Quotients of the mapping class group by power subgroups

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Dedicated to the memory of Ştefan Papadima (1953–2018).

ABSTRACT

We study the quotient of the mapping class group Mod_g^n of a surface of genus g with n punctures, by the subgroup $\mathrm{Mod}_g^n[p]$ generated by the pth powers of Dehn twists.

Our first main result is that $\mathrm{Mod}_g^1 / \mathrm{Mod}_g^1[p]$ contains an infinite normal subgroup of infinite index, and in particular is not commensurable to a higher rank lattice, for all but finitely many explicit values of p. Next, we prove that $\mathrm{Mod}_g^0 / \mathrm{Mod}_g^0[p]$ contains a Kähler subgroup of finite index, for every $p \geqslant 2$ coprime with six. Finally, we observe that the existence of finite-index subgroups of Mod_g^0 with infinite abelianization is equivalent to the analogous problem for $\mathrm{Mod}_g^0 / \mathrm{Mod}_g^0[p]$.

1. Introduction and statements

Throughout, S_g^n denotes the connected orientable surface of genus $g \geqslant 2$, with empty boundary and $n \geqslant 0$ marked points. Associated to S_g^n is the mapping class group Mod_g^n, namely the group of homeomorphisms of S_g^n up to isotopy. In the case when $n = 0$, we will drop it from the above notation and write S_g and Mod_g.

There is widespread interest in studying homomorphisms from mapping class groups to compact Lie groups, notably through the so-called *quantum representations* of mapping class groups. As it turns out, if G is a compact Lie group and $\mathrm{Mod}_g^n \to G$ is a homomorphism (with $g \geqslant 3$) then the image of every Dehn twist has finite order [2, Corollary 2.6]. With this motivation in mind, it is natural to study the structure of the group

$$\frac{\mathrm{Mod}_g^n}{\mathrm{Mod}_g^n[p]},$$

where $\mathrm{Mod}_g^n[p]$ denotes the (normal) subgroup of Mod_g^n generated by p-powers of Dehn twists. We remark that $\mathrm{Mod}_g^n / \mathrm{Mod}_g^n[p]$ is known to be infinite if $g = 2$ and $p \geqslant 4$ [19], or if $g \geqslant 3$ and $p \notin \{1, 2, 3, 4, 6, 8, 12\}$ [15].

1.1. Non-lattice properties

In [17], Pitsch and the second author used quantum topological techniques to prove that $\mathrm{Mod}_g / \mathrm{Mod}_g[p]$ is not commensurable to a higher rank lattice whenever $g \geqslant 4$ and $p \geqslant 2g - 1$. The first purpose of this note is to give the following uniform version of this result, in the case of once-punctured surfaces:

THEOREM 1. *Let $g, p \geqslant 2$, where $p \geqslant 4$ if $g = 2$, and $p \notin \{2, 3, 4, 6, 8, 12\}$ if $g \geqslant 3$. Then*

$$\frac{\mathrm{Mod}_g^1}{\mathrm{Mod}_g^1[p]}$$

has a descending normal series $Q_1 \trianglerighteq Q_2 \trianglerighteq \cdots$ such that Q_{i+1} has infinite index in Q_i for all $i \geqslant 1$.

The groups Q_i in the statement of Theorem 1 are obtained through a covering construction, similar in spirit to that of the 'non-geometric' injective homomorphism between mapping class groups of [1].

In light of Margulis' Normal Subgroup Theorem [23], Theorem 1 has the following immediate consequence:

COROLLARY 2. *Let $g, p \geqslant 2$, where $p \geqslant 4$ if $g = 2$, and $p \notin \{2, 3, 4, 6, 8, 12\}$ if $g \geqslant 3$. Then $\mathrm{Mod}_g^1 / \mathrm{Mod}_g^1[p]$ is not commensurable to a lattice in a semisimple Lie group of real rank $\geqslant 2$.*

Next, we will give the following analogue of the *Birman short exact sequence* for the groups $\mathrm{Mod}_g^1 / \mathrm{Mod}_g^1[p]$. Below, $\pi_1(S_g)[p]$ denotes the subgroup of $\pi_1(S_g)$ generated by p-powers of *simple* loops, that is, loops without transverse self-intersection.

PROPOSITION 3. *Let $g, p \geqslant 2$. There is a short exact sequence:*

$$1 \to \frac{\pi_1(S_g)}{\pi_1(S_g)[p]} \to \frac{\mathrm{Mod}_g^1}{\mathrm{Mod}_g^1[p]} \to \frac{\mathrm{Mod}_g}{\mathrm{Mod}_g[p]} \to 1.$$

Combining Proposition 3 with the construction of the groups Q_i of Theorem 1, we will immediately obtain

COROLLARY 4. *Let $g, p \geqslant 2$, where $p \geqslant 4$ if $g = 2$ and $p \notin \{2, 3, 4, 6, 8, 12\}$ if $g \geqslant 3$. Then*

$$\frac{\pi_1(S_g)}{\pi_1(S_g)[p]}$$

is not commensurable to a lattice in a semisimple Lie group of real rank $\geqslant 2$.

1.2. *Virtually Kähler quotients*

Farb [12] has asked whether Mod_g (with $g \geqslant 3$) is a *Kähler group*, that is, the fundamental group of a compact Kähler manifold. Our next result shows that a large class of quotient groups $\mathrm{Mod}_g/\mathrm{Mod}_g[p]$ have finite-index subgroups which are Kähler

THEOREM 5. *Suppose $p \geqslant 2$ and $\gcd(p, 6) = 1$. For every $g \geqslant 0$, the group $\mathrm{Mod}_g/\mathrm{Mod}_g[p]$ is virtually Kähler, that is, it has a Kähler subgroup of finite index.*

In order to prove Theorem 5, we adapt arguments of Pikaart–Jong [24] to exhibit a compact Kähler manifold whose fundamental group is a finite-index subgroup of $\mathrm{Mod}_g/\mathrm{Mod}_g[p]$.

As an immediate consequence, we obtain the following result (compare with Theorem 1).

