Aspherical groups and manifolds with extreme properties

Mark Sapir

Autrans, July 6, 2012

Monster Groups

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The cross-bred Monster There exists a finitely generated group that is both Tarski monster and Gromov monster.

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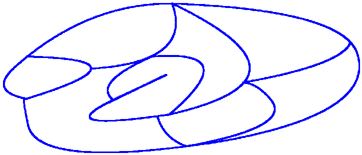
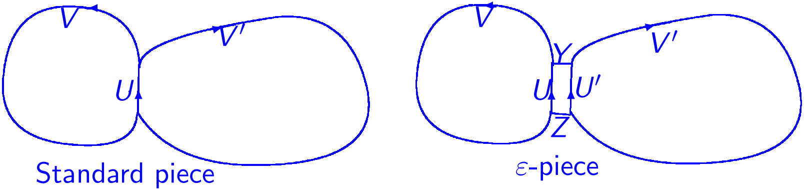


Small cancellation, Greendlinger lemma, Cartan-Hadamar

Definition of a piece: classical and modern



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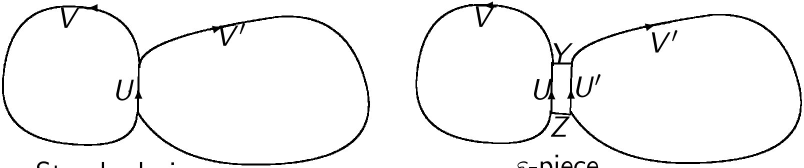
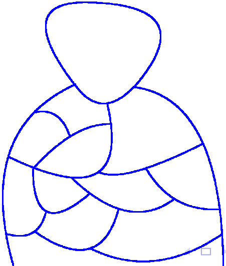


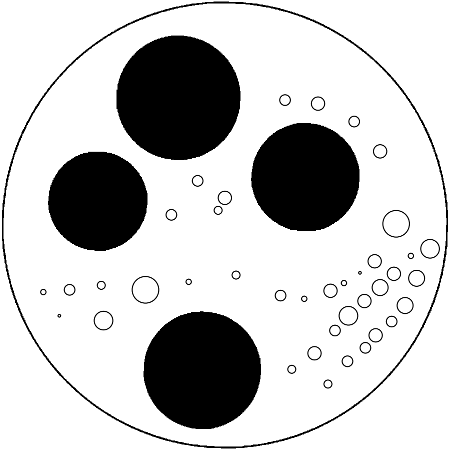
Standard piece

ε-piece

Greendlinger lemma:



Hyperbolic bullion



Cartan-Hadamard, Coulon's theorem

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Let $\delta \geq 0$. Let $\sigma > 10^{10}\delta$. Let X be a simply-connected length space. If every ball of radius σ is δ hyperbolic, then X is (globally) 500δ-hyperbolic.

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- Start with a free group $F = \langle x, y \rangle$. List all pairs of words (u_i, v_i) from F,
- Take the first pair (u_1, v_1). If they do not generate the whole group F or a cyclic group, impose two relations $p_1(u_1, v_1) = x, q_1(u_1, v_1) = y$. Produce a new group G_1.
- Take the second pair (u_2, v_2). If they do not generate the whole group G_1 or a cyclic group, impose two relations $p_2(u_1, v_1) = x, q_2(u_1, v_1) = y$. Produce a new group G_2. Make sure that G_2 is hyperbolic.
- Proceed by induction.

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The inductive limit $\varinjlim G_i = G$ is a Tarski monster.

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- Pick the first graph in the expander sequence, G_1.
- Consider a random labeling of edges of G_1 by letters $x_1^{\pm 1}, ..., x_k^{\pm 1}$.
- For every loop p of a generating set of the fundamental group $\pi_1(G_1)$ impose relation $\mathrm{label}(p) = 1$.
- Make sure that the resulting group is hyperbolic (that is true with probability > 0).
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The inductive limit $\varinjlim G_i$ is a Gromov monster. Note that the presentation is recursive.

Use of small cancellaion. The cross-bred monster.

To produce a cross-bred monster, alternate steps of the two recipes.

Small cancellation and asphericity

Every small cancellation group is aspherical, that is every map from the sphere S^2 to the presentation complex is homotopic to the constant map.

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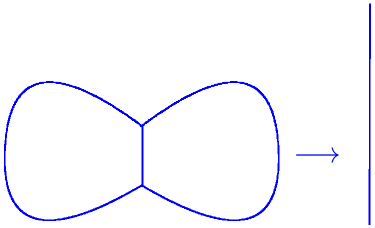
Combinatorial definition of asphericity. Peiffer moves.



