ON MAPPING CLASS GROUP QUOTIENTS BY POWERS OF DEHN TWISTS AND THEIR REPRESENTATIONS

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Dedicated to Vladimir G. Turaev on the occasion of his 65-th birthday

ABSTRACT. The aim of this paper is to survey some known results about mapping class group quotients by powers of Dehn twists, related to their finite dimensional representations and to state some open questions. One can construct finite quotients of them, out of representations with Zariski dense images into semisimple Lie groups. We show that, in genus 2, the Fibonacci TQFT representation is actually a specialization of the Jones representation. Eventually, we explain a method of Long and Moody which provides large families of mapping class group representations.
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1. Mapping class group quotients

1.1. **Introduction.** Set $\Sigma_{g,k}^r$ for the orientable surface of genus g with k boundary components and r marked points. We denote by $\Gamma_{g,k}^r$ the mapping class group of $\Sigma_{g,k}^r$, namely the group of isotopy classes of orientation-preserving homeomorphisms that fix pointwise the boundary components and preserve globally the set of marked points. The pure mapping class group $P\Gamma_{g,k}^r$ consists of those classes of homeomorphisms which fix pointwise both the boundary components and each of the marked points.

We set $\pi_{g,k}^r$ for the fundamental group of the surface $\Sigma_{g,k}^r$. Recall that, by the Dehn–Nielsen–Baer theorem Γ_g^1 is the group of orientation-preserving automorphisms of π_g, namely those which preserve the conjugacy class of the relator instead of reversing it. Further $\Gamma_g = \mathrm{Out}^+(\pi_g) = \mathrm{Aut}^+(\pi_g)/\mathrm{Inn}(\pi_g)$, where $\mathrm{Inn}(\pi_g)$ is the subgroup of inner automorphisms of π_g. There is a more general identification of algebraic and topological mapping class groups, as follows. Denote by γ_j and δ_s the loops around the punctures and respectively the boundary components and by $[z]$ the conjugacy class in $\pi_{g,k}^r$ of the element z. Let $\mathrm{Aut}^+(\pi_{g,k}^r; C_1, \ldots, C_s)$ stands for the subgroup of those automorphisms fixing globally each set of conjugacy classes in $C_1, C_2, \ldots, C_s$. Let P_r be the set of all peripheral conjugacy classes $[\gamma_j]$ and $\mathbf{P}_r$ be the vector consisting of these peripheral conjugacy classes. Similarly, P_k^∂ is the set of all boundary conjugacy classes $[\delta_j]$ and by $\mathbf{P}_k^\partial$ the vector consisting of these peripheral conjugacy classes. Then the Dehn–Nielsen–Baer theorem states that there is an isomorphism:

$$\Gamma_{g,k}^r \simeq \mathrm{Out}^+(\pi_{g,k}^r, P_r, \mathbf{P}_k^\partial) = \mathrm{Aut}^+(\pi_{g,k}^r; P_r, \mathbf{P}_k^\partial))/\mathrm{Inn}(\pi_{g,k}^r).$$

The notation is intended to specify that each boundary conjugacy class is fixed, while the peripheral conjugacy classes are only globally invariant. If we fix the base point of the fundamental group to be among the marked points, it will be automatically invariant by the pure mapping class group so that:

$$P\Gamma_{g,k}^{r+1} \simeq \mathrm{Aut}^+(\pi_{g,k}^r; \mathbf{P}_r, \mathbf{P}_k^\partial).$$

The main questions addressed here concern the finite-dimensional representations of mapping class groups. We shall motivate the introduction of the normal subgroups $\Gamma_{g,k}^p[p]$ generated by p-th powers T_γ^p of Dehn twists T_γ along simple curves γ on the surface. Furthermore, we introduce the family of characteristic quotients

$$\Gamma_{g,k}^r(p) = \Gamma_{g,k}^r/\Gamma_{g,k}^r[p].$$

More generally, if $G \subseteq \Gamma_{g,k}^r$ is a subgroup, then we denote by $G(p)$ the image of G within the quotient $\Gamma_{g,k}^r(p)$.

In the first half of this article we survey some of the known properties and state some questions concerning the groups $\Gamma_{g,k}^r(p)$, in relation with their representations. The main source of finite-dimensional representations for these groups are the modular tensor categories (see [67]), which arose from the seminal work of Witten ([71]), Reshetikhin and Turaev ([60]) on 3-manifold invariants. We discuss some properties of the family of representations associated with the groups $SU(2)/SO(3)$. In the last part we explain some algebraic/geometric constructions of mapping class group representations following Long and Moody.

1.2. **Compact representations of mapping class groups.**

The quotients $\Gamma_{g,k}^r(p)$ arise naturally when we study representations of mapping class groups into compact Lie groups.

Definition 1.1. *A representation $\rho : \Gamma_{g,k}^r \to G$ into a linear algebraic group G is called* unipotent-free *if the images of the Dehn twists are diagonalizable elements in $G_{\mathbb{C}}$.*

For instance, representations into a compact Lie group G are automatically unipotent-free representations.

Proposition 1.1. *Let G be a linear algebraic group. There exists some $p = p(G)$, such that any unipotent-free representation $\rho : \Gamma_{g,k}^r \to G$, for $g \geq 3$ factors through the quotient $\Gamma_{g,k}^r(p)$.*

Proof. The proof given by Aramayona–Souto ([3]) in the case where $\rho(\Gamma_{g,k}^r)$ contains no unipotent elements actually is valid for all unipotent-free representations. We sketch below the main argument, for the sake of completeness. Let $\rho : \Gamma_{g,k}^r \to G_{\mathbb{C}} \subset GL(V)$ be a unipotent-free representation. If γ is a simple curve on a $\Sigma_{g,k}^r$ then at least one component of the surface S obtained by cutting along γ has genus larger than or equal to 2. Note further that the Dehn twist T_γ along a boundary curve γ of the surface S belongs to the center of the mapping class groups $\Gamma(S)$. If W is an eigenspace for $\rho(T_\gamma)$, then W must be invariant by $\rho(\Gamma(S))$. Therefore we obtain a homomorphism $\Gamma(S) \to \mathbb{C}^*$ sending $x \in \Gamma(S)$ into $\det(\rho(x)|_W)$. Since $H_1(\Gamma(S), \mathbb{Q}) = 0$ we derive that this homomorphism has image contained in the group of roots of unity of order 10. In particular, eigenvalues of $\rho(T_\gamma)$ are roots of unity, whose order is bounded in terms of $\dim V$ alone. Since $\rho(T_\gamma)$ is diagonalizable, it has finite order dividing some p which only depends on $\dim V$.

Alternatively, this follows from Bridson's Theorem 2 from [7] which states if $\Gamma_{g,k}^r$ acts by isometries on complete CAT(0) spaces then Dehn twist act either as elliptics or neutral parabolics, the second case being prohibited by the hypothesis. □

Thus the study of unipotent-free representations of mapping class groups pops out the need of understanding the quotients $\Gamma_{g,k}^r(p)$. The simplest constructions of finite-dimensional mapping class group representations yield parabolic matrices for Dehn twists, as it is the case for the homological ones. The first interesting examples of unitary representations arise in topological quantum field theory, by means of the methods pioneered by Reshetikhin and Turaev and further Turaev and Viro. Later one observed that various classical constructions, as the Burau representations of braid groups and homological representations of coverings, as studied by Looijenga ([44]), can lead to unipotent-free representations. Note that, generically, the groups $\Gamma_{g,k}^r(p)$ are infinite (see [29]). Recently, new methods from combinatorial group theory permitted to prove that for any g, r there exists some k_0 such that $\Gamma_g^r(p)$ are acylindrically hyperbolic (see [15]) and also hierarchically hyperbolic (see [5]), when k_0 divides p.

1.3. **Finite index subgroups of $\Gamma_g(p)$.**

Let $\mathcal{I}_{g,k}$ be the k-th Johnson subgroup of Γ_g; in particular $\mathcal{I}_{g,1} = \mathcal{T}_g$ is the Torelli group and $\mathcal{I}_{g,2} = \mathcal{K}_g$ is the Johnson kernel. The following lemma was already used in [30] and we record it here for further use:

Lemma 1.1. *The Torelli group quotient $\mathcal{T}_{g,k}^r(p) \subseteq \Gamma_{g,k}^r(p)$ has finite index.*

Proof. Dehn twists have finite order dividing p in the image. Recall that $\Gamma_{g,k}^r$ is generated by Dehn twists and braids (half-Dehn twists). Dehn twists have finite order dividing p in $\Gamma_{g,k}^r(p)$. Now Dehn twists act as transvections on $H_1(\Sigma_{k,k}^r;\mathbb{Z})$ and they generate the group $\mathrm{Aut}^*(H_1(\Sigma_{k,k}^r;\mathbb{Z}))$ of automorphisms preserving the intersection form. Then p-th powers of Dehn twists generate the congruence subgroup

$$\ker(\mathrm{Aut}^*(H_1(\Sigma_{k,k}^r;\mathbb{Z})) \to \mathrm{Aut}^*(H_1(\Sigma_{k,k}^r;\mathbb{Z}/p\mathbb{Z}))).$$

The exact sequence

$$1 \to \mathcal{T}_{g,k}^r \to \Gamma_{g,k}^r \to \mathrm{Aut}^*(H_1(\Sigma_{k,k}^r;\mathbb{Z}))$$

induces an exact sequence

$$1 \to \mathcal{T}_{g,k}^r(p) \to \Gamma_{g,k}^r(p) \to \mathrm{Aut}^*(H_1(\Sigma_{k,k}^r;\mathbb{Z}/p\mathbb{Z})).$$

This implies that $T_{g,k}^r(p)$ is a finite index normal subgroup of $\Gamma_{g,k}^r(p)$. □

We need the following well-known lemma concerning nilpotent groups (see e.g. [14], Corollary 2.10):

Lemma 1.2. *Let N be a nilpotent group. Then N is finite if and only if $H_1(N)$ is finite.*

If G is a group we denote by $\gamma_k G$ the lower central series of G, $\gamma_1(G) = G$ and $\gamma_{k+1}(G) = [G, \gamma_k(G)]$. For the sake of simplicity we restrict now to the case of closed surfaces.

Proposition 1.2. *For every $k \geq 1$ and $g \geq 3$, the lower central series terms $\gamma_k\mathcal{T}_g(p)$ have finite index in $\mathcal{T}_g(p)$. In particular the images $\mathcal{I}_{g,k}(p)$ of the higher Johnson subgroups $\mathcal{I}_{g,k} \subset \mathcal{T}_g$ are also finite index subgroups of $\Gamma_g(p)$.*

Proof. The Torelli group $\mathcal{T}_g$ is generated by BP pairs, when $g \geq 2$. Lemma 1.1 implies that $\mathcal{T}_g(p)$ is finitely generated, because it is a finite index subgroup of the finitely generated group $\Gamma_g(p)$. In particular $H_1(\mathcal{T}_g(p))$ is of finite type. Since images of BP pairs into $\Gamma_g(p)$ have order p, we derive that the group $H_1(\mathcal{T}_g(p))$ has a generating set consisting of elements of order p and hence is finite. Thus $\mathcal{T}_g(p)/\gamma_k\mathcal{T}_g(p)$ is a finitely generated nilpotent group whose abelianization is finite. It is therefore finite, by Lemma 1.2.

Further, recall that the Johnson filtration is a central descending series with torsion-free quotients (see [40], Prop. 14.5) and in particular

$$\gamma_k(\mathcal{T}_g) \subseteq \mathcal{I}_{g,k}.$$

Now $\mathcal{T}_g/\gamma_k\mathcal{T}_g$ surjects onto $\mathcal{T}_g/\mathcal{I}_{g,k}$ and hence the latter is a nilpotent group. This implies that $\mathcal{T}_g(p)/\mathcal{I}_{g,k}(p)$ is also a nilpotent group. Its abelianization is a quotient of $H_1(\mathcal{T}_g(p))$ and hence $\mathcal{T}_g(p)/\mathcal{I}_{g,k}(p)$ is finite by Lemma 1.2. □

We will show later that the images $G(p)$ of reducible subgroups G of $\Gamma_{g,k}^r$ are of infinite index in $\Gamma_{g,k}^r(p)$. Moreover, there are also irreducible subgroups G of $\Gamma_{g,k}^r$ which not virtually abelian, such that $G(p)$ has infinite index in $\Gamma_{g,k}^r(p)$, as we shall see in Section 2.3. However, the following question seems relevant:

Question 1.1. *Are there irreducible subgroups G of Γ_g^r which are neither virtually abelian nor virtually conjugate into a subgroup of mapping classes which extend to a 3-manifold (e.g. a handlebody group) such that $G(p)$ is of infinite index in $\Gamma_g^r(p)$ for large enough p?*

Propositions 1.1 and 1.2 imply immediately the following generalization of the result obtained in [30] for quantum representations:

Proposition 1.3. *Let $\rho : \Gamma_g \to G$ be a unipotent-free representation into a Lie group G. Assume $g \geq 3$. Then $\rho(\gamma_k\mathcal{T}_g)$ and hence also $\rho(\mathcal{I}_{g,k})$ have finite index in $\rho(\Gamma_g)$.*

1.4. **Kähler groups and rank 1 representations.** It was proved in ([2], Thm. 5 see also Thm. 15 due to Pikaart and Jong) that:

Proposition 1.4. *For every $g \geq 2$ the groups $\Gamma_g^r(p)$ are virtually Kähler, indeed they have finite index subgroups which are fundamental groups of smooth complex projective varieties. In particular, the images $\mathcal{I}_{g,2}(p)$ of the Johnson kernel are Kähler groups, when p is odd.*

Remark 1.1. In recent work [21] the authors were able to show that $\Gamma_g^r(p)$ are actually Kähler.