COROLLARY 6. *Suppose $p \geqslant 2$ and $\gcd(p, 6) = 1$. For every $g \geqslant 0$, the group $\mathrm{Mod}_g / \mathrm{Mod}_g[p]$ is not a lattice in $\mathrm{SO}(n, 1)$, or more generally in a group of Hodge type.*

Another consequence is the following:

COROLLARY 7. *Let $g \geqslant 0$. Given $p \geqslant 2$ with $\gcd(p, 6) = 1$, a finite-index Kähler subgroup $J \subset \mathrm{Mod}_g / \mathrm{Mod}_g[p]$, a lattice $\Lambda \subset \mathrm{SO}(n, 1)$ and any homomorphism*

$$\Phi : \mathrm{Mod}_g / \mathrm{Mod}_g[p] \to \Lambda \subset \mathrm{SO}(n, 1), n > 2$$

then one of the following holds:

(1) $\Phi|_J$ *factors through* $\mathbb{Z}$*;*
(2) $\Phi|_J$ *factors through* $\pi_1(S_h)$*, for some* $h \geqslant 2$*;*
(3) $\Phi|_J$ *is trivial.*

In the first two cases $\Phi|_J$ *surjects onto* $\mathbb{Z}$ *and* $\pi_1(S_h)$*, respectively.*

REMARK 1. The proof of Theorem 5 will provide explicit finite-index Kähler subgroups J. Thus, given any finite-index subgroup $G \subset \mathrm{Mod}_g / \mathrm{Mod}_g[p]$, Corollary 7 applies to the subgroup $J \cap G$. As will become evident from our arguments, the groups J fall frequently in case (3) above.

One provides linear representations of the fundamental group of a Kähler manifold, like those obtained in Theorem 5, by considering variations of Hodge structures, in particular those obtained from families of complex algebraic varieties. We might wonder whether the restrictions of the quantum representations, known to factorize through $\mathrm{Mod}_g / \mathrm{Mod}_g[p]$, arise geometrically as above.

CONJECTURE 1. Complexified quantum representations of Mod_g are locally rigid.

If true, this conjecture would imply that the quantum representation at a prime level is a complex direct factor of a $\mathbb{Q}$-variation of Hodge structures, according to [**29**, Theorem 5, p. 56].

1.3. Abelianization of finite-index subgroups

A celebrated result of Powell [**25**] asserts that Mod_g has trivial abelianization whenever $g \geqslant 3$. In stark contrast, the situation for finite-index subgroups remains mysterious: a well-known question of Ivanov [**20**] asks whether every finite-index subgroup of Mod_g has finite abelianization. Our final result states that this question is equivalent to the analogous problem for the quotient subgroups $\mathrm{Mod}_g / \mathrm{Mod}_g[p]$. More concretely, we have

THEOREM 8. *Let* $K \subset \mathrm{Mod}_g$ *be a normal subgroup of finite index* n*. Then* $\dim H_1(K; \mathbb{Q}) > 0$ *if and only if the image* $K(n)$ *of* K *into* $\mathrm{Mod}_g / \mathrm{Mod}_g[n]$ *is of finite index* n *and* $\dim H_1(K(n); \mathbb{Q}) > 0$*.*

2. Preliminaries

In this section we introduce the basic definitions and notation needed for the rest of this note. We refer the reader to the standard text [**13**] for a comprehensive introduction to these topics.

As mentioned in the introduction, we denote by S_g^n a connected orientable surface of genus $g \geqslant 0$ with empty boundary and with $n \geqslant 0$ marked points. Let Mod_g^n be the mapping class group of S_g^n, namely the group of self-homeomorphisms of S_g^n up to homotopy. From now on, if $n = 0$ we will drop it from the notation and write S_g and Mod_g.

2.1. Power subgroups

We write T_c for the right Dehn twist about the (isotopy class of a) simple closed curve $c \subset S_g^n$. Given a number $p \in \mathbb{N}$, consider the subgroup $\mathrm{Mod}_g^n[p]$ generated by all p-powers of Dehn twists. Observe that $\mathrm{Mod}_g^n[p]$ is a normal subgroup, as

$$f T_c f^{-1} = T_{f(c)}$$

for every simple closed curve $c \subset S_g^n$ and every $f \in \mathrm{Mod}_g^n$.

The following result is due to Humphries [**19**] in the case $g = 2$, and to the second author [**15**] in the case $g \geqslant 3$:

THEOREM 9.   *Let $g \geqslant 2$ and $p \geqslant 0$. If $g = 2$, assume $p \geqslant 4$; if $g \geqslant 3$, assume that $p \notin \{2, 3, 4, 6, 8, 12\}$. Then $\mathrm{Mod}_g^n[p]$ has infinite index in Mod_g^n.*

Abusing notation, we will write

$$\mathrm{proj}_g^p : \mathrm{Mod}_g^n \to \frac{\mathrm{Mod}_g^n}{\mathrm{Mod}_g^n[p]}$$

for the natural projection. By Theorem 9, the image of proj_g^p is infinite whenever $g = 2$ and $p \geqslant 4$, or if $g \geqslant 3$ and $p \notin \{2, 3, 4, 6, 8, 12\}$.

2.2.   *The Birman short exact sequence*

There is an obvious surjective homomorphism

$$\mathrm{Mod}_g^1 \to \mathrm{Mod}_g$$

given by *forgetting* the marked point of S_g^1, call it x_0. This homomorphism is well known to fit in the so-called *Birman short exact sequence*:

$$1 \to \pi_1(S_g, x_0) \to \mathrm{Mod}_g^1 \to \mathrm{Mod}_g \to 1. \tag{1}$$

The injective homomorphism $p : \pi_1(S_g, x_0) \to \mathrm{Mod}_g^1$ of (1) is called the *point-pushing homomorphism*. Abusing notation, we will simply write $\pi_1(S_g)$ for $\pi_1(S_g, x_0)$; in addition, we will often not distinguish between $\pi_1(S_g)$ and its image in Mod_g^1 under the point-pushing homomorphism.

