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Manifolds Which Admit Maps with Finitely Many Critical Points Into Spheres of Small Dimensions

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Abstract. We construct, for $m \geq 6$ and $2n \leq m$, closed manifolds M^m with finite nonzero $\varphi(M^m, S^n)$, where $\varphi(M, N)$ denotes the minimum number of critical points of a smooth map $M \to N$. We also give some explicit families of examples for even $m \geq 6$ and $n = 3$, taking advantage of the Lie group structure on S^3. Moreover, there are infinitely many such examples with $\varphi(M^m, S^n) = 1$. Eventually, we compute the signature of the manifolds M^{2n} occurring for even n.

1. Motivation

We set $\varphi(M, N)$ for the minimum number of critical points of a smooth map $M \to N$ between compact manifolds, which extends the F-category defined and studied by Takens [24]. Following the work of Farber (see [7; 8]), we have:

$$\varphi(M, S^1) = \begin{cases} \varphi(M, \mathbb{R}) & \text{if } H^1(M, \mathbb{Z}) = 0, \\ 0 & \text{if } M \text{fibers over } S^1, \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

More precisely, for any nonzero class ξ in $H^1(M, \mathbb{Z})$, there exists a function $f : M \to S^1$ in the homotopy type prescribed by ξ with at most one critical point. This was extended in [8] to closed 1-forms in a prescribed nonzero class in $H^1(M, \mathbb{R})$ having at most one zero. The question on whether there is a closed nonsingular 1-form (i.e., a fibration over S^1 for integral classes) was answered by Thurston [25] in dimension 3 and Latour [17] for $\dim(M) \geq 6$. Notice that $\varphi(M, \mathbb{R}) \leq \dim M + 1$ (see [24]).

The aim of this paper is to show that there are examples of manifolds M^m with *nontrivial* (i.e., finite nonzero) $\varphi(M^m, S^{[m/2]-k})$, for $m \geq 6$, $m \geq 2k \geq 0$, where, when present, the superscripts denote the dimensions of the corresponding manifolds, and to describe how to construct all of them for $(m, n) = (6, 3)$.

Recall that in [1] the authors found that $\varphi(M^m, N^n) \in \{0, 1, \infty\}$ when $0 \leq m - n \leq 2$, except for the exceptional pairs of dimensions $(m, n) \in \{(2, 2), (4, 3), (4, 2)\}$. Further, if $m - n = 3$ and there exists a smooth function $M^m \to N^n$ with finitely many critical points, all of them cone-like, then $\varphi(M^m, N^n) \in \{0, 1\}$ except for the exceptional pairs of dimensions $(m, n) \in$

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$\{(5,2),(6,3),(8,5)\}$. On the other hand, in [11] the authors provided many nontrivial examples and showed that $\varphi(M^m, S^n)$ can take arbitrarily large even values for $m = 2n-2$, $n \in \{3,5,9\}$; these examples were classified in [10] for $n \in \{3,5\}$.

In the first part of the present paper, we approach this question by elementary methods. In [10] the first author outlined a method for constructing manifolds with finite $\varphi(M^6, S^3)$ using generalized *Hopf links,* which was further detailed in [4]. Our goal is to show that a slight extension of this construction provides nontrivial examples for *all dimensions* of the form $(m, [\frac{m}{2}] - k)$, where $m \geq 6$, $k \geq 0$, and in particular, we can find manifolds with $\varphi(M, N) = 1$ in this range of dimensions. In some sense, these provide other high-dimensional analogs of Lefschetz fibrations. The simplest approach comes from a closed formula computing the Euler characteristic $\chi(M^{2n})$ in terms of the combinatorial data used in the construction. We also give some explicit families of examples for dimensions $(m \geq 6, 3)$, taking advantage of the Lie group structure on S^3. In particular, we find that $\varphi_c(S^6, S^3) = \infty$, where φ_c counts the minimum number of critical points of smooth functions with only cone-like singularities. The last part is devoted to computation of signatures, which are obstructions to fibration over even-dimensional spheres. We obtain manifolds with boundary whose signatures are nonzero.

It would be interesting to know how accurate are our estimates—compare with the lower bounds for $\varphi(M^{2n-2}, S^n)$ obtained in [11]—in order to characterize the set of values taken by $\varphi(M^m, S^n)$.

Notice that no nontrivial examples are known for $m < 2n - 2$ and the present methods do not apply, though as polynomials maps with isolated singularities do exist for $m - n \geq 4$ [18].

2. Constructions of Manifolds with Finite φ and Statement of Results

2.1. Fibered Links and Local Models for Isolated Singularities

Recall, following Looijenga [18], that the isotopy class of the oriented submanifold $K = K^{m-n-1}$ of dimension $(m-n-1)$ of X^{m-1} with a trivial normal bundle is called *generalized Neuwirth–Stallings fibered* (or (X^{m-1}, K^{m-n-1}) is a generalized Neuwirth–Stallings pair) if, for some trivialization $\theta : N(K) \to K \times D^n$ of the tubular neighborhood $N(K)$ of K in X^{m-1}, the fiber bundle $\pi \circ \theta : N(K) - K \to S^{n-1}$ admits an extension to a smooth fiber bundle $f_K : X^{m-1} - K \to S^{n-1}$. Here $\pi : K \times (D^n - \{0\}) \to S^{n-1}$ is the composition of the radial projection $D^n - \{0\} \to S^{n-1}$ with the second factor projection. The data $(X^{m-1}, K, f_K, \theta)$ is then called an *open book decomposition* with binding K, whereas K is called a *fibered link.* This is equivalent to the condition that the closure of every fiber is its compactification by the binding link. When $X^{m-1} = S^{m-1}$, we have the classical notions of Neuwirth–Stallings fibrations and pairs.