An immediate consequence of the alternative proved by Delzant (see [17]) for Kähler groups states then:

Proposition 1.5. *Either any solvable quotient of any finite index subgroup of $\mathcal{I}_{g,2}(p)$ is virtually nilpotent or else $\mathcal{I}_{g,2}(p)$ has a finite index subgroup which surjects onto a nonabelian surface group.*

It is presently unknown which one of the two alternatives above holds. However, if Γ_g ($g \geq 3$) does not virtually surject onto $\mathbb{Z}$, then the second alternative cannot hold. We expect that the first alternative only could hold when all solvable quotients are actually finite. This could be proved if we could promote the virtual nilpotence above to a genuine nilpotence.

Proposition 1.6. *Let $f : \Gamma_g \to \Lambda$ be a homomorphism in a torsion-free uniform rank 1 lattice Λ in $SO(1,n)$, with $n \geq 3$. If $g \geq 3$, then f is trivial, i.e. with finite image.*

Proof. Since Λ is cocompact, the homomorphism f is unipotent-free. Therefore there exists some p such that f factors through $\Gamma_g(p)$. Recall that $I_{g,2}(p)$ is Kähler and hence the fundamental group of some compact Kähler manifold $\mathcal{X}$. Then $f|_{I_{g,2}(p)}$ is induced by a map $F : \mathcal{X} \to \mathbb{H}^{n+1}_{\mathbb{R}}/\Lambda$ into a hyperbolic space form. Eells–Sampson [20] proved that then the map F could be assumed to be a harmonic map. A result due to Carlson–Toledo (see [10], Thm. 7.1 and Cor. 3.7) shows that a harmonic map as above factors either through a circle or else through a compact Riemann surface. Thus $f|_{\mathcal{I}_{g,2}(p)}$ factors through $\mathbb{Z}$ or through $\pi_1(\Sigma_h)$.

Now, Dimca and Papadima proved in [19] that $H_1(\mathcal{I}_{g,2})$ is finitely generated and hence $H_1(\mathcal{I}_{g,2}(p))$ is finite, because it is generated by finitely many classes of Dehn twists. We derive that $f|_{\mathcal{I}_{g,2}(p)}$ is trivial and hence f is trivial. □

Question 1.2. *Are there any nontrivial (i.e. non virtually solvable image) homomorphisms $\Gamma_g \to SO(1,n)$, $n \geq 3$ and $g \geq 3$?*

Proposition 1.6 cannot extend to genus 2, as homomorphisms of Γ_2 do not necessarily factor through $\Gamma_2(p)$. However, there exists a nontrivial homomorphism $\Gamma_2(5) \to PU(1,4)$ arising in the Fibonacci TQFT whose image is not virtually solvable, which will be explained later.

2. Quantum representations

2.1. **Arithmetic groups and quantum representations.** We now consider the first examples of unitary (hence unipotent-free) representations of mapping class groups with infinite image (see [29]). Although we use the generic term *quantum* representations for a specific family ρ_p depending on an integer p, one should note that the algebraic machinery of modular tensor categories, in particular quantum groups, provide a large supply of such finite dimensional representations (see [67]). A comprehensive introduction to quantum representations can be found in [47].

Set $U = U(H)$ for the unitary group preserving a Hermitian form H. We suppose that H is associated to a non-degenerate sesquilinear form defined over a totally real number field $\mathbb{K}$. Let $\mathcal{O}_K$ be the ring of algebraic integers in $\mathbb{K}$. The group of integral points $\widetilde{\Lambda} = SU(\mathcal{O}_{\mathbb{K}})$ is a lattice in the group $S\mathbb{U} = Res_{\mathbb{K}/\mathbb{Q}}SU$ obtained by the Weil restriction of scalars from $\mathbb{K}$ to $\mathbb{Q}$. Specifically $S\mathbb{U} = \prod_\sigma SU(f^\sigma)$, where σ belongs to the set of real places of $\mathbb{K}$, i.e. embeddings $\sigma : \mathbb{K} \to \mathbb{R}$, up to conjugacy. Let $H_{g,p}$ be the Hermitian form on the space of conformal blocks $W_{g,p}$ in level p and $U_{g,p} = U(H_{g,p})$. We drop g and/or p when irrelevant.

In the case of the SU(2)/SO(3) theory we have a representation $\tilde{\rho}_p$ of a central extension $\widetilde{\Gamma_g}$ of the mapping class group by $\mathbb{Z}$ into the unitary group $U_{g,p}$, defined over a cyclotomic field $\mathbb{K}_p$.

Note that for odd p, $\mathbb{K}_p$ is the totally real cyclotomic field $\mathbb{Q}(\zeta_p+\overline{\zeta_p})$, when $p\equiv 3 (\mathrm{mod}\ 4)$ and $\mathbb{Q}(\zeta_{4p}+\overline{\zeta_{4p}})$, otherwise.

Now, a key property of this construction is that the corresponding projective representation ρ_p of Γ_g factors through $\Gamma_g(p)$, for odd p and through $\Gamma_g(2p)$, for even p. This is not surprising in view of Proposition 1.1, since these representations are finite-dimensional compact representations.

A deep theorem of Gilmer-Masbaum ([37]) actually says that the image $\widetilde{\mathcal{L}_p}=\widetilde{\rho}_p(\widetilde{\Gamma_g})\subset SU(\mathbb{K}_p)$ is integral when $p\equiv 3(\mathrm{mod}\ 4)$, namely it satisfies:

$$\widetilde{\mathcal{L}_p}\subseteq\widetilde{\Lambda_p}=SU(\mathcal{O}_p).$$

Note that $\widetilde{\Lambda_p}=S\mathbb{U}(\mathbb{Z})$ is a lattice within the linear algebraic group $S\mathbb{U}$ defined over $\mathbb{Q}$. There is a similar *projective* representation $\rho_p:\Gamma_g\to PU$, whose image is $\mathcal{L}_p=\rho_p(\Gamma_g)\subset\Lambda_p=PU(\mathcal{O}_p)$.

Knowing that the images $\widetilde{\mathcal{L}_p}$ of the mapping class groups are (generically) infinite (see [29]), Larsen and Wang proved that for prime $p\geq 5$ they are topologically dense in SU. Eventually the author showed in [30] that:

Proposition 2.1. *The image $\widetilde{\mathcal{L}_p}$ is Zariski dense in $S\mathbb{U}$, if the level p is prime, $p\geq 5$ and $g\geq 2$.*

There are several immediate questions which one could ask concerning the structure of the group $\mathcal{L}_p$ and whose answers might shed light on the structure of mapping class groups. The following seem to be unknown.

Question 2.1 (Arithmeticity)**.** *Is the group $\mathcal{L}_p$ arithmetic, namely of finite index in the higher rank lattice $\Lambda_p=PU_{g,p}(\mathcal{O}_p)$?*

Question 2.2 (Local rigidity)**.** *Is it true that quantum representations are locally rigid within $U_{g,p}$ for prime $p\geq 5$, $g\geq 2$? What about their rigidity in $U(N)\supset U_{g,p}$ or $GL(N,\mathbb{C})$?*

Question 2.3 (Injectivity)**.** *If $g\geq 2, p\geq 5$ is prime, then is $\mathcal{L}_p$ isomorphic to $\Gamma_g(p)$?*

It is proved in [33] that a positive answer to Question 2.3 implies a negative answer to Question 2.1. The arithmeticity of various monodromy groups was already intensively studied in the literature, starting with the non-arithmetic examples of Nori ([55]), the study of thin groups in [63], etc. Venkataramana (see [69]) solved the analog of conjecture 2.1 in the case of the Burau representation of braid groups $B_{n+1}=\Gamma_{0,1}^{n+1}$ at roots of unity of small order $d\leq\frac{n}{2}$, namely precisely the case where there are unipotents. The unipotent-free case is yet unsolved (see also [53]) even for the Burau representation. Moreover, another construction of mapping class group representations, which will be explained later in this article, was shown to provide finite-dimensional representations with arithmetic images in [38]. In this case also the existence of (many) unipotent elements was a key ingredient in the proof of arithmeticity.

Note that $\mathcal{L}_g=\rho_p(\Gamma_g)$ is also a linear group. Indeed $\mathcal{L}_g$ is the quotient of the linear group $\widetilde{\rho_p}(\widetilde{\Gamma_g})$ by the central finite subgroup $\widetilde{\rho_p}(Z(\widetilde{\Gamma_g}))$ where $Z(\widetilde{\Gamma_g})$ denotes the center of $\widetilde{\Gamma_g}$. Thus $\mathcal{L}_p$ is residually finite and hence Hopfian. Therefore the conjecture above is equivalent to the fact that $\ker\rho_p=\Gamma_g[p]$.

Recall that Bridson proved that Γ_g representations into $GL(g,\mathbb{C})$ are rigid because Γ_g has property FA_g. This question is also related to whether Γ_h has the property $(T,\mathbf{F})$, as introduced by Lubotzky and Zimmer, where $\mathbf{F}$ is the family of all finite-dimensional representations. A group Γ has property $(T,\mathbf{F})$ if the trivial representation is isolated in the set of unitary finite-dimensional representations. It is known that $SL(2,\mathbb{Z}\left[\frac{1}{p}\right])$ has property $(T,\mathbf{F})$ but not property T, because the congruence subgroup conjecture holds true for this group.

We have the following more general strong rigidity question, which is unlikely to have a positive answer:

Question 2.4 (Strong rigidity)**.** *Any homomorphism $f:\Gamma_g(p)\to G$ to a semi-simple Lie group G of Hermitian type factors through a homomorphism $S\mathbb{U}_{g,p}\to G$?*

By Simpson's results (see [66]) the validity of the local rigidity conjecture implies that quantum representations arise as factors of variations of Hodge structures (VHS) over $\mathbb{Q}$. Note that it is completely unknown whether a similar result holds for Γ_g, namely:

Question 2.5. *Let $\Gamma_g \to GL(n,\mathbb{C})$ be a locally rigid representation for $g \geq 3$. Then is the associated Γ_g-invariant flat bundle over the Teichmüller space T_g the vector bundle associated to a VHS?*

2.2. **Finite quotients through quantum representations.** If G is a group we denote by $\widehat{G}$ its profinite completion. It is known that $P\mathbb{U}_p(\mathbb{Z})$ has the congruence subgroup property CSP (see [30]), so that the profinite completion $\widehat{\Lambda_p}$ is isomorphic to $P\mathbb{U}(\widehat{\mathbb{Z}})$. Now, the Strong Approximation theorem due to Nori ([55]) and Weisfeiler ([70]) can be used to obtain information about the profinite completion of $\widehat{\mathcal{L}_p}$ and hence $\Gamma_g(p)$. First, let us recall the statement due to Nori for algebraic groups defined over $\mathbb{Q}$:

Theorem 2.1 ([56], Thm.5.4)**.** *Let G be a connected linear algebraic group G defined over $\mathbb{Q}$ and $\Lambda \subset G(\mathbb{Z})$ be a Zariski dense subgroup. Assume that $G(\mathbb{C})$ is simply connected. Then the completion of Λ with respect to the congruence topology induced from $G(\mathbb{Z})$ is an open subgroup in the group $G(\widehat{\mathbb{Z}})$ of points of G over the pro-finite completion $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$.*

Then, the Zariski density theorem 2.1 along with the Strong Approximation Theorem of Nori-Weisfeiler above imply that the image of the homomorphism

$$\hat{i} : \widehat{\mathcal{L}_p} \to P\mathbb{U}_p(\widehat{\mathbb{Z}})$$

is an open subgroup, where $\hat{i}$ denotes the map induced by inclusion $i : \mathcal{L}_p \to P\mathbb{U}_p(\mathbb{Z})$ at the level of profinite completions.