We will also need a description on the image of simple loops under the point-pushing homomorphism; we refer the reader to [**13**, Section 4.2] for a proof. Before stating it, we need some definitions. First, we recall that a *multitwist* in Mod_g^n is an element that may be written as a product of powers of Dehn twists along a set of pairwise-disjoint simple closed curves on S_g^n. Next, an element of $\pi_1(S_g)$ is *simple* if it can be realized on S_g as a loop without *transverse* self-intersections; in particular, note that any power of a simple loop is simple, according to the definition. Armed with these definitions, we have

PROPOSITION 10.   *Let $c \in \pi_1(S_g)$ be a simple loop. Then $p(c)$ is a multitwist in Mod_g^1.*

In fact, it is possible to give a more concrete description of $p(c)$ in the theorem above. Indeed, suppose that $c = a^n$, for some $a \in \pi_1(S_g)$ *primitive*. Let $a^{\pm}$ be the boundary components of a regular neighborhood of (a representative of) a in S_g^1. Then $p(a) = T_{a^+} T_{a^-}^{-1}$, and therefore $p(c) = T_{a^+}^n T_{a^-}^{-n}$.

2.2.1.   *Algebraic version of the Birman exact sequence.*   Given a group G, denote by $\mathrm{Aut}(G)$ its automorphism group, and by $\mathrm{Out}(G)$ its *outer* automorphism group, namely the group of conjugacy classes of automorphisms of G. The following is the celebrated *Dehn–Nielsen–Baer Theorem* (see, for instance, [**13**]):

THEOREM 11 (Dehn–Nielsen–Baer).   *For every $g \geqslant 1$,*

$$\mathrm{Mod}_g \cong \mathrm{Out}(\pi_1(S_g)).$$

In light of this theorem, the exact sequence (1) takes the following form:

$$1 \to \pi_1(S_g) \to \mathrm{Aut}(\pi_1(S_g)) \to \mathrm{Out}(\pi_1(S_g)) \to 1. \tag{2}$$

3. Normal subgroups via covers

In this section we will prove Theorem 1. For an element h of a group G denote by $\langle\langle h \rangle\rangle_G$ the smallest normal subgroup of G containing h; we will drop G from the notation when it is clear from the context. The main ingredient will be the following lemma:

LEMMA 12. *Suppose there is an injective homomorphism*

$$\phi : \mathrm{Mod}_g^1 \to \mathrm{Mod}_{g'}^1$$

and an element $h \in \mathrm{Mod}_g^1$ such that

(1) $\mathrm{proj}_g^p(h) \in \mathrm{Mod}_g^1 / \mathrm{Mod}_g^1[p]$ *has infinite order;*
(2) $\phi(h)$ *is a multitwist in $\mathrm{Mod}_{g'}^1$; in particular, $\phi(h^p) \in \ker(\mathrm{proj}_{g'}^p)$;*
(3) *the map*

$$\mathrm{proj}_{g'}^p \circ \phi : \mathrm{Mod}_g^1 \to \mathrm{Mod}_{g'}^1 / \mathrm{Mod}_{g'}^1[p]$$

has infinite image.

Then $\langle\langle \mathrm{proj}_g^p(h^p) \rangle\rangle$ is an infinite normal subgroup of $\mathrm{Mod}_g^1 / \mathrm{Mod}_g^1[p]$ of infinite index.

Proof. Write $N = \langle\langle \mathrm{proj}_g^p(h^p) \rangle\rangle$, which is obviously a normal subgroup of $\mathrm{Mod}_g^1 / \mathrm{Mod}_g^1[p]$. Observe that N is infinite, by (1). Next, $\phi(h)$ is a multitwist, by (2), so in particular $\phi(h^p) = \phi(h)^p \in \ker(\mathrm{proj}_{g'}^p)$. Consider

$$\mathrm{proj}_{g'}^p \circ \phi : \mathrm{Mod}_g^1 \to \mathrm{Mod}_{g'}^1 / \mathrm{Mod}_{g'}^1[p],$$

which again by (2) factors through the quotient

$$\left(\mathrm{Mod}_g^1 / \mathrm{Mod}_g^1[p] \right)/N.$$

Finally, this quotient is infinite since $\mathrm{proj}_{g'}^p \circ \phi$ has infinite image, by (3). $\square$

We can now prove Theorem 1:

Proof of Theorem 1. Let $g \geqslant 2$. We first explain how to construct the group Q_1 from the statement.

First, there exists $h_1 \in \pi_1(S_g)$ of infinite order in $\mathrm{Mod}_g^1 / \mathrm{Mod}_g^1[p]$: this is a consequence of work by Koberda–Santharoubane [**21**, Theorem 4.1] for large p, and Funar–Lochak [**16**, Proof of Proposition 3.2] for all p as in the hypotheses of Theorem 1.

We are going to produce an injective homomorphism $\mathrm{Mod}_g^1 \to \mathrm{Mod}_{g'}^1$ as in Lemma 12, suited to this element h_1. (We remark that our arguments are heavily inspired by the construction of the *non-geometric* injective homomorphism between mapping class groups of [**1**].)

First, a result of Scott [**27**] implies that there exists a finite-degree cover $S_{g''} \to S_g$ to which h *lifts simply*; that is, there exists some $n \in \mathbb{N}$ such that $h_1^n \in \pi_1(S_{g''})$ and h_1^n has no transverse self-intersections. Now, the intersection K of all subgroups of $\pi_1(S_g)$ of index $[\pi_1(S_g) : \pi_1(S_{g''})]$ is a characteristic subgroup of $\pi_1(S_g)$. Denote by $S_{g'} \to S_g$ the cover corresponding to $K = \pi_1(S_{g'})$, and observe that perhaps by raising h_1^n to a higher power, we can assume that $h_1^n \in \pi_1(S_{g'})$. We stress that, viewed as an element of $\pi_1(S_{g'})$, the element h_1^n would still be simple.