Recall now from [18; 15; 23] that open book decompositions $(S^{m-1}, K, f_K, \theta)$ give rise to isolated singularities $\psi_K : (D^m, 0) \to (D^n, 0)$ by means of the formula

$$\psi_K(x) = \begin{cases} \lambda(\|x\|) f_K(\frac{x}{\|x\|}) & \text{if } \frac{x}{\|x\|} \notin N(K), \\ \lambda(\|x\| \cdot \|\pi_2(\theta(\frac{x}{\|x\|}))\|) f_K(\frac{x}{\|x\|}) & \text{if } \frac{x}{\|x\|} \in N(K), \\ 0 & \text{if } x = 0, \end{cases}$$

where $\pi_2 : K \times D^n \to D^n$ is the projection on the second factor, and $\lambda : [0, 1] \to [0, 1]$ is any smooth strictly increasing map sufficiently flat at 0 and 1 such that $\lambda(0) = 0$ and $\lambda(1) = 1$. If K is in generic position, namely the space generated by vectors in $\mathbb{R}^m$ with endpoints in K coincides with the whole space $\mathbb{R}^m$, then $(d\psi_K)_0 = 0$, that is, ψ_K has rank 0 at the origin. We then call such ψ_K *local models* of isolated singularities.

Looijenga [18] proved that a Neuwirth–Stallings pair (S^{m-1}, L^{m-n-1}) can be realized by a *real polynomial* map if L is invariant and the open book fibration f_L is equivariant with respect to the antipodal maps. In particular, the connected sum $(S^{m-1}, K)\sharp((-1)^m S^{m-1}, (-1)^{m-n} K)$ is a Neuwirth–Stallings pair isomorphic to the link of a real polynomial isolated singularity $\psi_K : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$.

2.2. *Cut and Paste Local Models*

We can glue together a patchwork of such local models to obtain maps $M^m \to N^n$ with finitely many critical points. Let Γ be a bicolored decorated graph with vertices of two colors. Each black vertex v of Γ is decorated by a fibered link L_v^{m-n-1} of S^{m-1}. With every vertex v, there is associated an open book fibration $f_{L_v} : S^{m-1} - N(L_v) \to S^{n-1}$ that extends to a smooth local model map with one critical point $\psi_v = \psi_{L_v} : D_v^m \to D^n$. Its generic fibers are called local fibers. Each white vertex w is labeled by some $(m-n)$-manifold $F(w)$ whose boundary has as many connected components as the degree of w.

If there are no white vertices, then we glue together the disks D_v using the pattern of the graph Γ by identifying one component of $N(L_v)$ to one component of $N(L_w)$ if v and w are adjacent in Γ. The identification has to respect the trivializations $N(L_v) \to D^n$, and hence we can take them to be the same as in the double construction. Note that $N(L_v) = L_v \times D^n$, and thus identifications respecting the trivialization correspond to homotopy classes $[L, \mathrm{Diff}(D^n, \partial)]$.

Otherwise, we glue together the disks D_v and $F(w) \times D^n$ along part of their boundaries using the pattern of the graph Γ. We identify a component of $N(L_v)$ with a component of $\partial F(w) \times D^n$ whenever there is an edge between v and w such that the two trivializations of these manifolds do agree and the fibers of the open book and of the trivial fibration glue together. In such a case, $\partial F(w)$ and the link L_v should have the same number of components. When $F(v)$ is a bunch of cylinders, we recover the former construction. We then obtain a manifold with boundary $X(\Gamma)$ endowed with a smooth map $f_\Gamma : X(\Gamma) \to D^n$ whose singularities correspond to the black vertices.

The restriction of f_Γ to the boundary is a locally trivial F-fibration over S^{n-1}. Let now $\Gamma_1, \Gamma_2, \dots, \Gamma_p$ be a set of bicolored decorated graphs whose associated fibrations are *cobounding,* namely such that there exists a fibration over $S^n \setminus \bigsqcup_{i=1}^p D^n$, generally not unique, extending the boundary fibrations restrictions $\psi_{\Gamma_i} := f_{\Gamma_i}|_{\partial X(\Gamma)}$, $1 \le i \le p$. Any such set $\Gamma_1, \Gamma_2, \dots, \Gamma_p$ determines a closed manifold $M(\Gamma_1, \Gamma_2, \dots, \Gamma_p)$ endowed with a map with finitely many critical points into S^n.

In particular, we can realize the double of f_Γ by gluing together f_Γ and its mirror image. We could also generalize this to maps taking values into an arbitrary closed manifold N^n.