Note that the composition

$$\Gamma_g(p) \stackrel{\rho_p}{\to} \mathcal{L}_p \stackrel{i}{\to} \Lambda_p = P\mathbb{U}_p(\mathbb{Z})$$

cannot be an isomorphism onto a finite index subgroup of Λ_p, as $\Gamma_g(p)$ is not a higher rank lattice (see [33]). If $\mathcal{L}_p$ were a higher rank lattice then it should be an arithmetic subgroup of $P\mathbb{U}_p(\mathbb{Z})$ and hence of finite index. Moreover, in [2] it was proved that $\Gamma_g^1/\Gamma_g^1[p]$ has an infinite series of normal subgroups of infinite index in each other. This shows some evidence that $\Gamma_g(p)$ has many more finite quotients than the lattice $\Lambda_p = P\mathbb{U}_p(\widehat{\mathbb{Z}})$.

Question 2.6 (Arithmetic congruence kernel)**.** *Does the homomorphism induced at profinite completions*

$$\widehat{\Gamma_g(p)} \stackrel{\widehat{\rho_p}}{\to} \widehat{\mathcal{L}_p} \stackrel{\hat{i}}{\to} \widehat{\Lambda_p} = P\mathbb{U}_p(\widehat{\mathbb{Z}})$$

have infinite kernel? Is $\hat{i}$ injective?

If $\hat{i}$ were not injective, then Question 2.1 would have a positive answer.

Remark 2.1. If $\hat{i}$ were an isomorphism and $\mathcal{L}_p$ nonarithmetic, then the inclusion $i : \widehat{\mathcal{L}}_p \to P\mathbb{U}_p(\widehat{\mathbb{Z}})$ would provide additional counter-examples to a conjecture of Grothendieck. If $\mathcal{L}_p$ were locally rigid and nonarithmetic then one would expect it to be super-rigid. If $\mathcal{L}_p$ were not locally rigid then it would not have Kazhdan property T since the finite-dimensional unitary representations of groups with property T are locally rigid (see [59]). Thus $\Gamma_g(p)$ would not have property T.

This gives rise to a new profinite completion $\overline{\Gamma}_g$ of Γ_g, we will call the *q-congruence* completion. A principal q-congruence subgroup is the preimage of an open subgroup of $\prod_{\text{prime } p} \widehat{\Lambda}_p$. These are precisely the intersection of kernels of finitely many epimorphisms onto $P\mathbb{U}_{p_i}(\mathbb{Z}/q_i\mathbb{Z})$. A finite-index subgroup of Γ_g is a q-congruence subgroup if it contains a principal q-congruence subgroup.

As a consequence of the asymptotic faithfulness of the quantum representations by Andersen ([1]), and Freedman–Walker–Wang ([28]), we derive immediately:

Proposition 2.2. *The q-congruence topology is separated, namely $\Gamma_g \to \overline{\Gamma_g}$ is injective.*

This q-congruence topology can be further refined by allowing nonprime p. However, one should note that Proposition 2.1 is not anymore true, when p is not prime. Much more the representation ρ_p might even be reducible in this situation. In this case we need to consider an intermediary group which is the Zariski closure $\mathbb{L}_p$ of $\mathcal{L}_p$. Then $\mathbb{L}_p$ is a linear algebraic subgroup of $S\mathbb{U}_p$ defined over $\mathbb{Q}$. Moreover, the analog of the Gilmer-Masbaum integrality theorem might not hold when p is not prime. In this case $\widetilde{\Lambda_{g,p}}$ is a lattice in a product of p-adic groups.

The above construction is based on a single explicit family of quantum representations ρ_p indexed by the level p. The finite quotients constructed this way seem already form a meaningful and rich enough family (see [30, 50]). However, this is just the simplest possible TQFT, commonly associated with the SU(2)/SO(3) gauge groups. Stepping to arbitrary simple Lie groups, like the $SU(n)$ family might add further finite quotients. However, many of the technical results used above, like the Zariski density theorem 2.1 are yet to be developed in order to obtain clean statements.

Eventually we should note that the results obtained above for Γ_g could in principle be extended to $\Gamma^r_{g,k}$. The case of Γ^1_g was first considered by Koberda–Santharoubane ([42]) and further in [32, 21]. In particular we have linear algebraic groups $\mathbb{U}^1_{g,p}$, playing the role of $\mathbb{U}_p$ and projective representations $\rho_p : \Gamma^1_g \to P\mathbb{U}^1_{g,p}$. Notice that the group and the representation actually depend on the choice of a nonzero color of the marked point, which was chosen to be $p-3$ in [32].

Consider the Birman exact sequence for $g \geq 2$:

$$1 \to \pi_g \to \Gamma^1_g \to \Gamma_g \to 1.$$

The projective representations $\rho^1_p : \Gamma^1_g \to P\mathbb{U}^1_{g,p}$ factors through $\Gamma^1_g(p)$. It is proved in [2] that there is an analog of the Birman exact sequence:

$$1 \to \pi_g(p) \to \Gamma^1_g(p) \to \Gamma_g(p) \to 1$$

where $\pi_g(p) = \pi_g/\pi_g[p]$ is the quotient by the normal subgroup generated by the p-th powers of classes of the *simple* loops on the surface. It appears that ρ^1_p descends to a homomorphism $\rho^1_p : \pi_g(p) \to P\mathbb{U}^1_{g,p}$, whose image will be denoted by $\Pi_{g,p} \subset \Lambda^1_p = P\mathbb{U}^1_{g,p}(\mathbb{Z})$. The corresponding objects with $\tilde{}$ will be corresponding lifts to $S\mathbb{U}^1_{g,p}$.

Then the Zarisky density result in proposition 2.1 was extended in [32] to the punctured case, as follows:

Proposition 2.3. *The image $\rho^1_p(\widetilde{\Pi}_{g,p})$ is Zariski dense in $S\mathbb{U}^1_{g,p}$, if the level p is prime, $p \geq 5$ and $g \geq 2$.*

As a consequence, we obtained in [32] that:

Proposition 2.4. *q-congruence subgroups of Γ^1_g are congruence groups.*

Note that this method provided infinitely many finite simple quotients of π_g which are characteristic.

Question 2.7 (Zariski density)**.** *Is it true that the image of Γ^r_g in the corresponding unitary group $S\mathbb{U}^r_g$ is Zariski dense for nontrivial colors on the punctures and prime levels $p \geq 5$?*

2.3. **Handlebody subgroups.** Let $\mathcal{H}_g$ denote the mapping class group of the handlebody H_g of genus g. We also denote by $\mathcal{H}^r_g$ the mapping class group of the handlebody H_g with r marked points on the boundary. It is well-known that the restriction of homeomorphisms to the boundary induces an injective homomorphism

$$\mathcal{H}^r_g \to \Gamma^r_g$$

which permits to identify handlebody mapping class groups to subgroups of the mapping class groups.

The action of homomorphisms on the free group $\mathbb{F}_g = \pi_1(H_g)$ yields a surjective homomorphism

$$\mathcal{H}_g \to \mathrm{Out}(\mathcal{F}_g)$$

whose kernel $Tw(H_g)$ is the group of twists of the handlebody. The group $Tw(H_g)$ is generated by the Dehn twists along meridians, namely essential simple curves on the surface Σ_g bounding disks embedded into H_g. Let now $K_g = \ker(\pi_1(\Sigma_g) \to \pi_1(H_g))$ be the group of meridian curves. Then the Birman exact sequence induces a commutative diagram with exact rows and columns:

$$\begin{array}{ccccc} K_g & \to & Tw(H_g^1) & \to & Tw(H_g) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_g & \to & \mathcal{H}_g^1 & \to & \mathcal{H}_g \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{F}_g & \to & \mathrm{Aut}(\mathbb{F}_g) & \to & \mathrm{Out}(\mathbb{F}_g) \end{array}$$

An immediate consequence of proposition 2.3 and ([32], Remark 4.4) is the following:

Proposition 2.5. *The image $\tilde{\rho}_p^1(\mathcal{H}_g^1)$ is Zariski dense in $S\mathbb{U}_{g,p}^1$, if the level p is prime, $p \geq 5$ and $g \geq 2$. In particular, $\mathcal{H}_g^1$ surjects onto infinitely many simple nonabelian groups of the form $P\mathbb{U}_{g,p}^1(\mathbb{Z}/q\mathbb{Z})$, for large prime q. Thus $\mathcal{H}_g^1$ and Γ_g^1 are residually simple.*

We derive that the Frattini subgroup is trivial for Γ_g^1, complementing the result obtained in [51] for closed surfaces:

Corollary 2.2. *If $\Phi_f(G)$ denotes the intersection of all finite index maximal subgroups of G, then $\Phi_f(\Gamma_g^1) = 1$, for $g \geq 2$.*

By direct computation of the image of two twists along meridians which intersect in two points, as in [42], we derive that $\rho_p^1(K_g)$ and hence $\rho_p^1(Tw(\mathcal{H}_g^1))$ are infinite and hence they are topologically dense in $PU_{g,p}^1$. Then the method of [32] shows that the above result also holds for the twist groups:

Proposition 2.6. *The image $\rho_p^1(K_g)$ and so $\rho_p^1(Tw(\mathcal{H}_g^1))$ is Zariski dense in $P\mathbb{U}_{g,p}^1$, if the level p is prime, $p \geq 5$ and $g \geq 2$. In particular $Tw(\mathcal{H}_g^1))$ surjects onto infinitely many simple nonabelian groups of the form $P\mathbb{U}_{g,p}^1(\mathbb{Z}/q\mathbb{Z})$, for large prime q.*

This is not surprising, as conjecturally, every irreducible subgroup of Γ_g^r which is not virtually abelian should have a large image by ρ_p^r. However, the case of $\mathcal{H}_g^1$ is interesting by itself. In fact the family of finite quotients obtained above for large q do not separate the subgroup $\mathcal{H}_g^1$ within Γ_g^1, as both groups have the same images under ρ_p^1. Let us project down to $\mathcal{H}_g$ by means of the forgetful homomorphism $p : \Gamma_g^1 \to \Gamma_g$. Although $\rho_p(\Gamma_g)$ is Zariski dense in $P\mathbb{U}_{g,p}$, the image $\rho_p(\mathcal{H}_g)$ of the handlebody group is not. Indeed the restriction of ρ_p to $\mathcal{H}_g$ is not even irreducible, as it preserves the null-vector $w_{g,p} \in W_{g,p}$, which is the vector associated to the handlebody H_g. This holds more generally for any subgroup of mapping classes extending to a compact 3-manifold. In particular we have a map $\theta_p : \Gamma_g/\mathcal{H}_g \to PW_{g,p}$ defined by

$$\theta_p(x) = \rho_p(x) w_{g,p} \in PW_{g,p}$$

where PW denotes the projective space associated to the vector space W. The topological density of $\rho_p(\Gamma_g)$ implies that the image of θ_p is topologically dense in $PW_{g,p}$ and in particular it is infinite, when p is prime. A simpler argument consists in following the proof in [29] for the infiniteness of $\rho_p(\Gamma_g)$. There is a subrepresentation of $\rho_p|_{B_3}$ which can be identified with the Burau representation at a root of unity. But these representations have not finite orbits (see e.g. [31]). Since the map θ_p factors through $\Gamma_g(p)/\mathcal{H}_g(p)$, we derive:

Proposition 2.7. *The groups $\mathcal{H}_g^r(p)$ have infinite index in $\Gamma_g^r(p)$, when the latter are infinite.*

This method extends readily to prove the following:

Proposition 2.8. *Let $G \subset \Gamma_g$ be a reducible subgroup. Then $G(p)$ is of infinite index in $\Gamma_g(p)$, for prime $p \geq 5$, $g \geq 2$.*

Proof. It is enough to consider the case when G is the stabilizer of a simple closed nonperipheral curve γ on $\Sigma_{g,k}^r$. Then $\rho_p(G)$ is centralized by $\rho_p(T_\gamma)$ and hence its topological closure is a proper subgroup of $PU_{g,p}$. The map sending $x \in \Gamma_g/G$ to the class of $\rho_p(x)$ in the homogeneous space obtained by quotienting $PU_{g,p}$ by the closure of $\rho_p(G)$, has infinite image, by density. But this map factors through $\Gamma_g(p)/G(p)$, thereby proving the claim. □

Question 2.8. *If G is an irreducible subgroup of Γ_g which is neither virtually abelian nor conjugate into a subgroup of mapping classes extending to some compact orientable 3-manifold (like $\mathcal{H}_g$), does it have irreducible or even Zariski dense image in $S\mathbb{U}_{g,p}$, for large enough primes p? Moreover, in the case when it is Zariski dense, is $\rho_p(G)$ of finite index in $\rho_p(\Gamma_g)$? A particularly interesting case is $D_3I_{g,2}$, the second commutator group of the Johnson kernel $\mathcal{I}_{g,2}$. If $\varphi \in \Gamma_g$ is such that $\rho_p(\varphi)$ fixes the line spanned by the null-vector $w_{g,p}$, for infinitely many p, does it follow that $\varphi \in \mathcal{H}_g$?*

2.4. **Power quotients of surface groups.**

Proposition 2.9. *$\pi_g/\pi_g[p]$ is not boundedly generated for large p.*

Proof. A finitely generated infinite torsion group is not boundedly generated. As $\pi_g/\pi_g[p]$ surjects onto the Burnside group π_g/π_g^p associated to π_g, which is infinite for large p, we derive the claim. □

It is known (see e.g. discussion in [24]) that arithmetic groups have the Congruence Subgroup Property if and only if their profinite completion is boundedly generated as a profinite group, namely a finite product of pro-cyclic groups. In our case $\mathbb{U}_p(\mathbb{Z})$ have CSP, and thus they are boundedly generated, see also [11]. Note that Γ_g are not boundedly generated, as one proved that their pro-p completion is not a p-adic analytic group. We expect $\Gamma_g/\Gamma_g[p]$ to be not boundedly generated.