Next, since $\pi_1(S_{g'})$ is a characteristic subgroup of $\pi_1(S_g)$ we may define a homomorphism

$$\phi : \mathrm{Aut}(\pi_1(S_g)) \to \mathrm{Aut}(\pi_1(S_{g'}))$$

by assigning, to every $f \in \mathrm{Aut}(\pi_1(S_g))$, its restriction to $\pi_1(S_{g'})$. Note that ϕ is injective since $\pi_1(S_{g'})$ has finite index in $\pi_1(S_g)$ and surface groups have unique roots.

We now check that conditions (1)–(3) of Lemma 12 are satisfied for the homomorphism ϕ and the element $h = h_1^n$ constructed above. First, (1) holds since we chose h_1 of infinite order in $\mathrm{Mod}_g^1 / \mathrm{Mod}_g^1[p]$. Next, as h_1^n is a simple element of $\pi_1(S_{g'})$, Theorem 10 implies that $\phi(h_1^n)$ is a multitwist, yielding (2). Finally, in order to prove (3), observe that $\phi(\pi_1(S_{g'})) = \pi_1(S_{g'})$, by construction. In particular, $\phi(\pi_1(S_g))$ contains $\pi_1(S_{g'})$ as a subgroup of finite index and we can again apply the result of Koberda–Santharoubane [**21**, Theorem 4.1] used above to deduce that (3) holds. At this point, Lemma 12 tells us that the group

$$Q_1 := \langle\langle \mathrm{proj}_g^p(h^p) \rangle\rangle$$

is an infinite normal subgroup of infinite index of $\mathrm{Mod}_g^1 / \mathrm{Mod}_g^1[p]$.

In order to construct the rest of the groups in the desired descending normal series, we apply the above argument inductively. More concretely, since

$$\mathrm{proj}_{g'}^p \circ \phi : \mathrm{Mod}_g^1 \to \frac{\mathrm{Mod}_{g'}^1}{\mathrm{Mod}_{g'}^1[p]}$$

has infinite image, there exists $h_2 \in \pi_1(S_g)$ whose image in

$$(\mathrm{Mod}_g^1 / \mathrm{Mod}_g^1[p])/Q_1$$

has infinite order. Now, since $\phi(h_2) \in \phi(\pi_1(S_g))$ and the latter contains $\pi_1(S_{g'})$ as a subgroup of finite index, up to taking a power of h_2 we may assume that $\phi(h_2) \in \pi_1(S_{g'})$. We now apply the above arguments to $\phi(h_2)$ to construct an injective homomorphism

$$\phi_2 : \mathrm{Mod}_{g'}^1 \to \mathrm{Mod}_{g_2}^1$$

such that $\phi_2(\phi(h_2))$ is a multitwist; in particular, $\phi_2(\phi(h_2^p)) \in \ker(\mathrm{proj}_{g_2}^p)$. Again by the arguments above, the map

$$\mathrm{proj}_{g_2}^p \circ \phi_2 \circ \phi$$

has infinite image, and therefore $\langle\langle \mathrm{proj}_g^p(h_2^p) \rangle\rangle$ is an infinite normal subgroup of infinite index in $(\mathrm{Mod}_g^1 / \mathrm{Mod}_g^1[p])/Q_1$. We then set

$$Q_2 := \langle\langle \mathrm{proj}^p(h^p), \mathrm{proj}^p(h_2^p) \rangle\rangle.$$

Repeating this process we construct the desired descending normal series $Q_1 \trianglerighteq Q_2 \trianglerighteq \cdots$. $\quad\square$

4. A Birman exact sequence for quotient subgroups

Before proceeding with the proof of Proposition 3 we need the following result providing sufficient conditions for the exactness of a Birman sequence associated to quotient groups:

LEMMA 13. *Let $\Pi \subset \pi_1(S_g)$ be a characteristic subgroup, $K^1 \subset \mathrm{Mod}_g^1$ be a normal subgroup and $K \subset \mathrm{Mod}_g$ denote its image under the forgetful map. Assume that*

$$K^1 \subset \ker\left(\mathrm{Mod}_g^1 \to \mathrm{Aut}\left(\frac{\pi_1(S_g)}{\Pi}\right)\right) \tag{3}$$

and

$$\Pi \subset \pi_1(S_g) \cap K^1. \tag{4}$$

Then the following generalized Birman sequence is exact:

$$1 \to \frac{\pi_1(S_g)}{\Pi} \to \frac{\mathrm{Mod}_g^1}{K^1} \to \frac{\mathrm{Mod}_g}{K} \to 1.$$

Proof. By hypothesis, there exist well-defined homomorphisms:

$$\psi : \frac{\mathrm{Mod}_g^1}{K^1} \to \mathrm{Aut}\left(\frac{\pi_1(S_g)}{\Pi}\right) \tag{5}$$

and

$$\overline{\psi} : \frac{\mathrm{Mod}_g}{K} \to \mathrm{Out}\left(\frac{\pi_1(S_g)}{K}\right), \tag{6}$$

which fit in a commutative diagram

$$\begin{array}{ccc}
\frac{\mathrm{Mod}_g^1}{K^1} & \xrightarrow{\quad f \quad} & \frac{\mathrm{Mod}_g}{K} \\
\downarrow{\psi} & & \downarrow{\overline{\psi}} \\
\mathrm{Aut}\left(\frac{\pi_1(S_g)}{\Pi}\right) & \xrightarrow{\quad q \quad} & \mathrm{Out}\left(\frac{\pi_1(S_g)}{\Pi}\right).
\end{array} \tag{7}$$

In the above diagram, the homomorphism f is the one induced by the forgetful homomorphism of Section 2.2, and the homomorphism q is the obvious quotient map. We claim

CLAIM. The image of $\frac{\pi_1(S_g)}{\pi_1(S_g) \cap K^1} < \frac{\mathrm{Mod}_g^1}{K^1}$ under the homomorphism $q \circ \psi$ is trivial.