2.3. Constructions of Fibered Links in Dimensions $(2n, n)$*,* $n \ge 3$

Let us recall the construction from [10; 4]. It is known that, for $n \ge 3$, there is only one embedding of S^{n-1} in S^{2n-1}. The situation undergoes only little changes in the case of links. By Haefliger's classification theorem (see [13; 12]) the link $L = \bigsqcup_{j=0}^d S_j^{n-1}$ is uniquely determined, up to isotopy, by its linking matrix lk_L, and we denote it as $L_{\mathrm{lk}\, L}$. Note that the diagonal entries of lk_L are not defined, and by convention we set them to 0.

The *generalized Hopf links* with $d+1 \ge 2$ components are those links $L = \bigsqcup_{j=0}^d S_j^{n-1}$ for which the spheres $S_1^{n-1}, \dots S_d^{n-1} \subset S^{2n-1}$ are Hopf duals to a fixed *preferred* $S_0^{n-1} \subset S^{2n-1}$, namely their linking number $\mathrm{lk}(S_0, S_j) = \pm 1$ for $j \ge 1$. We will further suppose that $\mathrm{lk}(S_0, S_j) = 1$ for $j \ge 1$, so that the most important information is the linking submatrix lk_{L° of the sublink $L^\circ = \bigsqcup_{j=1}^d S_j^{n-1}$. Denote by $\widetilde{A}$ a $(-1)^n$-symmetric matrix obtained from a $d \times d$ matrix A by adding a first line and a first column of 1s with 0s on the diagonal.

In [10], it was observed that, for every integral $(-1)^n$-symmetric $d \times d$ matrix A with trivial diagonal, the link $L_{\widetilde{A}}$ has the property that its complement $S^{2n-1} \setminus N(L_{\widetilde{A}})$ naturally fibers over S^{n-1}. The fibers of this fibration are holed disks that intersect transversally every component S_j^{n-1} with $j \ge 1$ in one point, whereas their closure contains S_0^{n-1}. Note that this fibration comes along with a trivialization of the boundary: $\partial N(S_0^{n-1})$ is foliated by preferred longitudinal spheres, whereas $\partial N(S_j^{n-1})$ for $j \ge 1$ are foliated by preferred meridian spheres.

The fibration of $S^{2n-1} \setminus N(L_{\widetilde{A}})$ does not satisfy the last condition in the definition of a Neuwirth–Stallings pair. Although a link is always fibered if its complement fibers (not necessarily as an open book decomposition) when $n = 2$, by a suitable change of the framing, this is not so in higher dimensions. However, there is a simple way to convert transversal intersections of the fiber with S_j^{n-1} into one of binding type by doing surgery. Specifically, we denote by X_A^{2n-1} the result of gluing together $S^{2n-1} \setminus N(L_{\widetilde{A}})$ and $(d+1)$ solid tori $S^{n-1} \times D^n \bigsqcup_{j=1}^d D^n \times S^{n-1}$ such that:

(1) for $j = 0$, the solid torus $S^{n-1} \times D^n$ is glued along $\partial N(S_0^{n-1})$ such that $S^{n-1} \times \{pt\}$ correspond with the preferred longitude spheres;

(2) for $j \geq 1$, the jth copy of the solid torus $D^n \times S^{n-1}$ is glued along $\partial N(S_j^{n-1})$ such that $\{pt\} \times S^{n-1}$ correspond to the preferred meridian spheres.

The cores of the newly attached solid tori form a $(d+1)$-component link $K_A^{n-1} = \bigsqcup_0^d S^{n-1} \subset X_A^{2n-1}$. Though as X_A^{2n-1} might not be a sphere in general, (X_A^{2n-1}, K_A) is a generalized Neuwirth–Stallings pair. Note that the link complements $X_A \setminus N(K_A)$ and $S^{2n-1} \setminus L_{\widetilde{A}}$ are diffeomorphic and the corresponding fibrations match each other. Thus the fibers of the corresponding open book fibration $f_{K_A} : X_A \setminus K_A \to S^{n-1}$ are still holed disks. We warn the reader that the notions of longitude/meridian spheres do not correspond for the two link complements.

When X_A^{2n-1} is diffeomorphic to S^{2n-1}, we obtain a classical Neuwirth–Stallings pair (S^{2n-1}, K_A). Furthermore, X_A^{2n-1} is homeomorphic to a sphere S^{2n-1} if and only if A is unimodular, that is, $\det A = \pm 1$ [4]. This provides already examples of fibered links K_A in those dimensions when there are no exotic spheres, for instance, when $n = 3$. Moreover, when $n = 3$, every fibered link over S^2 is isotopic to some K_A [4; 10] since their fibers should be simply connected and hence holed disks. This is equally true for $n > 3$ if we restrict ourselves to those links whose components are spheres. However, when $n > 3$, links of isolated singularities might be nonsimply connected links.

