Morse complexes for manifolds with non-empty boundary, A_∞-structures and application to classical links

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The case of closed manifolds

Let M be a closed manifold and let $f : M \to \mathbb{R}$ be a Morse function. A (descending) *pseudo-gradient* for f is a vector field X such that:

1. (Lyapunov inequality) $X \cdot f < 0$ holds at any point apart from $\mathrm{crit} f$, the critical set of f;
2. (Non-degeneracy condition) $X \cdot f$ is a Morse function near $\mathrm{crit} f$.

Then, X is *hyperbolic* : each critical point p gives rise to an *unstable manifold* $W^u(p, X)$ and a stable manifold $W^s(p, X)$

René Thom, 1949

Sur une partition en cellules associée à une fonction sur une variété, C.R. Acad. Sci. Paris, t. 228, (14/21 mars 1949), 973-975.

$$M = \cup_{p \in \mathrm{crit}\, f} W^u(p, X)$$

Steve Smale, circa 1960

A pseudo-gradient is (now) said to be *Morse-Smale*, if the stable and unstable manifolds are mutually transverse.

On gradient dynamical systems (Annals 1961)

The Morse-Smale condition is generic.

Morse inequalities for a dynamical system (Bull. AMS 1960)

Under the Morse-Smale condition, the next inequalitiy holds true:

$$c_k - c_{k-1} + \ldots \geq b_k - b_{k-1} + \ldots$$

where c_k denotes the number of critical points of index k and b_k is the k-th Betti number of M (with any field of coefficients) .

When M is simply-connected and dim $M > 5$, there is a Morse function *with the minimal number of critical points consistent with the homology structure.* (Amer. J. of Math. 1962).

The latter paper is based on the same circle of ideas as in:

Generalized Poincaré's conjecture in dimensions greater than four, Ann. of Math. 74 (1961), 391-406.

The first Morse complex

It goes back to John Milnor in:
Lectures on the h-cobordism theorem, Princeton Univ. Press, 1965.

Theorem 7.4.

$C_* = (C_k, \partial)$ is a chain complex (i.e. $\partial^2 = 0$) and $H_k(C_*) \cong H_k(M; \mathbb{Z})$.

Here, C_k denotes the free $\mathbb{Z}$-module generated by the critical points of f of index k and, if (p, q) is a pair of critical points of respective index $(k, k-1)$,
$< \partial(p), q >$ ($=$ the component of ∂p on q) counts the signed number of connecting orbits from p to q of the considered Morse-Smale pseudo-gradient.

E. Witten, *Supersymmetry and Morse theory*, JDG, 1982

E. Witten uses a Morse function f for deforming the de Rham differential d_{DR} in the complex $\Omega^*(M)$ of differential forms on M:

The Witten differential

$$d_{\mathrm{Witten}} = \hbar e^{-f/\hbar} \, d_{\mathrm{DR}} \, e^{f/\hbar}$$

where $\hbar$ is a small positive constant.

Given a Riemannian metric μ, there is an associated Witten Laplacian

$$\Delta_{\mathrm{Witten}} = (d_{\mathrm{Witten}} + d^*_{\mathrm{Witten}})^2 \, .$$

When $\hbar \to 0$, some "gap" appears in the spectrum of Δ_{Witten}. The eigenspaces of the "small" eignevalues form a finitely generated sub-complex of $\Omega^*(M)$ in correspondence to the Morse complex of the $\nabla_\mu f$.

Witten's programme was achieved by B. Helffer & J. Sjöstrand (Comm. in PDE, 1985).

$\partial M \neq \emptyset$

Generic Morse function

The Morse function $f : M \to \mathbb{R}$ is said to be generic if :

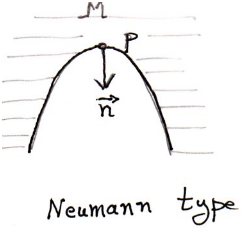
1. f is a Morse function whose critical points are in the interior of$M$,
2. $f_\partial := f|\partial M$ is Morse.

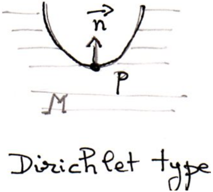
There are two types of critical points of f_∂:

Neumann / Dirichlet type

Let $p \in \partial M$ be a critical point of f_∂, $\vec{n}$ be a tangent vector at p pointing outward.