Proof of claim. First, the exactness of the Birman sequence (1) implies that

$$f\left(\frac{\pi_1(S_g)}{\pi_1(S_g) \cap K^1}\right)$$

is trivial, and therefore

$$(\overline{\psi} \circ f)\left(\frac{\pi_1(S_g)}{\pi_1(S_g) \cap K^1}\right)$$

is trivial too. Since the diagram (7) is commutative, we deduce that

$$\psi\left(\frac{\pi_1(S_g)}{\pi_1(S_g) \cap K^1}\right) < \ker(q) = \frac{\pi_1(S_g)}{\Pi}, \tag{8}$$

which in particular implies the desired result. □

From the claim above, we deduce that there is a diagram

$$\begin{array}{ccccccccc}
1 & \to & \pi_1(S_g) & \to & \mathrm{Mod}_g^1 & \to & \mathrm{Mod}_g & \to & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \frac{\pi_1(S_g)}{\pi_1(S_g) \cap K^1} & \to & \frac{\mathrm{Mod}_g^1}{K^1} & \to & \frac{\mathrm{Mod}_g}{K} & \to & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \frac{\pi_1(S_g)}{\Pi} & \to & \mathrm{Aut}\left(\frac{\pi_1(S_g)}{\Pi}\right) & \to & \mathrm{Out}\left(\frac{\pi_1(S_g)}{\Pi}\right) & \to & 1
\end{array}$$

with exact rows. From condition (4) we have a surjective homomorphism

$$\alpha : \frac{\pi_1(S_g)}{\Pi} \to \frac{\pi_1(S_g)}{\pi_1(S_g) \cap K^1}. \tag{9}$$

It follows that that upper leftmost vertical arrow in the diagram above, which is the projection homomorphism

$$\pi_1(S_g) \to \frac{\pi_1(S_g)}{\pi_1(S_g) \cap K^1}$$

factors as

$$\pi_1(S_g) \to \frac{\pi_1(S_g)}{\Pi} \to \frac{\pi_1(S_g)}{\pi_1(S_g) \cap K^1},$$

where the right homomorphism is the homomorphism (9). Now, note that the composition of the two leftmost vertical arrows in the diagram above is the natural projection

$$\pi_1(S_g) \to \frac{\pi_1(S_g)}{\Pi},$$

which then factors as the composition of homomorphisms:

$$\pi_1(S_g) \to \frac{\pi_1(S_g)}{\Pi} \xrightarrow{\alpha} \frac{\pi_1(S_g)}{\pi_1(S_g) \cap K^1} \xrightarrow{\psi} \frac{\pi_1(S_g)}{\Pi}.$$

Therefore the identity map

$$\mathrm{id} : \frac{\pi_1(S_g)}{\Pi} \to \frac{\pi_1(S_g)}{\Pi}$$

factors as

$$\frac{\pi_1(S_g)}{\Pi} \xrightarrow{\alpha} \frac{\pi_1(S_g)}{\pi_1(S_g) \cap K^1} \xrightarrow{\psi} \frac{\pi_1(S_g)}{\Pi}.$$

Since α is surjective, it follows that each of these maps is an isomorphism, and hence

$$\Pi = \pi_1(S_g) \cap K^1,$$

as desired. $\square$

Recall that $\pi_1(S_g)[p]$ denotes the subgroup of $\pi_1(S_g)$ generated by p-powers of *simple* elements of S_g; observe that $\pi_1(S_g)[p]$ is a characteristic subgroup of $\pi_1(S_g)$. We also need the following lemma:

LEMMA 14. *The natural action of* $\mathrm{Mod}_g^1[p]$ *on* $\frac{\pi_1(S_g)}{\pi_1(S_g)[p]}$ *is trivial.*

We will denote by $S_{g,n}$ the orientable surface of genus g with n boundary components.

Proof. We start with some well-known observations, which are similar to [8, Theorem 6.17]. Let a be a simple loop on S_g, viewed as an element of $\pi_1(S_g)$. If a separates S_g into two surfaces $S_{h,1}$ and $S_{g-h,1}$, then $\pi_1(S_g) = \pi_1(S_{h,1}) *_\mathbb{Z} \pi_1(S_{g-h,1})$, where $\mathbb{Z}$ is the cyclic group generated by a. The orientation of the surface defines a right side, say $S_{h,1}$ and a left side for a. Then the (right) Dehn twist T_a along a is the automorphism determined by

$$T_a(x) = \begin{cases} x, & \text{if } x \in \pi_1(S_{h,1}); \\ axa^{-1}, & \text{if } x \in \pi_1(S_{g-h,1}). \end{cases}$$

Thus

$$T_a^p(x) = \begin{cases} x, & \text{if } x \in \pi_1(S_{h,1}); \\ a^p x a^{-p}, & \text{if } x \in \pi_1(S_{g-h,1}) \end{cases}$$

so that $T_a^p(x) \in x\pi_g[p]$, for any x.

If a is non-separating, then we have a HNN splitting $\pi_1(S_g) = \pi_1(S_{g-1,2}) *_\mathbb{Z}$, where $\mathbb{Z}$ is generated by some element t corresponding to a primitive (hence simple) loop intersecting a once. Further the Dehn twist T_a along a is the automorphism determined by

$$T_a(x) = \begin{cases} x, & \text{if } x \in \pi_1(S_{g-1,2}); \\ ta, & \text{if } x = t. \end{cases}$$

Then

$$T_a^p(x) = \begin{cases} x, & \text{if } x \in \pi_1(S_{g-1,2}); \\ ta^p, & \text{if } x = t; \end{cases}$$

and so $T_a^p(x) \in x\pi_g[p]$, for any x. This finishes the proof of the lemma. $\qquad\square$

We are now in a position to prove Proposition 3 and Corollary 4.

