BRAIDED SURFACES AND THEIR CHARACTERISTIC MAPS

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Abstract. We show that branched coverings of surfaces of large enough genus arise as characteristic maps of braided surfaces that is, lift to embeddings in the product of the surface with $\mathbb{R}^2$. This result is nontrivial already for unramified coverings, in which case the lifting problem is well-known to reduce to the purely algebraic problem of factoring the monodromy map to the symmetric group S_n through the braid group B_n. In our approach, this factorization is often achieved as a consequence of a stronger property: a factorization through a free group. In the reverse direction we show that any non-abelian surface group has infinitely many finite simple non-abelian groups quotients with characteristic kernels which do not contain any simple loop and hence the quotient maps do not factor through free groups. By a pullback construction, finite dimensional Hermitian representations of braid groups provide invariants for the braided surfaces. We show that the strong equivalence classes of braided surfaces are separated by such invariants if and only if they are profinitely separated.

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1. Introduction

The question addressed in the present paper is the description of a particular case of 2-dimensional knots, called braided surfaces, up to fiber preserving isotopy.

Definition 1.1. A *braided surface* over some surface Σ is an embedding of a surface $j : S \to \Sigma \times \mathbb{R}^2$, such that the composition

$$S \overset{j}{\hookrightarrow} \Sigma \times \mathbb{R}^2 \overset{p}{\to} \Sigma$$

with the first factor projection p is a branched covering. Throughout this paper we only consider *locally flat PL* embeddings j. The composition $p \circ j$ is called the *characteristic map* of the braided surface S.

Braided surfaces over the disk were first considered and studied by Viro, Rudolph (see [39]) and later extensively studied by Kamada ([19]). A comprehensive survey of the subject could be found in the monograph [20]. Braided surfaces over the torus were introduced more recently in [30].

Equivalence classes of braided surfaces are in one-to-one correspondence with a subset of the set of representations of the punctured surface group into the braid group up to conjugacy and mapping class group action. Our aim is to give some insight about the structure of such discrete representation varieties.

Definition 1.2. A map $S \to \Sigma$ between surfaces is called a *2-prem* if it factors as above as $p \circ j$, where j is an embedding and p is the second factor projection $\Sigma \times \mathbb{R}^2 \to \Sigma$.

Whether all generic smooth maps are 2-prems seems widely open. We refer to the article [29] of Melikhov for the state of the art on this question. Although branched coverings are not generic the question of whether they are 2-prems seems natural. Our first result gives an affirmative answer in the asymptotic range:

Theorem 1.1. *There exists some $h_{n,m}$ such that any degree n ramified covering of the closed orientable surface of genus $g \geq h_{n,m}$ with m branch points occurs as a 2-prem.*

The key ingredient is the description of the mapping class group orbits on the space of surface group representations onto a finite group in the stable range, i.e. for large genus g. This was done by Dunfield and Thurston (see [8]) in the closed case and by Catanese, Lönne and Perroni (see [6]) in the branched case. Their results establish the classification of these orbits by means of some versions of homological Schur invariants. Note that the bound $h_{n,m}$ is not explicit.

The genuine classification of these orbits seems much subtler, see [26] for a survey of this and related questions. Livingston provided ([24]) examples of distinct orbits with the same homological Schur invariants.

Definition 1.3. The homological *Schur invariant* of a surjective homomorphism $f : \pi_1(\Sigma) \to G$ of a closed orientable surface group onto a group G is the image $sc(f) \in H_2(G)$ of the fundamental class $[\Sigma] \in H_2(\Sigma)$ by f. Recall that such a homomorphism $f : \pi_1(\Sigma) \to G$ is called *elementary* if it factors through a free group.

The null-homologous case, namely where the homological Schur invariant vanishes, corresponds to finding whether a surjective homomorphism $f : \pi_1(\Sigma) \to G$ of a closed surface group onto a finite group G inducing a trivial map in 2-homology is elementary. This amounts to estimating the minimal Heegaard genus for a 3-manifold group to which f extends, problem which was recently considered by Liechti and Marché for tori (see [22]).

These questions arose in relation with the equivalence problem for epimorphisms of free groups onto non-abelian simple groups. Let $\mathbb{F}_n$ denote the free group on $n \geq 3$ generators and $\mathrm{Out}(\mathbb{F}_n)$ its *outer automorphism* group. Wiegold conjectured (see [26]) that for any finite simple non-abelian group G the group $\mathrm{Out}(\mathbb{F}_n)$ acts transitively on the set of conjugacy classes of surjective homomorphisms $\mathbb{F}_n \to G$. A weaker statement which allows additional stabilizations is known to hold (see [28]). Also Gilman ([14]) and Evans ([10, 11]) proved that there exists a large orbit of $\mathrm{Out}(\mathbb{F}_n)$ on this set. In [2, 8] the authors proved that the action of the mapping class groups $\Gamma(\Sigma)$ on the set of conjugacy classes of surjective homomorphisms onto finite groups G also has at least one large orbit.

Wiegold's conjecture implies that there is no isolated orbit, namely there is no finite simple quotient G which is characteristic. Recall that a subgroup $H \subseteq G$ is a *characteristic* subgroup of G if it is invariant by all automorphisms of G. In this case G/H is called a *characteristic quotient* of G.

In [13] one proved that there exist finite simple non-abelian quotients of surface groups which are characteristic, by using quantum representations. Conjugacy classes of surjective homomorphisms onto characteristic quotients of $\pi_1(\Sigma)$ are therefore isolated orbits for the action of the mapping class group, contrasting with large orbits from [2, 8]. An easy consequence is that all these quotient epimorphisms are non-elementary, so that the classification of mapping class group orbits fundamentally differs from the stable one. This improves previous work of Livingston ([24, 25]) and Pikaart ([37])(see Propositions 5.1 and 5.2):

Theorem 1.2. *For any $g \geq 2$ there exist infinitely many simple non-abelian groups G and surjective homomorphisms of the closed genus g orientable surface onto G, such that the kernels are characteristic and do not contain any simple loop homotopy class. In particular, these homomorphisms are not elementary.*

Remark 1.1. Given an embedding $G \subset S_n$, if the surjective homomorphism $\pi_1(\Sigma) \to G$ is elementary, then f can be lifted to $\pi_1(\Sigma) \to B_n$. We don't know whether the non-elementary homomorphisms from Theorem 1.2 admit lifts to B_n, see also Remark 2.1.

Definition 1.4. Two braided surfaces $j_i : S \to \Sigma \times \mathbb{R}^2$, $i = 0, 1$ over Σ are *(Hurwitz) equivalent* if there exists some ambient isotopy $h_t : \Sigma \times \mathbb{R}^2 \to \Sigma \times \mathbb{R}^2$, $h_0 = id$ such that h_t is fiber-preserving and $h_1 \circ j_0 = j_1$. Recall that h_t is *fiber-preserving* if there exists a homeomorphism $\varphi_t : \Sigma \to \Sigma$ such that $p \circ h_t = \varphi_t \circ p$. There is no loss of generality to impose φ_t to be an isotopy of Σ. Assume that the branch loci of the branched coverings $j_i \circ p$ are the same finite set B. When φ_t can be taken to be isotopy fixing pointwise the branch locus B, we say that the braided surfaces are *strongly equivalent*. These definitions extend naturally to the case when these surfaces have boundary by requiring isotopies to fix the boundary points of $j_i(S)$.

In the last part of this paper we show that finite dimensional Hermitian representations of braid groups provide invariants for the strong equivalence classes of braided surfaces, by a standard pullback construction (see section 7), called spherical functions. We then show that the topological information underlying the spherical functions is of profinite nature (see Theorem 7.1 for the general statement):

Theorem 1.3. *Strong equivalence classes of braided surfaces are separated by some spherical function if and only if they are profinitely separated.*

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2. Braided surfaces

Consider a braided surface over a closed orientable surface Σ, namely a locally flat PL embedding of a closed orientable surface $j : S \to \Sigma \times \mathbb{R}^2$, such that the composition $p \circ j$ is a branched covering. We might consider, more generally, that S is embedded in a (orientable) plane bundle over Σ. However, the existence of nontrivial examples, for instance that some connected unramified covering of degree > 1 arise as a characteristic map, implies that the plane bundle should be trivial (see [9]).

A degree n branched covering $S \to \Sigma$ determines a homomorphism $f : \pi_1(\Sigma \setminus B, *) \to S_n$, where B is the branch locus of F, called *monodromy* homomorphism. Choose small simple loops γ_i each one encircling one branch point b_i of B, which will be called *peripheral* loops or homotopy classes in the sequel.

The *degree* n and *branch locus* B of a braided surface $j : S \to \Sigma \times \mathbb{R}^2$ is are, respectively, the degree and branch locus of its associated branched covering map $p \circ j$. Observe that the projection map

$$p|_{(\Sigma\times\mathbb{R}^2)\setminus j(S)} : (\Sigma \times \mathbb{R}^2) \setminus j(S) \to \Sigma$$

restricts to a locally trivial fiber bundle over $\Sigma - B$. The monodromy of this locally trivial fiber bundle is then a homomorphism

$$f : \pi_1(\Sigma \setminus B) \to B_n$$

into the braid group B_n on n-strands, which will be called the *braid monodromy* of the braided surface in the sequel.

Recall that two branched coverings $F_1, F_2 : S \to \Sigma$ are *equivalent* if there are homeomorphisms $\Phi : S \to S$ and $\phi : \Sigma \to \Sigma$ such that $F_1 \circ \Phi = \phi \circ F_2$. They are further *strongly equivalent* when there is some ϕ which is isotopic to the identity rel the branch locus.

Hurwitz proved that strong equivalence classes of branched coverings with given g genus of Σ, B and n, bijectively correspond to conjugacy classes of monodromy homomorphisms having nontrivial image on every peripheral loop. Denote by $\Gamma(\Sigma)$ the pure *mapping class group* of the possible punctured surface Σ. Moreover, equivalence classes of branched coverings bijectively correspond to orbits of the mapping class group $\Gamma(\Sigma \setminus B)$ on the set of conjugacy classes of monodromy homomorphisms as above.

We will show that a similar result holds in the case of braided surfaces. Let $\gamma \subset \Sigma \setminus B$ be an embedded loop. Its preimage $\ell_\gamma = p^{-1}(\gamma) \cap j(S)$ is a link in the open solid torus $p^{-1}(\gamma) \simeq \gamma \times \mathbb{R}^2$. The link ℓ_γ is the link closure $\widehat{b}$ of a braid $b \in B_n$ within the solid torus, because the projection $p|_{\ell_\gamma} : \ell_\gamma \to \gamma$ is an unramified covering. Note that the link ℓ_γ in the solid torus determines and is determined by the conjugacy class of b in B_n.

If γ were a peripheral loop, let us choose a bounding disk δ embedded in Σ, which contains a single point of B. Since S is compact, we can assume that $j(S) \subset \Sigma \times D^2$, where $D^2 \subset \mathbb{R}^2$ is a compact disk. Then $p^{-1}(\delta) \cap \Sigma \times D^2 \simeq \delta \times D^2$ is a manifold with corners diffeomorphic (after rounding the corners) with the 4-disk D^4. In particular, the solid torus link ℓ_γ determines a link in S^3 by means of the embedding $\ell_\gamma \subset \gamma \times D^2 \subset \partial D^4$.

Definition 2.1. A solid torus link $\ell \subset \gamma \times D^2$ is *completely split* if there exist disjoint disks $D_i^2 \subset D^2$ such that each connected component of L is contained within a solid torus $\gamma \times D_i^2$. The braid $b \in B_n$ is *completely splittable* if the corresponding link $\widehat{b} \subset \gamma \times D^2$ is completely split as a solid torus link and also trivial as a link in S^3. We denote by $\mathcal{A}_n \subset B_n \setminus \{1\}$ the set of completely splittable nontrivial braids.

If $\ell \subset \gamma \times D^2$ is a completely split unlink with components ℓ_i, then choose points $x_i \in D_i^2$ and let y be the single branch point belonging to $\mathrm{int}(\delta)$. Let $C(\ell_i)$ be the cone on ℓ_i with vertex $(y, x_i) \in \delta \times D^2$ and $C(\ell)$ be the union of $C(\ell_i)$. Since each component ℓ_i of ℓ is a trivial knot in $\partial(\delta \times D^2)$, the multi-cone $C(\ell)$ is the disjoint union of locally flat embedded disk in $\delta \times D^2$. Note that $C(\ell)$ is a braided surface over the disk δ with a single branch point $\{y\}$. By [20], Lemma 16.11) this is the unique braided surface over δ with branch point $\{y\}$ and boundary ℓ.

Lemma 2.1. *A braided surface S of degree n over Σ without branch points is determined up to equivalence rel boundary by its braid monodromy homomorphism $f : \pi_1(\Sigma) \to B_n$.*

Proof. A homomorphism f corresponds to a unique locally trivial fiber bundle over Σ with fiber $D^2 \setminus \{p_1, p_2, \ldots, p_n\}$ which is trivialized on the boundary ∂D^2 fiber bundle. One constructs the braided surface over a wedge of circles first and observe that it extends over the 2-cell which produces the surface Σ, as f is a group homomorphism. □

Theorem 2.1. *A homomorphism $f : \pi_1(\Sigma \setminus B) \to B_n$ arises as the braid monodromy of some braided surface of degree n with branch locus B if and only if f sends each peripheral loop γ_i into a completely splittable nontrivial braid $f(\gamma_i) \in \mathcal{A}_n \subset B_n$. Moreover, a braided surface S of degree n over Σ is determined up to strong equivalence by its braid monodromy $f : \pi_1(\Sigma \setminus B) \to B_n$.*

Proof. Consider disjoint disks δ_i bounded by peripheral loops γ_i, for all branch points and let X be their complement. Then $j(S) \cap p^{-1}(\delta_i)$ is a braided surface over the disk δ_i. By ([20], Lemma 16.12) $j(S) \cap p^{-1}(\delta_i)$ has a braid monodromy homomorphism f with $f(\gamma_i)$ completely splittable. This proves the necessity of our conditions.