2.5. **Infinite index subgroups of $\Gamma_g(p)$.** In [2] one proved that $\Gamma_g^1(p)$ have infinite nested sequences of normal subgroups, each one of infinite index into the previous one. This shows that $\Gamma_g^1(p)$ is far from being a higher rank lattice. This last result also holds for $\Gamma_g(p)$, although the former statement is unknown. However, we can infer from [5, 15] that $\Gamma_g(p)$ is SQ-universal, namely every countable group is a subgroup of some quotient of it.

By our results above we can see that every finite group is also a finite quotient of Γ_g (see [30, 50]). However this question is open for $\Gamma_g(p)$. An easy consequence of deep results of Margulis-Soifer is:

Proposition 2.10. *The groups $\Gamma_g(p)$, where $p \geq 5$, if p is odd, and $p/4 \geq 5$ when p is even, $g \geq 3$ and additionally $(g,p) \neq (2,24)$ admit (uncountably many) free infinite-index maximal subgroups.*

Proof. If p is generic then $\widetilde{\rho}_p(\widetilde{\Gamma_g})$ is a linear group which contains free non-abelian groups (see [31]). According to a result of Margulis and Soifer (see [48]) there exist uncountably many free (infinitely generated) subgroups $\mathbb{F}_\infty \subset \widetilde{\rho}_p(\widetilde{\Gamma_g})$ which are pro-finitely dense, namely they map surjectively onto every finite quotient of $\widetilde{\rho}_p(\widetilde{\Gamma_g})$. The image F_∞ of $\mathbb{F}_\infty$ into the quotient $\rho_p(\Gamma_g)$ of $\widetilde{\rho}_p(\widetilde{\Gamma_g})$ by a finite central group is still pro-finitely dense. We know that ρ_p factors through a homomorphism $\overline{\rho_p} : \Gamma_g(p) \to \rho_p(\Gamma_g)$. Let now W be a proper maximal subgroup of $\Gamma_g(p)$ containing $\overline{\rho_p}^{-1}(F_\infty)$. If W were of finite index in $\Gamma_g(p)$ then $\Gamma_g(p)/W$ would be a finite quotient of $\rho_p(\Gamma_g)$ in which F_∞ maps to the identity. This contradicts the fact that F_∞ is pro-finitely dense. Therefore, W is of infinite index in $\Gamma_g(p)$. □

Remark 2.2. The subgroups W from above are not normal subgroups of $\Gamma_g(p)$, in general. If W were normal, then the quotient $\Gamma_g(p)/W$ should be an infinite simple group, by the maximality of W.

2.6. **A nontrivial homomorphism of Γ_2 into a rank-1 lattice.**

2.6.1. *Hyperelliptic involutions.* The genus 2 closed orientable surface is a double covering of the sphere ramified at 6 points. The deck transformation group is generated by a hyperelliptic involution. From [6] all mapping classes in Γ_2 have $\mathbb{Z}/2\mathbb{Z}$-invariant representatives and isotopies can be promoted to $\mathbb{Z}/2\mathbb{Z}$-invariant isotopies, so that we have the following exact sequence:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \Gamma_2 \to \Gamma_0^6 \to 1$$

where the central kernel is generated by the hyperelliptic involution. Further Γ_0^6 is a quotient of B_6. Specifically, we have the usual presentation:

$$B_6 = \langle b_1, b_2, ..., b_5; b_ib_j = b_jb_i, |i-j| \geq 2, b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1}, i \leq 4\rangle.$$

The usual braid relations are recorded as `Braid`. Then we have the following quotient presentations:

$$\Gamma_0^6 = \langle b_1, b_2, ..., b_5; \texttt{Braid}, (b_1 b_2 \cdots b_5)^6 = 1, b_5 b_4 \cdots b_2 b_1^2 b_2 \cdots b_4 b_5 = 1\rangle.$$

We denote by $\Delta_6^2 = (b_1 b_2 \cdots b_5)^6$ the generator of the infinite cyclic center $Z(B_6)$ of B_6. The element $h_6 = b_5 b_4 \cdots b_2 b_1^2 b_2 \cdots b_4 b_5$ corresponds to the hyperelliptic involution. The spherical braid group $B_6(S^2)$ on 6 strands on the sphere is then given by

$$B_6(S^2) = \langle b_1, b_2, ..., b_5; \texttt{Braid}, b_5 b_4 \cdots b_2 b_1^2 b_2 \cdots b_4 b_5 = 1\rangle.$$

According to Faddell and Neuwirth ([22]) we have an exact sequence:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to B_6(S^2) \to \Gamma_0^6 \to 1$$

where the kernel $\mathbb{Z}/2\mathbb{Z}$ is central and generated by the image of Δ_6^2 in $B_6(S^2)$. On the other hand we have the following presentation of the mapping class group in genus 2 due to Birman and Hilden (see [6]):

$$\Gamma_2 = \langle b_1, b_2, ..., b_5; \texttt{Braid}, (b_1 b_2 b_3)^4 = b_5^2, [h_6, b_1] = 1, h_6^2 = 1\rangle.$$

We also have the following exact sequence:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \Gamma_2 \to \Gamma_0^6 \to 1$$

where the center $\mathbb{Z}/2\mathbb{Z}$ of Γ_2 is generated by the image of the hyperelliptic involution h_6. This exact sequence comes up with another presentation of Γ_2, as follows:

Lemma 2.1. *We have*

$$\Gamma_2 = \langle b_1, b_2, ..., b_5; \texttt{Braid}, (b_1 b_2 \cdots b_5)^6 = 1, [h_6, b_i] = 1, h_6^2 = 1\rangle.$$

Proof. Remark that we have the following relations in B_6:

$$\Delta_6^2 = h_6 (b_4 b_3 b_2 b_1)^5,$$

$$(b_4 b_3 b_2 b_1)^5 = [b_1 b_2 b_1 b_4^{-1}, b_1 b_2 b_3 b_4](b_1 b_2 b_3 b_4)^5,$$

and

$$(b_1 b_2 b_3 b_4)^5 = (b_1 b_2 b_3)^4 b_4 b_3 \cdots b_1^2 \cdots b_3 b_4 = (b_1 b_2 b_3)^4 b_5^{-2} [b_5, h_6] h_6.$$

Therefore the relation $\Delta_6^2 = 1$ is equivalent to the 3-chain relation $(b_1 b_2 b_3)^4 = b_5^2$, in the presence of the braid relations and the centrality of h_6. □

The subsurface $\Sigma_{1,2} \subset \Sigma_2$ inherits a hyperelliptic involution which exchanges the boundary circles, so that it is a double covering of the sphere ramified at 4 points. Then from ([6], see also [23], 9.4.1) we have an identification between $\Gamma_{1,2}$ and $B_4/Z(B_4)$. We derive the presentation:

$$\Gamma_{1,2} = \langle b_1, b_2, b_3; \texttt{Braid}, (b_1 b_2 b_3)^4 = 1\rangle.$$

2.6.2. *The Jones representation of* Γ_2. The Jones representation $J_q : B_g \to GL(5, \mathbb{Z}[q, q^{-1}])$ is the representation of the Hecke algebra at q corresponding to the rectangular Young diagram associated to the partition 2^3. Specifically we have:

$$J_q(b_1) = \begin{pmatrix} -1 & 0 & 0 & 0 & q \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & q \end{pmatrix}, \; J_q(b_2) = \begin{pmatrix} q & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 \\ 0 & q & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}, \; J_q(b_3) = \begin{pmatrix} -1 & 0 & 0 & q & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & q & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

$$J_q(b_4) = \begin{pmatrix} q & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & q \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & q \end{pmatrix}, \; J_q(b_5) = \begin{pmatrix} -1 & q & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}, \; J_q(\delta_6) = q^2 \begin{pmatrix} 0 & 0 & 1 & 0 & q \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Here $\delta_6 = b_1b_2b_3b_4b_5$. Observe that this is slightly modified with respect to the original one from ([41], section 10, p. 362), such that the eigenvalues of $J_q(b_i)$ are 1 and q, with multiplicities 3 and 2 respectively, for reasons which will be become clear later.

The Jones representation is of *Hecke type*, namely it factors through the Hecke algebra $H(q,6)$ defined as the quotient algebra:

$$H(q,n) = \mathbb{C}[B_n]/(b_i^2 + (1-q)b_i - q).$$

2.6.3. *Fibonacci representations in genus 2.* The Fibonacci TQFT is the $SO(3)$-TQFT at $p=5$. In order to fix completely the theory we have to choose a primitive 10-th root of unity A. There are only two colors $\{0,2\}$ and thus we can compute explicitly the dimension of the space of conformal blocks $W_g(2^k)$ of the surface of genus g with k boundary components labeled by the color 2. Recall that the boundary components labeled with the color 0 could be filled in by disks. Then

$$\dim W_{g,5}(2^k) = 5^{\frac{g-1}{2}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^{g+n-1} + (-1)^g \left(\frac{1-\sqrt{5}}{2}\right)^{g+n-1} \right).$$

If we wish to specify the number of boundary components labeled by 2 we will write $\tilde{\rho}_{g,5;k}$ for the corresponding representation in genus g and level 5.

Lemma 2.2. *The representation $\rho_{g,5;k}$ is irreducible, as soon as $\dim W_{g,5}(2^k) \geq 1$.*

Proof. The irreducibility of representations arising in the $SU(2)$-TQFT for all roots of unity of order $4p$, with prime p was proved by Roberts ([61]). The proof works ad-literam for the $SO(3)$-TQFT at roots of unity of order $2p$, with prime p. A different proof is provided in ([27], Prop. 6.4) at $p=5$. □

Therefore the representation $\tilde{\rho}_{2,5}$ is irreducible. Composition with the obvious surjection $B_6 \to \Gamma_2$ provides a projective representation still denoted $\rho_2 : B_6 \to PU(W_2, H_A)$, where H_A is the Hermitian form associated to the primitive root of unity A. By the formula above we have $\dim W_2 = 5$.

The first main result of this section is:

Proposition 2.11. *The representation $\rho_{2,5} : B_6 \to PU(W_2, H_A)$ is equivalent to the projectivisation of the Jones representation $-J_q$ at a primitive 10-th root of unity $q = -A^8$.*

Proof. It is proved in ([27], Thm. 6.2) that not only $\rho_{2,5}$ is irreducible but its image is topologically dense within $PU(W_2, H_A)$, when $A = \exp\left(\frac{6\pi i}{10}\right)$ (see also [43] for the more general case). Notice that in this case H_A is positive definite and the group $PU(W_2, H_A)$ can be identified with the compact group $PU(5)$. This implies that the image of $\rho_{2,5}$ is Zariski dense in $PU(W_2, H_A)$ for every 10-th primitive root of unity A.

The first observation is that the Zariski density of the image of ρ_2 implies that the image of the hyperelliptic involution is trivial, so that ρ_2 factors through Γ_0^6.

Recall now that we have a linear representation $\widetilde{\rho}_{2,5} : \widetilde{\Gamma}_2 \to U(W_2, H_A)$ of a central extension of Γ_2 by $\mathbb{Z}$ which lifts $\rho_{2,5}$. The description of $\widetilde{\Gamma}_2$ in terms of group presentations was given by Gervais in [36]: we only have to replace the chain relation by $(b_1b_2b_3)^4 = z^{12}b_5^2$, where z is a central infinite-order additional generator. According to Lemma 2.1 this amounts to the following presentation:

$$\widetilde{\Gamma}_2 = \langle b_1, b_2, ..., b_5, z; \texttt{Braid}, (b_1b_2\cdots b_5)^6 = z^{12}, [z, b_i] = 1, [h_6, b_i] = 1, h_6^2 = 1\rangle.$$

The pull-back of the central extension $\widetilde{\Gamma}_2 \to \Gamma_2$ by the homomorphism $B_6 \to \Gamma_2$ is a central extension of B_6 by $\mathbb{Z}$. Arnold ([4], see also [68], Thm. 4.3) proved that the cohomology of braid groups stabilizes $H^k(B_n;\mathbb{Z}) = H^k(B_{2k-2};\mathbb{Z})$ for $n \geq 2k-2$, so that $H^2(B_n;\mathbb{Z}) = H^2(B_2;\mathbb{Z}) = 0$, for $n \geq 2$. This proves that linear representations of central extensions of B_n by $\mathbb{Z}$ lift to linear representations of B_n. Actually the presentation we gave for $\widetilde{\Gamma}_2$ makes it clear that the tautological map on generators is a well-defined homomorphism $B_6 \to \widetilde{\Gamma}_2$. In particular there is a lift $\hat{\rho}_{2,5}$:

$B_6 \to U(W_2, H_A)$ of $\widetilde{\rho}_2$. Lemma 2.2 shows that this linear representation is irreducible. Moreover this representation verifies $\hat{\rho}_{2,5}(h_6) = 1$.