Proof of Proposition 3. We want to apply the result of Lemma 13 to $\Pi = \pi_1(S_g)[p]$ and $K^1 = \mathrm{Mod}_g^1[p]$. Using the description of the image of the point-pushing homomorphism for simple loops (see comment after Proposition 10), we know that $\pi_1(S_g)[p] \subset \pi_1(S_g) \cap \mathrm{Mod}_g^1[p]$, namely condition (4) holds. Further Lemma 14 shows that condition (3) is also satisfied and hence the claim follows. $\qquad\square$

Proof of Corollary 4. In the proof of Theorem 1, all of the elements h_i constructed belong to $\pi_1(S_g)$. Then the subgroups

$$\overline{Q}_i = Q_i \cap \frac{\pi_1(S_g)}{\pi_1(S_g)[p]}$$

are normal subgroups of $\frac{\pi_1(S_g)}{\pi_1(S_g)[p]}$ thus forming a descending normal series. Each $\overline{Q}_i$ is infinite and of infinite index into $\overline{Q}_{i-1}$ by the proof above, based on the Koberda–Santharoubane infiniteness statement [21, Theorem 1.1] . We have:

$$\langle\langle \mathrm{proj}_g^p(h_1^p), \ldots, \mathrm{proj}_g^p(h_j^p) \rangle\rangle \subset \frac{\pi_1(S_g)}{\pi_1(S_g)[p]}.$$

In fact, the groups

$$\langle\langle \mathrm{proj}_g^p(h_1^p), \ldots, \mathrm{proj}_g^p(h_j^p) \rangle\rangle$$

are characteristic subgroups of $\frac{\pi_1(S_g)}{\pi_1(S_g)[p]}$, and in particular they are also normal subgroups in $\frac{\mathrm{Mod}\,g^1}{\mathrm{Mod}_g^1[p]}$. $\qquad\square$

REMARK 2. The subgroups

$$\langle\langle \mathrm{proj}_g^p(h_1^p), \ldots, \mathrm{proj}_g^p(h_j^p) \rangle\rangle_{\mathrm{Mod}_g^1} \cap \pi_1(S_g)$$

are normal subgroups in Mod_g^1, in particular they form a series of characteristic subgroups of $\pi_1(S_g)$.

5. Virtually Kähler groups

In this section we prove Theorem 5. We refer the reader to [13, 18], as well as to the references below, for standard facts about moduli space.

The analytic moduli space $\mathcal{M}_g^{an}$ of smooth genus g algebraic curves is an orbifold with orbifold fundamental group Mod_g. It is a classical result of Serre [28] that $\mathcal{M}_g^{an}$ has finite-degree unramified coverings which are topological manifolds, e.g. those obtained by quotienting Teichmüller space by the stabilizer of an abelian level $k \geqslant 3$ structure, namely

$$\ker(\mathrm{Mod}_g \to \mathrm{Aut}(H_1(S_g, \mathbb{Z}/k\mathbb{Z})).$$

There is a standard Kähler structure on Teichmüller space, namely the one induced by the Weil–Petersson metric. In particular, each of the finite-degree unramified covers of $\mathcal{M}_g^{an}$ mentioned above also admits a Kähler structure. However, it should be noted that this fact does not imply that Mod_g is virtually Kähler, as the action of the relevant group on the appropriate cover of $\mathcal{M}_g^{an}$ is not cocompact.

The Deligne–Mumford compactification $\overline{\mathcal{M}}_g^{an}$ of $\mathcal{M}_g$ is the analytic space underlying the moduli stack $\overline{\mathcal{M}}_g$ of stable curves of genus g. Looijenga [22] and later several other authors proved that $\overline{\mathcal{M}}_g^{an}$ also admits finite Galois covers which are topological manifolds.

Deligne and Mumford [9] introduced the moduli stack $_G\mathcal{M}_g$ of smooth genus g algebraic curves with a Teichmüller structure of level G, where G is any finite characteristic quotient G of $\pi_1(S_g)$, and considered its underlying moduli space $_G\mathcal{M}_g^{an}$, which in turn parametrizes smooth curves with a G-structure. They further defined the compactification $_G\overline{\mathcal{M}}_g^{an}$ of $_G M_g^{an}$ as the normalization of $\overline{\mathcal{M}}_g$ with respect to $_G\mathcal{M}_g^{an}$.

For suitable choices of the finite group G, the analytic space $_G\overline{\mathcal{M}}_g^{an}$ is a compact smooth complex manifold (see [4, 5, 22, 24]). Moreover, the map forgetting the level structure $_G\overline{\mathcal{M}}_g^{an} \to \overline{\mathcal{M}}_g^{an}$ is a Galois covering ramified along the divisor at infinity. In order to prove Theorem 5 we will compute $\pi_1(_G\overline{\mathcal{M}}_g^{an})$ in such cases. For the sake of conciseness, most arguments borrowed from [4, 5, 22, 24] are only sketched below.

We consider, after Pikaart–Jong [24], the family of characteristic subgroups

$$\pi_g(k, p) = \gamma_{k+1}(\pi_1(S_g)) \cdot \pi_1(S_g)^p,$$

where $\pi_1(S_g)^p$ denotes the subgroup of $\pi_1(S_g)$ generated by all pth powers, and γ_k denotes the kth term of the lower central series of a group G, namely the one defined as $\gamma_1(G) = G$ and $\gamma_{k+1}(G) = [\gamma_k(G), G]$, for $k \geqslant 1$. We stress that the groups $\pi_g(k, p)$ are finite-index subgroups of $\pi_1(S_g)$, as nilpotent groups with generators of finite order are finite. We set

$$\mathrm{Mod}_g(k, p) = \ker(\mathrm{Mod}_g \to \mathrm{Out}(\pi_1(S_g)/\pi_g(p, k)).$$

Now, the following result is due to Pikaart–Jong [24]:

THEOREM 15 [24]. *The analytic space $_{\pi_g(p,k)}\overline{\mathcal{M}}_g^{an}$ is smooth if $k \geqslant 4$, $p \geqslant 3$ and $\gcd(p, 6) = 1$, if $k = 1$ and $g = 2$, if $k = 2$ and p is odd, if $k = 3$ and p is either odd or divisible by 4.*

Note that $\pi_1(_{\pi_g(k,p)}\mathcal{M}_g^{an}) = \mathrm{Mod}_g(k, p)$. In order to compute the fundamental group $\pi_1(_{\pi_g(k,p)}\overline{\mathcal{M}}_g^{an})$ of the compactification it suffices to know the monodromy along the boundary on the relative fundamental group of the universal curve over $\mathcal{M}_g^{an}$. In fact the orbifold homotopy class of a loop in $\mathcal{M}_g^{an}$ coincides with the class of the monodromy of the universal curve along that loop, when the orbifold fundamental group of $\mathcal{M}_g^{an}$ is identified with Mod_g.