Furthermore, since the connected sum $X_A^{2n-1} \sharp \overline{X_A^{2n-1}}$ is diffeomorphic to S^{2n-1} for any n, in [4, Corollary 4.2], it was obtained that the links of the form $K_{A \oplus -A} \subset S^{2n-1}$ are fibered for any $n > 3$ if A is unimodular. Notice that the number of components in this construction satisfies $d \equiv 1 \pmod 4$. Further, we also have that $\sharp_{\theta_{2n-1}} X^{2n-1}$ is diffeomorphic to S^{2n-1}, where θ_{2n-1} denotes the order of the group of homotopy spheres in dimension $(2n-1)$. The connected sum construction by Looijenga [18] shows that $K_{\bigoplus_1^{\theta_{2n-1}} A}$ is fibered for any $n > 3$ when A is unimodular.

We can therefore use fibered links of the form $K_A^{n-1} \subset S^{2n-1}$, which we call *generalized Hopf links*. The cut-and-paste procedure from Section 2.2 then produces manifolds with boundary $X^{2n}(\Gamma)$ endowed with maps $\psi_\Gamma : X^{2n}(\Gamma) \to D^n$ with finitely many critical points. The generic fiber of ψ_Γ is $\sharp_g S^1 \times S^{n-1}$, where g is the rank of $H_1(\Gamma)$. If we allow orientation-reversing gluing homeomorphisms, then we can also obtain nonorientable fibers homeomorphic to a twisted S^{n-1}-fibration over the circle.

The restriction of ψ_Γ to the boundary is a $\sharp_g S^1 \times S^{n-1}$-fibration over S^{n-1}. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ be a set of graphs associated with a family of cobounding fibrations, namely such that there exists a fibration over $D^n \setminus \bigsqcup_{i=1}^{p-1} D^n$ extending the boundary fibrations restrictions of f_{Γ_i}, $1 \leq i \leq p$. We remark that $H_1(\Gamma_i)$ should be isomorphic. Then we can glue together ψ_{Γ_j} to obtain some manifold $M(\Gamma_1, \ldots, \Gamma_p)$ endowed with a smooth map with finitely many critical points onto S^n.

When $n = 3$, all 6-manifolds M^6 admitting a smooth map $M^6 \to S^3$ with finitely many cone-like singularities arise by this construction.

2.4. *Fibered Links in Dimensions* $(2n+1, n)$*, Where* $n \geq 2$

We can construct a much larger family of examples from existing ones by means of a method of Looijenga [18] to construct nontrivial local isolated singularities. Specifically, we consider the *spinning* of Hopf links in a similar manner as the spinning of a knot. Consider a link $L = \bigsqcup_{j=0}^{d} S_j^{n-1} \subset S^{2n-1}$ with a choice of one component S_i^{n-1} to be spin off. We isotope L so that all components but S_i^{n-1} lie in the interior of the upper half-space $H_+^{2n-1} := \{(x_1, \dots, x_{2n-1}) \in \mathbb{R}^{2n-1}; x_{2n-1} \geq 0\}$, whereas the intersection of S_i^{n-1} with the lower half-space consists of a hemisphere. We now spin H_+^{2n-1} in $\mathbb{R}^{2n}$ around $\mathbb{R}^{2n-2}$ so that each point $(x_1, \dots, x_{2n-1}) \in H_+^{2n-1}$ sweeps out the circle $(x_1, \dots, x_{2n-2}, x_{2n-1}\cos\theta, x_{2n-1}\sin\theta)$, $\theta \in [0, 2\pi]$. The spinning orbits of the hemisphere along $\bigsqcup_{j\neq i} S_j^{n-1}$ form a link of the form $SL = S_i^n \bigsqcup_{j\neq i}(S^1 \times S_j^{n-1}) \subset S^{2n}$ (for more details on the knot counterpart, see [9]). When L is a fibered link, the spinning links SL are all fibered. In particular, this is the case when $L = K_A$. If $F^n = S^n \setminus \bigsqcup_{j=0}^{d} D_j^n$ is the fiber of L, then $SF^{n+1} = S^{n+1} \setminus (D_i^{n+1} \bigsqcup_{j\neq i}^{d} S^1 \times D_j^n)$ is the fiber of SL.

Note that we can iterate this procedure k times, and by choosing each time the same spinning component we obtain links of the form $S^{n+k-1} \bigsqcup_1^d (S^1)^k \times S^{n-1} \subset S^{2n+k-1}$.

2.5. *Fibered Links in Dimensions* $(2n, k)$ *and* $(2n+1, k)$*, Where* $n \geq k \geq 2$

The *rank* of a critical point is the rank of the differential at that point. Given a smooth map $\psi : (D^m, 0) \to (D^k, 0)$, $k \geq 2$, with an isolated singularity at 0 of rank zero, we consider the map $\Pi\psi : (D^m, 0) \to (D^{k-1}, 0)$ obtained by composing ψ with the projection $\Pi : D^k \to D^{k-1}$. This is again a smooth map with an isolated singularity at the origin of rank zero.

According to [14; 3], the local Milnor fiber $F_{\Pi\psi}$ of $\Pi\psi$ around 0 is homeomorphic to $F_\psi \times [0,1]$ if ψ is a real polynomial.

Starting from a smooth map $\psi_L : (D^{2n}, 0) \to (D^n, 0)$ as constructed in Section 2.3 out of a generalized Hopf link L in generic position, we deduce by iterated projections smooth maps with an isolated singularity at the origin $\Pi^k\psi : (D^{2n}, 0) \to (D^{n-k}, 0)$ in all dimensions $(2n, n-k)$ with $0 \leq k \leq n-1$. We call the links $K^{n+k-1} \subset S^{2n-1}$ obtained from these maps *generalized Hopf links* in dimensions $(2n, n-k)$.