The classification of 5-dimensional irreducible representations of B_6 was given by Formanek ([25], see also [26] for a systematic description). Following ([26], Thm. 14) they are of Hecke type, namely they factor through the Hecke algebra $H(q, 6)$ for some q. Moreover, up to tensoring with a 1-dimensional representation these are equivalent to the specialization of either the Burau representation which corresponds to the Young diagram associated to the partition 21^4 or else to another representation which corresponds to the Young diagram associated to the partition 2^3.

In order to decide which one appears one has to compare the eigenvalues of the Dehn twists along with their multiplicities. In the case of $\hat{\rho}_2(b_i)$ they are $(1, 1, 1, A^8, A^8)$, up to a scalar, for the Burau representation the eigenvalues are $(-1, -1, -1, -1, q)$, while in the case of the Jones representation J_q they are $(-1, -1, -1, q, q)$. It follows that $q = -A^8$ is a primitive 10-th root of unity and that $\hat{\rho}_2$ cannot be equivalent to the Burau representation. We derive that $\hat{\rho}_2$ is equivalent to $-J_{-A^8}$. Notice also that the Jones representation factors through $\Gamma_{0,6}$ and Dehn twists are of order 5. $\square$

The non-degenerate Hermitian form H_A has signature $(5, 0)$, when $A = \exp\left(\pm\frac{6\pi i}{10}\right)$ and signature $(1, 4)$, when $A = \exp\left(\pm\frac{2\pi i}{10}\right)$, respectively. In particular we obtain a homomorphism $f : \Gamma_2(5) \to PU(1, 4)$.

In [2] the authors proved that $\mathcal{K}_g(p) = I_{g,2}(p)$ is a Kähler group, for any $g \geq 2$ and odd p. In particular, $\mathcal{K}_2(p) = \mathcal{T}_2(p)$ is a Kähler group. Instead of restricting f to $\mathcal{T}_2(5)$ we will work directly with $\Gamma_2(5)$, knowing that in [21] it was proved that $\Gamma_g(p)$ is a Kähler group. Let $X_2(5)$ be a complex projective variety with fundamental group $\Gamma_2(5)$. The constructions in [2, 21] show that we can take for $X_2(5)$ a compactified moduli space of curves with level structure, and in particular $\dim_{\mathbb{C}} X_2(5) = 3$. Consider next a finite index torsion-free subgroup $\Lambda \subset SU(1, 4)(\mathcal{O}_{10})$ and let J be its preimage within $\Gamma_2(5)$.

According to Eells–Sampson $f : J \to \Lambda$ is induced by a harmonic map $F : X_2(5) \to Z$, where Z is the compact complex hyperbolic manifold $H^4_{\mathbb{C}}/\Lambda$. Moreover, Carlson–Toledo proved in ([10], Thm. 7.2) that either F has rank at most 2, or else we can take F to be holomorphic or anti-holomorphic. It seems that F can be taken to be holomorphic or anti-holomorphic, some evidence being provided by the following:

Proposition 2.12. *The virtual cohomological dimension of $f(\Gamma_2(5))$ is at least 4.*

Proof. Consider the stabilizer of a nontrivial nonseparating simple closed curve γ on Σ_2. This is isomorphic to $\Gamma_{1,2}/\langle ab^{-1}\rangle$, where a, b denote the Dehn twists along the boundary components. If we label γ by the color 2 then we obtain a subspace V of the space of conformal blocks in genus 2, which is invariant by the action of the stabilizer. Its orthogonal $V^\perp$ with respect to the Hermitian form H_A is the 2-dimensional subspace corresponding to the label 0 of γ. Since $\Gamma_{1,2}$ is isomorphic to $B_4/Z(B_4)$ we obtain two representations of $\beta : B_4 \to GL(V)$, and $\gamma : B_4 \to GL(V^\perp)$.

Now, both β and γ are factors of the restriction of $\hat{\rho}_{2,5}$ to B_4. The restriction of a Hecke type representation of B_n to the subgroup $B_{n-1} \subset B_n$ splits into irreducible components indexed by the Young subdiagrams with one box less. It follows that the restriction of J_{-A^8} to B_5 is the irreducible representation with Young diagram $2^2 1$ and the restriction to B_4 has two irreducible components corresponding to 21^3 (of dimension 3) and 2^2 (of dimension 2). Notice that indeed these two representations are irreducible of the same dimension as the ones for generic q (see [26]). It follows that β is equivalent to the Burau representation β_{-A^8} while γ is equivalent to the Hecke type representation associated to the partition 2^2.

Recall also that the curves labeled 0 in conformal blocks can be filled in by disks. This means that the projectivization of γ factors through the mapping class group of the torus. Thus γ factors through the folding homomorphism $i : B_4 \to B_3$ given by $i(b_1) = b_1, i(b_2) = b_2, i(b_3) = b_1$. It is known that $\ker i \subset B_4$ is the free group on two generators $b_1 b_3^{-1}, b_2 b_1 b_3^{-1} b_2^{-1}$. Therefore the B_3 representation obtained from γ is the restriction of $-J_{-A^8}$ to B_3, namely the Burau representation β_q of B_3 at the primitive 10-th root of unity $q = -A^8$.

Recall that the reduced Burau representation β_q is given by:

$$\beta(b_1) = \begin{pmatrix} q & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \beta(b_2) = \begin{pmatrix} 1 & 0 & 0 \\ q & q & -1 \\ 0 & 0 & -1 \end{pmatrix}, \beta(b_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -q & q \end{pmatrix}.$$

Following ([31], Prop.3.1) the image of $\gamma(B_4) = \beta_q \circ i(B_4) \subset U(V^\perp, H_A)$ is $B_3/B_3[5]$, where $B_3[5]$ is the normal subgroup of B_3 generated by b_i^5. Moreover this group is isomorphic to $GL(2, \mathbb{Z}/5\mathbb{Z})$, of order 600.

Further, β is equivalent to the Burau representation β_q of B_4 at q. Moreover this representation preserves the Hermitian form H_q, whose signature is $(3,0)$ when $q = \exp\left(\frac{\pm 6\pi i}{10}\right)$ (i.e. $A = \exp\left(\frac{\pm 2\pi i}{10}\right)$) and of signature $(1,2)$ when $q = \exp\left(\frac{\pm 2\pi i}{10}\right)$ (i.e. $A = \exp\left(\frac{\pm 6\pi i}{10}\right)$). The real points of the linear algebraic group obtained by the restriction of scalars $\mathbb{Q}(q+\overline{q}) : \mathbb{Q}$ is therefore isomorphic to the product $U(3) \times U(1,2)$. Therefore $\beta_q(B_4) \subset U(V)$ is a discrete subgroup.

McMullen proved in [53] that in this case the image of the Burau representation of B_4 at $q = \exp\left(\frac{6\pi i}{10}\right)$ is a lattice in $PU(1,2) = PU(V, H_q)$. This is a cocompact arithmetic lattice. In particular the image of $f(\Gamma_2(5))$ contains a cocompact lattice in $PU(1,2)$ whose virtual cohomological dimension is 4. Now, the virtual cohomological dimensions decreases when passing to subgroups (see [9], VIII, 11, ex.1, Prop. 2.4) and hence the claim follows. $\square$

3. Local rigidity of Weil representations

The simplest test for the rigidity questions is the case when the representations have finite images. Among those, the Weil representations were intensively studied (see [35] for a brief history). Weil representations could be defined also by geometric quantization or in the quantum groups framework. They were rediscovered within the framework of Chern–Simons theory with abelian gauge group $U(1)$.

Let $k \geq 2$ be an integer, and denote by $\langle , \rangle$ the standard bilinear form on $(\mathbb{Z}/k\mathbb{Z})^g \times (\mathbb{Z}/k\mathbb{Z})^g \to \mathbb{Z}/k\mathbb{Z}$. The Weil representation we consider is a representation in the unitary group of the complex vector space $\mathbb{C}^{(\mathbb{Z}/k\mathbb{Z})^g}$ endowed with its standard Hermitian form. Notice that the canonical basis of this vector space is canonically labeled by elements in $\mathbb{Z}/k\mathbb{Z}$.

It is well-known that $Sp(2g, \mathbb{Z})$ is generated by the matrices having one of the following forms: $\begin{pmatrix} \mathbf{1}_g & B \\ 0 & \mathbf{1}_g \end{pmatrix}$ where $B = B^\top$ has integer entries, $\begin{pmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{pmatrix}$ where $A \in GL(g, \mathbb{Z})$ and $\begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix}$.

We can now define the Weil representations on these generating matrices as follows:

$$\rho_{g,k}\begin{pmatrix} \mathbf{1}_g & B \\ 0 & \mathbf{1}_g \end{pmatrix} = \mathrm{diag}\left(\exp\left(\frac{\pi\sqrt{-1}}{k}\langle m, Bm\rangle\right)\right)_{m \in (\mathbb{Z}/k\mathbb{Z})^g}, \tag{1}$$

where diag stands for diagonal matrix with given entries;

$$\rho_{g,k}\begin{pmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{pmatrix} = (\delta_{A^\top m, n})_{m,n \in (\mathbb{Z}/k\mathbb{Z})^g}, \tag{2}$$

where δ stands for the Kronecker symbol;

$$\rho_{g,k}\begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix} = k^{-g/2} \exp\left(-\frac{2\pi\sqrt{-1}\langle m, n\rangle}{k}\right)_{m,n \in (\mathbb{Z}/k\mathbb{Z})^g}. \tag{3}$$

For even k these formulas define a unitary projective representation $\rho_{g,k}$ of $Sp(2g, \mathbb{Z})$ in $U(k^g)/R_8$, where $R_8 \subset U(1) \subset U(\mathbb{C}^N)$ is the subgroup of scalar matrices whose entries are roots of unity of order 8. For odd k the same formulas define representations of the theta subgroup $Sp(2g, 1, 2)$. There is however an extension of this representation to the whole symplectic group $Sp(2g, \mathbb{Z})$, as defined by Murakami, Ohtsuki and Okada in [54]. Notice that by construction $\rho_{g,k}$ factors through $Sp(2g, \mathbb{Z}/2k\mathbb{Z})$ for even k and through the image of the theta subgroup in $Sp(2g, \mathbb{Z}/k\mathbb{Z})$ for odd k.

In [35] we proved that the projective Weil representation $\rho_{g,k}$ of $Sp(2g, \mathbb{Z})$, for $g \geq 3$ and even k does not lift to linear representations of $Sp(2g, \mathbb{Z})$, namely it determines a generator of

$H^2(Sp(2g,\mathbb{Z}/2k\mathbb{Z});\mathbb{Z}/2\mathbb{Z})$ and hence a (universal) central extension $\tilde{Sp}(2g,\mathbb{Z}/2k\mathbb{Z})$ of $Sp(2g,\mathbb{Z}/2k\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$. In fact $H_2(Sp(2g,\mathbb{Z}/k\mathbb{Z}))=\mathbb{Z}/2\mathbb{Z}$, if and only if k is divisible by 4, while for other cases, it vanishes (see [34]). For odd k it was already known that Weil representations did not detect any non-trivial element, i.e. that the projective representation $\rho_{g,k}$ lifts to a linear representation. By pulling-back the central extension of $\tilde{Sp}(2g,\mathbb{Z}/2k\mathbb{Z})$ to Γ_g we obtain a central extension $\tilde{\Gamma}_g$ by $\mathbb{Z}/2\mathbb{Z}$, endowed with a linear representation $\rho_{g,k}^{U(1)}$ into $U(k^g)$. We then have an exact sequence

$$1\to\Gamma_g((k))\to\tilde{\Gamma}_g\to\tilde{Sp}(2g,\mathbb{Z}/k\mathbb{Z})\to 1$$

where

$$\Gamma_g((2k))=\ker(\Gamma_g\to Sp(2g,\mathbb{Z}/2k\mathbb{Z}))$$

is the so-called abelian level $2k$ mapping class group.