Let C be a complex stable genus g curve with singular points $p_1, \ldots, p_m$. The stable curve C is the singular fiber of a local universal deformation family of stable curves over the polydisk in $\mathbb{C}^{3g-3}$. Choose local coordinates z_i such that $z_j = 0$ describes stable curves for which p_j is a singular point. Then, the universal family has smooth fibers outside the discriminant locus given by the equation

$$z_1 z_2 \cdots z_m = 0.$$

Vanishing cycles in the universal family correspond to a set $c_1, c_2, \ldots, c_m$ of simple closed curves on the generic smooth fiber, which are shrunk to the singular points $p_1, p_2, \ldots, p_m$ of the singular fiber. If U denotes the complement of the discriminant, then $\pi_1(U)$ is the abelian group generated by classes $[\gamma_j]$ of loops γ_j encircling the complex hyperplane $z_j = 0$ exactly once, and in the positive direction. Moreover, the monodromy ρ of the universal curve is given by

$$\rho([\gamma_i]) = T_{c_i},$$

where T_{c_i} denotes the right Dehn twist along the curve c_i. In other words, the orbifold homotopy classes of the loops $[\gamma_i]$ in $\mathcal{M}_g^{an}$ are the classes T_{c_i}.

Let $D \subset \overline{\mathcal{M}}_g^{an}$ be the divisor at infinity. Recall [**9**, Theorem 5.2] that D is a normal crossing divisor in the orbifold sense, see [**3**, Chapter XII]) for a detailed description both from orbifold and stack point of view. Note that U is homeomorphic to the intersection of $\overline{\mathcal{M}}_g^{an} - D$ with a neighborhood of the point $[C]$ within $\overline{\mathcal{M}}_g^{an}$. A similar description exists for neighborhoods of points of $_{\pi_g(k,p)}\overline{\mathcal{M}}_g^{an}$ lying above $[C]$.

We will use then the following lemma (see [**14**, Section 7]):

LEMMA 16. *Let Y be a connected barycentrically subdivided locally finite complex, and let K be a subcomplex such that, for each vertex v of K, the intersection $S(v)$ of $Y - K$ with the open star of v in Y is non-empty and connected. Then the homomorphism $\pi_1(Y - K) \to \pi_1(Y)$ induced by the natural inclusion is surjective. Moreover, its kernel is the normal closure of those elements in $\pi_1(Y - K)$ that can be represented by loops in $\cup_v S(v)$.*

With the above lemma to hand, we now choose a triangulation of $\overline{\mathcal{M}}_g^{an}$ for which D is a subcomplex, which in turn induces a triangulation of $_{\pi_g(k,p)}\overline{\mathcal{M}}_g^{an}$. We want to use Lemma 16, by taking for Y the triangulation $_{\pi_g(k,p)}\overline{\mathcal{M}}_g^{an}$, and for K the triangulation of the divisor at infinity.

Note that the $S(v)$ are pairwise disjoint in the statement of Lemma 16. As such, each $S(v)$ is homeomorphic to U. It remains to identify the classes of the lifted loops γ_j in $\pi_1(_{\pi_g(k,p)}\overline{\mathcal{M}}_g^{an})$. Equivalently, we need to find the kernel of the local monodromy representation for $_{\pi_g(k,p)}\overline{\mathcal{M}}_g^{an}$ over a neighborhood of $[C]$ in $\overline{\mathcal{M}}_g^{an}$. This has been already done in [**24**, Theorem 3.1.3]:

THEOREM 17 [**24**]. *If $k \geqslant 4$ and $\gcd(p, 6) = 1$, then the kernel of the monodromy is generated by the classes of the form $T_{c_i}^p$.*

Before proving our result, we will need the following lemma:

LEMMA 18. *Let $K < \mathrm{Mod}_g$ be a normal subgroup, with the property that $T_c^n \in K$, for every simple closed curve c. Set $\langle\langle T_c^n \rangle\rangle_K$ for the normal subgroup of K generated by the nth powers of all Dehn twists. Then $\langle\langle T_c^n \rangle\rangle_K = K \cap \mathrm{Mod}_g[n]$.*

Proof of Lemma 18. We have $\langle\langle T_c^n \rangle\rangle_K \subset K \cap \mathrm{Mod}_g[n]$. For the reverse inclusion consider $x \in K \cap \mathrm{Mod}_g[n]$. Thus $x = \prod g_i T_{c_i}^{\varepsilon_i n} g_i^{-1}$, where $g_i \in \mathrm{Mod}_g$, $\varepsilon_i \in \{-1, 1\}$. But $g_i T_{c_i}^{\varepsilon_i n} g_i^{-1} = T_{g_i(c_i)}^{\varepsilon_i n}$. Recall that $T_c^n \in K$, for every simple closed curve c. Therefore x is the product of the nth powers of Dehn twists $T_{g_i(c_i)}^{\varepsilon_i n}$, each element being in K. This means that $x \in \langle\langle T_c^n \rangle\rangle_K$.   □

We are finally in a position for proving the main result of this section:

Proof of Theorem 5. By Lemma 16 the group $\pi_1(_{\pi_g(k,p)}\overline{\mathcal{M}}_g^{an})$ is isomorphic to the quotient

$$\mathrm{Mod}_g(k, p) / \langle\langle T_\gamma^p \rangle\rangle_{\mathrm{Mod}_g(k,p)}.$$