Assume that ψ_L is the local model associated with a fibered generalized Hopf link L with $(d+1)$ components in generic position. Then the local fiber F_{ψ_L} is diffeomorphic to an n-disk with d handles of index $(n-1)$ attached along trivially embedded and unlinked spheres $S^{n-2} \subset \partial D^n$.

The link $L_{\Pi\psi_L} \subset S^{2n-1}$ associated with $\Pi\psi_L$ is the union of local fibers $f_L^{-1}(\overline{\Pi}^{-1}(0))$, where $\overline{\Pi} : S^{n-1} \to D^{n-1}$ is the projection. Now $\overline{\Pi}^{-1}(0) = \{n, s\}$

is a pair of points, the north and the south pole of S^{n-1} with respect to the projection $\overline{\Pi}$. Therefore $L_{\Pi\psi_L}$ is the closure of the union of the two local fibers $f_L^{-1}(n)$ and $f_L^{-1}(s)$ of f_L, that is, their union with L.

The link $L_{\Pi\psi_L} \subset S^{2n-1}$ associated with $\Pi\psi_L$ is

$$\begin{aligned}
&S^{2n-1} \cap (\Pi\psi_L)^{-1}(0) \\
&\quad = S^{2n-1} \cap \psi_L^{-1}(\Pi^{-1}(0)) = S^{2n-1} \cap \psi_L^{-1}([sn]) \\
&\quad = [(S^{2n-1} \setminus N(L)) \cap f_L^{-1}([sn])] \cup [N(L) \cap \psi_L^{-1}([s0) \cup \{0\} \cup (0n])] \\
&\quad = [(S^{2n-1} \setminus N(L)) \cap f_L^{-1}(\{s, n\})] \cup L \cup [N(L) \cap \psi_L^{-1}([s0) \cup (0n])],
\end{aligned}$$

as $\psi_L|_{S^{2n-1}\setminus N(L)} = f_L|_{S^{2n-1}\setminus N(L)}$. Note that $\psi_L|_{N(L)} \neq f_L|_{N(L)}$ as $\psi_L(L) = 0$ whereas $f_L(L) \subseteq S^{n-1}$. Since $N(L) \cap \psi_L^{-1}([s0))$ is homeomorphic with $N(L) \cap f_L^{-1}(s)$ and $N(L) \cap \psi_L^{-1}((0n])$ is homeomorphic with $N(L) \cap f_L^{-1}(n)$, we obtain that the link $L_{\Pi\psi_L} \subset S^{2n-1}$ associated with $\Pi\psi_L$ is homeomorphic with the closure of the union of the two local fibers $f_L^{-1}(n)$ and $f_L^{-1}(s)$ of f_L, that is, their union with L.

Furthermore, the open book fibration $f_{L_{\Pi\psi}} : S^{2n-1} \setminus L_{\Pi\psi} \to S^{n-2}$ is obtained as $f_{L_{\Pi\psi}}(x) = R\overline{\Pi} f(x)$, where $R : D^{n+1} \setminus \{0\} \to S^{n-2}$ is the radial projection. If $x \in S^{n-2}$, then let $\gamma_x \subset S^{n-1}$ be the great arc passing through n, s and $x = \overline{\Pi}^{-1}(x) \in S^{n-1}$. Then the local fiber $F_{\Pi\psi_L}$ of $\Pi\psi_L$ is the union of fibers $f_L^{-1}(\gamma_x)$. It follows that $F_{\Pi\psi_L}$ is homeomorphic to $F_{\psi_L} \times [0, 1]$.

By induction the local fiber of $\Pi^k f$ is an $(n+k)$-disk with d handles of index $(n-1)$ attached along trivially embedded and unlinked $S^{n-2} \subset \partial D^{n+k}$. It follows that the local fiber $F_{\Pi^k f} = \sharp_{\partial d} S^{n-1} \times D^{k+1}$, where $\sharp_\partial$ denotes the boundary connected sum of manifolds with boundary. In particular, the corresponding link $L_{\Pi^k f} \subset S^{2n-1}$ is diffeomorphic to a connected sum $\sharp_{j=1}^d S^{n-1} \times S^k$. Note that the link $L_{\Pi^k f}$ is connected when $k \geq 1$.

It follows that, for $k \geq 1$, any decorated graph Γ that occurs in the previous construction consists of two black vertices and an edge joining them or else a single white vertex connected to several black vertices. Note that the gluing map in the former case is highly not unique, the result depending on the corresponding element of mapping class group of $\sharp_{j=1}^d S^k \times S^{n-1}$.