Proposition 3.1. *If $g\geq 3$ the $U(1)$ representations $\rho_{g,k}^{U(1)}:\tilde{\Gamma}_g\to U(k^g)\subset GL(n,\mathbb{C})$ are locally rigid as $GL(n,\mathbb{C})$ representations.*

Proof. We have the five term exact sequence in cohomology:

$$0\to H^1(\tilde{Sp}(2g,\mathbb{Z}/2k\mathbb{Z}),\mathfrak{gl}_n^{\Gamma_g((2k))})\to H^1(\tilde{\Gamma}_g,\mathfrak{gl}_n)\to H^1(\Gamma_g((2k)),\mathfrak{gl}_n)^{\tilde{Sp}(2g,\mathbb{Z}/2k\mathbb{Z})}\to$$
$$\to H^2(\tilde{Sp}(2g,\mathbb{Z}/k\mathbb{Z}),\mathfrak{gl}_n^{\Gamma_g((2k))})\to H^2(\tilde{\Gamma}_g,\mathfrak{gl}_n).$$

Here we use the cohomology with twisted coefficients, where the action of $\tilde{\Gamma}_g$ on the Lie algebra $\mathfrak{gl}_n$ is by $Ad\circ\rho_{g,k}^{(U(1)}$. Since the action of $\Gamma_g((2k))$ is trivial we have

$$H^1(\Gamma_g((2k)),\mathfrak{gl}_n)=\mathrm{Hom}(\Gamma_g((2k)),\mathfrak{gl}_n)=0,\text{when } g\geq 3.$$

In fact $\mathfrak{gl}_n$ is considered here with its structure of abelian group and thus any homomorphism $\Gamma_g((2k))\to\mathfrak{gl}_n$ factors through $H_1(\Gamma_g((2k)))$. Now, McCarthy proved in [52] that $H^1(\Gamma_g((k)))=0$, for every k and $g\geq 3$. This is actually true for any finite index subgroup of Γ_g which contains the Torelli group. See for instance [64] for the precise description of the finite group $H_1(\Gamma_g((k)))$.

Then we have:

$$\begin{aligned}H^1(\tilde{Sp}(2g,\mathbb{Z}/k\mathbb{Z}),\mathfrak{gl}_n^{\Gamma_g((2k)))}) &= H^1(\tilde{Sp}(2g,\mathbb{Z}/2k\mathbb{Z}),\mathfrak{gl}_n)=\\ &= H^1(\tilde{Sp}(2g,\mathbb{Z}/2k\mathbb{Z}),\mathfrak{gl}_{k^g})\oplus H^1(\tilde{Sp}(2g,\mathbb{Z}/2k\mathbb{Z}),\mathfrak{gl}_{n-k^g}).\end{aligned}$$

Now $H^1(\tilde{Sp}(2g,\mathbb{Z}/2k\mathbb{Z}),\mathfrak{gl}_{n-k^g})=0$ by the universal coefficients theorem, since the action is trivial and $H^1(\tilde{Sp}(2g,\mathbb{Z}/2k\mathbb{Z}))=0$, when $g\geq 3$, because this is a universal central extension. Eventually, the five-term exact sequence above implies that:

$$H^1(\tilde{\Gamma}_g,\mathfrak{gl}_n)=0$$

so that $\rho_{g,k}^{U(1)}$ is locally rigid in $GL(n,\mathbb{C})$, following Weil's criterion. □

Proposition 3.2. *The representation $\rho_{2,k}^{U(1)}:\tilde{\Gamma}_2\to U(k^2)\subset GL(n,\mathbb{C})$ at genus $g=2$ is not locally rigid, if $k\geq 4$ is even or divisible by 3 and $n>k^2$.*

Proof. Notice that $H^1(\Gamma_2((k)))$ is non-trivial when k is divisible by 2 or 3 (see [52]). In particular it contains a factor $\mathbb{Z}$.

From ([46], p.37-38 and more generally Prop. 10.1 from [9]) we have $H^1(F,\mathfrak{gl}_m)=0$ for any representation of a finite group F in the $GL(m,\mathbb{C})$. In fact finite groups are reductive and hence they are rigid. In particular, we have $H^1(\tilde{Sp}(4,\mathbb{Z}/2k\mathbb{Z}),\mathfrak{gl}_n)=0$.

Lemma 3.1.

$$H^2(\tilde{Sp}(2g,\mathbb{Z}/2k\mathbb{Z}),\mathfrak{gl}_n)=0,\ g\geq 2,k\neq 2.$$

Proof. This follows from the following classical fact (see Prop. 2.1 of [9]): If G is a finite group and M is a G-module which is also a $\mathbb{K}$-vector space for a field $\mathbb{K}$ whose characteristic does not divide the order of G then $H^j(G,M)=0$, when $j>0$. In particular this is true in characteristic zero. □

The five term exact sequence above shows that

$$H^1(\tilde{\Gamma}_2, \mathfrak{gl}_n) \cong H^1(\tilde{\Gamma}_2((k)), \mathfrak{gl}_n))^{\tilde{Sp}(4,\mathbb{Z}/2k\mathbb{Z})}$$

But now

$$H^1(\tilde{\Gamma}_2((2k)), \mathfrak{gl}_n)^{\tilde{Sp}(4,\mathbb{Z}/2k\mathbb{Z})} \cong \mathrm{Hom}(\Gamma_2((2k)), \mathfrak{gl}_n)^{\tilde{Sp}(4,\mathbb{Z}/2k\mathbb{Z})} \supset \mathfrak{gl}_n^{\tilde{Sp}(4,\mathbb{Z}/2k\mathbb{Z})}$$

since $H_1(\Gamma_2((2k))) \supset \mathbb{Z}$. In particular if $n > k^2$ then the unitary representation of $\tilde{Sp}(4, \mathbb{Z}/2k\mathbb{Z})$ in $\mathfrak{gl}_n$ keeps invariant the orthogonal of $\mathfrak{gl}_{k^2}$ within $\mathfrak{gl}_n$. This implies that $H^1(\tilde{\Gamma}_2, \mathfrak{gl}_n) \neq 0$, so that the representations $\rho_{2,k}^{U(1)}$ are not locally rigid. □

Remark 3.1. Since $Sp(2g, \mathbb{Z})$, $g \geq 2$ have property F, their unitary representations have finite images and thus they are discrete. In particular any small deformation of the representation $\rho_{g,k}$ is still a discrete representation in $U(k^g)$. Therefore Selberg's proof from [65] can be used to show that the images are isomorphic. Since $Sp(2g, \mathbb{Z})$ are linear reductive (see [46]) all its linear representations, in particular the $U(1)$ representations $\rho_{g,k} : Sp(2g, \mathbb{Z}) \to U(k^g) \subset GL(n, \mathbb{C})$, are locally rigid as representations in $GL(n, \mathbb{C})$, when $g \geq 2$ and $n \geq k^g$. The linear reductivity is a consequence of the Margulis super-rigidity. In the unitary case this also follows from the easier fact that $Sp(2g, \mathbb{Z})$ has property T.

Remark 3.2. It seems that the $SU(2)/SO(3)$ quantum representations ρ_p having finite image are locally rigid, if $g \geq 3$. It suffices to show that

$$H^1(\ker \rho_p, \mathfrak{gl}_n) = 0.$$

At 4-th roots of unity (and hence $p = 8$) this could follow from the description due to Masbaum and Wright (see [49, 73]) of $\ker \rho_8$, and the fact that finite-index subgroups of Γ_g containing the Johnson kernel $\mathcal{K}_g$ have finite abelianization, according to Putman (see [57]). For $p = 12$ this might use the results from [72].

4. Tangent representations from moduli spaces

4.1. **Mapping class groups as (outer) automorphisms groups.** Now, let $\Gamma_{g,k}^{r,1} \subset \Gamma_{g,k}^{r+1}$ denote the index $r+1$ subgroup of mapping classes of those homeomorphisms which fix one marked point. Then we have the more general statement:

$$\Gamma_{g,k}^{r|1} = \mathrm{Aut}^+(\pi_{g,k}^r; P_r, \mathbf{P}_k^\partial).$$

Here P_r consists of the r conjugacy classes of peripheral loops with the exception of the one around the marked basepoint.

Then we have the following commutative diagram consisting of two exact sequences corresponding to Birman's exact sequence, connected by isomorphisms provided by the Dehn–Nielsen–Baer theorem:

$$\begin{array}{ccccccccc} 1 & \to & \pi_g^r/Z(\pi_g^r) & \to & P\Gamma_g^{r+1} & \to & P\Gamma_g^r & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \pi_g^r/Z(\pi_g^r) & \to & \mathrm{Aut}^+(\pi_g^r; \mathbf{P}_r) & \to & \mathrm{Out}^+(\pi_{g,k}^r; \mathbf{P}_r) & \to & 1 \end{array}$$

We have also a similar commutative diagram in the non pure case:

$$\begin{array}{ccccccccc} 1 & \to & \pi_g^r/Z(\pi_g^r) & \to & \Gamma_g^{r|1} & \to & \Gamma_g^r & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \pi_g^r/Z(\pi_g^r) & \to & \mathrm{Aut}^+(\pi_{g,k}^r; P_r) & \to & \mathrm{Out}^+(\pi_{g,k}^r, P_r) & \to & 1 \end{array}$$

Here $Z(G)$ denotes the center of the group G. Now, the group π_g^r is either a free group of rank $2g + r - 1$, if $r > 0$ or else a surface group. In particular it is centerless when $2g + r - 2 > 0$.

Consider a surface with one boundary component $\Sigma_{g,1}^r$ and take the basepoint to be on the boundary component. Therefore, the basepoint is automatically invariant by the pure mapping class group. It follows that we also have the alternative description:

$$\Gamma_{g,1}^r = \mathrm{Aut}^+(\pi_{g,1}^r; [\partial\Sigma_{g,1}^r], P_r).$$

Notice that homeomorphisms of $\Sigma_{g,1}^r$ automatically preserve the orientation. Denote by

$$\tau : \Gamma_{g,1}^r \to \mathrm{Aut}^+(\pi_{g,1}^r; [\partial\Sigma_{g,1}^r], P_r)$$

the natural isomorphism, which generalizes the usual Artin representation. The following is rather well-known:

Lemma 4.1. *There is an isomorphism between $\Gamma_{g,1}^{r|1}$ and the semi-direct product $\pi_{g,1}^r \rtimes_\tau \Gamma_{g,1}^r$, if and $2g+r-1>0$, which restricts to an isomorphism between the pure mapping class group $P\Gamma_{g,1}^{r+1}$ and the semi-direct product $\pi_{g,1}^r \rtimes_\tau P\Gamma_{g,1}^r$.*

Proof. The embedding of $\Sigma_{g,1}^r$ into $\Sigma_{g,1}^{r+1}$ as the complement of a punctured annulus $\Sigma_{0,2}^1$ induces injective homomorphisms $\pi_1(\Sigma_{g,1}^r, *) \to \pi_1(\Sigma_{g,1}^{r+1}, *)$ and $\Gamma_{g,1}^r \to \Gamma_{g,1}^{r|1}$. Here $*$ is a basepoint on the boundary component of the punctured annulus. This provides a splitting of the Birman exact sequence above. Moreover, the action of the subgroup $\Gamma_{g,1}^r$ on the subgroup $\pi_1(\Sigma_{g,1}^r, *)$ coincides with τ. Therefore $\Gamma_{g,1}^{r|1}$ is isomorphic to the given semi-direct product. $\square$

It is easy to see that there is a more general version, in which we consider mapping class groups instead of pure ones (see e.g. [18]). The corresponding semi-direct product is now isomorphic to the stabilizer of the last puncture in the mapping class group of the surface with one extra puncture, provided the surface has boundary.