Using Lemma 18 we have

$$\langle\langle T_\gamma^p \rangle\rangle_{\mathrm{Mod}_g(k,p)} = \mathrm{Mod}_g(k,p) \cap \mathrm{Mod}_g[p]$$

so that

$$\pi_1\big({}_{\pi_g(k,p)}\overline{\mathcal{M}}_g^{an}\big) = \frac{\mathrm{Mod}_g(k,p)}{\mathrm{Mod}_g(k,p) \cap \mathrm{Mod}_g[p]}$$

is a subgroup of finite index in $\frac{\mathrm{Mod}_g}{\mathrm{Mod}_g[p]}$. This proves the claim. $\qquad\square$

Proof of Corollary 6. Since $\frac{\mathrm{Mod}_g}{\mathrm{Mod}_g[p]}$ is virtually Kähler, the results are consequences of corresponding statements of Carlson–Toledo [**7**] for lattices in $\mathrm{SO}(n,1)$, and of Simpson [**29**] for groups of Hodge type. $\qquad\square$

Before proceeding with the proof we will need the following result due to Carlson–Toledo (see [**7**, Theorem 7.1 and Corollary 3.7]):

THEOREM 19 [**7**]. *Let* $F : \mathcal{X} \to \mathbb{H}_\mathbb{R}^{n+1}/\Lambda$ *be a harmonic map from a compact Kähler manifold* $\mathcal{X}$ *to a hyperbolic space form* $\mathbb{H}_\mathbb{R}^{n+1}/\Lambda$, *where* $\Lambda \subset SO(1,n)$, *for some* $n > 2$, *is a lattice. Then the harmonic map* F *factors either through a circle or else through a compact Riemann surface.*

Proof of Corollary 7. Let J be a finite-index Kähler subgroup of $\frac{\mathrm{Mod}_g}{\mathrm{Mod}_g[p]}$; for instance, after the proof of Theorem 5, we may set

$$J = \frac{\mathrm{Mod}_g(k,p)}{\mathrm{Mod}_g(k,p) \cap \mathrm{Mod}_g[p]}.$$

Let $\mathcal{X}$ be a compact Kähler manifold with fundamental group J, and

$$f : \frac{\mathrm{Mod}_g}{\mathrm{Mod}_g[p]} \to \Lambda$$

a homomorphism into a lattice $\Lambda \subset SO(1,n)$, for some $n > 2$. Note that f is induced by a map $F : \mathcal{X} \to \mathbb{H}_\mathbb{R}^{n+1}/\Lambda$ into a hyperbolic space form. Eells–Sampson [**10**] proved that then the map F could be assumed to be a harmonic map. Further, by Theorem 19 the harmonic map F factors through a circle or a compact Riemann surface. Thus f factors through $\mathbb{Z}$ or through $\pi_1(\Sigma_h)$. In the first case $f(J)$ is either trivial, or isomorphic to $\mathbb{Z}$. In the second case, the subgroup $f(J) \subset \pi_1(\Sigma_h)$ is finitely generated and hence also a surface group, by a result of Scott [**27**].

Now, if J were rationally perfect, then $f(J)$ would be finite, by a result of Scott [**27**], since surface groups with finite abelianization are trivial. A recent result of Ershov–He [**11**, Theorem 1.9] shows that $H_1(J;\mathbb{Q}) = 0$ for any finite-index subgroup $J \subset \mathrm{Mod}_g$ with the property that $J \subset \gamma_k \mathcal{I}_g$ and $4g \geqslant 8k - 4$, where $\mathcal{I}_g$ denotes the Torelli group. In particular, this is so if J contains the kth Johnson subgroup

$$\ker\left(\mathrm{Mod}_g \to \mathrm{Out}\left(\frac{\pi_1(S_g)}{\gamma_{k+1}(\pi_1(S_g))}\right)\right).$$

Note that $\mathrm{Mod}_g(k,p)$ contains the kth Johnson subgroup, by definition. $\qquad\square$

REMARK 3. All examples worked out in [**5**, Section 3] provide finite ramified coverings with finite or trivial $\pi_1({}_G\overline{\mathcal{M}}_g^{an})$.

REMARK 4. The same method yields (many) other virtually Kähler quotients of Mod_g. For instance we could take for odd p the normal subgroup generated by pth powers of separating

bounded simple closed curves of genus 1 and the $3p$th powers of other Dehn twists, see ([**5**, Proposition 2.8], $k = 7$) and also [**4**, Proposition 3.12] for other possibilities.

6. *Abelianization of finite-index subgroups*

Finally, we prove Theorem 8.

Proof of Theorem 8. As before, let T_c denote the Dehn twist along the simple closed curve c. By hypothesis $T_c^n \in K$. Let $p : K \to H_1(K, \mathbb{Q})$ be the abelianization map. According to Bridson [**6**] and Putman [**26**] we have $p(T_c^n) = 0$, which implies that p factors through $K/\langle\langle T_c^n \rangle\rangle_K$. Observe now that the inclusion $K \to \mathrm{Mod}_g$ induces an injective homomorphism

$$K(n) = \frac{K}{\langle\langle T_c^n \rangle\rangle_K} = \frac{K}{K \cap \mathrm{Mod}_g[n]} \hookrightarrow \frac{\mathrm{Mod}_g}{\mathrm{Mod}_g[n]}.$$

Moreover, we also have that

$$\left[\frac{\mathrm{Mod}_g}{\mathrm{Mod}_g[n]} : K(n) \right] = [\mathrm{Mod}_g : K] = n,$$

and we are done. $\square$

Finally, we remark that the normality assumption for K in Proposition 8 is not essential. In fact, we have:

PROPOSITION 20. *Suppose there exists $K \subset \mathrm{Mod}_g$ of finite index with $\dim H_1(K; \mathbb{Q}) > 0$, then there exists a normal subgroup $H \subset \mathrm{Mod}_g$, also of finite index, with $\dim H_1(H; \mathbb{Q}) > 0$.*

Proof. It is known that if $H \subset K$ is a subgroup of finite index then the map $H_1(H, \mathbb{Q}) \to H_1(K, \mathbb{Q})$ is surjective. This follows from the existence of the tranfer map in homology (see [**26**, Lemma 2.1]). It suffices now to consider H a normal subgroup of Mod_g contained in K. $\square$

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