Primes, Knots and Po

in honor of the N-th birthday of Po!

Smooth homology n-spheres bounding smooth contractible $n+1$-manifolds

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- The total space is a contractible 4-manifold.
- The boundary is a non-simply connected homology 3-sphere.

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Discuss: simple connectivity versus geometric simple connectivity.

Doubling manifolds

Both Po's examples and mine had the further feature that when you doubled them on their boundary—i.e., put two copies of them together by identifying their boundaries—you got a closed differentiable manifold diffeomorphic to the four-dimensional sphere,

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which means that there is a smooth involution of S^4 (switching the two copies of the doubled manifold) with non-simply-connected fixed point set, and therefore 'exotic' in the sense that the involution is not equivalent to a linear involution.

Doubling manifolds

I remember that at the time—the late fifties of the past century—I was very much in awe of the magical construction of R.H. Bing who showed that the double of the closure of the *bad* component of the complement of the Alexander horned sphere in S^3 is again (topologically) S^3 giving, therefore, a thoroughly wild and untamable involution of S^3.

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The surprising re-creation of S^4 by sewing together the boundaries of two manifolds continued as a theme of Po's work, as in the marvelous theorem that he proved with Francois Laudenbach in the early seventies.

Knot theory a link to many other—seemingly far-flung—aspects of mathematics

When I was trying to get a feel for number theory, I found that a certain analogy between the knot theory that I knew as a topologist and the phenomenology of prime numbers (that I was trying to become at home with) was exceedingly helpful, as a bridge.

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I've returned to it often as a learning device and it seems that it might allow two-way traffic, from knots to primes, and from primes to knots. In celebration of Po for this conference, let me explain very briefly what this analogy consists of.

A single knot:

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$$K \hookrightarrow S^3.$$

The knot complement

$$X = X_K := \ S^3 - K \ \hookrightarrow \ S^3.$$

Alexander duality establishes a $\mathbb{Z}$-duality between

$$H^1(X; \mathbf{Z})$$

and

$$\partial : H_2(S^3, K; \mathbf{Z}) \xrightarrow{\simeq} H_1(K; \mathbf{Z}) = \mathbf{Z},$$

giving us a canonical isomorphism:

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Abelian coverings

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- that all finite abelian covering spaces of S^3 branched at the knot, but unramified outside it have *cyclic* groups of deck transformations,
- that these cyclic groups have canonical compatible generators,
- and that $X^{\mathrm{ab}} \to X$, the maximal abelian covering space of X, has group of deck transformations Γ canonically isomorphic to $\mathbb{Z}$.

The fundamental group of the knot

Or equivalently, setting

$$\Pi_K := \pi_1(X, x),$$

with suitable base point x—the *fundamental group of the knot*—we have

$$\Pi^{ab} := \Pi/[\Pi, \Pi] \simeq \mathbb{Z}.$$

Some vocabulary to begin the analogy

Up to isotopy, the knot complement X_K may be viewed as compact manifold with torus boundary, $T_K = \partial X_K$, and within that torus—up to homotopy—there's a normal meridian loop generating an infinite cycle group

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In anticipation of our comparison we might call

$$\mathcal{D}_K = \pi_1(T_K) = \mathbb{Z} \times \mathbb{Z}$$

the *decomposition group* of the knot, and

$$\mathcal{I}_K = \pi_1(N_K) = \mathbb{Z}$$

the *inertia subgroup*.

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Discuss the issue of profinite completion. When does the profinite completion of $(*)$ determine the knot K (or link)?

A single prime number p in the integers $\mathbb{Z}$

The algebra here is just given by the natural "reduction mod p" homomorphism

$$\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} = \mathbf{F}_p.$$

We will avoid the prime $p = 2$.

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our analogy begins by thinking of $\mathcal{K} := \mathrm{Spec}(\mathbb{F}_p)$ as 'like' the knot K and $\mathcal{S} := \mathrm{Spec}(\mathbb{Z})$ as 'like" the three-dimensional sphere S^3.

The "geometry" of the prime p

For every positive integer n, up to isomorphism, there is a unique field of cardinality p^n, $\mathbb{F}_{p^n}$ given as a field extension $\mathbb{F}_{p^n}/\mathbb{F}_p$ which is Galois, cyclic, and degree n.