Conversely, the multi-cone $C(\ell_{\gamma_i})$ over the link ℓ_{γ_i} provides a braided surface over δ_i. The homomorphism $f:\pi_1(\Sigma - \cup\delta_i) \to B_n$ provides by Lemma 2.1 a unique embedding $j: S' \to \Sigma \times \mathbb{R}^2$ which has no branch points. We then glue together S' and the cones C_i along their boundaries, in order to respect the projection map p. As the glued surface S is unique, the braided surface is determined up to strong equivalence by f (see also [19] and [20, Thm. 17.13]). □

As an immediate consequence we have:

Corollary 2.1. The degree n branched covering $S \to \Sigma$ with branch locus B is the characteristic branched covering of a braided surface over Σ if and only if its monodromy homomorphism $f:\pi_1(\Sigma \setminus B) \to S_n$ can be lifted to a homomorphism $F:\pi_1(\Sigma \setminus B) \to B_n$ such that F sends peripheral loops into completely splittable nontrivial braids.

Proof. Theorem 2.1 yields a braided surface lifting some branched covering $S' \to \Sigma$ of degree n, branch locus B and the prescribed monodromy f. As the ramification degrees are determined by the cycle structure of the permutations corresponding to the peripheral loops this branched covering can be identified with $S \to \Sigma$. □

When $B = \emptyset$ we retrieve the following result of Hansen ([15, 16]):

Corollary 2.2. The degree n unramified covering $S \to \Sigma$ factors as the composition

$$S \xhookrightarrow{j} \Sigma \times \mathbb{R}^2 \xrightarrow{p} \Sigma$$

of some embedding j and the second factor projection p, if and only if its monodromy map $f:\pi_1(\Sigma) \to S_n$ lifts to a homomorphism $\pi_1(\Sigma) \to B_n$.

Another consequence is

Corollary 2.3. Degree n braided surfaces on Σ with branch locus B, up to strong equivalence rel boundary are in one-to-one correspondence with the set $M_{B_n}(\Sigma, B)$ of homomorphisms $\pi_1(\Sigma \setminus B) \to B_n$ sending peripheral loops into $\mathcal{A}_n$ modulo the conjugacy action by B_n. Furthermore, the classes of these braided surfaces up to equivalence rel boundary are in one-to-one with the set $\mathcal{M}_{B_n}(\Sigma, B)$ of orbits of the mapping class group $\Gamma(\Sigma \setminus B)$ action on $M_{B_n}(\Sigma, B)$ by left composition. In particular, braided surfaces provide a topological interpretation for the space of double cosets $B_\infty \backslash B_\infty^k / B_\infty$, studied by Pagotto in [32, 33].

Remark 2.1. Symmetric groups and braid groups form nested sequences $\subset S_n \subset S_{n+1} \subset \cdots$ $\subset B_n \subset B_{n+1} \subset \cdots$, where inclusions are induced by adding one more strand on the right. Inclusions are compatible with the projections $p_n : B_n \to S_n$. We note that the answer to the lifting question for homomorphism $f:\pi_1(\Sigma \setminus B) \to S_n$ is independent on the chosen value for n. This follows from the existence of a group homomorphism $p_{n+1}^{-1}(S_n) \to B_n$ induced by the map removing the last strand from the right, which sends completely splittable braids into completely splittable braids.

Remark 2.2. The braided surfaces whose characteristic branched covering is a simple branched covering are analogous to achiral Lefschetz fibrations. The monodromy around a branch point is given by a band, namely a standard generator of the braid group or its inverse.

Remark 2.3. Recovering braided surfaces from their characteristic maps is just an instance of more general questions about compactifications of fibre bundles. There are examples of smooth maps between closed manifolds in specific dimensions having only finitely many critical points (see e.g. [12]). Characterizing the fibre bundle arising in the complementary of the critical locus and how they determine the original maps might have far-reaching implications.

3. Lifting homomorphisms and the proof of Theorem 1.1

3.1. **The stable lifting problem.** A basic problem in algebra and topology is, for a given surjective homomorphism $p : \tilde{G} \to G$, to characterize those group homomorphisms $f : J \to G$ which admit a lift to $\tilde{G}$, namely a homomorphism $\varphi : J \to \tilde{G}$ such that $p \circ \varphi = f$. In the simplest case when J is a free group any homomorphism is liftable. The next interesting case is $J = \pi_1(\Sigma_g)$, where Σ_g denote the genus g closed orientable surface and $g \geq 2$. The lifting question might appear under a slightly more general form, by requiring that $(\varphi(\gamma_i))_{i=1,\ldots,m} = (c_i)_{i=1,\ldots,m} \in \widetilde{G}$, for a set of elements $\gamma_i \in J$, $c_i \in \widetilde{G}$.

Let Σ' be a closed orientable surface and Σ a surface, possibly punctured. Denote by $\Sigma \sharp \Sigma'$ the connected sum. There is a natural map $\Sigma \sharp \Sigma' \to \Sigma$, called *pinch* which consists of crushing the complement of an open disk in Σ' to a point. The operation which replaces Σ by $\Sigma \sharp \Sigma'$ will be called a (genus) *stabilization*.

Although in general it seems difficult to lift homomorphisms f (see [29, 34]) there is only a homological obstruction to lift f, if we allow the surface be stabilized, as it will be explained below.

Let $\Sigma_h \backslash B$ be a stabilization of the surface $\Sigma_g \backslash B$, $h \geq g+1$ and let $P : \pi_1(\Sigma_h \backslash B) \to \pi_1(\Sigma_g \backslash B)$ be the homomorphism induced by the pinch map. If $f : \pi_1(\Sigma_g \backslash B) \to G$ is a homomorphism, we call the composition $f \circ P$ a (genus) *stabilization* of f. We further say that $f : \pi_1(\Sigma_g \setminus B) \to G$ *stably lifts* along $p : \tilde{G} \to G$ if it has some stabilization $f' = f \circ P : \pi_1(\Sigma_h \setminus B) \to G$ which lifts to $\tilde{G}$.

3.2. **Lifting in the unramified case.** We start with an outline of the proof of Theorem 1.1 in the unramified case. A homomorphism $\pi_1(\Sigma_g) \to G$ corresponds to a homotopy class of based maps $f : \Sigma_g \to K(G, 1)$, thereby defining the Schur class $sc(f) = f_*([\Sigma_g]) \in H_2(G)$.

Recall that two surjective homomorphisms $f, f' : \pi_1(\Sigma_g) \to G$ are *equivalent* if there exists an automorphism $\Theta \in \mathrm{Aut}^+(\pi_1(\Sigma_g))$ such that $f' = f \circ \Theta^{-1}$. Here $\mathrm{Aut}^+(\pi_1(\Sigma_g))$ is the group of automorphisms of the fundamental group which are induced by homeomorphisms preserving the orientation and fixing a point of the surface Σ_g. Alternatively, these are those automorphisms of $\pi_1(\Sigma_g)$ which act trivially on $H_2(\pi_1(\Sigma_g))$. Now, Zimmermann ([43], see also [23]) proved that group epimorphisms have stabilizations which are equivalent if and only if their classes in the second homology agree.

This implies that an epimorphism stably lifts to $\tilde{G}$ if and only if its Schur class in $H_2(G)$ lies in the image of $H_2(\tilde{G})$. Indeed every class in $H_2(\tilde{G})$ is the Schur class of some homomorphism $\pi_1(\Sigma_{\tilde{g}}) \to \tilde{G}$, and moreover it is not hard to find a $\tilde{g}$ and such a homomorphism which is surjective (see Lemma 3.1).

Dunfield and Thurston (see [8]) improved this result when the group G is finite. They showed that there exists $g(G)$ with the property that any two surjective homomorphisms $f, f' : \pi_1(\Sigma_g) \to G$ with $g \geq g(G)$ having the same Schur class in $H_2(G)$ are already equivalent under the action of $\Gamma(\Sigma_{g,1}) \times G$, where G acts by inner automorphisms by right composition. The same argument as above shows that for large enough $g \geq g(G)$ a surjective homomorphism $f : \pi_1(\Sigma_g) \to G$ lifts to $\tilde{G}$ if and only if $sc(f)$ lies in the image of $H_2(\tilde{G})$. Eventually, when $G \subseteq S_n$ and $\tilde{G}$ is the preimage of G within the braid group B_n, one shows that $H_2(\tilde{G}) \to H_2(G)$ is surjective (see Lemma 3.2). This proves our claim.

The rest of this section is devoted to make this strategy work for the ramified case as well.

3.3. **Schur invariants for punctured surfaces.** We now describe a construction of homological Schur invariants for homomorphisms $\pi_1(\Sigma_g \backslash B) \to G$. At first let D^2 be a disk embedded in Σ_g containing the punctures $B = \{b_1, b_2, \dots, b_m\}$ and γ_i be a based loop encircling once the puncture b_i, so that γ_i are pairwise disjoint except for their base-point. Identify then $\Sigma_g \setminus B$ with the boundary union of $\Sigma_g \setminus D^2$ and $D^2 \setminus B$, so that there is a fixed system of curves γ_i on $\Sigma_g \setminus B$.

Consider the surface with boundary Σ obtained from Σ_g after removing pairwise disjoint open small disks around each puncture b_i, namely replacing the puncture b_i with a boundary component $\mathbf{b}_i$. Let also Σ° be the result of cutting Σ along the curves γ_i and discarding the annuli bounded by $\mathbf{b}_i$ and γ_i.

Given the elements $\mathbf{c} = (c_i)_{i=1,\dots,m} \in G^m$ we represent them as homotopy classes of based oriented loops ℓ_i embedded within the space $K'(G,1) = K(G,1) \times \mathbb{R}^5$, which are disjoint except for their base-point. Let also L_i be disjoint embedded oriented loops in $K'(G,1)$ such that each pair ℓ_i and L_i bounds an embedded annulus A_i in $K'(G,1)$. Let $L_{\mathbf{c}}$ be the union of L_i.

A homomorphism $f : \pi_1(\Sigma_g \setminus B) \to G$ such that $f(\gamma_i) = c_i \in G$, for every i, provides a continuous based map $\phi : \Sigma^\circ \to K'(G,1)$, which is unique up to homotopy. Then $\phi(\gamma_i)$ is based homotopic to ℓ_i. By adjoining these homotopies we can arrange that $\phi(\gamma_i) = \ell_i$. By gluing the annuli A_i we obtain a based map $\phi : \Sigma \to K'(G,1)$ which sends $\partial\Sigma$ homeomorphically onto $L_{\mathbf{c}}$. Two based homotopies between $\phi(\gamma_i)$ and ℓ_i define a map from a 2-sphere (with poles identified) into $K'(G,1)$ which must extend to the 3-disk, since $\pi_2(K'(G,1)) = 0$. This implies that

$$\phi_*([\Sigma, \partial\Sigma]) \in H_2(K'(G,1), L_{\mathbf{c}})$$

is a well-defined homology class in the relative homology, independent on the various choices made in the construction.

Of course a homomorphism f as above could only exist if $\prod_{i=1}^m c_i$ belongs to the commutator subgroup $[G,G]$, which we assume to be the case from now on.

Definition 3.1. Let $\mathbf{c} = (c_i)_{i=1,\dots,m} \in G^m$ with $\prod_{i=1}^m c_i \in [G,G]$, and choose a system of curves γ_i and a link $L_{\mathbf{c}}$ as above. We denote by $H_2(G; \mathbf{c})$ the group $H_2(K'(G,1), L_{\mathbf{c}})$ and say that $sc(f) = \phi_*([\Sigma, \partial\Sigma]) \in H_2(G; \mathbf{c})$ is the *Schur class* of f.

From the exact sequence of the pair $(K'(G,1), L_{\mathbf{c}})$ we derive the exact sequence:

$$0 \to H_2(G) \to H_2(G, \mathbf{c}) \to \mathbb{Z}^m \to H_1(G)$$

As the map ϕ is a degree one map on the circles γ_i, the image of $sc(f)$ in $H_1(L_{\mathbf{c}}) = \mathbb{Z}^m$ is $(1, 1, \dots, 1)$ and hence the rightmost map sends $(1, 1, \dots, 1) \in \mathbb{Z}^m$ into $0 \in H_1(G)$, by exactness. Classes in $H_2(G, \mathbf{c})$ whose image is $(1, 1, \dots, 1) \in \mathbb{Z}^m$ will be called *primitive*.

A similar invariant, denoted $\varepsilon(f)$, was defined by Catanese, Lönne and Perroni in [5]. Our invariant $sc(f)$ is non-canonical, in the sense that it depends on the choice of the curves γ_i and $L_{\mathbf{c}}$, while $\varepsilon(f)$ is canonical. The construction of $\varepsilon(f)$ proceeds as above, working with all possible values of $\mathbf{c}$ at once. The target group in [5] would naturally be $H_2(K(G,1), K(G,1)^{(1)})$, where $K(G,1)^{(1)}$ is the 1-skeleton of $K(G,1)$. However, the classes so obtained in $H_2(K(G,1), K(G,1)^{(1)})$ are only well-defined when we pass to a suitable quotient of it identifying classes of surfaces Σ whose boundaries are only freely homotopic in $K(G,1)$.

