A combinatorial approach to Heegaard Floer invariants

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Motivation (4D)

Much insight into smooth 4-manifolds comes from PDE techniques: *gauge theory* and *symplectic geometry*.

- **Donaldson** (1980's): used the Yang-Mills equations to get constraints on the intersection forms of smooth 4-manifolds, etc. Solution counts $\longrightarrow$ *Donaldson invariants*, able to detect exotic smooth structures on closed 4-manifolds.
- **Seiberg-Witten** (1994): discovered the monopole equations, which can replace Yang-Mills for most applications, and are easier to study. Solution counts $\longrightarrow$ *Seiberg-Witten invariants*.
- **Ozsváth-Szabó** (2000's): developed Heegaard Floer theory, based on counts of holomorphic curves in symplectic manifolds. Their *mixed HF invariants* of 4-manifolds are conjecturally the same as the Seiberg-Witten invariants, and can be used for the same applications (in particular, to detect exotic smooth structures).

Motivation (3D)

All three theories (Yang-Mills, Seiberg-Witten, Heegaard-Floer) also produce invariants of closed 3-manifolds, in the form of graded Abelian groups called *Floer homologies*. These have applications of their own, e.g.:

- What is the minimal genus of a surface representing a given homology class in a 3-manifold? (**Kronheimer-Mrowka, Ozsváth-Szabó**)
- Does a given 3-manifold fiber over the circle? (**Ghiggini, Ni**)

In dimension 3, the Heegaard-Floer and Seiberg-Witten Floer homologies are known to be isomorphic: work of **Kutluhan-Lee-Taubes** and **Colin-Ghiggini-Honda**, based on the relation to **Hutchings**'s ECH.

There also exist Floer homologies for knots (or links) $K \subset S^3$. Applications: knot genus, fiberedness of the knot complement, concordance, unknotting number, etc.

Motivating problem

All these gauge-theoretic or symplectic-geometric invariants are defined by counting solutions of nonlinear elliptic PDE's. As such, they are very difficult to compute.

Question

Are the (Yang-Mills, Seiberg-Witten, Heegaard-Floer) invariants of knots, 3-manifolds, and 4-manifolds *algorithmically computable*? In other words, do they admit *combinatorial* definitions?

The answer is now Yes for the Heegaard-Floer invariants (mod 2).

This is based on joint work with / work of (various subsets of) : **Robert Lipshitz, Peter Ozsváth, Jacob Rasmussen, Sucharit Sarkar, András Stipsicz, Zoltan Szabó, Dylan Thurston, Jiajun Wang**.

Outline

1. Summary of Heegaard Floer theory
2. Algorithms for the HF invariants of knots and links in S^3
3. Presenting 3-manifold and 4-manifolds in terms of links in S^3
4. From HF link invariants to HF invariants of 3- and 4-manifolds
5. An algorithm for the HF invariants of 3- and 4-manifolds
6. Open problems

Summary of Heegaard Floer theory

For simplicity, we work over the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

Ozsváth-Szabó (2000)

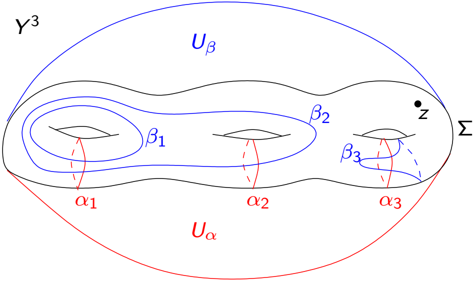
Y^3 closed, oriented 3-manifold $\rightarrow$ $\mathbf{HF}^-(Y), \widehat{HF}(Y), HF^+(Y)$ = modules over the power series ring $\mathbb{F}[[U]]$ (variants of Heegaard Floer homology).

Start with a marked Heegaard diagram for Y:

- Σ = surface of genus g
- $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\}$ collection of disjoint, homologically linearly independent, simple closed curves on Σ, specifying a handlebody U_α
- $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\}$ a similar collection, specifying U_β
- a basepoint $z \in \Sigma - \cup\alpha_i - \cup\beta_i$

such that $Y = U_\alpha \cup_\Sigma U_\beta$.