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$$\mathrm{Spec}(\mathbb{F}_{p^n}) \to \mathbb{F}_p$$

is a cyclic (unramified!) Galois cover with Galois group canonically $\mathbb{Z}/n\mathbb{Z}$.

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An algebraic closure $\bar{\mathbb{F}}/\mathbb{F}_p$ is an appropriate union of these field extensions, and its Galois group is (canonically) isomorphic to $\hat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$.

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From the étale homotopy perspective, $\mathrm{Spec}(\bar{\mathbb{F}})$ is contractible, and therefore $\mathcal{K} := \mathrm{Spec}(\mathbb{F}_p)$ is homotopically a $K(\hat{\mathbb{Z}}, 1)$-space.

The "geometry" of the ring of integers $\mathbb{Z}$

The theory for $\mathcal{S} := \mathrm{Spec}(\mathbb{Z})$ requires some class field theory, as reformulated in the vocabulary of étale (and some other Grothendieckian) cohomology theories.

The "geometry" of the ring of integers $\mathbb{Z}$

The theory for $\mathcal{S} := \mathrm{Spec}(\mathbb{Z})$ requires some class field theory, as reformulated in the vocabulary of étale (and some other Grothendieckian) cohomology theories.

Firstly, $\mathcal{S}$ is simply connected, in the sense that every connected finite cover of $\mathcal{S}$ is ramified.

The "geometry" of the ring of integers $\mathbb{Z}$

Moreover, $\mathcal{S}$ enjoys a three-dimensional 'Poincaré-type' duality theorem for étale and flat cohomology with values in the multiplicative group $\mathbf{G}_m$ in the sense that

- $H^i(\mathcal{S}, \mathbf{G}_m)$ is (canonically) equal to $\{\pm 1\}, 0, 0, \mathbf{Q}/\mathbb{Z}$, and 0 for $i = 0, 1, 2, 3$, and > 3 respective;

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- If F is a finite flat group scheme over $\mathcal{S}$ and $F^* := Hom(F, \mathbf{G}_m)$ its (Cartier) dual finite flat group scheme, then cup-product induces a perfect pairing

$$H^i(\mathcal{S}, F) \otimes H^{3-i}(\mathcal{S}, F^*) \longrightarrow H^3(\mathcal{S}, \mathbf{G}_m) = \mathbf{Q}/\mathbb{Z}.$$

The "geometry" of the ring of integers

In a word, $\mathcal{S}$ is morally 2-connected and enjoys a 3-dimensional Poincaré duality "oriented" by the coefficient sheaf $\mathbf{G}_m$.

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and form the 'knot complement'

$$\mathcal{X} := \mathcal{S} - \mathcal{K} = \mathrm{Spec}(\mathbb{Z}[1/p]) \hookrightarrow \mathcal{S}.$$

The “fundamental group of the knot”

Set

$$\Pi_{\mathcal{K}} := \pi_1^{et}(\mathcal{X}, x),$$

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$\Pi_{\mathcal{K}}$ is the quotient of the Galois group of the maximal extension of the field of rational functions in its algebraic closure that is unramified except at p and infinity.

We have:

$$\Pi_{\mathcal{K}}^{ab} := \Pi_{\mathcal{K}}/[\Pi_{\mathcal{K}}, \Pi_{\mathcal{K}}] \simeq \mathbb{Z}_p^*,$$

and if $\mathcal{X}^{\mathrm{ab}} \to \mathcal{X}$ is the maximal unramified abelian (connected) cover, then we can also say

$$\text{“Gal}(\mathcal{X}^{\mathrm{ab}}/\mathcal{X})\text{”} = \Pi_{\mathcal{K}}^{ab} = \mathbb{Z}_p^*.$$

Discuss the **Frobenius conjugacy class**

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Let q be a prime different from p and and consider the map induced on *fundamental groups* of

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The *conjugacy class* of the image of the generator in $\Pi_{\mathcal{K}}$ is called the **Frobenius conjugacy class** of q in $\Pi_{\mathcal{K}}$.

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Cebotarev arrangements