This equivalence relation between surface groups homomorphisms readily extends to surjective homomorphisms $f : \pi_1(\Sigma_g \setminus B) \to G$ with prescribed peripheral monodromy $f(\gamma_i) = c_i$, for $i = 1, \dots, m$. Then two homomorphisms f and f' as above are equivalent if there exists some $\Theta \in \mathrm{SAut}^+(\pi_1(\Sigma_g \setminus B))$ such that $f' = f \circ \Theta^{-1}$. Here $\mathrm{SAut}^+(\pi_1(\Sigma_g \setminus B))$ denotes the group of automorphisms of the group $\pi_1(\Sigma_g \setminus B)$ which are induced by homeomorphisms preserving the orientation of $\Sigma_g \setminus B$ fixing a point of the surface and preserving pointwise the punctures along with the peripheral monodromy, that is $f \circ \Theta^{-1}(\gamma_i) = c_i$, for $i = 1, \ldots, m$. Observe that $\mathrm{SAut}^+(\pi_1(\Sigma_g \setminus B))$ contains the automorphisms of $\pi_1(\Sigma_g \setminus B)$ whose classes belongs to the subgroup $\Gamma(\Sigma_{g,1}) \subset \Gamma(\Sigma_g \setminus B, *)$ of those mapping classes of homeomorphisms which are the identity on $D^2 \setminus B \subset \Sigma_g \setminus B$. Then the Schur class $sc(f) \in H_2(G, \mathbf{c})$ of a homomorphism f is invariant with respect to the left action by $\mathrm{SAut}^+(\pi_1(\Sigma_g \setminus B))$.

When $B = \emptyset$ the equivalence relation is compatible with the G-conjugacy. Consider the set

$$M_G(\Sigma_g) = \mathrm{Hom}^{\mathrm{s}}(\pi_1(\Sigma_g), G)/G$$

of G-conjugacy classes of *surjective* homomorphisms f. There is an obvious action of the mapping class group $\Gamma(\Sigma_g)$ on $M_G(\Sigma_g)$ by left composition. We say that G-conjugacy classes of homomorphisms are equivalent if they belong to the same $\Gamma(\Sigma_g)$-orbit. Conjugacy in G acts trivially on $H_2(G)$. Livingston ([23], see also [43]) has proved that G-conjugacy classes of surjective homomorphisms are stably equivalent if and only if their classes in $H_2(G)$ agree.

In the punctured case we fix an m-tuple $\mathbf{c} \in G^m$ and its conjugacy class with respect to the diagonal action:

$$G \cdot \mathbf{c} = \{(ac_i a^{-1})_{i=1,\cdots,m}, \ |a \in G\} \subset G^m.$$

We then consider the set of surjective homomorphisms mod conjugacy:

$$M_G(\Sigma_g, B, \mathbf{c}) = \{f \in \mathrm{Hom}^{\mathrm{s}}(\pi_1(\Sigma_g \setminus B), G) | (f(\gamma_i))_{i=1,\cdots,m} \in G \cdot \mathbf{c}\}/G$$

where $\mathrm{Hom}^{\mathrm{s}}$ denotes the surjective homomorphism. The *pure* mapping class group $\Gamma(\Sigma_g \setminus B)$ (which fixes the punctures b_i pointwise) has a left action on $\mathrm{Hom}(\pi_1(\Sigma_g \setminus B), G)/G$ which keeps the subspace $M_G(\Sigma_g, B, \mathbf{c})$ invariant. Conjugacy classes are said equivalent if they determine the same element in the orbit set:

$$\mathcal{M}_G(\Sigma_g, B, \mathbf{c}) = \Gamma(\Sigma_g \setminus B) \backslash M_G(\Sigma_g, B, \mathbf{c}).$$

Observe that the conjugacy by $a \in G$ sends isomorphically $H_2(G; \mathbf{c})$ onto $H_2(G, a\mathbf{c}a^{-1})$, in particular $sc(f)$ does not descends to $M_G(\Sigma_g, B, \mathbf{c})$. However we can identify those pairs of elements in the union of groups $H_2(G; \mathbf{b})$, where $\mathbf{b} \in G \cdot \mathbf{c}$ which are related by some conjugacy isomorphism. The result is a quotient of $H_2(G; \mathbf{c})$ which was explicitly described by Catanese, Lönne and Perroni in [5]. Moreover, the image of $sc(f)$ in this quotient group is the same as their ε invariant which is defined on $M_G(\Sigma_g, B, \mathbf{c})$.

3.4. **Stable equivalence for punctured surfaces.** The previous result of Livingston and Zimmermann on G-conjugacy classes was extended to the punctured case by Catanese, Lönne and Perroni in [6]. Specifically, the G-conjugacy classes from $M_G(\Sigma_g, B, \mathbf{c})$ are stably equivalent if and only if their ε-invariants agree. There is a corresponding result for genuine homomorphisms, as follows:

Proposition 3.1. *Surjective homomorphisms of surface groups with the same puncture set B and boundary holonomy $\mathbf{c} \in G^m$ are stably equivalent if and only if their Schur classes in $H_2(G, \mathbf{c})$ agree.*

Proof. One can derive this from the corresponding stability result in [6]. However, there is a direct proof following the lines of the closed case (see [23]). First, the class $sc(f)$ is preserved by stabilizations. Further, if $\Omega_n(X, A)$ is the dimension n orientable bordism group associated to the pair (X, A) of CW complexes, then seminal work of Thom implies that the natural homomorphism

$$\Omega_n(X, A) \to H_n(X, A)$$

is an isomorphism if $n \leq 3$ and an epimorphism if $n \leq 6$ (see e.g. [40], Thm. IV.7.37). In particular, the classes in $H_2(G, p(\mathbf{c}))$ correspond to bordism classes of maps $f : (\Sigma, \partial\Sigma) \to (K'(G,1), L_{\mathbf{c}})$.

The maps f and $f' : (\Sigma', \partial\Sigma') \to (K'(G,1), L_{\mathbf{c}}))$ are bordant if they extend to a 3-manifold. This means that that there exists a 3-manifold M whose boundary splits as $\partial M = \partial_+ M \cup \partial_0 M \cup \partial_- M$, where $\partial_+ M = \Sigma$ and $\partial_- M = \Sigma'$, and a map $F : (M, \partial_0 M) \to (K'(G,1), L_{\mathbf{c}})$, which restricts to $\partial_\pm M$ to f and f'. We can assume that $\partial_0 M$ is a trivial cobordism and moreover $F : \partial_0 M \to L_{\mathbf{c}}$ is a product projection.

Take then a Heegaard surface $(\Sigma'', \partial\Sigma'')$ of the triad $(M, \partial_+ M, \partial M_-)$, as in [4]. This means that $\partial\Sigma''$ is the union of core circles of $\partial_0 M$ and Σ'' decomposes M into two compression bodies H and H'. We can obtain such Heegaard decompositions by extending smoothly to M a function which takes constant values on $\partial_\pm M$ and perturb it away from the boundary to become Morse. Assuming that Σ and Σ' are connected we obtain a Heegaard surface after attaching index one handles away from $\partial_+ M$.

It follows that the map induced by F_* on the image of $\pi_1(\Sigma'')$ within $\pi_1(M)$ is a common stabilization of the homomorphisms f and f', up the the action of the gluing homeomorphism of the two compression bodies H and H'. □

We now prove:

Lemma 3.1. *Let $\tilde{G}$ be a finitely generated group, $\tilde{\mathbf{c}} \in \tilde{G}^p$ and $a \in H_2(\tilde{G}, \tilde{\mathbf{c}})$. Then there is a compact surface Σ and a surjective homomorphism $\phi : \pi_1(\Sigma) \to \tilde{G}$ such that $\phi_*([\Sigma, \partial\Sigma]) = a$ and $(f(\gamma_i))_{i=1,\cdots,p} = \tilde{\mathbf{c}} \in \tilde{G}^p$.*

Proof. Let first $a = 0$ and $\tilde{\mathbf{c}}$ empty. For large enough n there exists a surjective homomorphism $\psi : \mathbb{F}_n \to \tilde{G}$. Consider then $\phi_0 = \psi \circ i_*$, where the homomorphism $i_* : \pi_1(\Sigma_n) \to \pi_1(H_n) = \mathbb{F}_n$ is induced by the inclusion i of Σ_n into the boundary of the genus n handlebody H_n. Note that i_* is a surjection. Then $\phi_{0*}([\Sigma_n]) = 0$, as ϕ_0 factors through a free group.

Let now $a \in H_2(\tilde{G}, \tilde{\mathbf{c}})$ be arbitrary. Pick up a homomorphism $\psi_a : \pi_1(\Sigma_{m,p}) \to \tilde{G}$ realizing the class a, so that $(\psi_a(\gamma_i))_{i=1,\cdots,p} = \tilde{\mathbf{c}} \in \tilde{G}^p$. By crushing the genus n separating loop on $\Sigma_{n+m,p}$ to a point we obtain a surjective homomorphism $\pi : \pi_1(\Sigma_{n+m,p}) \to \pi_1(\Sigma_n) * \pi_1(\Sigma_{m,p})$ onto the fundamental group of the join $\Sigma_n \vee \Sigma_{m,p}$. Consider further the homomorphism $\psi_0 * \psi_a : \pi_1(\Sigma_n) * \pi_1(\Sigma_{m,p}) \to \tilde{G}$.

Then the composition $\phi_a : \pi_1(\Sigma_{n+m,p}) \to \tilde{G}$, $\phi_a = (\psi_0 * \psi_a) \circ \pi$ is surjective. Further, by Mayer-Vietoris we have

$$H_2(\Sigma_n \vee \Sigma_{m,p}, \partial\Sigma_{m,p})) = H_2(\Sigma_n) \oplus H_2(\Sigma_{m,p}, \partial\Sigma_{m,p})$$

and $\pi_*([\Sigma_{n+m,p}, \partial\Sigma_{n+m,p}]) = ([\Sigma_n], [\Sigma_{m,p}, \partial\Sigma_{m,p}])$. This implies that $\phi_{a*}([\Sigma_{m+n,p}]) = a$. Thus ϕ_a satisfies our requirements. □

Proposition 3.2. *Consider $\tilde{\mathbf{c}} \in \tilde{G}^p$. The surjective homomorphism $f : \pi_1(\Sigma_g \setminus B) \to G$ stably lifts to a (surjective) homomorphism $\varphi : \pi_1(\Sigma_h \setminus B) \to \tilde{G}$ satisfying the constraints $\varphi(\gamma_i) = \tilde{c}_i$, for $1 \leq i \leq p$, if and only if there exists a class in $a \in H_2(\tilde{G}, \tilde{\mathbf{c}})$ such that*

$$p_*(a) = sc(f) = f_*([\Sigma, \partial\Sigma]) \in H_2(G, p(\tilde{\mathbf{c}})).$$

Proof. If ϕ is the map provided by Lemma 3.1 above, then $p \circ \phi$ and f are two surjective homomorphisms having the same Schur class. By the previous Proposition 3.1 they have equivalent stabilizations. This shows that f is stably equivalent with a liftable homomorphism and hence stably liftable. □

3.5. **Finite target groups.** In case when the group G is *finite* there is an improvement of the stable equivalence of surface group epimorphisms, following the Dunfield-Thurston Theorem ([8], Thm.6.20) and we can state:

Proposition 3.3. *Let G be a finite group. There exists $g(G,m)$ with the property that any two surjective homomorphisms $f,f':\pi_1(\Sigma_g\setminus B)\to G$ with $g\geq g(G,m)$ and $f(\gamma_i)=f'(\gamma_i)=c_i\in G$ having the same class in $H_2(G,\mathbf{c})$ are equivalent under the action of $\mathrm{SAut}^+(\pi_1(\Sigma_g\setminus B))$.*

Proof. The proof from ([8] Thm. 6.20 and 6.23) extends without major modifications. In fact if $g>|G|$ any homomorphism $f:\pi_1(\Sigma_g\setminus B)\to G$ is a stabilization and this produces a surjective homomorphism induced by stabilization

$$M_G(\Sigma_g,B,\mathbf{c})\to M_G(\Sigma_{g+1},B,\mathbf{c})$$

It follows that the cardinal of the orbits set is eventually constant. On the other hand, by Proposition 3.1, the orbits set eventually injects into $H_2(G,\mathbf{c})$. □

Proposition 3.4. *Let G be a finite group G, $p:\tilde{G}\to G$ be a surjective homomorphism and $\mathbf{c}\in G^m$ such that $\prod_i c_i\in[G,G]$.*

1. *There exist lifts $\tilde{\mathbf{c}}\in\tilde{G}^m$ such that $p(\tilde{\mathbf{c}})=\mathbf{c}$ and $\prod_i\tilde{c}_i\in[\tilde{G},\tilde{G}]$.*
2. *Given a lift $\tilde{\mathbf{c}}$ as in the previous item, there exists some $g(G,m,\tilde{G},\tilde{\mathbf{c}})$ such that every surjective homomorphism $f:\pi_1(\Sigma_g\setminus B)\to G$ with $f(\gamma_i)=c_i\in G$, $g\geq g(G,m,\tilde{G},\tilde{\mathbf{c}})$, for which there exists some class in $a\in H_2(\tilde{G};\tilde{\mathbf{c}})$ satisfying*
$$p_*(a)=sc(f)\in H_2(G,\mathbf{c})$$
lifts to $\varphi:\pi_1(\Sigma_g\setminus B)\to\tilde{G}$ with the constraints $(\varphi(\gamma_i))_{i=1,\cdots,m}=\tilde{\mathbf{c}}\in\tilde{G}^m$.