A marked Heegaard diagram



$$U_\alpha = \Sigma \cup \bigcup_{i=1}^{g} D_i^\alpha \cup B^3, \quad \partial D_i^\alpha = \alpha_i$$

$$U_\beta = \Sigma \cup \bigcup_{i=1}^{g} D_i^\beta \cup B^3, \quad \partial D_i^\beta = \beta_i.$$

Heegaard Floer homology

Consider the tori

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$$

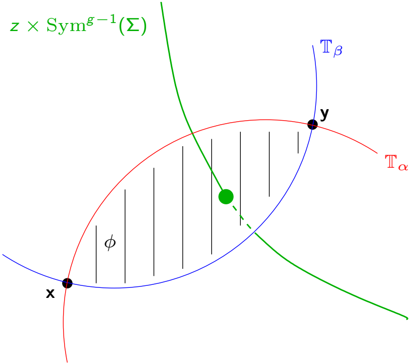
$$\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$$

inside the symmetric product $\mathrm{Sym}^g(\Sigma) = (\Sigma \times \cdots \times \Sigma)/S_g$, viewed as a symplectic manifold.

The Heegaard Floer complex $\mathbf{CF}^-(Y)$ is freely generated (over $\mathbb{F}[[U]]$) by intersection points

$$\mathbf{x} = \{x_1, \ldots, x_g\} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$$

with $x_i \in \alpha_i \cap \beta_j \subset \Sigma$. The differential is given by counting pseudo-holomorphic disks in $\mathrm{Sym}^g(\Sigma)$ with boundaries on $\mathbb{T}_\alpha, \mathbb{T}_\beta$. These are solutions to the nonlinear Cauchy-Riemann equations, and depend on the choice of a generic family of *almost complex structures* J on the symmetric product.



$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(\mathbf{x}, \mathbf{y})} n(\phi, J) \cdot U^{n_z(\phi)} \mathbf{y},$$

where $\pi_2(\mathbf{x}, \mathbf{y})$ is the space of relative homology classes of Whitney disks from $\mathbf{x}$ to $\mathbf{y}$, and $n(\phi, J) \in \mathbb{F} = \{0, 1\}$ is the count of holomorphic disks in the class ϕ.

The variable U keeps track of the quantity $n_z(\phi)$, the intersection number between ϕ and the divisor $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma)$.

One can show that $\partial^2 = 0$. We define:

- $\mathbf{HF}^- = H_*(\mathbf{CF}^-)$;
- $\widehat{HF} = H_*(\mathbf{CF}^-/(U=0))$;
- $HF^+ = H_*(U^{-1}\mathbf{CF}^-/\mathbf{CF}^-)$.

Ozsváth and Szabó proved that these are invariants of the 3-manifold Y. Typically, $\widehat{HF}$ has less information than $\mathbf{HF}^-$ and HF^+ (although it suffices for most 3D applications).

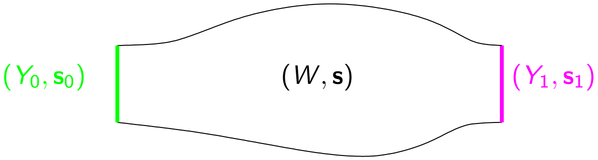
For example, $\mathbf{HF}^-(S^3) = \mathbb{F}[[U]], \widehat{HF}(S^3) = \mathbb{F}, HF^+(S^3) = \mathbb{F}[U^{-1}]$.

All three variants ($\circ = +, -, \text{hat}$) split according to Spin^c-structures on Y:

$$HF^\circ(Y) = \bigoplus_{\mathbf{s} \in \mathrm{Spin}^c(Y)} HF^\circ(Y, \mathbf{s})$$

Heegaard Floer theory as a TQFT

The three variants of Heegaard Floer homology are functorial under Spin^c-decorated cobordisms.

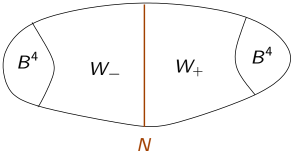


$$F^{\circ}_{W,\mathbf{s}} : HF^{\circ}(Y_0, \mathbf{s}_0) \longrightarrow HF^{\circ}(Y_1, \mathbf{s}_1)$$

The maps are defined by counting *pseudo-holomorphic triangles* in the symmetric product, with boundaries on three different tori.