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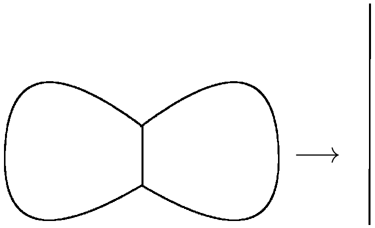

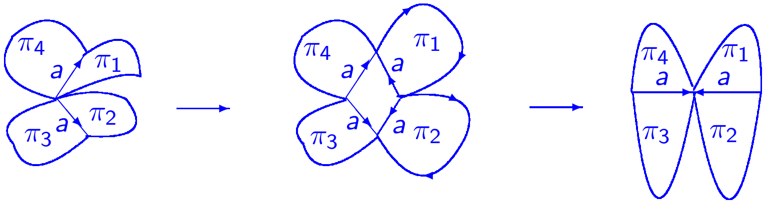

The main result

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Corollary. There exists a closed compact Riemannian aspherical 5-manifold M^5 such that the universal cover $\tilde{M}^5$

- contains an expander,
- has infinite asymptotic dimension,
- does not coarsely embed into a Hilbert space,
- does not satisfy the Baum-Connes conjecture with coefficients,
- admits a free action by a Tarski monster.
- ...

Note that we can also assume that the universal cover of $M^5 \times T^2$ is homeomorphic to $\mathbb{R}^7$.

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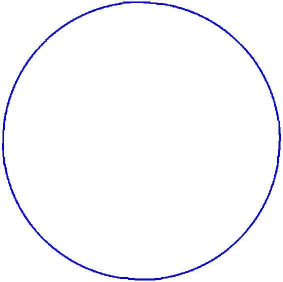

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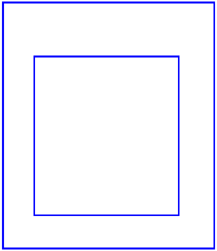

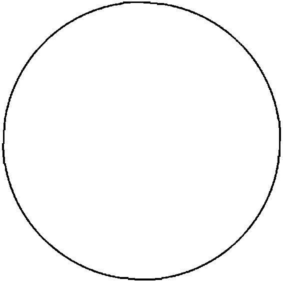

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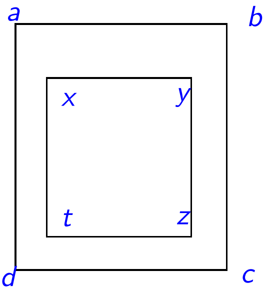

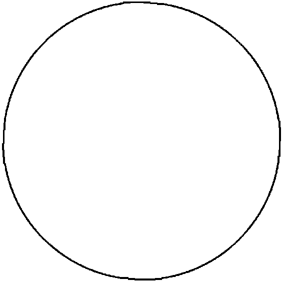

Davis' trick continued.

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$$C = \langle a, b, c, d, x, y, z, t \mid a^2 = b^2 = c^2 = d^2 = x^2 = y^2 = z^2 = t^2 = 1,$$

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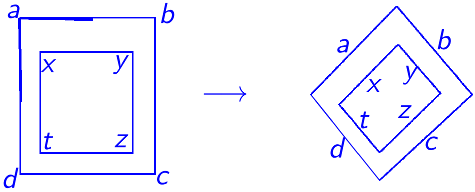

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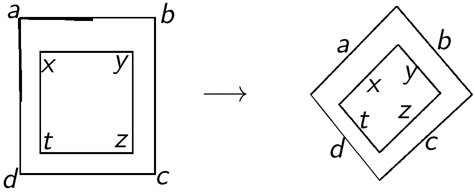

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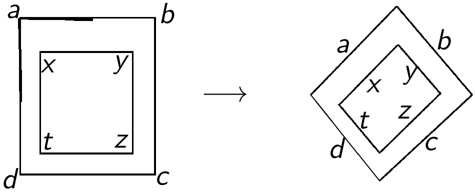

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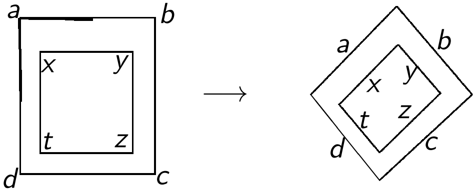