Proof. Let K denote the kernel of the surjection $\tilde{G}\to G$. Choose any lift $\tilde{\mathbf{c}}\in\tilde{G}^m$. Then the product of its components differs from a product of commutators by some element $k\in K$. We can correct this by replacing the lift $\tilde{c_1}$ by $k^{-1}\tilde{c_1}$. This proves the first claim.

Consider the finite set of all pairs $(\mathbf{c},s)$, where $\mathbf{c}\in G^m$ is an m-tuple which admits a lift $\tilde{\mathbf{c}}\in\tilde{G}^m$ and some class $a_{\mathbf{c}}\in H_2(\tilde{G};\tilde{\mathbf{c}})$ projecting onto the primitive class $s\in H_2(G,\mathbf{c})$.

By Lemma 3.1 there exists a punctured surface $\Sigma_{k(a_{\mathbf{c}})}\setminus B$ and a continuous map defined on the compact surface with boundary $(\Sigma',\partial\Sigma')$ which compactifies it, say $(\Sigma',\partial\Sigma')\to(K'(\tilde{G},1),L_{\mathbf{c}})$, which induces a surjective homomorphism $\phi:\pi_1(\Sigma_{k(a_{\mathbf{c}})}\setminus B)\to\tilde{G}$ such that $\phi_*([\Sigma',\partial\Sigma'])=a_{\mathbf{c}}$. Let then $g_0=g_0(G,m,\tilde{G},\tilde{\mathbf{c}})$ be the maximum of all $k(a_{\mathbf{c}})$.

If $g\geq g_0$ we stabilize ϕ to be defined on $\Sigma_g\setminus B$. Let then $\varphi=p\circ\phi$. Then φ is a surjective homomorphism onto G and

$$\varphi_*([\Sigma',\partial\Sigma'])=f_*([\Sigma,\partial\Sigma])\in H_2(G,p(\tilde{\mathbf{c}}))$$

Now, from Proposition 3.3. there exists some $g(G)$ such that for $g\geq g(G)$ any two surjective homomorphisms φ and f as above are equivalent up to the action of $\mathrm{SAut}^+(\pi_1(\Sigma_g\setminus B))$. We can take $g(G,m,\tilde{G},\tilde{\mathbf{c}})=\max(g(G),g_0(G,m,\tilde{G},\tilde{\mathbf{c}}))$. The action of $\mathrm{SAut}^+(\pi_1(\Sigma_g\setminus B))$ preserves the set of homomorphisms $\pi_1(\Sigma_g\setminus B)\to G$ which admit a lift to $\tilde{G}$ with the given constraints, thereby proving our claim. □

A directly related question is whether a surjective homomorphism $f:\pi_1(\Sigma_g)\to G$ with vanishing Schur class factors through a free group $\mathbb{F}$, in which case of course it can be lifted along any epimorphism $p:\tilde{G}\to G$, for any group $\tilde{G}$. If this happens, the homomorphism f will be called *free*, or *elementary*. As can be inferred from the previous results we have:

Proposition 3.5. *Let G be a finite group. Then there is some $g(G)$ such that for any $g \geq g(G)$ every surjective homomorphism $f : \pi_1(\Sigma_g) \to G$ with $f_*([\Sigma_g]) = 0 \in H_2(G)$ is elementary.*

Proof. We can take $\widetilde{G}$ to be a fixed free group surjecting onto G and use Proposition 3.4. □

We now need some preliminary results concerning braid groups.

Lemma 3.2. *Let $G \subseteq S_n$ be a finite group and $\tilde{G} \subset B_n$ be the preimage of G by the projection homomorphism $p : B_n \to S_n$. Then the map $p_* : H_2(\tilde{G}) \to H_2(G)$ is surjective.*

Proof. The kernel of p is the pure braid group P_n on n strands. The five term exact sequence in homology reads:

$$H_2(\tilde{G}) \to H_2(G) \to H_1(P_n)_G \to H_1(\tilde{G}) \to H_1(G) \to 0$$

On one hand $H_2(G)$ is a torsion group, as G is finite. Furthermore $H_1(P_n)$ is the free abelian group generated by the set $S(n)$ of classes A_{ij}, $1 \leq i < j \leq n$ and the action of S_n is

$$\sigma \cdot A_{ij} = A_{\min(\sigma(i),\sigma(j)),\max(\sigma(i),\sigma(j))}.$$

By ([3], II.2.ex.1) the module of co-invariants $H_1(P_n)_G = \mathbb{Z}S(n)_G$ is isomorphic to the free abelian group $\mathbb{Z}[S(n)/G]$. In particular any homomorphisms $H_2(G) \to H_1(P_n)_G$ must be trivial. Then the exact sequence above implies the claim. □

3.6. **Lifting permutations to completely splittable braids.**

Lemma 3.3. *Every m-tuple $\sigma \in S_n^m$ satisfying $\prod_i \sigma_i \in [S_n, S_n]$ has a lift $\tilde{\sigma} \in B_n^m$ with the properties:*

1. *$\tilde{\sigma} \in \mathcal{A}_n^m \subset B_n^m$;*
2. *$\prod_i \tilde{\sigma}_i \in [B_n, B_n]$;*
3. *Suppose that $\{r_1, r_2, \ldots, r_\nu, t_1, t_2, \ldots, t_\nu\} \subseteq \{1, 2, \ldots, m\}$ is a subset of the set of indices with the property $\sigma_{r_s} = \sigma_{t_s}^{-1}$, for $1 \leq s \leq \nu$. Then we can choose $\tilde{\sigma}_i$ such that additionally:*
$$\tilde{\sigma}_{r_s} = \tilde{\sigma}_{t_s}^{-1}, \text{ for } 1 \leq s \leq \nu$$

Proof. Let b_i, $1 \leq i \leq n-1$, denote the standard generators of the braid group B_n. Recall that the exponent sum $e : B_n \to \mathbb{Z}$ is the unique homomorphism taking values $e(b_i) = 1$ on the standard generators. Set also τ_j for the transposition $(j, j+1)$ of S_n.

We first show that any permutation σ has a lift $\tilde{\sigma} \in \mathcal{A}_n$, such that $e(\tilde{\sigma}) = \mu$, where μ is any prescribed element of $\{-1, 1\}$, when σ is odd and $\mu = 0$, otherwise. This is obvious for $n = 2$. We proceed by induction when $n > 2$, by assuming the claim for $n-1$. Any $\sigma \in S_n$ which does not belong to S_{n-1} can be written as $\sigma = \alpha\tau_{n-1}\beta$, where $\alpha, \beta \in S_{n-1}$. Pick-up some $\mu \in \{-1, 0, 1\}$ which is compatible with the parity of σ, as asked above. Choose an arbitrary lift $\tilde{\beta} \in B_n$. By the induction hypothesis we can find a lift $\widetilde{\beta\alpha} \in \mathcal{A}_{n-1}$ with prescribed exponent sum $\nu \in \{-1, 0, 1\}$, depending on the parity of the permutation $\beta\alpha \in S_n$. We then define the lift:

$$\tilde{\sigma} = \tilde{\beta}^{-1} \cdot \widetilde{\beta\alpha} \cdot \tau_{n-1}^{\delta} \cdot \tilde{\beta}$$

where we set:

$$\delta = \begin{cases} \mu, & \text{if } \mu \in \{-1, 1\}; \\ -\nu, & \text{if } \mu = 0. \end{cases}$$

Now $\tilde{\sigma}$ is conjugate to $\widetilde{\beta\alpha} \cdot \tau_{n-1}^{\delta}$ which is a stabilization of $\widetilde{\beta\alpha}$ and hence it has the same link closure as the latter. Therefore $\tilde{\sigma} \in \mathcal{A}_n$, proving the induction step and hence the claim. Moreover, we can take $\widetilde{\sigma^{-1}} = \tilde{\beta}^{-1} \cdot \tau_{n-1}^{-\delta} \cdot (\widetilde{\beta\alpha})^{-1} \cdot \tilde{\beta}$, as a lift of σ^{-1}, which still belongs to $\mathcal{A}_n$.

We can actually find explicit lifts $\tilde{\sigma}$, as follows. Recall that the half-braid $b_{i,j}$ is defined as:

$$b_{i,j} = b_i b_{i+1} \cdots b_{j-2} b_{j-1} b_{j-2}^{-1} \cdots b_{i+1}^{-1} b_i^{-1}, \text{for } i < j$$

$$b_{i,j} = b_{j,i}^{-1}, \text{for } i > j$$

To every permutation cycle $c = (i_1, i_2, \ldots, i_k) \in S_n$ and map $\varepsilon : \{i_1, i_2, \ldots, i_{k-1}\} \to \{\pm 1\}$, to be called cycle signature, we associate a signed mikado braid, as follows:

$$\beta(c, \varepsilon) = b_{i_1,i_2}^{\varepsilon(i_1)} b_{i_2,i_3}^{\varepsilon(i_2)} \cdots b_{i_{k-1},i_k}^{\varepsilon(i_{k-1})} \in B_n$$

Now, every permutation $\sigma \in S_n$ is the product of disjoint cycles, say $\sigma = c_1 c_2 \cdots c_s$. Pick-up a cycle signature ε_i for each cycle c_i. We then set:

$$\beta(\sigma, (\varepsilon_i)) = \beta(c_1, \varepsilon_1)\beta(c_2, \varepsilon_2) \cdots \beta(c_s, \varepsilon_s)$$

Observe that $\beta(c, \varepsilon)$ and $\beta(c', \varepsilon')$ commute with each other if the cycles c and c' are disjoint. This implies that:

$$p(\beta(\sigma, (\varepsilon_i)) = \sigma$$

Note that the closure of $\beta(c, \epsilon)$ is a trivial link, for any cycle c. In fact, we can assume up to a conjugacy, that the cycle c has the form $(1, 2, \ldots, k)$, so that up to a conjugacy in B_n we have:

$$\beta(c, \epsilon) = b_1^{\varepsilon(1)} b_2^{\varepsilon(2)} \cdots b_{k-1}^{\varepsilon(k-1)} \in B_n$$

Now we see that this is an iterated stabilization of a trivial braid and hence its closure is a trivial link, regardless of the cycle signature. Moreover, the closure of a product of such braids $\beta(c_i, \epsilon_i)$ associated to disjoint cycles c_i is split, each cycle providing a single component of the link. Thus $\beta(\sigma, (\varepsilon_i))$ is a completely split unlink.

We can always choose the cycle signature ε of a given cycle c such that $e(\beta(c, \epsilon)) = 0$, when c has odd length and $e(\beta(c, \epsilon)) = 1$, otherwise. By changing the cycle signature above to its negative $-\varepsilon$ we can also find a cycle signature such that $e(\beta(c, -\epsilon)) = -1$, if the length of c is even. If σ is the product of disjoint cycles c_i we can find some cycle signatures ε_i such that $e(\beta(\sigma, (\varepsilon_i)) \in \{-1, 0, 1\}$, by summing up factors with $e(\beta(c_i, \varepsilon_i) \in \{-1, 0, 1\}$.

Let now $\sigma_i \in S_n$ be a collection of permutations such that $\prod_i \sigma_i \in [S_n, S_n]$. We then set $\tilde{\sigma}_i = \beta(\sigma_i, (\varepsilon_{ij}))$ for suitable cycle signature maps ε_{ij}, such that $e(\beta(\sigma_i, (\varepsilon_{ij})) \in \{-1, 0, 1\}$ and also $e(\prod_i \beta(\sigma_i, (\varepsilon_{ij})) \in \{-1, 0, 1\}$.

If $\sigma_{r_s} = \sigma_{t_s}^{-1}$, then the decompositions into cycles correspond bijectively to each other. For each cycle $c_{r_s j}$ of length k_j arising in the decomposition of σ_{r_s} the cycle $c_{t_s j} = c_{r_s j}^{-1}$ appears in the decomposition of $\sigma_{t_s}^{-1}$. We then set:

$$\varepsilon_{t_s j}(q) = \varepsilon_{r_s j}(k_j - q)$$

This implies that

$$\beta(c_{t_s j}, (\varepsilon_{t_s j})) = \beta(c_{r_s j}, (\varepsilon_{r_s j}))^{-1}$$

and hence:

$$\tilde{\sigma}_{r_s} = \beta(\sigma_{t_s}, (\varepsilon_{t_s j})) = \beta(\sigma_{r_s}, (\varepsilon_{r_s j}))^{-1} = \tilde{\sigma}_{t_s}^{-1}$$

Since $\prod_i \sigma_i$ is an even permutation, $e(\prod_i \tilde{\sigma}_i)$ must be even and hence it must vanish, since it belongs to $\{-1, 0, 1\}$. This implies that $\prod_i \tilde{\sigma}_i \in [B_n, B_n]$, proving the claim. □

Remark 3.1. We have a large freedom in the choice of lifts $\tilde{\sigma} \in \mathcal{A}_n$. Indeed any braid conjugate to $\beta(\sigma, (\varepsilon_i))$ is in $\mathcal{A}_n$ and the corresponding product still belongs to the commutator subgroup of B_n.