Mixed HF invariants of 4-manifolds (Ozsváth-Szabó, 2001)

X^4 = closed, oriented, smooth 4-manifold with $b_2^+(X) > 1$



N = admissible cut

$b_2^+(W_\pm) > 0$

$\delta H^1(N) = 0 \subset H^2(X)$

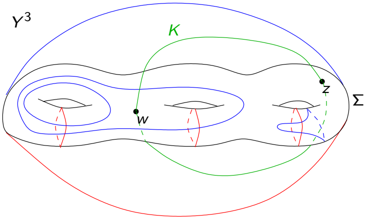
$$\begin{tikzcd}
1 \in \mathbb{F}[[U]] = HF^-(S^3) \arrow[r, "F^-_{W_-}"] \arrow[dr, dashed] \arrow[dddrr, bend right] & HF^-(N) & \\
& HF^{\mathrm{red}}(N) \arrow[u] \arrow[dr, dashed] & \\
& HF^+(N) \arrow[u] \arrow[r, "F^+_{W_+}"] & HF^+(S^3) = \mathbb{F}[U^{-1}] \\
& & \Psi_{X,\mathbf{s}} \in \{0, 1\}
\end{tikzcd}$$

detects exotic smooth structures; conjecturally $= SW_{X,\mathbf{s}}$

Going back to a marked Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, if one specifies another basepoint w, this gives rise to a knot $K \subset Y$.

Ozsváth-Szabó, Rasmussen (2003)

Counting pseudo-holomorphic disks and keeping track of the quantities $n_z(\phi)$ and $n_w(\phi)$ in various ways yields different versions of *knot Floer homology* $HFK^\circ(Y, K)$.



One can also break the knot (or link) into more segments, by using more basepoints.

Simplest version of knot Floer homology: $\widehat{HFK}(S^3, K)$

We count only pseudo-holomorphic disks with $n_z(\phi) = n_w(\phi) = 0$:

$$\widehat{HFK}(S^3, K) = \bigoplus_{m,s \in \mathbb{Z}} \widehat{HFK}_m(S^3, K, s)$$

Its Euler characteristic is the Alexander polynomial of the knot:

$$\sum_{m,s \in \mathbb{Z}} (-1)^m q^s \cdot \mathrm{rk}\, \widehat{HFK}_m(S^3, K, s) = \Delta_K(q).$$

The genus of a knot

$$g(K) = \min\{g \mid \exists \text{ embedded, oriented}, \Sigma^2 \subset S^3 \text{ of genus } g,\ \partial\Sigma = K\}$$

can be read from $\widehat{HFK}$ (**Ozsváth-Szabó**, 2004):

$$g(K) = \max\{s \geq 0 \mid \exists\ m,\ \widehat{HFK}_m(S^3, K, s) \neq 0\}.$$

More about $\widehat{HFK}(S^3, K)$

In particular:

K is the unknot $(g(K) = 0) \iff \widehat{HFK}(S^3, K) \cong \mathbb{F}_{(0,0)}$.

By a result of **Ghiggini** (2006), knot Floer homology has enough information to also detect the right-handed trefoil, the left-handed trefoil, and the figure-eight knot.

Ni (2006) showed that $S^3 \setminus K$ fibers over the circle if and only if $\oplus_m \widehat{HFK}_m(S^3, K, g(K)) \cong \mathbb{F}$.

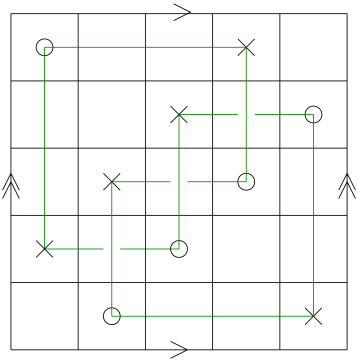
Theorem (M.-Ozsváth-Sarkar, 2006)

All variants of Heegaard Floer homology for links $L \subset S^3$ are algorithmically computable.

Every link in S^3 admits a *grid diagram*; that is, an n-by-n grid in the plane with O and X markings inside such that:

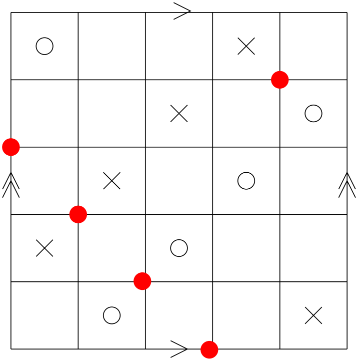
- Each row and each column contains exactly one X and one O;
- As we trace the vertical and horizontal segments between O's and X's (verticals on top), we see the link L.