4.2. **Geometric actions of (outer) automorphisms groups on moduli spaces.** Let π be a finitely generated group and G a connected Lie group. We denote by $\mathrm{Hom}(\pi, G)$ the space of representations of π. The group $\mathrm{Aut}(\pi)$ acts on $\mathrm{Hom}(\pi, G)$ by right composition:

$$(\varphi \cdot \rho)(x) = \rho(\varphi^{-1}(x)), \text{ for } \varphi \in \mathrm{Aut}(\pi), \rho \in \mathrm{Hom}(\pi, G), x \in \pi.$$

This is a real algebraic action. Let now $\mathfrak{M}_{\pi,G}$ be the character variety of representations $\pi \to G$, or the GIT quotient $\mathrm{Hom}(\pi, G)/G$. Then the action

$$\mathrm{Aut}(\pi) \times \mathrm{Hom}(\pi, G) \to \mathrm{Hom}(\pi, G)$$

above passes to a quotient action of

$$\mathrm{Out}(\pi) \times \mathfrak{M}_{\pi,G} \to \mathfrak{M}_{\pi,G}.$$

Let F be a finitely generated group. Fix a surjective homomorphism $\rho : \pi \to F$ whose kernel $\ker \rho$ is denoted by K and consider its stabilizer, i.e. the subgroup of those elements whose induced action on F via ρ is trivial:

$$\mathrm{Aut}(\pi, \rho) = \{\varphi; \rho(\varphi(x)) = \rho(x), \text{for any } x \in \pi\} \subset \mathrm{Aut}(\pi).$$

Note that $\mathrm{Inn}(K) \subset \mathrm{Aut}(\pi, \rho)$. The image of $\mathrm{Aut}(\pi, \rho)$ in $\mathrm{Out}(\pi)$ will be denoted as $\mathrm{Out}(\pi, \rho)$. However $\mathrm{Inn}(\pi)$ does not preserve ρ. In order to fix this problem consider the following quotient:

$$\widetilde{\mathrm{Out}(\pi, \rho)} = \mathrm{Aut}(\pi, \rho)/\mathrm{Inn}(K).$$

Then $\widetilde{\mathrm{Out}(\pi, \rho)}$ has a well-defined action on $\mathfrak{M}_{\pi,G}$ and keeps the class $[\rho]$ invariant. Note that we have an exact sequence:

$$1 \to F \to \widetilde{\mathrm{Out}(\pi_g, \rho)} \to \mathrm{Out}(\pi, \rho) \to 1.$$

For any homomorphism $r : F \to G$ the group $\mathrm{Aut}(\pi, r\circ\rho)$ fixes $r\circ\rho \in \mathrm{Hom}(\pi, G)$. Therefore there is an induced action at the level of Zariski tangent spaces. This provides a linear representation of $\mathrm{Aut}(\pi_g, r \circ \rho)$ on the Zariski tangent space $T_\rho\mathrm{Hom}(\pi, G)$ at $r \circ \rho$, which will be called the *tangent representation* at $r \circ \rho$. Recall that Weil identified $T_{r\circ\rho}\mathrm{Hom}(\pi, G)$ with the space of twisted 1-cocycles $Z^1(\pi, \mathfrak{g}_{Ad\, r\circ\rho})$ with coefficients in the Lie algebra $\mathfrak{g}$ twisted by the composition of the adjoint representation Ad of G with $r \circ \rho$. This linear representation

$$\mathrm{Aut}(\pi, r \circ \rho) \to GL(Z^1(\pi, \mathfrak{g}_{Ad\, r\circ\rho}))$$

could be defined directly at the level of twisted cocycles $\psi : \pi \to \mathfrak{g}_{Ad\, r\circ\rho}$, as a right composition.

We explained above that $\widetilde{\mathrm{Out}}(\pi, r\circ\rho)$ acts on $\mathfrak{M}_{\pi,G}$ and stabilizes the class $[r\circ\rho]$ of $r\circ\rho$. We derived then a linear action of $\widetilde{\mathrm{Out}}(\pi, r\circ\rho)$ on the Zariski tangent space $T_{[\rho]}\mathfrak{M}_{\pi,G}$. By Weil, this amounts to a linear representation:

$$\widetilde{\mathrm{Out}}(\pi, r\circ\rho) \to GL(H^1(\pi, \mathfrak{g}_{Ad\, r\circ\rho})).$$

For non-reductive G, for instance when $G = \mathbb{C}^*$, we have to modify slightly this setting, as it will be explained below.

This setting also extends to families of representations using intermediary quotients. Let us consider the map $\iota_F : \mathrm{Hom}(F, G) \to \mathrm{Hom}(\pi, G)$, given by $\iota_F(r) = r\circ\rho$. We denote by $V_F = \iota_F(\mathrm{Hom}(F, G)) \subset \mathrm{Hom}(\pi, G)$ the closed subset consisting of all those ρ with $\rho(\pi_g)$ isomorphic to a quotient of F. For any homomorphism $r : F \to G$ we have $\mathrm{Aut}(\pi_g, r\circ\rho) \subset \mathrm{Aut}(\pi_g, \rho)$. The group action of $\mathrm{Aut}(\pi_g, \rho)$ on $\mathrm{Hom}(\pi_g, G)$ keeps globally invariant the subvariety V_F. Note that V_F is not pointwise invariant. Consider the Gunning sheaf $TV_F = \cup_{\rho\in V_F} T_\rho \mathrm{Hom}(\pi_g, G)$. As an immediate consequence $\mathrm{Aut}(\pi_g, \rho)$ acts both on TV_F and the pull-back $\iota_F^* TV_F$

$$\mathrm{Aut}(\pi_g, \rho) \times \iota_F^* TV_F \to \iota_F^* TV_F.$$

We have a similar action $\iota_F : \mathfrak{M}_{F,G} \to \mathfrak{M}_{\pi,G}$ whose image $\iota_F(\mathfrak{M}_{F,G})$ is endowed with a Gunning sheaf $T\mathfrak{M}_{F,G} = \cup_{\rho\in\mathfrak{M}_{F,G}} T_\rho \mathfrak{M}_{\pi_g,G}$ and a fiber-preserving action:

$$\widetilde{\mathrm{Out}}(\pi_g, \rho) \times \iota_F^* T\mathfrak{M}_{F,G} \to \iota_F^* T\mathfrak{M}_{F,G}.$$

We ignored the fact that dimensions of the fibers could be of non-constant dimension. If we restrict to the non-singular locus of the varieties $\mathfrak{M}_{F,G}$ or V_F, then Gunning sheaves restrict to fiber bundles. On any open contractible (in the usual topology) non-singular subset $U \subset \mathfrak{M}_{F,G}$ or V_F respectively we obtain linear representations

$$U \times \mathrm{Aut}(\pi, \rho) \to GL(Z^1(\pi, \mathfrak{g}_{Ad\, r\circ\rho}))$$

and

$$U \times \widetilde{\mathrm{Out}}(\pi, \rho) \to GL(H^1(\pi, \mathfrak{g}_{Ad\, r\circ\rho}))$$

respectively, parameterized by U.

4.3. **Finite representations.** The group $\mathrm{Aut}^+(\pi_g)$ of orientation-preserving automorphisms of π_g acts on $\mathrm{Hom}(\pi_g, G)$ by right composition and this action passes to a quotient action of Γ_g on $\mathfrak{M}_{g,G}$.

Let now F be a finite quotient of $\rho : \pi_g \to F$. The subgroup $\mathrm{Aut}^+(\pi_g, \rho)$ is of finite index in $\mathrm{Aut}^+(\pi_g)$. If we fix an embedding $F \subset G$ then $\mathrm{Aut}^+(\pi_g, \rho)$ is the stabilizer of ρ on $\mathrm{Hom}(\pi_g, G)$. Its image $\Gamma_g(\rho)$ in Γ_g is also the stabilizer of the class $[\rho]$ of ρ in $\mathfrak{M}_{g,G}$.

Consider the exact sequence associated to ρ:

$$1 \to K \to \pi_g \to F \to 1$$

where F is finite. We are given a representation $r : F \to GL(V)$ which induces the structure of π_g-module on V. Without loss of generality we can suppose that V is from now on an *irreducible* F-module. For the sake of simplicity we consider first that V is a complex vector space.

Following [38] we call ρ *redundant* if it factors through $\pi_g \to \mathbb{F}_g$ and if the kernel of the homomorphism $\mathbb{F}_g \to F$ contains a free generator. Here $\mathbb{F}_g$ is the free group of g generators and the homomorphism $\pi_g \to \mathbb{F}_g$ can be taken as the one induced by the inclusion of the surface Σ_g as the boundary of a handlebody with g handles.

Furthermore $F \subset G$ is *adjoint* if the composition $F \to GL(\mathfrak{g})$ by the adjoint representation $Ad : G \to GL(\mathfrak{g})$ is an irreducible representation.

Proposition 4.1. *Suppose that ρ is a finite adjoint redundant representation of π_g. Then the tangent action at $T_{[\rho]}\mathfrak{M}_{g,G}$ is an arithmetic group of symplectic/orthogonal or linear type.*

This is a consequence of the main result of [38]. Specifically, one decomposes the semisimple algebra $\mathbb{Q}[F]$ into simple algebras:

$$\mathbb{Q}[F] = \mathbb{Q} \oplus \bigoplus_{i=1}^{p} A_i$$

where each A_i is a ring of matrices $m_i \times m_i$ over a division algebra D_i with center a number field L_i. Each A_i corresponds to a nontrivial irreducible $\mathbb{Q}$-representation of F. Then the authors of [38] constructed representations of (a finite-index subgroup of) $\Gamma_g(\rho)$ into the algebraic group of V_i-automorphisms $\mathrm{Aut}_{A_i}(A_i^{2g-2}, \langle -, - \rangle)$ of A_i^{2g-2} endowed with a skew-Hermitian sesquilinear A_i-valued form. Then the image of this representation is a finite index subgroup of the arithmetic group $\mathrm{Aut}_{\mathfrak{D}_i}(\mathfrak{D}_i^{2g-2})$, where $\mathfrak{D}_i \subset A_i$ is the image of $\mathbb{Z}[F]$ by the projection onto A_i and is an order in A_i.

Proposition 4.2. *Assume that V is a nontrivial F-module. Then we have an isomorphism*

$$H^1(\pi_g, V) \to \mathrm{Hom}_{\mathbb{C}[F]}(V, V)^{(2g-2)\dim V}.$$

Proof. The five-term exact sequence reads:

$$H^1(F, V^K) \to H^1(\pi_g, V) \to H^1(K, V)^F \to H^2(F, V^K).$$

As in the proof of lemma 3.1 above we use the vanishing of the higher cohomology of a finite group with coefficients in a $\mathbb{Q}$-vector space (Prop. 2.1 of [9]), in order to derive that the restriction homomorphism $H^1(\pi_g, V) \to H^1(K, V)^F$ is an isomorphism.

A classical result from [12] gives a description of the F-module $H_1(K; \mathbb{Q})$. Another proof is given in [38]. In the case when π_g were replaced by a free group this was a classical result by Gaschútz. Specifically, for every $g \geq 2$ we have an isomorphism of F-modules:

$$H_1(K; \mathbb{Q}) \to \mathbb{Q}^2 \oplus \mathbb{Q}[F]^{2g-2}.$$

Some remarks are in order to understand the action of F on the module $H^1(K, V)$. Indeed F acts on K by conjugacy and on V through ρ. Classes in $H^1(K, V)$ are represented by homomorphisms $f : K \to V$, since V is a trivial K-module, and for $\gamma \in F$, $x \in K$ we have:

$$\gamma \cdot f(x) = \rho(\gamma) f(\tilde{\gamma}^{-1} x \tilde{\gamma})$$

where $\tilde{\gamma} \in \pi_g$ is an arbitrary lift of γ. In particular the class of f is F-invariant if for any $\gamma \in F$ and $x \in K$ we have:

$$f(\tilde{\gamma} x \tilde{\gamma}^{-1}) = \rho(\gamma) f(x)$$

By the previous description of the F-action on $H^1(K, V)$ and the Chevalley-Weil description of $H^1(K; \mathbb{C})$ we derive an isomorphism

$$H^1(\pi_g, V) \to \mathrm{Hom}_{\mathbb{C}[F]}(\mathbb{C}[F]^{2g-2} \oplus \mathbb{C}^2, V).$$

On the other hand, for simple $\mathbb{C}[F]$-modules V and W we have $\mathrm{Hom}_{\mathbb{C}[F]}(W, V) = 0$, unless V and W are isomorphic, from Schur's lemma. As a consequence of Maschke's theorem $\mathbb{C}[F] = \mathbb{C} \oplus \bigoplus_{i=1}^{m} V_i^{\dim(V_i)}$, where V_i are all irreducible $C[F]$-modules. It follows that

$$\mathrm{Hom}_{\mathbb{C}[F]}(\mathbb{C}[F]^{2g-2} \oplus \mathbb{C}^2, V) = \mathrm{Hom}_{\mathbb{C}[F]}(V, V)^{(2g-2)\dim V}.$$

□

Now we have an action of $\mathrm{Aut}^+(\pi_g, \rho)$ on $H^1(\pi_g, V)$ induced by the left composition, which we denote by $\phi : \mathrm{Aut}^+(\pi_g, \rho) \to GL(H^1(\pi_g, V))$. Notice however that inner automorphisms do not necessarily act trivially. First, not all inner automorphisms are in $\mathrm{Aut}^+(\pi_g, \rho)$. Second, if the conjugacy ι_α by $\alpha \in \pi_g$ does belong to $\mathrm{Aut}^+(\pi_g, \rho)$, then its image is the automorphism:

$$\phi(\iota_\alpha) = r\rho(\alpha)$$

Since elements in $\mathrm{Aut}^+(\pi_g,\rho)$ which project onto the same element of $\Gamma_g(\rho)$ differ by an inner automorphism from $\mathrm{Aut}^+(\pi_g,\rho)$, it follows that we have an induced representation into a quotient group:

$$\Phi : \Gamma_g(\rho) \to GL(H^1(\pi_g, V))/r(F).$$

This is particularly simple when F is abelian, since $r(F)$ must be a group of scalar matrices and so we obtain a projective representation. In the case considered by [38] the authors rather considered punctured surfaces in order to work directly with the mapping class group $\Gamma_g^1 \subset \mathrm{Aut}^+(\pi_{\mathrm{g}})$. We have an exact sequence

$$1 \to \pi_g \to \Gamma_g^1 \to \Gamma_g \to 1$$

and the representation Φ lifts to

$$\Phi : \Gamma_g^1(\rho) \to GL(H^1(\pi_g, V)).$$

The argument from ([38], section 8.2) shows that its restriction to a suitable finite-index subgroup of $\Gamma_g^1(\rho)$ factors through Γ_g, so that Φ lifts to a genuine representation after restriction to a finite index subgroup of $\Gamma_g(\rho)$.