3.7. **End of proof of Theorem 1.1.** We further have the following result, which along with Corollary 2.1 proves Theorem 1.1. Several arguments of the proof have essentially been discussed in [1].

Theorem 3.1. *There exists some $h_{n,m}$ such that if $g \geq h_{n,m}$, then every homomorphism $f : \pi_1(\Sigma_g \setminus B) \to S_n$, admits a lift $\varphi : \pi_1(\Sigma_g \setminus B) \to B_n$ satisfying $\varphi(\gamma_i) \in \mathcal{A}_n$.*

The case $n = 3$ and $B = \emptyset$ was solved in [17].

Proof. The group S_n is generated by two elements a and b, for instance a n-cycle and a transposition. Set B' for the result of adding 4 more points $p_{m+1}, p_{m+2}, p_{m+3}, p_{m+4}$ to B. There is a natural surjection $\pi_1(\Sigma_g \setminus B') \to \pi_1(\Sigma_g \setminus B)$ which corresponds to removing the extra punctures. Define the lift $f' : \pi_1(\Sigma_g \setminus B') \to S_n$ of f by asking that the monodromy σ_{m+i} around a loop encircling once counterclockwise p_{m+i}, for $1 \leq i \leq 4$ be a, a^{-1}, b and b^{-1}, respectively. By construction f' is a surjective homomorphism onto S_n.

By Lemma 3.3 we can choose for each $(m+4)$-tuple $\sigma \in S_n^{m+4}$ some lift $\tilde{\sigma} \in \mathcal{A}_n^{m+4} \subset B_n^{m+4}$ with $\prod_i \tilde{c}_i \in [B_n, B_n]$ and $\tilde{\sigma}_{m+1}\tilde{\sigma}_{m+2} = 1$, $\tilde{\sigma}_{m+3}\tilde{\sigma}_{m+4} = 1$. Let $h_{n,m}$ be the maximum of $g(S_n, m+4, B_n, \tilde{\sigma})$, over all $\sigma \in S_n^{m+4}$.

We claim that we can lift f' to B_n with the constraints $\tilde{\sigma} \in B_n^{m+4}$. By Proposition 3.4, it suffices to prove that the homomorphism $p_* : H_2(B_n, \tilde{\mathbf{c}}) \to H_2(S_n, \mathbf{c})$ surjects onto the primitive classes. We have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & H_2(B_n) & \to & H_2(B_n, \tilde{\mathbf{c}}) & \to & \mathbb{Z}^{m+4} & \to & H_1(B_n) & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & H_2(S_n) & \to & H_2(S_n, \mathbf{c}) & \to & \mathbb{Z}^{m+4} & \to & H_1(S_n) & \to 0
\end{array}$$

where the rightmost vertical arrow is the homomorphism induced by the projection $H_1(B_n) \to H_1(S_n)$, which is surjective. Note that the third vertical arrow is $H_1(L_{\tilde{\mathbf{c}}}) \to H_1(L_{\mathbf{c}})$, which is an isomorphism. Then the five-lemma reduces the surjectivity claim to the surjectivity of $p_* : H_2(B_n) \to H_2(S_n)$, which was proved in Lemma 3.2.

Therefore f' lifts to a homomorphism $\varphi' : \pi_1(\Sigma_g \setminus B') \to B_n$. By removing from $\Sigma_g \setminus B$ two disks containing the pairs p_{m+1}, p_{m+2} and p_{m+3}, p_{m+4} respectively, we obtain a surface with boundary, whose fundamental group injects into $\pi_1(\Sigma_g \setminus B')$. The homomorphism φ' takes trivial values on the loops around each of the two holes. Therefore φ' induces a homomorphism of the fundamental group of the surface obtained by capping off the boundary components by disks, namely a homomorphism $\varphi : \pi_1(\Sigma_g \setminus B) \to B_n$. This is the desired lift for f. $\square$

4. Thickness of elementary surface group homomorphisms

4.1. **Elementary homomorphisms and 3-manifolds.** For the sake of simplicity, we stick in this section to the unramified case $B = \emptyset$. Let $f : \pi_1(\Sigma_g) \to G$ be a surjective homomorphism. Assume that $f_*([\Sigma_g]) = 0 \in H_2(G)$. Then there exists some 3-manifold M^3 with boundary Σ_g such that f extends to $F : \pi_1(M^3) \to G$ (see the proof of Propositions 3.1 and 4.1).

Definition 4.1. The *thickness* $t(f)$ of the surjective homomorphism $f : \pi_1(\Sigma_g) \to G$ with $sc(f) = 0$ is the smallest value of n for which there exists a 3-manifold M^3 with boundary Σ_g and Heegaard genus $g + n$ such that f extends to $F : \pi_1(M^3) \to G$.

Other meaningful version might be the rank of the homology or the rank of $\pi_1(M^3)$, the hyperbolic volume (when $g = 1$) of M^3 or any other complexity function on 3-manifolds.

This situation generalizes the case of the commutator width of elements in $[G, G]$. On the other hand it is an analog of Thurston's norm on the homology $H_2(M)$ of a 3-manifold.

Proposition 4.1. *Let $f : \pi_1(\Sigma_g) \to G$ be a homomorphism satisfying $f_*([\Sigma_g]) = 0 \in H_2(G)$. The minimal genus h for which there exists a stabilization $f' : \pi_1(\Sigma_h) \to G$ which is elementary equals the minimal Heegaard genus a 3-manifold M^3 with boundary Σ_g such that f extends to a homomorphism $\pi_1(M^3) \to G$.*

Proof. The arguments come from Livingston's proof ([23, 8]) of the stable equivalence of homomorphisms. Observe that there is a map $F : \Sigma_g \to BG$ inducing f at the fundamental group level. Our assumptions and Thom's solution to the Steenrod realization problem implies that there is some 3-manifold M^3 with boundary Σ_g such that F extends to a map still denoted by the same letter $F : M^3 \to BG$. It follows that f factors as the composition

$$\pi_1(\Sigma_g) \to \pi_1(M^3) \stackrel{F_*}{\to} G$$

where $\pi_1(\Sigma_g) \to \pi_1(M^3)$ is the homomorphism induced by the inclusion.

Let Σ_k be a Heegaard surface in M^3, bounding a handlebody H_k of genus k on one side and a compression body $H_{k,g}$ on the other side. Recall that a compression body $H_{k,g}$ is a compact orientable irreducible 3-manifold obtained from $\Sigma_k \times [0,1]$ by adding 2-handles with disjoint attaching curves, so that $\pi_1(\Sigma_k) \to \pi_1(H_{k,g})$ is surjective. Alternatively we can see $H_{k,g}$ as the result of adding to $\Sigma_g \times [0,1]$ a number of 1-handles, so that $\pi_1(H_{k,g}) = \pi_1(\Sigma_g) * \mathbb{F}_{k-g}$. We then have $\pi_1(M^3) = \pi_1(H_k) *_{\pi_1(\Sigma_k)} \pi_1(H_{k,g})$ where all homomorphisms are induced by the inclusions.

Observe that f is given by the composition:

$$\pi_1(\Sigma_g) \to \pi_1(H_{k,g}) \to \pi_1(M^3) \stackrel{F_*}{\to} G$$

where the first two arrows are inclusion induced homomorphisms. Consider now the homomorphism f' defined by the composition

$$\pi_1(\Sigma_k) \to \pi_1(H_{k,g}) \to \pi_1(M^3) \stackrel{F_*}{\to} G$$

where the first two arrows are inclusion induced homomorphism. Since F_* extends f, it follows that f' is a stabilization of f (see also [8] section 6.15). On the other hand f' factors through the free group $\pi_1(H_k)$. It follows that h is bounded by the Heegaard genus, $h \leq k$.

Conversely, let $f' : \pi_1(\Sigma_h) \to G$ be a stabilization of f which factors through a free group $\mathbb{F}$, namely we can write it as $f' = q' \circ \rho$, where $q' : \mathbb{F} \to G$ and $\rho : \pi_1(\Sigma_h) \to \mathbb{F}$.

Recall the following lemma due to Zieschang, Stallings and Jaco (see [42], [21, Lemma 3.2]) in the form presented by Liechti and Marché ([22], Lemma 3.5):

Lemma 4.1. *Let Σ_h be a surface bounding a handlebody H_h and $\mathbb{F}$ a free group. Then any homomorphism $\rho : \pi_1(\Sigma_h) \to \mathbb{F}$ factors as $q \circ i_* \circ \phi_*$, where ϕ_* is an automorphism of $\pi_1(\Sigma_h)$ preserving the orientation, $i : \pi_1(\Sigma_h) \to \pi_1(H_h)$ is the inclusion and $q : \pi_1(H_h) \to \mathbb{F}$ is a homomorphism.*

Write then $\rho = q \circ i_* \circ \phi_*$ as in Lemma 4.1 and define the manifold $M^3 = H_h \cup_\phi H_{h,g}$, where the gluing homeomorphism ϕ induces the automorphism ϕ_*. It then follows that f' factors through $\pi_1(M^3) = \pi_1(H_h) *_{\pi_1(\Sigma_h)} \pi_1(H_{h,g})$. Since Σ_h is a Heegaard surface in M^3 we derive that $k \leq h$. □

Corollary 4.1. There is some h_n such that whenever $g \geq h_n$ and $f : \pi_1(\Sigma_g) \to G \subseteq S_n$ is a homomorphism with $f_*([\Sigma_g]) = 0 \in H_2(G)$, then f is equivalent to a homomorphism which factors through $\pi_1(H_g)$.

4.2. **Expressing thickness algebraically.** The next result aims at formulating an algebraic formula for $t(f)$, similar to Hopf's formula for the second homology. Consider a standard presentation of the group $\pi_1(\Sigma_g)$ using the generators system $\{a_i, b_i\}_{i\in\{1,2,\ldots,g\}}$ of the form:

$$\pi_1(\Sigma_g) = \langle \{a_i, b_i\}_{i\in\{1,2,\ldots,g\}} | \prod_{i=1}^{g} [a_i, b_i] = 1\rangle$$

Then one identifies a homomorphism $f : \pi_1(\Sigma_g) \to G$ with a labeled set $S = \{\alpha_i = f(a_i), \beta_i = f(b_i)\}_{i\in\{1,2,\ldots,g\}}$ of elements of G satisfying the condition:

$$\prod_{i=1}^{g} [\alpha_i, \beta_i] = 1 \in G \tag{4.1}$$

Let $G = \mathbb{F}/R$ be a presentation of the group G, where $\mathbb{F}$ is a free group and R the normal subgroup generated by the relators. For every labeled set $\tilde{S} = \{(\tilde{\alpha}_i, \tilde{\beta}_i)\}_{i\in\{1,2,\ldots,g\}}$ of lifts of S to $\mathbb{F}$ we set:

$$ocl(\tilde{S}) = \min\{n | \text{there exist } f_j \in \mathbb{F}, r_j \in R, j = 1, 2, \ldots, n \text{ with} \prod_{i=1}^{g} [\tilde{\alpha}_i, \tilde{\beta}_i] = \prod_{j=1}^{n} [r_j, f_j]\} \tag{4.2}$$

Note that such n exists. Indeed the Hopf formula provides us with an isomorphism

$$H_2(G) = \frac{[\mathbb{F}, \mathbb{F}] \cap R}{[\mathbb{F}, R]}$$

Under this identification the $f_*([\Sigma_g])$ is represented by the class of the element $\prod_{i=1}^{g} [\tilde{\alpha}_i, \tilde{\beta}_i] \in [\mathbb{F}, \mathbb{F}] \cap R$. As $f_*([\Sigma_g])$ vanishes by our assumptions $\prod_{i=1}^{g} [\tilde{\alpha}_i, \tilde{\beta}_i] \in [\mathbb{F}, R]$.

Eventually we define:

$$ocl(f) = \min\{ocl(\tilde{S}) | \tilde{S} \text{ lifts } S\} \tag{4.3}$$

We then have the following:

Proposition 4.2. *The minimal number of stabilizations needed for making f elementary is $ocl(f) - g$.*

Proof. By Proposition 4.1 the minimal $h = g+n$ which appears above is the minimal Heegaard genus of a manifold M^3 with boundary Σ_g such that f extends to some homomorphism $F_* : \pi_1(M^3) \to G$. It remains to prove that the smallest Heegaard genus coincides with $ocl(f)$. The proof goes similarly with that given by Liechti-Marché [22] for the case of a bordant torus.

Let Σ_h a Heegaard surface in M^3. Take a standard system of generators of $\pi_1(\Sigma_h)$ of the form $\{a_j, b_j\}_{j=1,\ldots,h}$ such that all b_j bound disks in the handlebody H_h. By adjoining 2-handles to $\Sigma_h \times [0, 1]$ along $b_j \times \{0\}$, for all $j \notin \{1, 2, \ldots, g\}$, we obtain a compression body $H_{h,g}$ and we can write $M^3 = H_{h,g} \cup_\phi \overline{H_h}$, for some gluing homeomorphism ϕ. We have then surjective homomorphisms $\pi_1(\Sigma_h) \to \pi_1(H_h)$ and $\pi_1(\Sigma_h) \to \pi_1(H_{h,g})$, while $\pi_1(M^3) = \pi_1(H_{h,g}) *_{\pi_1(\Sigma_h)} \pi_1(H_h)$.