A grid diagram for the trefoil



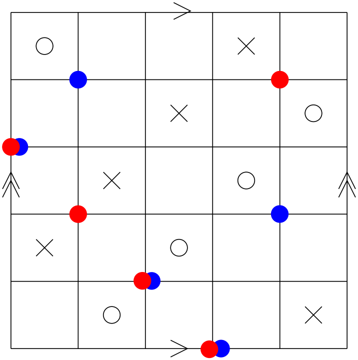
Grid diagrams are particular examples of Heegaard diagrams: if we identify the opposite sides to get a torus, we let: $\alpha =$ horizontal circles, $\beta =$ vertical circles; the X's and O's are the basepoints.

Generators



The generators $\mathbf{x} = \{x_1, \ldots, x_n\}$ are n-tuples of points on the grid (one on each vertical and horizontal circle).

Generators



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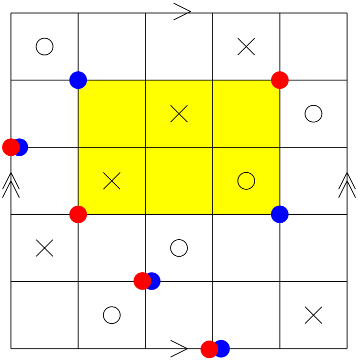
Differentials



In a grid diagram, isolated pseudo-holomorphic disks are in 1-to-1 correspondence to empty rectangles on the grid torus (that is, having no red or blue dots inside).

Why does this work?

Key fact: torus admits a metric of nonnegative curvature. (It could also work on a sphere.)

How about HF for 3-manifolds?

A typical 3-manifold does not admit a Heegaard diagram of genus 0 or 1. (Only S^3, $S^1 \times S^2$ and lens spaces do.)

However, on a surface of higher genus we can move all negative curvature to a neighborhood of a point $\longrightarrow$ algorithm for computing $\widehat{HF}$ of any 3-manifold (**Sarkar-Wang**, 2006), based on *nice diagrams*.

Similarly, one can compute the cobordism maps $\hat{F}_{W,\mathbf{s}}$ for any simply connected W (**Lipshitz-M.-Wang**, 2006). These suffice to detect exotic smooth structures on some 4-manifolds with boundary, but not on any closed 4-manifolds.

Other related results:

- One can give combinatorial proofs of invariance for knot Floer homology, even over $\mathbb{Z}$ (**M.-Ozsváth-Szabó-D.Thurston**, 2006) and for $\widehat{HF}$ of 3-manifolds (**Ozsváth-Stipsicz-Szabó**, 2009).
- One can compute $H_*(\mathbf{CF}^-/(U^2=0))$ (**Ozsváth-Stipsicz-Szabó**, 2008)
- Alternate combinatorial descriptions of knot Floer homology: **Ozsváth-Szabó** (2007), **Baldwin-Levine** (2011); and of $\widehat{HF}(Y^3)$: **Lipshitz-Ozsváth-D.Thurston** (2010).

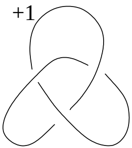
To get algorithms for all HF invariants, we need to present 3- and 4-manifolds in terms of links in S^3...

Surgery presentations of 3-manifolds

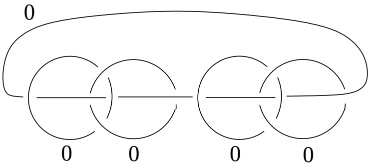
Any closed 3-manifold is integral surgery on a link in S^3 (**Lickorish-Wallace**, 1960):

$$Y = (S^3 \setminus \mathrm{nbhd}(L)) \cup_\phi (\mathrm{nbhd}(L)).$$

For example,



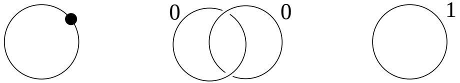
is the Poincaré sphere



is $S^1 \times \Sigma_2$

Kirby diagrams of 4-manifolds

By Morse theory, a closed 4-manifold can be broken into a 0-handle, some 1-handles (dotted circles), some 2-handles (numbered circles), some 3-handles (automatic), and a 4-handle. For example:



$S^1 \times S^3$ $\quad$ $S^2 \times S^2$ $\quad$ $\mathbb{CP}^2$

Heegaard Floer homology and integer surgeries on knots

Theorem (Ozsváth-Szabó, 2004)

There is an (infinitely generated) version of the knot Floer complex, $\mathcal{A}(K)$, such that

$$\mathbf{HF}^-(S^3_n(K)) = H_*(\mathcal{A}(K) \xrightarrow{\Phi^K_n} \mathcal{A}(\emptyset))$$

where in $\mathcal{A}(K)$ we count holomorphic disks and in Φ^K_n we count holomorphic triangles.
Furthermore, the inclusion of $\mathcal{A}(\emptyset)$ into the mapping cone complex

$$\mathcal{A}(K) \xrightarrow{\Phi^K_n} \mathcal{A}(\emptyset)$$

induces (on homology) the map

$$F^-_W : \mathbf{HF}^-(S^3) \longrightarrow \mathbf{HF}^-(S^3_n(K))$$

corresponding to the surgery cobordism (2-handle attachment along K).

Definition of $\mathcal{A}(K) = \prod_{s \in \mathbb{Z}} \mathcal{A}_s(K)$

Given a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ for K, each intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ has an Alexander grading $A(\mathbf{x}) \in \mathbb{Z}$, such that

$$\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \;\Rightarrow\; A(\mathbf{x}) - A(\mathbf{y}) = n_z(\phi) - n_w(\phi).$$

For $s \in \mathbb{Z}$, define

$$\begin{aligned} E_s(\phi) &= \max(s - A(\mathbf{x}), 0) - \max(s - A(\mathbf{y}), 0) + n_z(\phi) \\ &= \max(A(\mathbf{x}) - s, 0) - \max(A(\mathbf{y}) - s, 0) + n_w(\phi). \end{aligned}$$

In particular, $E_s = n_w$ for $s \gg 0$ and $E_s = n_z$ for $s \ll 0$.

Then $\mathcal{A}_s$ is generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with the differential

$$\partial \mathbf{x} = \sum_{\mathbf{y}} \sum_{\phi \in \pi_2(\mathbf{x}, \mathbf{y})} n(\phi, J) \cdot U^{E_s(\phi)} \mathbf{y}.$$

The Link Surgery Formula

Theorem (M.-Ozsváth, 2010)

If $L = K_1 \cup K_2 \subset S^3$ is a link, $\mathbf{HF}^-$(surgery on L) is isomorphic to the homology of the surgery complex

$$\begin{tikzcd} \mathcal{A}(L) \arrow[r] \arrow[d] \arrow[dr] & \mathcal{A}(K_1) \arrow[d] \\ \mathcal{A}(K_2) \arrow[r] & \mathcal{A}(\emptyset) \end{tikzcd}$$

where the edge maps count holomorphic triangles, and the diagonal map counts holomorphic quadrilaterals.

This can be generalized to links with any number of components. The higher homotopies involve counting higher holomorphic polygons. Further, the inclusion of the subcomplex corresponding to $L' \subseteq L$ corresponds to the cobordism maps given by surgery on $L - L'$. (Any 2-handle attachment can be presented in this way.)

General algorithms

Theorem (M.-Ozsváth-D.Thurston, 2009)

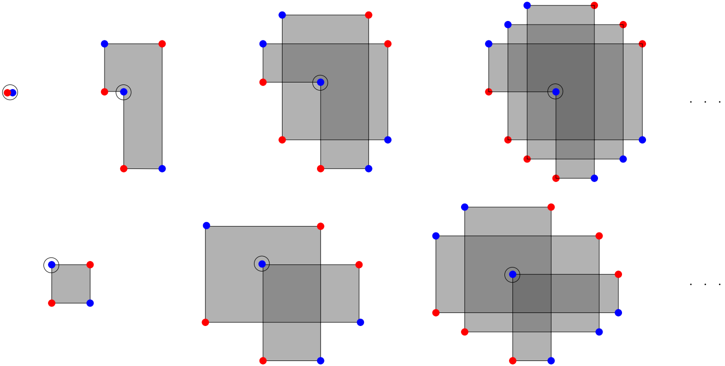
All versions of HF for 3-manifolds are algorithmically computable. So are the mixed invariants $\Psi_{X,\mathbf{s}}$ for closed 4-manifolds X with $b_2^+(X) > 1$ and $\mathbf{s} \in \mathrm{Spin}^c(X)$.