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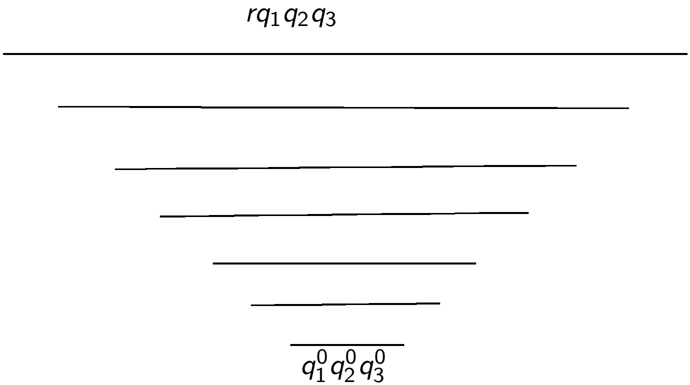
The proof of the Higman embedding theorem. 1

Let $\Gamma = \langle X \mid R \rangle$, R recursive. Here is a computation of a Turing machine accepting a word $r \in R$. We are going to turn it into a tesselated disc.



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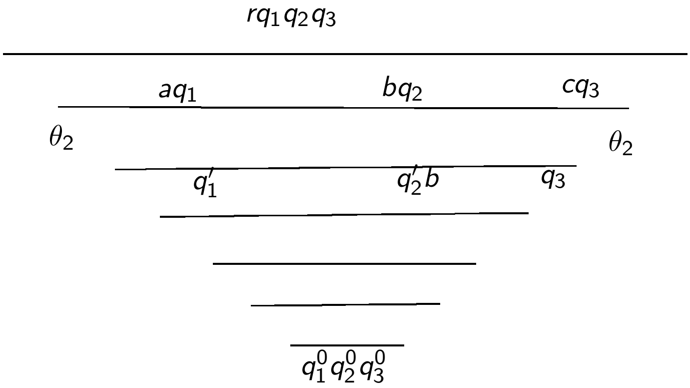

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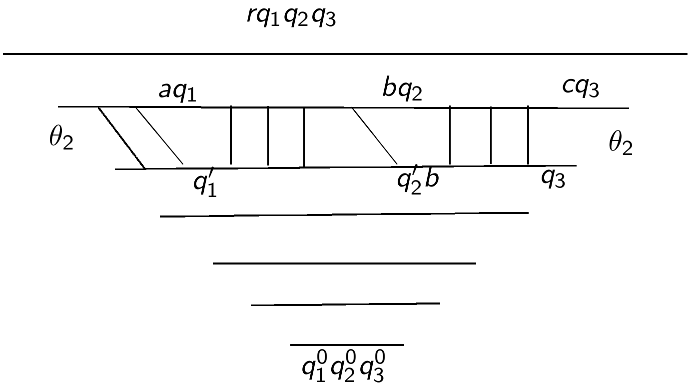

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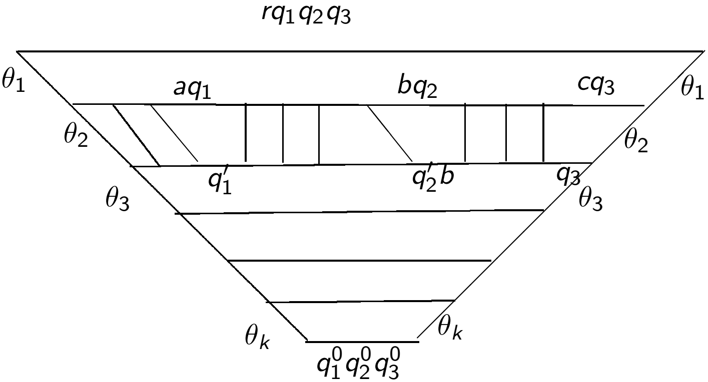

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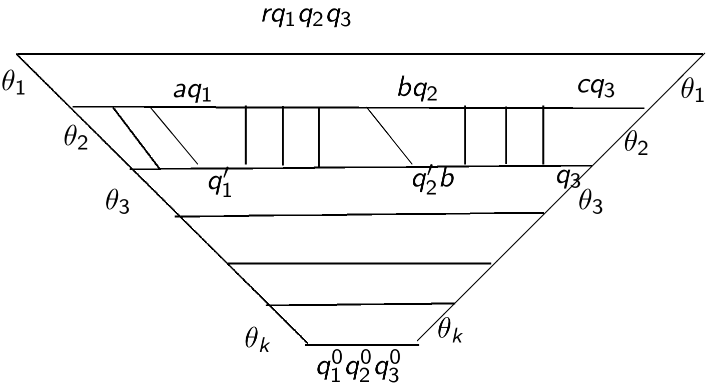

r is accepted if and only if $rq_1q_2q_3$ is conjugated to $q_1^0q_2^0q_3^0$. The conjugator is the history of computation.