Denote by $\theta_* : \pi_1(\Sigma_h) \to \pi_1(M^3)$ the inclusion induced homomorphism. We observed in the proof of Proposition 4.1 above that $F_* \circ \theta_* : \pi_1(\Sigma_h) \to G$ is a stabilization of f. Its key property is that

$$F_* \circ \theta_*(b_i) = 1, \text{ if } i \notin \{1, 2, \ldots, g\}$$

The homomorphism θ factors through the free group $\pi_1(H_h)$. Therefore $F_*\circ\theta_* : \pi_1(\Sigma_h) \to G$ lifts to a homomorphism $\tilde{f} : \pi_1(\Sigma_h) \to \mathbb{F}$. Consider the images $\tilde{\alpha}_j = \tilde{f}\circ i_*(a_j), \tilde{b}_j = \tilde{f}\circ i_*(b_j)$ of the generators above into $\mathbb{F}$. As $\tilde{f}\circ i_*$ is a homomorphism, we have the relation:

$$\prod_{i=1}^{g}[\tilde{\alpha}_i, \tilde{\beta}_i] = \prod_{i=g+1}^{h}[\tilde{\beta}_i, \tilde{\alpha}_i] \tag{4.4}$$

As $\tilde{\beta}_i \in R$, we derive that

$$ocl(f) + g \leq h \tag{4.5}$$

Conversely, if we have elements $\tilde{\alpha}_j, \tilde{\beta}_j$ satisfying equation (4.4), then we can define a homomorphism $\tilde{f} : \pi_1(\Sigma_h) \to \mathbb{F}$, by $\tilde{f}(\alpha_j) = \tilde{\alpha}_j, \tilde{f}(\beta_j) = \tilde{\beta}_j$, $1 \leq j \leq h$. By Lemma 4.1 such a homomorphism factors through $\pi_1(H_h)$, namely is a composition

$$\tilde{f} = q \circ i_* \circ \phi_*^{-1}$$

where i_* is as above, ϕ_* is an automorphism of $\pi_1(\Sigma_h)$ and $q : \pi_1(H_h) \to \mathbb{F}$ is some homomorphism.

Let $f' : \pi_1(\Sigma_h) \to G$ be the composition of the projection $\mathbb{F} \to G$ with $\tilde{f}$. Then $f'\circ\phi_*(b_i) = 1$, for $i = 1, 2, \ldots, h$. On the other hand, as $\tilde{\beta}_j \in R$, for $j > g$, f' factors also through $\pi_1(H_{h,g})$. This implies that f' extends to a homomorphism $F_* : \pi_1(M^3) \to G$, where $M^3 = H_{h,g} \cup_\phi \overline{H_h}$. Eventually, note that the restriction of F_* to the image of $\pi_1(\Sigma_g)$ within $\pi_1(M^3)$ is f. We constructed an extension of f to a 3-manifold of Heegaard genus at most h and thus we proved the reverse inequality

$$ocl(f) + g \geq h \tag{4.6}$$

□

Remark 4.1. Consider two surjective homomorphisms $f_j : \pi_1(\Sigma_{g_j}) \to G$. We know that f_j are stably equivalent if and only

$$f_{1*}[\Sigma_1] = f_{2*}[\Sigma_2] \in H_2(G)$$

If $S_j = \{\alpha_i, \beta_i\}_{i\in I_j}$ are images of generators of $\pi_1(\Sigma_j)$ by f_j and $\tilde{S}_j = \{\tilde{\alpha}_i, \tilde{\beta}_i\}_{i\in I_j}$ are lifts to $\mathbb{F}$, we can define

$$ocl^s(\tilde{S}_1, \tilde{S}_2) = \min\{n | \prod_{i\in I_1}[\tilde{\alpha}_i, \tilde{\beta}_i] \prod_{j=1}^{n-g_1}[r_j, f_j] = \prod_{i\in I_2}[\tilde{\alpha}_i, \tilde{\beta}_i] \prod_{j=1}^{n-g_2}[r'_j, f'_j],\ f_j, f'_j \in \mathbb{F}, r_j, r'_j \in R\}$$

Eventually we set:

$$ocl^s(f_1, f_2) = \min\{ocl(\tilde{S}_1, \tilde{S}_2) | \tilde{S}_j \text{ lifts } S_j\}$$

If f_1 and f_2 are stably equivalent, then $ocl^s(f_1, f_2)$ equals the minimal genus of a Heegaard splitting separating the boundaries of a 3-manifold to which f_j extend and also the minimal number of stabilizations yielding equivalent representations in G. The proof is identical.

Remark 4.2. The branched surface case of homomorphisms $f : \pi_1(\Sigma_g \setminus B) \to G$ with prescribed images of peripheral loops follows directly from the closed surface treated above, without essential modifications.

5. Nontrivial thickness and proof of Theorem 1.2

5.1. **Finite simple non-abelian characteristic quotients.** In [13] the authors proved that the obvious extension of Wiegold's conjecture to surface groups does not hold.

Let Σ_g^1 denote the once punctured closed orientable surface of genus g. TQFTs provide the so-called quantum representations of punctured mapping class groups

$$\rho_p : \Gamma(\Sigma_g^1) \to P\mathbb{G}_p,$$

for prime $p \equiv 3 \pmod 4$, into the integral points of a projective pseudo-unitary group $P\mathbb{G}_p$ defined over $\mathbb{Q}$. It is proved in [13] that for large enough p (or $p < 100$) the restriction to $\rho_p(\pi_1(\Sigma_g))$ is a Zariski dense subgroup of $P\mathbb{G}_p$. By the Nori-Weisfeiler strong approximation Theorem we obtain (see [13, Theorem 1.4]):

Theorem 5.1. *For large prime $p \equiv 3 \pmod 4$ and large enough primes q the reduction mod q of the quantum representation ρ_p exists and has the following properties:*

1. *its restriction to $\pi_1(\Sigma_g)$ is a surjective homomorphism $f_{p,q} : \pi_1(\Sigma_g) \to P\mathbb{G}_p(\mathbf{F}_q)$ onto group of points of $P\mathbb{G}_p$ over the finite field $\mathbf{F}_q$;*
2. *the finite groups $P\mathbb{G}_p(\mathbf{F}_q)$ are finite simple groups of Lie type;*
3. $\ker f_{p,q}$ *is a characteristic subgroup of $\pi_1(\Sigma_g)$.*

Remark 5.1. For all but finitely many q the finite groups $P\mathbb{G}_p(\mathbf{F}_q)$ are isomorphic to either $PSL(N_{g,p}, \mathbf{F}_q)$ or to projective unitary groups $PU(\mathbf{F}_{q^2})$. If $q-1$ is coprime with $N_{g,p}$, then $PSL(N_{g,p}, \mathbf{F}_q)$ has vanishing Schur multiplier $H_2(PSL(N_{g,p}, \mathbf{F}_q)) = 0$.

We now first show that such quotient homomorphism should be non-elementary:

Proposition 5.1. *If $f : \pi_1(\Sigma_g) \to G$ is a surjective homomorphism onto a characteristic finite non-trivial quotient G, then f is not elementary, namely its thickness is positive.*

Proof. From Lemma 4.1 f is elementary iff there exists some automorphism ϕ such that $f = h \circ i_* \circ \phi^{-1}$, where $i_* : \pi_1(\Sigma_g) \to \pi_1(H_g) = \mathbb{F}_g$ is the inclusion induced homomorphism and $h : \mathbb{F}_g \to G$. If α is an oriented simple closed curve on Σ_g let $\overline{\alpha}$ denote the conjugacy class of α in $\pi_1(\Sigma_g)$. Let α be a non-separating simple closed curve on Σ_g which bounds a properly embedded disk in H_g. Then $i_*(\overline{\alpha}) = 1$, so that $\phi(\overline{\alpha}) \in \ker f$.

Since $\ker(f)$ is a characteristic subgroup of $\pi_1(\Sigma_g)$, we also have $\psi(\overline{\alpha}) \subset \ker(f)$ for any automorphism $\psi \in \mathrm{Aut}(\pi_1(\Sigma_g))$. However, any non-separating simple closed curve γ on Σ_g is the image of α by some homeomorphism of the surface. In particular, the conjugacy classes of the simple closed curves from a standard generator system of $\pi_1(\Sigma_g)$ are contained into $\ker f$. This implies that f would be constant, which is a contradiction, thereby proving the claim. ☐

5.2. **Non-geometric quotients.** We can slightly improve the result above, for the specific case of the homomorphisms $f_{p,q}$ from [13].

Proposition 5.2. *Let $f_{p,q} : \pi_1(\Sigma_g) \to P\mathbb{G}_p(\mathbf{F}_q)$ be the homomorphisms above, defined by prime $p \equiv 3 \pmod 4$, and prime q large enough, depending on p. Then* $\ker f_{p,q}$ *is non-geometric, namely it contains no simple closed curve.*

Proof. The image of the homotopy class of a based simple closed curve γ into $\Gamma(\Sigma_g^1)$ is the product of the two commuting left Dehn twists along the curves γ^+ and γ^- obtained by pushing slightly the curve γ off the base point towards the left and right, respectively. Therefore the order of $\rho_p(\gamma)$ is equal to the order of the image of a Dehn twist by the representation ρ_p, which is known to be p (see [13]).

Consider the projective matrices $\rho_p(\gamma)^m$, for $1 \leq m < p$, where γ belongs to a finite set of representatives of the set of simple closed on Σ_g^1 up to the mapping class group action. Then, for all large enough primes q the reduction mod q of these projective matrices are non-trivial. Thus, for every simple closed curve γ the elements $f_{p,q}(\gamma)$ have order p. In particular, the kernel of $f_{p,q}$ is non-geometric. $\square$

Theorem 1.2 is a consequence Theorem 5.1 along with Propositions 5.1 and 5.2.

It is not known what is the largest possible stabilizer of the kernel of an elementary homomorphism. The following is relevant:

Proposition 5.3. *Let $\Gamma(H_g^1)$ be the mapping class group of the punctured handlebody H_g^1. If $f : \pi_1(\Sigma_g) \to G$ is a surjective elementary homomorphism onto a finite quotient whose kernel is invariant by the handlebody subgroup $\Gamma(H_g^1)$, then there exists a characteristic finite quotient of $\mathbb{F}_g$.*

Proof. If f is elementary, then up to composing with an automorphism ϕ of $\pi_1(\Sigma_g)$, it factors through $i_* : \pi_1(\Sigma_g) \to \pi_1(H_g) = \mathbb{F}_g$, namely $f = h \circ i_* \circ \phi$, for some homomorphism $h : \pi_1(H_g) \to G$.

Recall from [18] that the mapping class group $\Gamma(H_g^1)$ of the once punctured (or marked) handlebody embeds into the mapping class group $\Gamma(\Sigma_g^1)$ of its boundary surface. Moreover, Luft ([27]) showed that the action in homotopy provides an exact sequence:

$$1 \to Tw(H_g^1) \to \Gamma(H_g^1) \to \mathrm{Aut}^+(\mathbb{F}_g) \to 1$$

whose kernel is the group of twists, generated by the Dehn twists along meridians of Σ_g^1 (i.e. curves bounding disks in H_g^1).

As f is invariant by $\Gamma(H_g^1)$, the exact sequence above shows that the homomorphism h is also invariant by $\mathrm{Aut}^+(\mathbb{F}_g)$. The same argument also work for the full automorphism group. $\square$

Remark 5.2. Let $P : \pi_1(\Sigma_{g+1}) \to \pi_1(\Sigma_g)$ be the map induced by a pinch map $P : \Sigma_{g+1} \to \Sigma_g$. Let γ be a simple closed curve on Σ_g based at the point of Σ_g corresponding to the image of the pinched handle. The based homotopy class of γ^p can be realized by p parallel copies of γ which only intersect at the base point. Observe that γ^p is the image by the pinch map of a simple closed loop $\widehat{\gamma}$ in Σ_{g+1}. If f is a homomorphism into a group G and $f(\gamma)$ has order p then $f \circ P(\widehat{\gamma}) = 1$ and hence $f \circ P$ has not anymore non-geometric kernel.

Remark 5.3. The method used in [13] also provides epimorphisms $f : \pi_1(\Sigma_g \setminus B) \to G$ onto finite simple non-abelian groups G, whose kernels are $\Gamma(\Sigma_g \setminus B)$-invariant.

6. Stabilizing cohomology groups

We now consider approximated lifts of homomorphisms into S_n. Let $\gamma_0 G = G$, $\gamma_{k+1}G = [\gamma_k G, G]$ denote the lower central series of the group G. It is well-known that PB_n is residually torsion-free nilpotent, namely $\bigcap_{k=0}^{\infty} \gamma_k PB_n = 1$ and $A_k = \frac{\gamma_{k-1}PB_n}{\gamma_k PB_n}$ are finitely generated torsion-free abelian groups. We denote $B_n^{(k)}$ the quotient $B_n/\gamma_k PB_n$. We then have a series of *abelian extensions*

$$1 \to \gamma_k PB_n/\gamma_{k+1}PB_n \to B_n^{(k+1)} \to B_n^{(k)} \to 1$$

The question whether a homomorphism $f_k : \pi_1(\Sigma_g) \to B_n^{(k)}$ admits a lift to $f_{k+1} : \pi_1(\Sigma_g) \to B_n^{(k+1)}$ can be reformulated in purely cohomological terms. For every $k \geq 1$ there exist examples of homomorphisms f_k which admit no lift. Our goal here is to show that the lifting is always possible when $k = 0$.