- p is said of Neumann type if $< df(p), \vec{n} >$ is negative (denoted by $p \in$crit$^N f_\partial$).
- p is said of Dirichlet type $< df(p), \vec{n} >$ is positive (denoted by $p \in$crit$^D f_\partial$).





Witten's programme makes sense for compact manifolds with non-empty boundary by considering the complex of differential forms with Neumann or Dirichlet vanishing conditions along ∂M.

It was solved by different authors, in different settings:

Chang – Liu (1995)

Helffer – Nier (2006)

Kolan – Prokhorenkov – Shubin (2009)

A semi-classical limit of Δ_{Witten} with Neumann boundary conditions gives rise to a finite dimensional complex (for d_{Witten}) one base of which is in 1:1 correspondence to $\mathrm{crit}\, f \cup \mathrm{crit}^N f_\partial$ and whose homology is isomorphic to $H_*(M;\mathbb{R})$.

When taking the Dirichlet boundary conditions the limit gives rise to a complex based on $\mathrm{crit}\, f \cup \mathrm{crit}^D f_\partial$ which calculates the relative homology $H_*(M, \partial M; \mathbb{R})$.

Morse complex of Neumann type

Observation.

There exists a pseudo-gradient X^N, said to be *N-adapted*, such that:

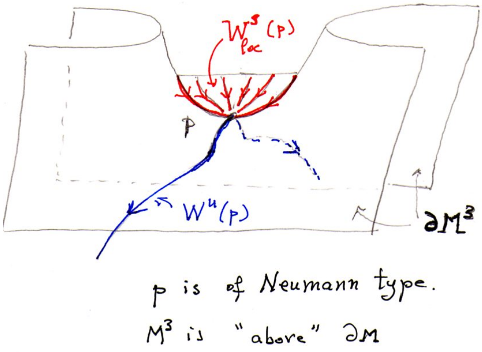
1. $X^N \cdot f < 0$ apart from $\mathrm{crit}\, f \cup \mathrm{crit}^N f_\partial$,
2. X^N is pointing inward at every point of ∂M except near $\mathrm{crit}^N f_\partial$ where X^N is tangent to the boundary.

As a consequence, the flow of X^N is positively complete and each zero p has a global unstable manifold and a local stable manifold. For $p \in crit^N f_\partial$, one has $W^u(p, X^N) \cong \mathbb{R}^{\mathrm{ind}(p,f_\partial)}$ and $W^s_{loc}(p, X^N) \cong \mathbb{R}_+^{1+\mathrm{coind}(p,f_\partial)}$.
Moreover, X^N may be chosen *Morse-Smale, i.e.*:

$$W^u(p) \pitchfork W^s_{loc}(q)$$

for every pair in $\mathrm{crit} f \cup \mathrm{crit}^N f_\partial$. Thus, a counting of connecting orbits is possible.

Global stable manifold



Theorem (L., 2011)

Counting the connecting orbits gives rise to a chain complex $C_*(f, X^N)$ based on $\mathrm{crit} f \cup \mathrm{crit}^N f_\partial$ whose homology is isomorphic to $H_*(M;\mathbb{Z})$.

Morse complex of Dirichlet type

There exists a vector field X^D, said to be *D-adapted*, such that:

1. $X^D \cdot f > 0$ apart from $\mathrm{crit}\, f \cup \mathrm{crit}^D f_\partial$,
2. X^D is pointing inward at every point of ∂M except near $\mathrm{crit}^D f_\partial$ where X^D is tangent to the boundary.

Remark. X^D is N-adapted to $-f$. So, by a Poincaré duality argument, it is natural to get the following:

Theorem (L., 2011)

Counting the connecting orbits gives rise to a chain complex $C_*(f, X^D)$ which in degree k is based on $\mathrm{crit}_k f \cup \mathrm{crit}^D_{k-1} f_\partial$ whose homology is isomorphic to the relative homology $H_*(M, \partial M; \mathbb{Z})$.

According to K. FUKAYA

Case of closed manifolds

K. Fukaya, *A_∞-structures and Floer homologies*, 1993.

$\partial M \neq \emptyset$ (joint work with C. Blanchet, in progress)

Theorem. *There exists a natural structure of A_∞-algebra on each of the above-mentioned complexes*

$$m_d : C_* \otimes \ldots \otimes C_* = C_*^{\otimes d} \to C_*$$

such that $m_1 = \partial$ and m_2 induces the cup-product in homology.

