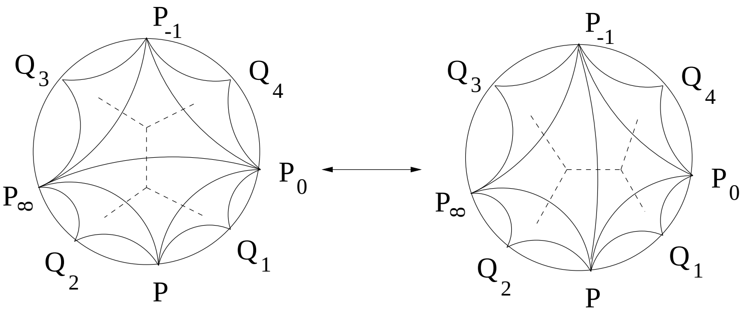
A result of Negami ([16]) states that any two triangulations of a surface can be be connected by a sequence of embedded flips, namely those for which the associated quadrilateral is embedded in the surface, at least when the triangulations have a sufficiently large number of triangles (a linear function in the genus). In particular, it suffices to consider only embedded flips in the sequel.

Proposition 2.4 *An embedded flip acts on the edge coordinates of a fatgraph as follows:*



where $\phi(z) = \log(1 + e^z)$. Here it is understood that the coordinates associated to the edges not appearing in the picture remain unchanged.

Proof. The flip on the graph corresponds to the following flip of ideal triangulations:



Then the coordinates a, b, c, d, z using the left-hand-side graph are the following cross-ratios: $a = [Q_3, P_\infty, P_0, P_{-1}]$, $b = [Q_4, P_{-1}P_\infty, P_0]$, $c = [Q_1, P_0, P_\infty, P]$, $d = [Q_2, P, P_0, P_\infty]$, $z = [P_{-1}, P_\infty, P, P_0]$. Let a', b', c', d', z' be the coordinates associated to the respective edges from the right-hand-side graph, which can again be expressed as cross-ratios as follows: $a' = [Q_3, P_\infty, P, P_{-1}]$, $b' = [Q_4, P_{-1}, P, P_0]$, $c = [Q_1, P_0, P_{-1}, P]$, $d = [Q_2, P, P_{-1}, P_\infty]$, $z = [P_\infty, P, P_0, P_{-1}]$. One uses for simplifying computations the half-plane model where, up to a Möbius transformation, the points P_{-1}, P_∞, P, P_0 are sent respectively into $-1, \infty, e^z$ and 0. The flip formulas follow immediately. □

Remark 2.9 *Similar computations hold for Penner's λ-coordinates on the decorated Teichmüller spaces. However the transformations of $\mathbf{R}^{6g-6+2s}$ obtained using λ-coordinates are* rational *functions.*
Notice that the action of a non-embedded flip can be computed in the same way (see [4]). For instance, if the edges corresponding to the labels a and c coincide then we have to replace $a + \phi(z)$ by $a + 2\phi(z)$; if the edges corresponding to the labels a and b (respectively d) coincide then we replace $a + \phi(z)$ by $a + z$.

Let us denote by $Aut^\omega(\mathbb{R}^m)$ the group of real analytic automorphisms of $\mathbb{R}^m$.

Corollary 2.1 *1. We have a faithful representation $\rho : \mathcal{M}_{g,s} \to Aut^\omega(\mathbb{R}^{6g-6+3s})$ induced by the $P_{g,s}$ action on the Teichmüller space $\mathcal{T}_{g,s;or}$ if $(g, s) \neq (1, 1)$.*

2. The groupoid $P_g^s \subset P_{g,s}$ leaves invariant the Teichmüller subspace $\mathcal{T}_g^s \subset \mathcal{T}_{g,s;or}$. Therefore the formula given in proposition 2.4 above for the flip actually yields a representation of P_g^s into $Aut^\omega(\mathbb{R}^{6g-6+2s})$. The restriction to the mapping class groups is a faithful representation $\rho : \mathcal{M}_g^s \to Aut^\omega(\mathbb{R}^{6g-6+2s})$ if $(g, s) \neq (1, 1)$, and a faithful representation of $\mathrm{PSL}(2, \mathbb{R})$ *when $(g, s) = (1, 1)$.*

Proof. The representation of $\mathcal{M}_{g,s}$ (respectively $\mathcal{M}_g^s$) is injective because the mapping class group acts effectively on the Teichmüller space. Therefore if the class of any (marked) Riemann surface is preserved by a homeomorphism then this homeomorphism is isotopic to the identity.

The invariance of the subspace $\mathcal{T}_g^s \subset \mathcal{T}_{g,s;or}$ by flips is geometrically obvious, but we write it down algebraically for further use. This amounts to check that the linear equations $t_\gamma = 0$, for $\gamma \in F_\Gamma$ are preserved. Let γ be a left-hand-turn path, which intersects the part of the graph shown in the picture, say along the edges labelled a, z, b. Then the flip of γ intersects the new graph along the edges labelled by $a + \phi(z)$ and $b - \phi(-z)$. The claim follows from the equality $z = \phi(z) - \phi(-z)$. The remaining three cases reduces to the same equation. □

Remark 2.10 *There is a* Pt_g^s*-action on the Teichmüller space but it is not free, and actually factors through* P_g^s.

