The pentagon equation and mapping class group representations

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"Geometry, Topology and Group Theory" in honor of the 80th birthday of Valentin Poenaru

Autrans, July 1-6, 2012

Motivation: the Yang–Baxter Equation and braid group representations

Definition

An *R-matrix* is a solution of the *Yang–Baxter Equation*

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$$

$$R \in \mathrm{End}(V^{\otimes 2}), \quad R_{12} := R \otimes \mathrm{id}_V, \quad R_{23} := \mathrm{id}_V \otimes R$$

Theorem (Jones, Turaev)

Let $R \in \mathrm{Aut}(V^{\otimes 2})$ be an invertible R-matrix. Then, for any $n \in \mathbb{Z}_{>0}$, there exists a canonical representation of the braid group $\rho_n \colon B_n \to \mathrm{Aut}(V^{\otimes n})$ such that $\sigma_1 \mapsto R \otimes \mathrm{id}_{V^{\otimes(n-2)}}$.

Question: how about mapping class groups?

The Pentagon Equation

Definition

A *T-matrix* is a solution of the *Pentagon Equation*

$$T_{12}T_{13}T_{23} = T_{23}T_{12}, \quad T \in \mathrm{End}(V^{\otimes 2})$$

Example

Let B be a bialgebra with product $m\colon B^{\otimes 2} \to B$ and coproduct $\Delta\colon B \to B^{\otimes 2}$. Then

$$T^{(B)} := (\mathrm{id}_B \otimes m)(\Delta \otimes \mathrm{id}_B) \in \mathrm{End}(B^{\otimes 2})$$

is a T-matrix.

Theorem (Militaru)

Let $T \in \mathrm{Aut}(V^{\otimes 2})$ be a T-matrix with $\dim(V) < \infty$. Then, there exists a unique finite-dimensional Hopf algebra H such that T is essentially $T^{(H)}$.

Need for extra properties

Definition

A T-matrix $T \in \mathrm{End}(V^{\otimes 2})$ is *semisymmetric* if there exists a *symmetry* $A \in \mathrm{Aut}(V)$ and a *projective factor* $\zeta \in \mathbb{C}_{\neq 0}$ such that $A^3 = \mathrm{id}_V$ and $T(A \otimes \mathrm{id}_V)ST = \zeta A \otimes A$, where $S \in \mathrm{Aut}(V^{\otimes 2})$, $x \otimes y \mapsto y \otimes x$.

Remark

No finite-dimensional T-matrix can be semisymmetric.

Theorem

Let $T \in \mathrm{Aut}(V^{\otimes 2})$ be a semisymmetric T-matrix. Then, for any hyperbolic surface $S_{g,s}$ of genus g and s punctures, there exists a canonical projective representation of the mapping class group $\rho_{g,s}\colon \Gamma_{g,s} \to \mathrm{Aut}(V^{\otimes n_{g,s}})$, $n_{g,s} := 4g - 4 + 2s$, such that the image of the Dehn twist along any non-separating simple closed curve is conjugated to $T \otimes \mathrm{id}_{V^{\otimes n_{g,s}-2}}$.

Example from Quantum Teichmüller theory

Let p and q be the (normalized) Heisenberg operators

$$pf(x) := \frac{1}{2\pi i} f'(x), \quad qf(x) := xf(x)$$

For $\hbar \in \mathbb{R}_{>0}$, Faddeev's quantum dilogarithm function is defined by

$$\Phi_\hbar(x) = (\bar{\Phi}_\hbar(x))^{-1} = \exp\left(\int_{\mathbb{R}+i\epsilon} \frac{e^{-i2xz}}{4\sinh(zb)\sinh(zb^{-1})z} dz\right)$$

where $\hbar = 4(b + b^{-1})^{-2}$. Choosing $V = L^2(\mathbb{R})$ $(V^{\otimes n} := L^2(\mathbb{R}^n))$,

$$T = e^{i2\pi p_1 q_2} \bar{\Phi}_\hbar(q_1 + p_2 - q_2) \in \mathrm{Aut}(V^{\otimes 2})$$

is a unitary semisymmetric T-matrix with

$$A = e^{i\pi(\alpha^2-1)/3} e^{i3\pi q^2} e^{i\pi(p+q)^2} e^{i2\pi\alpha p}, \quad \zeta = e^{-i\pi(\hbar^{-1}+\alpha^2)/3}, \quad \alpha \in \mathbb{R}$$

Definition

Let X be a set.

- A map $t\colon X^2 \to X^2$ is a *set-theoretical T-matrix* if $t_{23} \circ t_{13} \circ t_{12} = t_{12} \circ t_{23}$
- A set-theoretical T-matrix is *semisymmetric* if there exists a *symmetry* $a\colon X \to X$, such that $a^3 = \mathrm{id}_X$ and and $t \circ s \circ (a \times \mathrm{id}_X) \circ t = a \times a$, where $s\colon X^2 \ni (x, y) \mapsto (y, x) \in X^2$.

If $t\colon X^2 \ni (x, y) \mapsto (xy, x * y) \in X^2$ is a set-theoretical T-matrix, then

- $(x, y) \mapsto xy$ is associative.
- t is semisymmetric with symmetry $a\colon X \to X$ iff $x * y = a^{-1}(a(y)a^{-1}(x))$, $(x * y)a(xy) = a(x)$, and $x * (yz) = (x * y)((xy) * z)$.

Set-theoretical solutions from groups with addition

Definition

A group G is called *group with addition* if it has a commutative and associative binary operation $(x, y) \mapsto x + y$ with respect to which the group product is distributive.

Examples

- $\mathbb{Q}_{>0}$ or $\mathbb{R}_{>0}$
- $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ with tropical addition $\max(x, y)$
- $\left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \middle| \ x, z \in \mathbb{R}_{>0}, y \in \mathbb{R} \right\}$

Theorem

Let G be a group with addition, $c \in G$ a central element, and $X = G^2$. Then, there exists a semisymmetric set-theoretical T-matrix $t\colon X^2 \ni (x, y) \mapsto (x \cdot y, x * y) \in X^2$ with symmetry $a\colon X \ni (x_1, x_2) \mapsto (cx_1^{-1}x_2, x_1^{-1}) \in X$ and $x \cdot y = (x_1y_1, x_1y_2 + x_2)$.