For instance, m_3 yields a Morse approach to the *Massey product*.

Axioms of A_∞-Algebra

1. $m_1 \circ m_1 = 0$
2. $m_2 \circ (m_1 \otimes 1 + 1 \otimes m_1) = m_1 \circ m_2$;
   m_2 is a chain morphism; so, it induces a product at homology level.
3. The defect for m_3 of being a chain morphism equals the defect for m_2 of being associative, *i.e.*
   $$m_1 \circ m_3 - m_3 \circ (m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1) = m_2 \circ (m_2 \otimes 1 - 1 \otimes m_2);$$
4. etc.
5. Grading (next slide).

The Massey product

If u, v, w are three cycles such that $m_2(u, v) = \partial a$ and $m_2(v, w) = \partial b$, then $m_3(u, v, w) - (m_2(a, w) - m_2(u, b))$ is a cycle.

Grading

After shifting the grading by $n = \dim M$, (set $A_* := C_{*+n}$), m_d becomes a morphism of degree $(d-2)$. For instance, when $n = 3$, if u, v, w are of degree 2 (*i.e.* they belong to A_{-1}), then $m_3(u, v, w)$ belongs to C_1.

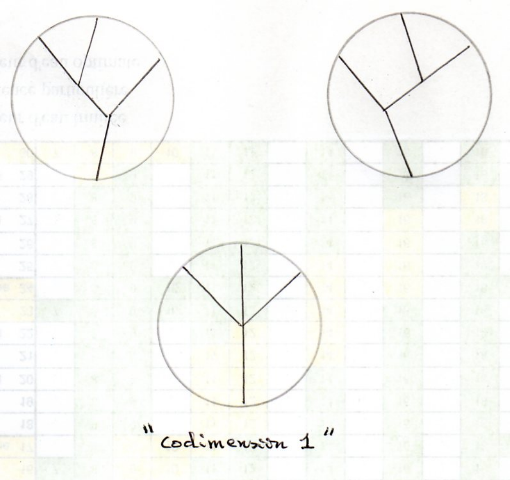
Idea for the proof (case of $C_*(f, X^D)$). It is based on the next remark:

Stability by perturbation

If (f', X'^D) is close to (f, X^D), then the associated complexes are canonically isomorphic. In general position, the invariant manifolds of the first pair are transverse to those of the second pair. Moreover, *multi-intersections* in Fukaya's sense may be taken.

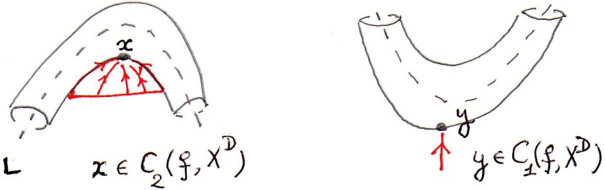
Multi-intersections in Fukaya's sense

Multi-intersections are modeled on *Fukaya's rooted trees*. Here are those with three leaves.



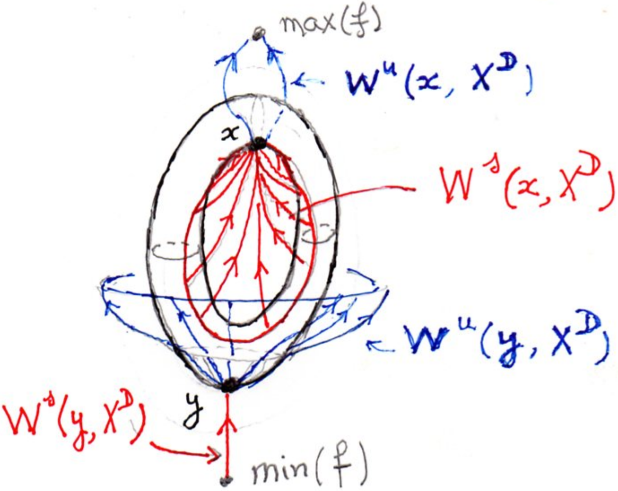
Examples

Let $(\mathbb{S}^3, h)$ be the 3-sphere equipped with the standard height function (one max, one min.). Let $L \subset \mathbb{S}^3$ be a generic link: $h|L$ is a Morse function. Consider $M = \mathbb{S}^3 \smallsetminus \operatorname{int} N(L)$ where $N(L)$ is a "standard" tubular neighborhood of L. Set $f := h|M$, choose a D-adapted vector field X^D, and calculate $C_*(f, X^D)$.
Near each maximum of $h|L$, there is one critical point of f_∂ of Dirichlet type; it has degree 2.