2.6. *Statement of Results*

Our first result shows that all these examples are nontrivial:

THEOREM 2.1. *Let $\Gamma_1, \Gamma_2, \dots, \Gamma_p$ be bicolored graphs decorated by generalized Hopf links in dimensions $(2n, n-k)$ as in Section 2.5 such that the fibrations $f_{\Gamma_1}, f_{\Gamma_2} \dots, f_{\Gamma_p}$ cobound. When $n-k$ is even, we assume that the total number s of black vertices of the graphs $\Gamma_1, \Gamma_2, \dots, \Gamma_p$ is odd. Then we have the inequalities*

$$1 \leq \varphi(M^{2n}(\Gamma_1, \Gamma_2, \dots, \Gamma_p), S^{n-k}) \leq s. \tag{2}$$

REMARK 2.1. The fibrations with fiber F over S^{n-1}, $n \geq 3$, are classified by their characteristic elements in the group $\pi_{n-2}(\mathrm{Diff}(F))$. A collection of fibrations cobound if the sum of their characteristic elements is trivial. This provides abundant examples verifying the assumptions of the theorem for odd $n-k$. Notice that, for even $n-k$, it is not clear that there exists a collection $\Gamma_1, \Gamma_2, \dots, \Gamma_p$ of bicolored decorated graphs with odd total number of vertices in order to be able to use Theorem 2.1 for finding nontrivial examples.

Let now φ_c count the minimum number of critical points of smooth maps with only *cone-like* singularities (see [15]).

THEOREM 2.2. *If* $\varphi_c(M^6, S^3)$ *is finite nonzero, then* M *is diffeomorphic to* $M^6(\Gamma_1, \dots, \Gamma_p)$ *for some decorated bicolored graphs* Γ_i. *In particular,* $\pi_1(M)$ *is a* (*closed*) *3-manifold group.*

Moreover, if $\pi_1(M^6) = 1$ *and* $\chi(M) \geq -1$, *then either* $\varphi_c(M^6, S^3) = 0$, *or* $\varphi_c(M^6, S^3) = \infty$.

Since S^6 does not fiber over S^3 (see, e.g., [1]), we derive the following:

COROLLARY 2.1. *We have* $\varphi_c(S^6, S^3) = \infty$.

We think that it is possible to classify all manifolds M^6 with finite $\varphi_c(M^6, S^3)$.

We further show that this method can indeed produce explicit examples with φ equal to one in all dimensions. We state our result below separately for odd and even dimensions, as the combinatorial data is slightly different.

THEOREM 2.3. *Suppose that* $n \geq 3$ *and the decorated graph is as follows*:

(1) *for* $k = 0$, *a tree* Γ_0 *with one black vertex decorated by a generalized Hopf link and several white vertices decorated by disks*;
(2) *for* $k \geq 1$, *the graph* Γ_0 *has a single black vertex* v *decorated by a generalized Hopf link* $L_{\prod^k L}$, *where* L *is an* $(n-1)$*-dimensional generalized Hopf link with* $d+1 \geq 5$ *components and a white vertex, the two vertices being connected by an edge. The white vertex* w *is decorated by* $F_w = \sharp_{\partial d} D^n \times S^k$.

Then

$$\varphi(M^{2n}(\Gamma_0), S^{n-k}) = 1.$$

THEOREM 2.4. *Suppose that* $n \geq 3$ *and the decorated graph is as follows*:

(1) *for* $k = 0$, *the graph* Γ_0 *is a tree consisting of one black vertex decorated by the fibered link* SK_A *adjacent to* $d+1 \geq 2$ *white vertices, one of which being decorated by the disk* D^{n+1}, *and the remaining white vertices being decorated by* $S^1 \times D^n$;
(2) *for* $k \geq 1$, *the graph* Γ_0 *has a single black vertex* v *decorated by* $L_{\prod^k SL}$, *where* L *is an* $(n-1)$*-dimensional generalized Hopf link with* $d+1 \geq 5$ *components and a white vertex, the two vertices being connected by an edge. The white vertex* w *is decorated by the manifold* $F_w = (\sharp\partial_{j=1}^d D^n \times S^{k+1})\sharp_\partial(\sharp\partial_{j=1}^d S^k \times D^{n+1})$.

Then

$$\varphi(M^{2n+1}(\Gamma_0), S^n) = 1.$$

The gluing map between the decoration and the local fiber associated with the black vertex will be specified in the proof.

The only drawback of this method is that we have no explicit description of the manifolds of the form $M^m(\Gamma_1, \ldots, \Gamma_p)$. Using different tools, we can provide a first sample of easy to understand examples in arbitrary high dimensions, which might be interesting by themselves, as follows.

PROPOSITION 2.1. *We have*

$$1 \leq \varphi(S^4 \times S^4 \times \cdots \times S^4, S^3) \leq 2^m$$

when we have m factors S^4. Moreover, we have

$$\begin{aligned} 1 &\leq \varphi((\sharp_{r_1} S^2 \times S^2) \times (\sharp_{r_2} S^2 \times S^2) \times \cdots \times (\sharp_{r_m} S^2 \times S^2), S^3) \\ &\leq 2^m (r_1 + 1) \cdots (r_m + 1). \end{aligned}$$

The existence of the Hopf fibration $S^3 \to S^2$ implies the following:

COROLLARY 2.2. *We have*

$$\varphi((\sharp_{r_1} S^2 \times S^2) \times (\sharp_{r_2} S^2 \times S^2) \times \cdots \times (\sharp_{r_m} S^2 \times S^2), S^2) \leq 2^m (r_1 + 1) \cdots (r_m + 1).$$

When $m = 1$ and $r_1 = 1$, the left-hand side vanishes. It seems that otherwise it is positive.