The proof of the Higman embedding theorem. 2

We need to hide the history. Apply the Davis' idea. Consecutive petals of the flower are mirror images of each other, glued by k-strips.

The proof of the Higman embedding theorem. 2

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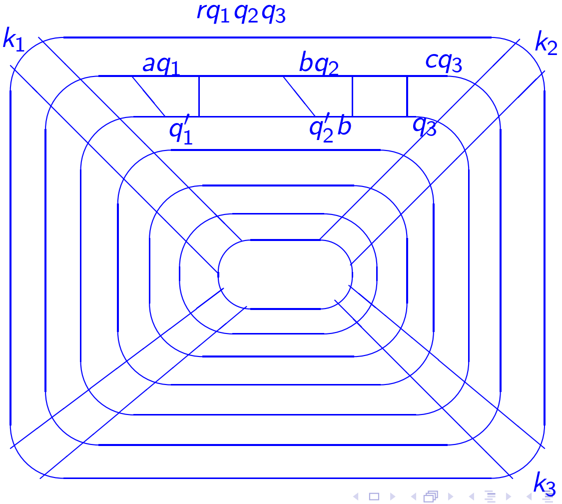

The proof of the Higman embedding theorem. 3

The word $(r^{(1)}q_1^{(1)}q_2^{(1)}q_3^{(1)})(\overline{r^{(2)}q_1^{(2)}q_2^{(2)}q_3^{(2)}})\ldots(\overline{r^{(12)}q_1^{(12)})q_2^{(12)}q_3^{(12)}})$ is 1 in the group if an only if r is accepted by the Turing machine.

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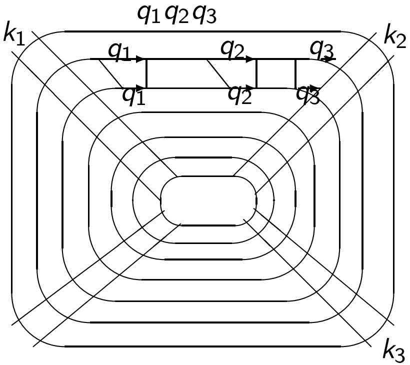

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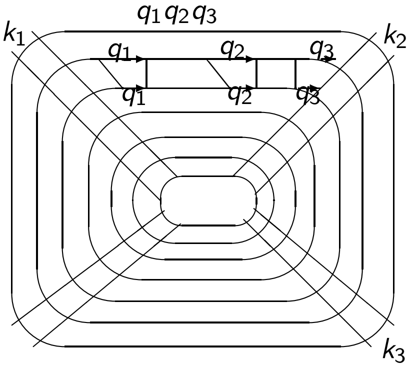

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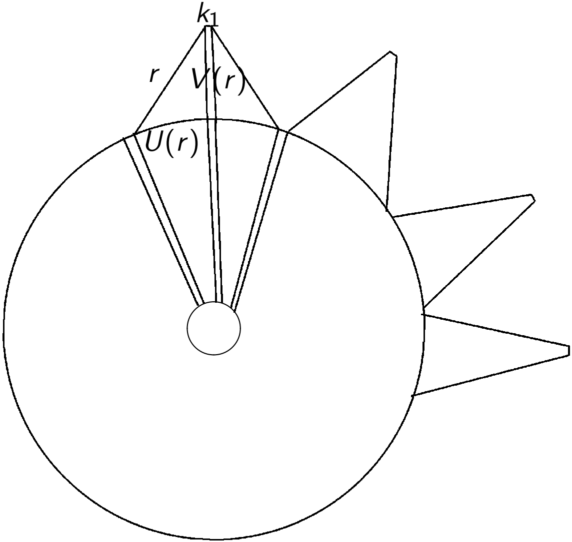
Problem. The group G is almost never aspherical

Indeed, the letters of r commute with θ. Consider the closed cylinder with top and bottom circle containing the diagram for $r = 1$ in G, the side tesselated by the commutativity cells. It is a map from the sphere S^2 into the representation complex.

The real embedding

Replace a sunflower with a rose:



How to prove asphericity of G

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