In order to do that, we first show that the pinching map P induces an injection at cohomological level. For the sake of simplicity we only consider the unramified case, but the result works in full generality. Specifically, let $f:\pi_1(\Sigma_g)\to G$ be a surjective homomorphism and A be a finitely generated G-module, say by means of a homomorphism $\tau: G\to \mathrm{Aut}(A)$. Then A inherits a $\pi_1(\Sigma_g)$-module structure through $\tau\circ f$. Let now $P:\pi_1(\Sigma_{g+1})\to\pi_1(\Sigma_g)$ be the pinch map, which is given in convenient basis $\{\alpha_i,\beta_i\}_{i=1,\dots,g+1}$ and $\{a_i,b_i\}_{i=1,\dots,g}$ by

$$P(\alpha_i)=a_i, P(\beta_i)=b_i,\; i\le g,\; P(\alpha_{g+1})=P(\beta_{g+1})=1$$

Then $f\circ P:\pi_1(\Sigma_{g+1})\to G$ provides a $\pi_1(\Sigma_{g+1})$-module structure on A by means of $\tau\circ f\circ P$. Our main result is the following:

Proposition 6.1. *The homomorphism $P^*: H^2(\pi_1(\Sigma_g),A)\to H^2(\pi_1(\Sigma_{g+1}),A)$ is injective.*

Proof. Consider a normalized 2-cocycle $w:\pi_1(\Sigma_g)\times\pi_1(\Sigma_g)\to A$, namely such that $w(x,1)=w(1,x)=0$, whose cohomology class lies in $\ker P^*$. Then the image 2-cocycle P^*w is given by $P^*w(x,y)=w(P(x),P(y))$. By hypothesis it is exact, namely of the form

$$P^*w(x,y)=\delta\phi(x,y)=x\cdot\phi(y)-\phi(xy)+\phi(x)$$

where $\phi:\pi_1(\Sigma_{g+1})\to A$ is a 1-cochain. Let $H=\langle\alpha_i,\beta_i,i\in\{1,2,\dots,g\}\rangle$ and $K=\langle\alpha_{g+1},\beta_{g+1}\rangle$ be the subgroups of $\pi_1(\Sigma_{g+1})$ generated by the respective elements and note that they are free groups. We observe that whenever $x\in H$ and $u,v\in K$ we have:

$$\phi(xu)=x\cdot\phi(u)+\phi(x)-w(P(x),P(u))=x\cdot\phi(u)+\phi(x)$$
$$\phi(ux)=u\cdot\phi(x)+\phi(u)-w(P(u),P(x))=\phi(x)+\phi(u)$$
$$\phi(uv)=u\cdot\phi(v)+\phi(u)-w(P(u),P(v))=\phi(v)+\phi(u)$$

The last equation implies that

$$\phi(u^{-1})=-\phi(u),\ \text{for } u\in K,$$

so that

$$\phi([\alpha_{g+1},\beta_{g+1}])=0$$

We aim at analyzing the restriction of ϕ to H. Set

$$R=\prod_{i=1}^{g}[\alpha_i,\beta_i]\in H$$

Then

$$0=\phi(1)=\phi(R[\alpha_{g+1},\beta_{g+1}])=R\cdot\phi(R)=P(R)\cdot\phi(R)=\phi(R)$$

Consider further $x\in H$. By above we have

$$\begin{aligned}
\phi(xRx^{-1}) &= xR\cdot\phi(x^{-1})+\phi(x)-P^*w(xR,x^{-1})\\
&= xR\cdot(x^{-1}\cdot(P^*w(x,x^{-1})-\phi(x))+\phi(x)-P^*w(xR,x^{-1})\\
&= xRx^{-1}\cdot P^*w(x,x^{-1})-xRx^{-1}\cdot\phi(x)+\phi(x)-P^*w(xR,x^{-1})\\
&= xRx^{-1}\cdot P^*w(x,x^{-1})-P^*w(xR,x^{-1})
\end{aligned}$$

the last equality following from the fact that $P(xRx^{-1})=1$ thereby xRx^{-1} is acting trivially on A. Recall that w is a 2-cocycle and hence satisfies

$$x\cdot w(y,z)-w(xy,z)+w(x,yz)-w(x,y)=0$$

Its pullback verifies then

$$\begin{aligned}
\phi(xRx^{-1}) &= xRx^{-1}\cdot P^*w(x,x^{-1})-P^*w(xR,x^{-1})=\\
&= P^*w(xRx^{-1},x)-P^*w(xRx^{-1},1)=w(P(xRx^{-1}),P(x))=0
\end{aligned}$$

It follows that

$$\phi(xR^{-1}x) = 0$$

Let $L \triangleleft H$ be the normal subgroup generated by R within H. Every element of L can be written as a product of conjugates of R and R^{-1} within H. If $x, y \in L$ and $\phi(x) = \phi(y) = 0$, then

$$\phi(xy) = x \cdot \phi(y) + \phi(x) - w(P(x), P(y)) = 0$$

because $P(x) = P(y) = 1$. By induction on the number of conjugates, we derive that $\phi|_L : L \to A$ is trivial.

Now, if $x = uy$, where $u \in L$, then

$$\phi(x) = \phi(uy) = u\phi(y) + \phi(u) - w(P(u), P(y)) = \phi(y)$$

because $P(u) = 1$. It follows that ϕ is constant on right cosets of L, so that ϕ induces a well-defined map $\overline{\phi} : H/L \to A$. Moreover the restriction of the homomorphism $P : \pi_1(\Sigma_{g+1}) \to \pi_1(\Sigma_g)$ to H induces an isomorphism of $\overline{P} : H/L \to \pi_1(\Sigma_g)$.

Observe that the 1-chain $\overline{\phi}$ satisfies for all $x, y \in H/L$

$$\delta\overline{\phi}(x, y) = P^* w(\tilde{x}, \tilde{y}) = w(\overline{P}(x), \overline{P}(y))$$

where $\tilde{x}, \tilde{y}$ are lifts in H of x, y, respectively. It follows that w is exact, as claimed. □

Then using Proposition 6.1 we obtain a conceptual (without calculation) proof of the following:

Proposition 6.2. *Any homomorphism $f : \pi_1(\Sigma_g) \to S_n$ has a lift $f_1 : \pi_1(\Sigma_g) \to B_n^{(1)}$.*

Proof. Let $\mathcal{E}_1$ denote the extension with abelian kernel:

$$1 \to A_1 \to B_n^{(1)} \to S_n \to 1$$

whose characteristic class $c_{\mathcal{E}_1}$ is denoted $e_1 \in H^2(S_n, A_1)$.

Observe first that f admits a lift f_1 to $B_n^{(1)}$ if and only if the pull-back extension $f^*\mathcal{E}_1$ admits a section s over $\pi_1(\Sigma_g)$. This amounts to saying that $f^*\mathcal{E}_1$ is a split extension which is equivalent with $c_{f^*\mathcal{E}_1} = f^*e_1 = 0 \in H^2(\pi_1(\Sigma_g), A_1)$, where A_1 has a $\pi_1(\Sigma_g)$-module structure induced by f.

On the other hand Theorem 3.1 shows that after sufficiently many stabilizations $f \circ P_{g,h}$ lifts to a homomorphism F into B_n, where $P_{g,h} : \pi_1(\Sigma_{g+h}) \to \pi_1(\Sigma_g)$ is the pinch map of the last h handles. Let $Q^{(1)} : B_n \to B_n^{(1)}$ be the quotient by $\gamma_1 PB_n$. Then $Q^{(1)} \circ F$ is a lift of $f \circ P_{g,h}$ to $B_n^{(1)}$.

The previous argument implies that $(f \circ P_{g,h})^*\mathcal{E}_1$ is a split extension over $\pi_1(\Sigma_{g+h})$ and hence $P_{g,h}^* \circ f^*e_1 = 0 \in H^2(\pi_1(\Sigma_{g+h}), A_1)$. Proposition 6.1 implies that $f^*e_1 = 0 \in H_2(\pi_1(\Sigma_g), A_1)$ and thus the claim follows. □

Remark 6.1. The lifts f_1 of f modulo A_1-conjugacy are in one-to-one correspondence with the section s of $f^*\mathcal{E}_1$, and thus they form an affine space with underlying vector space $H^1(\pi_1(\Sigma_g), A_1)$. It seems possible that for any f there exists some lift f_1 of f which further can be lifted to $B_n^{(2)}$.

7. Spherical functions and proof of Theorem 1.3

7.1. Pullback spherical functions from Lie groups. A key algebraic object in this section is the representation space

$$M_G(\Sigma, B) \subseteq \mathrm{Hom}(\pi_1(\Sigma \setminus B), G)/G,$$

containing the subspace $M_G(\Sigma, B, \mathbf{c})$ of classes of representations with prescribed conjugacy classes of peripheral loops. There are analogous moduli spaces of mapping class group orbits:

$$\mathcal{M}_G(\Sigma, B) = \Gamma(\Sigma \setminus B)\backslash M_G(\Sigma, B).$$

We observed in the first section that the corresponding discrete spaces for $G = B_n$ correspond to (strong) equivalence classes of braided surfaces. One should note that $G = \Gamma(S)$ corresponds to achiral Lefschetz fibrations with fiber S. Our aim is to construct functions on these spaces, corresponding in particular to invariants of braided surfaces.

In order to treat the unbranched case $B = \emptyset$ we observe that we have a natural embedding:

$$M_G(\Sigma, \emptyset) \subset \mathrm{Hom}(\pi_1(\Sigma \setminus \{p\}), G)/G,$$

which provides functions on $M_G(\Sigma, \emptyset)$ by restricting functions defined on the right hand side space.

We construct spherical functions on representation spaces associated to discrete groups by pullback of spherical functions defined on Lie groups. Let $R : G \to \mathfrak{G}$ be a homomorphism representation of G into the Lie group $\mathfrak{G}$. To any $f : \pi_1(\Sigma \setminus B) \to G$ we associate the homomorphism $R_*(f) = R \circ f : \pi_1(\Sigma \setminus B) \to \mathfrak{G}$. This induces a map

$$R_* : M_G(\Sigma, B) \to M_{\mathfrak{G}}(\Sigma, B)$$

Obviously the map R_* only depends on the class of R inside $\mathrm{Hom}(G, \mathfrak{G})/\mathfrak{G}$. Now, the representation variety $M_{\mathfrak{G}}(\Sigma, B)$ was the subject on intensive study, when $\mathfrak{G}$ is a Lie group.

If $B \neq \emptyset$, then $M_G(\Sigma, B) = G^m/G$, where m is the rank of the free group $\pi_1(\Sigma \setminus B)$ and G acts diagonally by conjugation on G^m. Note that $M_G(\Sigma, B)$ can also be identified with the double coset space $G\backslash G^{m+1}/G$, where G is diagonally embedded in G^{m+1}.

Let now introduce some terminology from representation theory. If ρ is a unitary representation of a group H in a Hilbert space V, then a *matrix coefficient* is the function $\phi(x) = \langle \rho(x)v, w\rangle$, where $v, w \in V$. Let $L(H)$ be the vector space of complex functions on H. If $K \subseteq H$ is a subgroup, we denote by $L(K\backslash H/K) \subset L(H)$ the subspace of functions which are bi-K-invariant, namely such that $\phi(k_1 x k_2) = \phi(x)$, for $k_i \in K, x \in H$. A matrix coefficient $\phi(x) = \langle \rho(x)v, w\rangle$ is bi-K-invariant if v, w belong to the space of K-invariants vectors V^K.

Observe first that in the case when V is finite dimensional complex vector space, the same formula define a bi-K-invariant function, even if $\langle\ ,\ \rangle$ is only a Hermitian form on V, not necessarily positive definite. The functions obtained this way will be called *K-spherical functions* on H; we will add *unitary* if we want to specify that the Hermitian form is positive definite. Carrying this construction for the pair $K = G$ and $H = G^k$ we obtain a family of complex functions on $G\backslash G^k/G$, called spherical functions. The main question addressed here is to what extent the spherical functions separate points of representation spaces.

7.2. Compact Lie groups. We can organize spherical function by using the map R_* associated to a representation of G into some compact group $\mathfrak{G}$ in order to pullback spherical functions from $\mathfrak{G}^k$. The following should be well-known, but for lack of references, we sketch the proof:

Proposition 7.1. *Let $K \subset H$ be either finite groups or compact connected Lie groups. Then the unitary K-spherical functions on H separate the points of $K\backslash H/K$.*

Proof. Let $L(H/K)$ be the Hilbert space of complex valued functions on H/K, which is endowed with the tautological left action by H.

Consider first the case when H is finite. Write $L(H/K)$ as a sum of irreducible representations V_j, along with their multiplicities m_j:

$$L(H/K) = \oplus_{j\in J} V_j^{m_j}$$

By Wielandt's lemma (see e.g. [7], Thm.3.13.3), the number of K-orbits in H/K is equal to $\sum_{j\in J} m_j^2$, so that

$$\dim L(K\backslash H/K) = \sum_{j\in J} m_j^2$$

Now the Frobenius reciprocity (see [41], Thm. 1.4.9) gives us

$$m_j = \dim V_j^K$$

Consider the matrix coefficients associated to irreducible representations of $\mathfrak{G}$ into finite dimensional vector spaces V and vectors v, w arising from a basis of V^K. According to ([7], Lemma 3.6.3) matrix coefficients of this form are orthonormal in $L(H)$. Since they are $\sum_{j\in J} m_j^2$ elements of $L(K\backslash H/K)$, it follows that they form a basis of $L(K\backslash H/K)$. In particular, the basis functions separate points of $K\backslash H/K)$, as the set of all functions in $L(K\backslash H/K)$ does separate.