COROLLARY 2.3. *There exist examples with nontrivial $\varphi(M^{2n}, S^3)$ for every $n \geq 2$.*

This is a consequence of Theorem 2.1 and the proof of Proposition 2.1.

The second part of this paper aims at a deeper understanding of these examples when n is even and, in particular, approaching the case where $n - k$ is even in Theorem 2.1.

A necessary condition for M^{2n} to admit a fibration over S^k is that $\chi(M^{2n}) = 0$ when k is odd and $\chi(M^{2n}) \equiv 0 \pmod 2$ for even k. When n is even, there are stronger requirements for a manifold to be a fibration over S^n. Recall that the *signature* of the compact oriented M is set to be zero unless its dimension is multiple of 4; in the latter case, the signature of the symmetric bilinear form on the middle dimension cohomology is given by the cup product evaluated on the fundamental class. A classical theorem due to Chern, Hirzebruch, and Serre [6] states that whenever we have a fibration $E \to B$ with fiber F of oriented compact manifolds such that the action of $\pi_1(B)$ on the cohomology $H^*(F)$ is trivial, then the signature is multiplicative, namely

$$\sigma(M) = \sigma(B)\sigma(F).$$

In particular, this happens when $\pi_1(B)$ is trivial. This is known not to be true for general fibrations as, for instance, in the case of the Atiyah–Kodaira fibrations (see

[5; 16]), which are fibrations of some 4-manifolds of signature 256 over surfaces. In particular, if $\sigma(M) \neq 0$, then $\varphi(M, S^p) \geq 1$ for any p and thus also for even values of p.

Our next goal is the explicit computation of $\sigma(M(\Gamma_1, \Gamma_2, \ldots, \Gamma_p))$. Observe that, for even n, we have $\sigma(M^{2n}(\Gamma_1, \Gamma_2, \ldots, \Gamma_p)) \equiv s \pmod 2$.

THEOREM 2.5. *For even n, there exist graphs Γ decorated by generalized Hopf links in dimensions $(2n, n-k)$ as in Section 2.5 such that*

$$\sigma(M(\Gamma)) \neq 0.$$

3. Proofs of Theorems 2.1, 2.2, 2.3, and 2.5

3.1. *Preliminaries on Fibered Generalized Hopf Links in Dimensions* $(2n, n)$

Denote by K_i, $0 \leq i \leq d$, the components of K_A indexed as the components of $L_{\widetilde{A}}$. Note that unlike arbitrary fibered links K_A also have a *canonical framing* in X_A, namely a set of isotopy classes of parallel copies $K_i^\sharp \subset \partial N(K_i)$ obtained by intersecting the generic fiber of the given open book decomposition with the boundary of the link complement. In particular, it makes sense to consider the diagonal of the linking matrices of K_A with entries $\mathrm{lk}(K_i^\sharp, K_i)$. We can actually identify the link K_A when A is unimodular as follows.

LEMMA 3.1. *If A is unimodular, then $K_A = L_{A^*}$, where the linking matrix in the canonical framing A^* is the $(-1)^n$-symmetric matrix with entries*

$$A^*_{ij} = \begin{cases} (A^{-1})_{ij} & \text{if } 1 \leq i, j \leq d, \\ -\sum_{k=1}^d (A^{-1})_{kj} & \text{if } i = 0, 1 \leq j \leq d, \\ \sum_{k=1}^d \sum_{l=1}^d (A^{-1})_{kl} & \text{if } i = j = 0. \end{cases}$$

Proof. Let X_s denote the result of filling all but the sth boundary components using surgery as before. Then X_s is $(n-2)$-connected, and the Mayer–Vietoris sequence reads

$$\begin{aligned} &H_{n-1}\Big(\bigsqcup_{j=0, j\neq s}^{d} K_i \times \partial D^n\Big) \\ &\quad \to H_{n-1}(S^{2n-1} \setminus N(L_{\widetilde{A}})) \oplus H_{n-1}\Big(\bigsqcup_{j=0, j\neq s}^{d} K_i \times D^n\Big) \\ &\quad \to H_{n-1}(X_s) \to 0. \end{aligned}$$

If X_A is homeomorphic to a sphere $H_{n-1}(X_s) \cong \mathbb{Z}$, then the linking number $\mathrm{lk}(K_j, K_s)$ in X_A is the image of the class of K_j in $\mathbb{Z}$. Moreover, $H_{n-1}(S^{2n-1} \setminus N(L_{\widetilde{A}})) \cong \bigoplus_{j=0}^d \mathbb{Z}\mu_j$, where the classes μ_j correspond to the meridians spheres around each boundary component. Let δ_j denote the generator of $H_{n-1}(K_j \times D^n)$. We give K_j the orientation induced as a boundary component of the fiber (which disagrees with the convention in [4]).