Thus, in principle one can now detect exotic smooth structures on 4-manifolds combinatorially.

Idea: represent the 3-manifold (or 4-manifold) in terms of a link, take a grid diagram for the link, and use the Link Surgery Formula.

We know that isolated holomorphic disks correspond to empty rectangles:

Isolated holomorphic triangles correspond to *snails*



Here, the larger circles represent fixed points on the grid (destabilization points).

Higher polygons

Let G be a grid diagram. The counts of higher pseudo-holomorphic polygons on $\mathrm{Sym}^n(G)$ depend on the choice of a generic almost complex structure J on $\mathrm{Sym}^n(G)$.

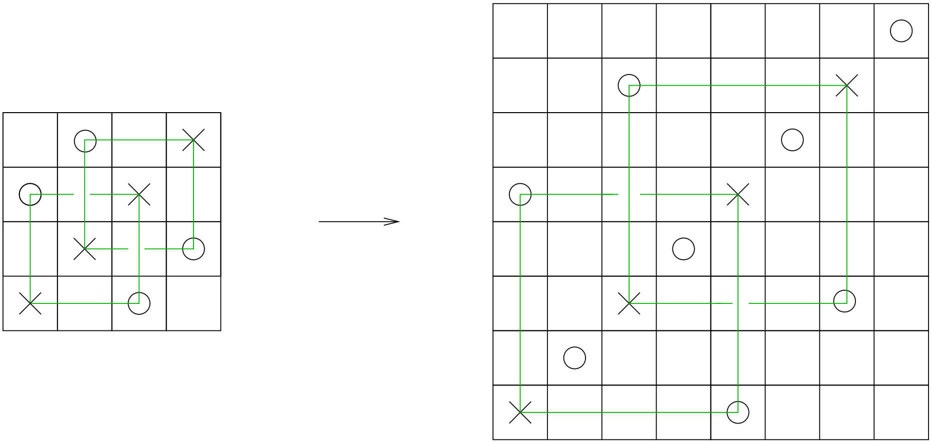
However, they are required to satisfy certain constraints, coming from positivity and Gromov compactness. We define a *formal complex structure* on G to be any count of higher polygons that satisfies these constraints.

Formality Theorem

On a *sparse* grid diagram, any two formal complex structures are homotopic; hence they produce chain homotopic surgery complexes, and the same mixed invariants.

Conjecturally, this is true for any grid.

A sparse double



The algorithm

Represent the 3-manifold as surgery on a link, and choose a grid diagram for the link.

Pick any formal complex structure (on the sparse double), construct a surgery complex using the respective counts. By the Formality Theorem, we know this gives the same answer as a true almost complex structure, so it does give HF of the surgery.

Inclusions of subcomplexes produce cobordism maps (2-handle additions). Building on this, we get a combinatorial description of the mixed invariants of closed 4-manifolds.

Effectiveness

For closed 4-manifolds, not very effective (the $K3$ surface needs a grid of size at least 88).

For 3-manifolds and some 4-dimensional cobordisms (e.g. surgeries on knots in S^3), hard but doable.

For knots and links in S^3, very useful!

Computer programs (**Beliakova-Droz**) can calculate HF of knots with grid number up to 13.

Grid diagrams and HF theory yield a nice algorithm for detecting the genus of a knot and, in particular, whether a knot is unknotted ($g(K) = 0$).

Open problems

1. Develop more efficient algorithms.
2. Extend the algorithms to the integer-valued invariants (rather than mod 2).
3. Give combinatorial proofs of invariance.
4. Prove the Seiberg-Witten / Heegaard-Floer equivalence in 4D.
5. Understand the relationship of Heegaard-Floer theory to Yang-Mills (instanton) theory, and to π_1.
6. Use the (links $\rightarrow$ 3-manifolds $\rightarrow$ 4-manifolds) strategy to turn other link invariants into 3- and 4-manifold invariants.