The case when F is an abelian group and V a 1-dimensional irreducible representation of it has been considered by Looijenga in [44] where the associated representations are called Prym representations. This has to be connected with previous construction by Gunning (see [39]) in genus 2 and later extended by Chueshev (see [13]) to all genera, which is based on Prym differentials.

4.4. **Magnus representations for free groups.** In the case when $\pi = \mathbb{F}_n$ is a free group, there exists a simple description of these representations. Specifically, we first consider $V = \mathbb{Z}[\mathbb{F}_n]$ as a left $\mathbb{F}_n$-module. Then

$$H^1(\mathbb{F}_n, \mathbb{Z}[\mathbb{F}_n]) = I(\mathbb{F}_n) = \ker(\mathbb{Z}[\mathbb{F}_n] \to \mathbb{Z})$$

On the other hand we have an isomorphism

$$I(\mathbb{F}_n) \to (\mathbb{Z}[\mathbb{F}_n])^n.$$

given by the Fox derivatives. Specifically, if the x_i form a free basis of $\mathbb{F}_n$ then we send $x \in \mathbb{F}_n$ into $(\frac{\partial x^{-1}}{\partial x_i})_{i=1,n}$, where the Fox derivatives $\frac{\partial}{\partial x_i} : \mathbb{F}_n \to \mathbb{Z}[\mathbb{F}_n]$ form a basis of the space of 1-cocycles and they are determined by:

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij}.$$

Now any automorphism φ of $\mathbb{F}_n$ induces an automorphism of $I(\mathbb{F}_n)$; under the previous isomorphism this automorphism is described as an element of $GL(n, \mathbb{Z}[\mathbb{F}_n]) \subset GL(V^{\oplus n})$ and is given by the matrix

$$\overline{\left(\frac{\partial \varphi(x_i)}{\partial x_i}\right)} \in GL(n, \mathbb{Z}[\mathbb{F}_n])$$

where $\overline{A}$ is the involution of $\mathbb{Z}[\mathbb{F}_n]$ sending each $x \in \mathbb{F}_n$ into x^{-1}.

In particular, given a surjective homomorphism $\rho : \mathbb{F}_n \to F$ we derive a representation

$$\mathrm{Aut}(\mathbb{F}_n, \rho) \to GL(H^1(\mathbb{F}_n, \mathbb{Z}[F]))$$

which is obtained from the Magnus representation in $GL(n, \mathbb{Z}[\mathbb{F}_n])$ by evaluating each entry via $\rho : \mathbb{Z}[\mathbb{F}_n] \to \mathbb{Z}[F]$. A similar description holds when we choose a family V_F of representations $r : F \to GL(V)$, in which case the tangent representation

$$\mathrm{Aut}(\mathbb{F}_n, \rho) \to GL(H^1(\mathbb{F}_n, V_{r\circ\rho}))$$

is obtained by evaluating the Magnus representation entries at points of V_F.

5. Long-Moody twisted cohomological induction

5.1. **The construction.** Long and Moody considered in [45] a very general recipe for constructing braid group representations (see also [8]). We generalize their construction here to general automorphisms groups.

Data. Let π be a group, in our case it will be a closed surface group or a free group. Let now B be a group related to the automorphisms group $\mathrm{Aut}(\pi)$, in the sense that it is endowed with a homomorphism $\tau : B \to \mathrm{Aut}(\pi)$.

Our data consists of a (finite dimensional)*B-equivariant linear representation*, namely $\rho : \pi \to GL(V)$ coming along with a linear representation $\beta : B \to GL(V)$ such that ρ is equivariant with respect to the source and target actions τ and β:

$$\beta(b)\rho(f) = \rho(\tau(b)f)\beta(b), \text{for any } b \in B, f \in \pi$$

(Equivariant) twisted cohomological induction. To every B-equivariant representation:

$$(\rho : \pi \to GL(V), \beta : B \to GL(V), \tau : B \to \mathrm{Aut}(\pi))$$

we can associate a new representation

$$\beta^+ : B \to GL(V^+), \text{ where } V^+ = H^1_\rho(\pi, V)$$

by the explicit formula:

$$(\beta^+(b)\psi)(f) = \beta(b)\left(\psi(\tau^{-1}(b)(f))\right)$$

for every $\psi \in Z^1_\rho(\pi, V), f \in \pi, b \in B$.

Proposition 5.1. *The twisted cohomological induction is well-defined.*

Proof. We first have to verify that $\beta^+(b)\psi \in Z^1_\rho(\pi, V)$:

$$\begin{aligned}
(\beta^+(b)\psi)(fg) &= \beta(b)\left(\psi(\tau^{-1}(b)(fg))\right) = \beta(b)\left(\psi(\tau^{-1}(b)(f)\cdot\tau^{-1}(b)(g))\right) = \\
&= \beta(b)\left(\psi(\tau^{-1}(b)(f) + \rho(\tau^{-1}(b)f)\psi(\tau^{-1}(b)(g)\right) = \\
&= \beta^+(b)\psi(f) + \beta(b)\rho(\tau^{-1}(b)f)\psi(\tau^{-1}(b)(g) = \\
&= \beta^+(b)\psi(f) + \rho(f)\beta(b)\psi(\tau^{-1}(b)(g) = \beta^+(b)\psi(f) + \beta^+(b)\psi(g).
\end{aligned}$$

Moreover this representation on $Z^1_\rho(\pi, V)$ descends to $H^1_\rho(\pi, V)$. Indeed, if $\psi \in B^1_\rho(\pi, V)$, say $\psi(g) = \rho(g)v - v$, for any $g \in \pi$ for some $v \in V$, then

$$\begin{aligned}
(\beta^+(b)\psi)(g) &= \beta(b)\left(\psi(\tau^{-1}(b)(g))\right) = \beta(b)(\rho(\tau^{-1}(b)g)v - v) = \\
&= \beta(b)\rho(\tau^{-1}(b)(g))v - \beta(b)v = \rho(g)\beta(b)v - \beta(b)v \in B^1_\rho(\pi, V).
\end{aligned}$$

□

Lemma 5.1. *A pair $(\rho : \pi \to GL(V),\ \beta : B \to GL(V))$ satisfying the B-equivariance is equivalent to a representation $\boldsymbol{\beta} : \pi \rtimes_\tau B \to GL(V)$ of the semi-direct product group $\pi \rtimes_\tau B$ obtained by using the action of B on π by means of τ.*

Proof. Indeed $\boldsymbol{\beta}|_\pi = \rho$, while $\boldsymbol{\beta}|_{s(B)} = \beta$, where $s : B \to \pi \rtimes_\tau B$ is a section of the split extension. □

Remark 5.1. If π is either a free group or a surface group and β is unitary then, generically β^+ is unitary (see [45], Thm. 2.8).

A linear representation is cohomological if it can be obtained by iterated Long-Moody induction from the trivial representation.

Question 5.1. *It is true that any quantum representation of the mapping class group $\Gamma_{g,1}$, $g \geq 3$, is a subrepresentation of a cohomological representation?*

Here by a quantum representation we mean a representation obtained from a modular tensor category with zero anomaly, e.g. obtained from the Turaev–Viro construction.

Proposition 5.2. *The Fibonacci representation $\rho_{2,5}$ of Γ_2 can be obtained from the trivial representation of the braid groups by cohomological induction.*

Proof. Indeed from ([45], Cor.2.10) we know that all Jones representations of Hecke algebras associated to Young diagrams with two rows can be obtained by cohomological induction. Our description of $\rho_{2,5}$ as a representation of the braid group B_6 from Section 2.6 completes the proof of the claim. □

5.2. **Examples of cohomological representations.** *Braid group representations.* Long and Moody used this method to define from a series of representations $\rho_n : B_n \to GL(V_n)$ of the braid groups B_n a new series of linear representations $\rho_{n+1}^{+} : B_n \to GL(V_{n+1}^{n})$ (see [45], Thm.2.1). Note the shift in the subscript. We identify B_n and the mapping class group of the 2-disk with n punctures. The stabilizer of the (first) puncture is isomorphic to the semi-direct product $F_n \rtimes_\tau B_n \subset B_{n+1}$, where τ denotes the Artin representation $\tau : B_n \to \mathrm{Aut}(F_n)$. Then twisted cohomological induction yields a representation $\rho_{n+1}^{+} : B_n \to GL(H^1_{\rho_{n+1}}(\pi, V_{n+1}))$. As π is the free group on n generators, the standard free resolution reads (see [9], I.4.4, IV.2, ex.3):

$$0 \to \mathbb{Z}[\pi]^n \to \mathbb{Z}[\pi] \to \mathbb{Z} \to 0.$$

Therefore $H^1_\rho(\pi, V)$ is isomorphic to $V^{\oplus n}$. With this identification at hand one could write explicitly β^+ in terms of generators and the values of β (see [45], Thm.2.2).

It is already noticed that there are several embeddings of some semi-direct product $\pi \rtimes B_n$ within B_{n+1}. Above we considered the pure braid local system in which π is freely generated by $g_1 = \sigma_1^2$, $g_2 = \sigma_2\sigma_1^2\sigma_2^{-1}$, $g_3 = \sigma_3\sigma_2\sigma_1^2\sigma_2^{-1}\sigma_3^{-1}$, …, $g_n = \sigma_n\sigma_{n-1}\cdots\sigma_2\sigma_1^2\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}\sigma_n^{-1}$. The action of B_n, which is generated by $\sigma_2, \sigma_3, \ldots, \sigma_n$ normalizes the subgroup π, and the conjugacy action is identified to the action of B_n on the fundamental group π of the punctured disk.

If we set $g_1 = (\sigma_2\sigma_3\cdots\sigma_n)^n$ and then inductively $g_{i+1} = \sigma_i g_i \sigma_i^{-1}$ then the subgroup π generated by $g_1, g_2, \ldots, g_n$ is also free of rank n and the subgroup B_n generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ also normalizes π. This provides the inner automorphism local system $\pi \rtimes B_n$. Moreover, as we have an obvious map $p : \pi \rtimes B_n \to \mathbb{Z} \rtimes B_n$, we can use an arbitrary representation $\beta_n : B_n \to GL(V_n)$ and consider $(\beta_n \circ p)^+ : B_n \to GL(V_n^{\oplus n})$.

Mapping class group representations. According to Lemma 4.1, $\Gamma_{g,1}^{r|1} = \pi_{g,1}^r \rtimes \Gamma_{g,1}^r$. The Long-Moody twisted cohomological induction machinery provides then for any representation $\beta : \Gamma_{g,1}^{r+1} \to GL(V)$ another linear representation

$$\beta^+ : \Gamma_{g,1}^r \to GL(V^{\oplus 2g+r}).$$

Finite index subgroups of mapping class groups. Consider a homomorphism $\rho : \pi \to GL(V)$ and $B = \mathrm{Aut}^+(\pi, \rho)$, with the usual action on π and the trivial action β on V. Then β^+ is the tangent action of B on $\mathrm{Hom}(\pi, GL(V))$ at ρ.

Surface braid groups. We can consider the braid group $B(\Sigma_{g,1}, r) = \ker(\Gamma_{g,1}^r \to \Gamma_{g,1})$ on the surface $\Sigma_{g,1}$ on r strands. The isomorphism from Lemma 4.1 provides an isomorphism between the stabilizer of the last strand in $B(\Sigma_{g,1}, r+1)$ and the semi-direct product $\pi_{g,1}^r \rtimes B(\Sigma_{g,1}^r)$.

Magnus representations of $\mathrm{Aut}(\mathbb{F}_n)$. In the case when $\pi = \mathbb{F}_n$ and $\rho : \pi \to F$ has a characteristic kernel K, Magnus constructed a crossed-homomorphism $\mathrm{Aut}(\mathbb{F}_n) \to GL(n, \mathbb{Z}[F])$ whose restriction

$$\mathrm{Aut}(\mathbb{F}_n, \rho) \to GL(n, \mathbb{Z}[F])$$

is the homomorphism described in section 4.4 (see [62]). Note that Magnus' homomorphism coincides with the morphism β^+ provided by the construction above to the data (ρ, β), where β is the left action of $\mathrm{Aut}(\mathbb{F}_n)$ on $V = \mathbb{Z}[F]$, after identifying $GL(n, \mathbb{Z}[F])$ with a subgroup of $GL(V^{\oplus n})$. According to ([62], Prop. 3.4)

$$\ker \beta^+ = \ker\left(\mathrm{Aut}(\mathbb{F}_n) \to \mathrm{Aut}\left(\frac{\mathbb{F}_n}{[K,K]}\right)\right).$$

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