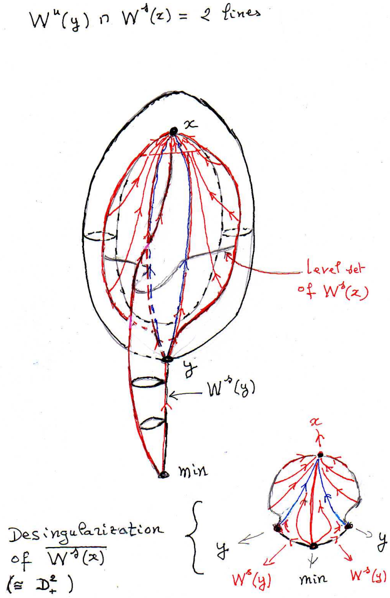


Near each minimum of $h|L$, there is one critical point of f_∂ of Dirichlet type; it has degree 1.

Example 1: the trivial knot

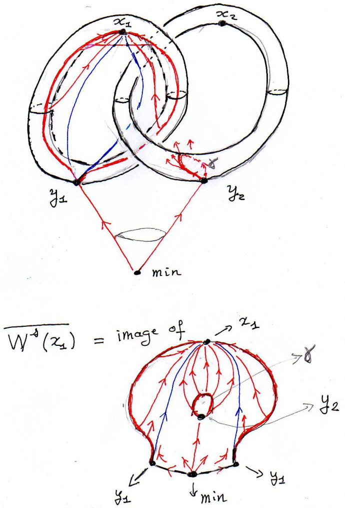


And, with another D-adapted pseudo-gradient:



Example 2: the Hopf link

From the picture below it follows: $m_2(x_1, x_2) = y_1 - y_2$.



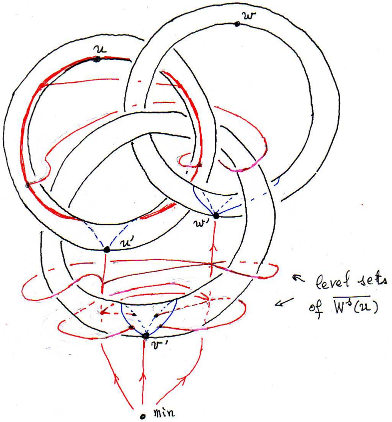
Example 3: the Borromean link

Consider now the "standard" Borromean link (*i.e.*: $h|L$ has 3 max. and 3 min.). Let u, v, w be these generators of $C_2(f, X^D)$ and let u', v', w' be these generators of $C_1(f, X^D)$.

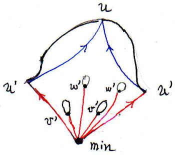
The figures below imply the following formulas:

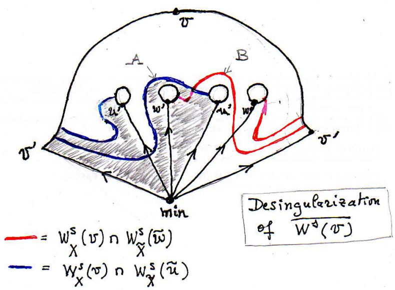
1. $\partial^D(u) = \partial^D(v) = \partial^D(w) = 0$
2. $m_2 = 0$ (due to no linking between components of L)
3. $m_3(u, v, w)$ is a 1-cycle whose homology class is non-trivial:

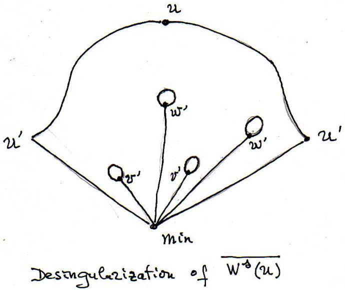
$$m_3(u, v, w) = u' - w'.$$



Desingularization of $\overline{W^s(u)}$




v
A
B
u'
w'
u'
w'
v'
v'
min
Desingularization of $\overline{W^s(v)}$
$= W^s_X(v) \cap W^s_{\tilde{X}}(\tilde{w})$
$= W^s_X(v) \cap W^s_{\tilde{X}}(\tilde{u})$


u
w'
u'
u'
w'
v'
v'
min
Desingularization of $\overline{W^s(u)}$