The proof in the case where H is a compact Lie group follows the same lines as in the finite case, now using instead the Peter-Weyl theorem. For instance, matrix coefficients are dense in the space $L^2(G)$, the H-spherical functions are spanning the space of L^2-class functions while $L^2(G) = \oplus W^{\dim W}$ is the direct sum of all irreducible representations W of G with multiplicity equal to their dimension. We leave details to the reader. □

This method provides an infinite family of spherical functions for the case where $M_{\mathfrak{G}}(\Sigma, B) = \mathfrak{G}\backslash\mathfrak{G}^k/\mathfrak{G}$, when $B \neq \emptyset$ and $\pi_1(\Sigma \setminus B)$ is a free group of rank $k-1$. Specifically we consider the set $V_i, i \in \widehat{\mathfrak{G}}$ of all isomorphisms types of irreducible representations of $\mathfrak{G}$. Then $V_{i_1} \otimes \cdots V_{i_k}$ form a representation of $\mathfrak{G}^k$. We should restrict to those unitary representations for which $V_{i_1} \otimes \cdots \otimes V_{i_k}$ has a fixed $\mathfrak{G}$-vector. For each $u, v \in B_I$ in some basis B_I of the space of $\mathfrak{G}$-invariants $H^0(\mathfrak{G}, V_{i_1} \otimes \cdots \otimes V_{i_k})$ we have the spherical function

$$\phi_{u,v,I}(x) = \langle V_{i_1} \otimes \cdots \otimes V_{i_k}(x)u, v\rangle$$

where $I = (i_1, \ldots, i_k)$. The (infinite) set of all such functions will separate points of $M_{\mathfrak{G}}(\Sigma, B) = \mathfrak{G}\backslash\mathfrak{G}^k/\mathfrak{G}$. It is now easy to construct a single function taking values in the series in several variables with matrix coefficients:

$$\Phi(x) = \sum_{I,(u,v)} \frac{1}{I!} (\phi_{I,u,v}(x))_{u,v\in B_I}) X^I$$

A direct consequence of Proposition 7.1 is:

Proposition 7.2. *Assume that $\mathfrak{G}$ is a compact Lie group. Then Φ is a complete invariant for $M_{\mathfrak{G}}(\Sigma, B)$, namely it separates its points: $\Phi(x) = \Phi(y)$ iff $x = y$.*

Neretin ([31]) considered the case $\mathfrak{G} = SU(2)$ and expressed (a modified version of) the algebraic function Φ as a determinant. In this case we know that B_I is indexed by the set of partitions $\alpha = (\alpha_{st})_{s,t=1,\ldots,k}$ with

$$\sum_t \alpha_{st} = i_s$$

Then we consider

$$\Phi_N = \sum_{I,(\alpha_{st})} \frac{1}{\alpha!\beta!} \prod_{s,t} \mathbf{x}^\alpha \mathbf{y}^\beta (\phi_{I,\alpha,\beta})$$

where we set $\mathbf{x}^\alpha = \prod_{s,t} x_{st}^{\alpha_{st}}$, $\alpha! = \prod_{s,t} \alpha_{st}!$. Then the closed formula of [31] reads:

$$\Phi_N(A) = \det(1 - AXA^\perp Y)^{-1/2}$$

for $A \in SU(2)^k$, where $X = (X_{ij})$, $Y = (Y_{ij})$ are matrices of blocks of the form $X_{ij} = \begin{pmatrix} 0 & x_{ij} \\ -x_{ij} & 0 \end{pmatrix}$, $Y_{ij} = \begin{pmatrix} 0 & y_{ij} \\ -y_{ij} & 0 \end{pmatrix}$, and x_{ij}, y_{ij} are variables.

In particular we obtain:

Proposition 7.3. *To any representation $R : G \to SU(2)$ of the group G we have associated a polynomial valued invariant map $\Phi_R : M_G(\Sigma, B) \to \mathbb{C}[X, Y]$, given by:*

$$\Phi_R(a) = \det(1 - R(A)XR(A)^\perp Y).$$

In particular, this holds when the group G is the braid group B_3 and R is the Burau representation for a parameter within the unit circle $U(1)$, the map Φ_R providing then invariants of braided surfaces of degree 3 with nontrivial branch locus.

Often we can reduce the matrix-valued function Φ to a finite polynomial in more variables. In fact, for any $\mathfrak{G}$ as above $M_\mathfrak{G}(\Sigma, B)$ is homeomorphic to a finite CW complex. In particular it admits an embedding $\varphi : M_\mathfrak{G}(\Sigma, B) \to \mathbb{R}^n$. The components of φ form therefore a complete invariant for $M_\mathfrak{G}(\Sigma, B)$ and so there is a much simpler invariant than Φ. Nevertheless we lack an exact form of φ, in general.

In many interesting cases $\mathfrak{G}\backslash\mathfrak{G}^k/\mathfrak{G}$ has the structure of an (affine) algebraic variety over $\mathbb{C}$. Thus we can expect to have a nice algebraic embedding φ. Such an embedding can be obtained from a basis of the algebra of regular functions on $M_\mathfrak{G}(\Sigma, B)$.

This is the case of $\mathfrak{G} = U(n)$, for instance. Let $A = (A_1, A_2, \ldots, A_k) \in U(n)^k$, $I = \{i_1, i_2, \ldots, i_j\}$, $1 \le i_1 < i_2 < \ldots < i_j \le k$ and $\varepsilon : I \to \{1, \star\}$. We denote by

$$A^{I;\varepsilon} = A_{i_1}^{\varepsilon(i_1)} A_{i_2}^{\varepsilon(i_2)} \cdots A_{i_j}^{\varepsilon(i_j)}$$

where $A^\star$ denotes $(A^{-1})^T$. Procesi proved in ([38], Thm. 11.2) that the set of trace functions

$$\{\mathrm{tr}(A^{I;\varepsilon}) |\ I \subseteq \{1, 2, \ldots, k\},\ \varepsilon : I \to \{1, \star\}\}$$

over all possible I and ε represent a basis of the algebra of regular functions on $U(n)\backslash U(n)^k/U(n)$. Let x_i be noncommutative variables,

$$X^{I;\varepsilon} = x_{i_1}^{\overline{\varepsilon}(i_1)} x_{i_2}^{\overline{\varepsilon}(i_2)} \cdots A_{x_j}^{\overline{\varepsilon}(i_j)}$$

where $\overline{\varepsilon}(i) = \varepsilon(i)$, when the later equals 1 and -1, otherwise. Then the noncommutative Laurent polynomial

$$\Psi(A) = \sum_{I,\varepsilon} \mathrm{tr}(A^{I;\varepsilon}) X^{I;\varepsilon}$$

separates points of $U(n)\backslash U(n)^k/U(n)$.

Proposition 7.4. *To any unitary representation $R : G \to U(n)$ of the group G we have associated a noncommutative Laurent polynomial valued invariant map Ψ_R on $M_G(\Sigma, B)$, given by:*

$$\Psi_R(a) = \Psi(R(a))$$

In particular, this holds when the group $G = B_n$ and the R is the Burau representation for a parameter within the unit circle $U(1)$, the map Ψ_R providing then invariants of braided surfaces of degree n with nontrivial branch locus.

Remark 7.1. There is a similar result for the noncommutative polynomial

$$\Psi'_R(A) = \sum_{I \subseteq \{1,2,\ldots,k\}} \mathrm{tr}(A^I) X^I$$

associated to a linear representation $R : G \to GL(n)$ of the group G, which now separates points of $GL(n)\backslash GL(n)^k/GL(n)$, following ([38], section 3).

7.3. **Spherical functions for discrete groups.** Consider now the case of a discrete group G. As observed above it is enough to consider that $B \neq \emptyset$, so that $M_G(\Sigma, B) = G\backslash G^k/G$ is a space of cosets. In contrast to the case of a compact group G, now spherical functions do not necessarily separate points of $M_G(\Sigma, B)$.

For a discrete group H we denote by $\widehat{H}$ its *profinite* completion. There is a natural map $i : H \to \widehat{H}$ which is injective if and only if H is residually finite. If $K \subseteq H$ is a subgroup, we denote by $\overline{K}$ the closure of $i(K)$ into $\widehat{H}$. The map i induces a map between cosets

$$\iota : K\backslash H/K \to \overline{K}\backslash \widehat{H}/\overline{K}$$

Definition 7.1. Two cosets of $K\backslash H/K$ are *profinitely separated* if their images by ι are distinct.

One case of interest is when $K = G$ is embedded diagonally within $H = G^k$. It is easy to see that $\widehat{G^k}$ is isomorphic to $\widehat{G}^k$ and we will identify them in the sequel. If G is embedded diagonally into G^k, then its closure $\overline{G}$ into $\widehat{G}^k$ is isomorphic to the image of $\widehat{G}$ into $\widehat{G}^k$ by its diagonal embedding. Then the map ι from above

$$\iota : G\backslash G^k/G \to \widehat{G}\backslash \widehat{G}^k/\widehat{G}$$

sends a double coset mod G into its class mod $\widehat{G}$. This notion encompasses more classic notions, as the conjugacy separability of the group G, when we take $k = 2$ above.

The main result of this section is:

Theorem 7.1. *Assume that H is finitely generated and $K \subseteq H$ is a subgroup. Two cosets of $K\backslash H/K$ are separated by some Hermitian spherical function if and only if they are profinitely separated.*

Proof. Let x and y be cosets which cannot be distinguished by spherical functions associated to Hermitian representations of H, and in particular by spherical functions associated to finite representations. Let now F be a finite quotient of H, K_F be the image of K in F. Proposition 7.1 shows that spherical functions associated to linear representations of F separately precisely the points of $K_F\backslash F/K_F$. Then the images of x and y should coincide in $K_F\backslash F/K_F$, for any F and hence $\iota(x) = \iota(y)$.

Conversely, a finite dimensional Hermitian representation V of H is defined over some finitely generated ring $\mathcal{O} \subset \mathbb{C}$. By enlarging $\mathcal{O}$ we can suppose that $\langle \, , \rangle$ has entries from $\mathcal{O}$. We can assume, by further enlarging $\mathcal{O}$, that there is a basis B of V^H consisting of vectors whose coordinates belong to $\mathcal{O}$.

Suppose that for some H-invariant vectors u and v the spherical function $\phi_{u,v}$ separates the cosets x and y. We can take then $u, v \in \mathcal{O}\langle B\rangle$. Further, for all but finitely many prime ideals $\mathfrak{p}$ in $\mathcal{O}$, we have

$$\phi_{u,v}(x) \not\equiv \phi_{u,v}(y) (\text{mod } \mathfrak{p}) \in \mathcal{O}/\mathfrak{p}.$$

Now let W be the reduction mod $\mathfrak{p}$ of the H representation on V. These are finite representations and the invariant subspace W^K contains the reduction mod $\mathfrak{p}$ of V^K. Denote by $\overline{w}$ the reduction mod $\mathfrak{p}$ of the vector $w \in \mathcal{O}\langle B\rangle$. It follows that $\overline{u}$ and $\overline{v}$ belong to W^H. As spherical functions are bilinear, for any $z \in H$ we have:

$$\phi_{\overline{u},\overline{v}}(z) \equiv \phi_{u,v}(z) \in \mathcal{O}/\mathfrak{p}.$$

In particular, the spherical function $\phi_{\overline{u},\overline{v}}$ associated to a finite representation distinguishes x from y. This implies that x and y are profinitely separated. □

Remark 7.2. The profinite separability of all cosets in $B_n\backslash B_n^k/B_n$ and their mapping class group generalizations seems to be widely open.

7.4. **Hurwitz equivalence.** To step from strong equivalence to the usual (i.e. Hurwitz) equivalence amounts of studying the action of $\Gamma(\Sigma\backslash B)$ on the vector space of functions on $M_G(\Sigma, B)$. However the previous approach using pull-backs of spherical functions from compact Lie groups leads to a dead end. In fact, we have the following result due to Goldman for $SU(2)$ and to Pickrell and Xia for a general compact group:

Theorem 7.2 ([35, 36])**.** *If $\mathfrak{G}$ is a compact connected Lie group and $\Sigma \setminus B$ is hyperbolic then the action of $\Gamma(\Sigma \setminus B)$ on $M_{\mathfrak{G}}(\Sigma_g, B)$ is ergodic with respect to the quasi-invariant measure.*

In particular there are no continuous functions on $M_{\mathfrak{G}}(\Sigma_g, B)$ which are invariant under the $\Gamma(\Sigma \setminus B)$ action, other than the constants. Pull-backs of spherical functions associated to compact groups could only provide constant functions on $\mathcal{M}_G(\Sigma, B)$. In order to get further insight by this method we have to step to non-compact Lie groups and the corresponding higher Teichmüller theory. As in the previous section, components of $\mathcal{M}_{\mathfrak{G}}(\Sigma, B)$ which have a CW complex structure, as Hitchin components, will provide functions on the corresponding subsets of $\mathcal{M}_G(\Sigma, B)$.

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