Remark 2.11 *Assume that there exists an element* $\mathbf{r} \in \mathcal{T}_g^s$*, which is fixed by* some $\psi \in \mathcal{M}_g^s$*, i.e.* $\varphi(\psi)(\mathbf{r}) = \mathbf{r}$*. Then* $\mathbf{r}$ *is contained in some codimension two analytic submanifold* $Q_g^s \subset \mathcal{T}_g^s$*, and for a given* $\mathbf{r}$ *its isotropy group is finite. This is a reformulation of the fact that* $\mathcal{M}_g^s$ *acts properly discontinuously on the Teichmüller space with finite isotropy groups corresponding to the Riemann surfaces with non-trivial automorphism groups (biholomorphic). Moreover the locus of Riemann surfaces with automorphisms is a proper complex subvariety of the Teichmüller space, corresponding to the singular locus of the moduli space of curves.*

2.4 Deformations of the mapping class group representations

We want to consider deformations of the tautological representation $\rho = \rho_0$ of $\mathcal{M}_g^s$ obtained in the previous section. We first restrict ourselves to deformations $\rho_h : \mathcal{M}_g^s \to Aut^\omega(\mathbb{R}^{6g-6+2s})$ satisfying the following requirements:

1. The deformation ρ_h extends to the Ptolemy groupoid P_g^s. In particular ρ_h is completely determined by $\rho_h(F)$ and $\rho_h(\tau_{(ij)})$.
2. The image of a permutation $\rho_h(\tau_{(ij)})$ is the automorphism of $\mathbb{R}^{6g-6+2s}$ given by the permutation matrix $P_{(ij)}$, which exchanges the i-th and j-th coordinates.
3. The action of a flip $F_h = \rho_h(F)$ on the edge coordinates has the form given in proposition 2.4, but using the function ϕ_h, instead of ϕ. Moreover $\lim_{h\to 0} \phi_h = \log(1 + e^z)$.
4. The linear subspace $\mathcal{T}_g^s \subset \mathcal{T}_{g,s;or}$ is invariant by ρ_h.

Proposition 2.5 *The real function* $\phi : \mathbb{R} \to \mathbb{R}$ *yields a deformation of the tautological Ptolemy groupoids representations above if and only if it satisfies the following functional equations:*

$$\phi(x) = \phi(-x) + x. \tag{1}$$

$$\phi(x + \phi(y)) = \phi(x + y - \phi(x)) + \phi(x). \tag{2}$$

$$\phi\left(\phi\left(x + \phi(y)\right) - y\right) = \phi(-y) + \phi(x). \tag{3}$$

Proof. The first equation is equivalent to the invariance of the linear equations defining the cusps. The other two equations follow from the cumbersome but straightforward computation of terms involved in the pentagon equation. □

Chekhov, Fock ([5]) and Kashaev ([10]) found the existence of nontrivial deformations, as follows:

Proposition 2.6 *The meromorphic function*

$$\phi_h(z) = -\frac{\pi h}{2} \int_{-\infty}^{\infty} \frac{e^{\sqrt{-1}tz}}{\sinh \pi t \sinh \pi h t}\, dt$$

(where the integral is computed along a contour going along the real axis and bypassing the origin from above) verifies the functional equations above and $\lim_{h\to 0} \phi_h(z) = \phi(z)$.

This function is Faddeev's quantum dilogarithm. Recently, Bai showed ([1]) that this solution is essentially unique, under suitable conditions.

References

[1] Hua Bai, *A uniqueness property for the quantization of Teichmüller spaces*, math.GT/0509679.

[2] G. V. Belyi, *On Galois Extensions of a maximal cyclotomic polynomial*, Math. USSR Izvestia 14 (1980), 247-256.

[3] F.Bonahon, *Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form*, Ann. Fac. Sci. Toulouse Math. (6) 5 (1996), no. 2, 233–297.

[4] F.Bonahon, Xiaobo Liu, *Representations of the quantum Teichmuller space, and invariants of surface diffeomorphisms*, math.GT/0407086.

[5] L.Chekhov, V.Fock, *Quantum Teichmüller spaces*, Theoret. Math. Phys. 120 (1999), no. 3, 1245–1259.

[6] L.Chekhov, R.C.Penner, *On Quantizing Teichmüller and Thurston theories*, math.AG/0403247.

[7] V.Fock, *Dual Teichmüller spaces*, math.dg-ga/9702018.

[8] J.Harer *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent.Math. 84(1986), 157-176.

[9] A.Hatcher, *On triangulations of surfaces*, Topology Appl. 40(1991), 189-194.

[10] R. Kashaev, *Quantization of Teichmüller spaces and the quantum dilogarithm*, Lett. Math. Phys. 43 (1998), no. 2, 105–115.

[11] R. Kashaev, *The pentagon equation and mapping class groups of surfaces with marked points*, Theoret. and Math. Phys. 123 (2000), no. 2, 576–581.

[12] Xiaobo Liu, *The quantum Teichmuller space as a noncommutative algebraic object*, math.GT/0408361.

[13] L. Mosher, *Mapping class groups are automatic*, Ann. of Math. 142(1995), 303–384.

[14] L. Mosher, *A user's guide to the mapping class group: once punctured surfaces*, Geometric and computational perspectives on infinite groups, 101–174, DIMACS, 25, A.M.S., 1996.

[15] M.Mulase, M.Penkava, *Ribbon graphs, quadratic differentials on Riemann surfaces, and algebraic curves defined over* $\overline{\mathbf{Q}}$, Asian J. Math. 2 (1998), no. 4, 875–919.

[16] S.Negami, *Diagonal flips in triangulations of surfaces*, Discrete Math. 135(1994), 225-232. [

[17] R.Penner, *The decorated Teichmüller space of punctured surfaces*, Commun. Math. Phys 113(1987), 299-339.

[18] R.Penner, *Universal constructions in Teichmüller theory*, Adv.Math 98(1993), 143-215.

[19] W.Thurston, *Three-dimensional geometry and topology*, Vol. 1., Ed. Silvio Levy, Princeton Mathematical Series, 35, 1997.

[20] W.Thurston, *Minimal stretch maps between hyperbolic surfaces*, preprint 1986, available at `math.GT/9801039`.