If $s \neq 0$, then it follows (see the computations from [4], proof of Lemma 3.4) that we have the presentation

$$\begin{aligned}H_{n-1}(X_s) = \bigoplus_{j=0}^{d} \mathbb{Z}\langle \mu_j \rangle \bigoplus_{i=0, i\neq s}^{d} \mathbb{Z}\langle \delta_i \rangle \Big/ \\ \left(\mathbb{Z}\left\langle \delta_0 + \sum_{j=1}^{d} \mu_j \right\rangle \bigoplus_{1\leq i\leq d, i\neq s} \mathbb{Z}\langle \mu_i - \delta_i \rangle \right. \\ \left. \oplus\, \mathbb{Z}\mu_0 \bigoplus_{1\leq i\leq d, i\neq s} \mathbb{Z}\left\langle \sum_{j=1}^{d} A_{ij}\mu_j \right\rangle \right).\end{aligned}$$

Further, the homomorphism $ev : H_{n-1}(X_s) \to \mathbb{Z}$ given on the generators by

$$ev(\mu_i) = (A^{-1})_{is}, \quad 1 \leq i \leq d, \qquad ev(\mu_0) = 0,$$
$$ev(\delta_i) = (A^{-1})_{is}, \quad 1 \leq i \leq d, i \neq s, \quad \text{and} \quad ev(\delta_0) = -\sum_{i\neq s} (A^{-1})_{is}$$

is well defined, and it is an isomorphism since A is invertible over $\mathbb{Z}$. The class of K_j and respectively $K_s^\sharp$ in $H_{n-1}(X_s)$ is represented by μ_j if $j \neq s$, and hence

$$\mathrm{lk}(K_j, K_s) = A^*_{js}, \quad j \neq 0, \qquad \mathrm{lk}(K_s^\sharp, K_s) = A^*_{ss}.$$

Further, the class of K_0 is represented by $-\sum_{j=1}^{d} \mu_j$, and hence

$$\mathrm{lk}(K_0, K_s) = -\sum_{j=1}^{d} A^*_{js} = A^*_{0s}.$$

If $s = 0$, then we have a similar presentation of $H_{n-1}(X_0)$:

$$H_{n-1}(X_0) = \frac{\bigoplus_{j=0}^{d} \mathbb{Z}\langle \mu_j \rangle \bigoplus_{i=1}^{d} \mathbb{Z}\langle \delta_i \rangle}{\bigoplus_{1\leq i\leq d} \mathbb{Z}\langle \mu_i - \delta_i \rangle \bigoplus_{1\leq i\leq d} \mathbb{Z}\langle \mu_0 + \sum_{j=1}^{d} A_{ij}\mu_j \rangle}.$$

Further, the homomorphism $ev : H_{n-1}(X_0) \to \mathbb{Z}$ given on the generators by

$$ev(\mu_i) = -\sum_{j=1}^{d} (A^{-1})_{ij}, \quad 1 \leq i \leq d, \qquad ev(\mu_0) = 1,$$

and

$$ev(\delta_i) = -\sum_{j=1}^{d} (A^{-1})_{ij}(A^{-1})_{is}, \quad 1 \leq i \leq d,$$

is also an isomorphism. We derive:

$$\mathrm{lk}(K_j, K_0) = -\sum_{i=1}^{d} (A^{-1})_{ji} = A^*_{j0}, \quad j \neq 0,$$

$$\mathrm{lk}(K_0^\sharp, K_0) = \sum_{j=1}^{d} \sum_{k=1}^{d} (A^{-1})_{jk} = A^*_{00}. \qquad \square$$

3.2. *Proof of Theorem 2.1*

We only need to prove that $M(\Gamma_1, \dots, \Gamma_p)$ does not fiber over S^{n-k}. For simplicity of exposition, we will only consider the case where there are no insertion of trivial fiber bundles here and hence we can drop the decoration. Note that this implies that Γ_i only contain black vertices and that there are no univalent vertices of Γ_i.

Now the Euler characteristic χ is multiplicative in locally trivial fiber bundles, namely for any locally trivial fibration $\pi : E \to B$ with fiber F, we have $\chi(E) = \chi(B)\chi(F)$. This is well known to hold in the case where the action of $\pi_1(B)$ on the cohomology $H^*(F)$ is trivial, in particular, where $\pi_1(B) = 0$. The standard argument to prove this uses spectral sequences. Nevertheless, the multiplicativity of the Euler characteristic holds in full generality as soon as E, F, and B are finite CW complexes, by induction on the number of cells of the basis. This is obviously true when B has only one cell, in which case E is a product. Assume that the multiplicativity is true for fiber bundles over CW complexes with at most N cells, and consider a complex B with $N + 1$ cells. Let e^n be an n-cell of B. The restriction $\pi^{-1}(B - e^n) \to B - e^n$ is a fiber bundle, so that $\chi(\pi^{-1}(B - e^n)) = \chi(B - e^n)\chi(F)$. By excision we have $H^*(E, \pi^{-1}(B - e^n)) = H^*(e^n \times F, \partial e^n \times F)$. This implies that $\chi(E, \pi^{-1}(B - e^n)) = (-1)^n \chi(F)$, and hence $\chi(E) = \chi(\pi^{-1}(B - e^n) + \chi(E, \pi^{-1}(B - e^n)) = \chi(B)\chi(F)$. This proves the induction step.

Thus a necessary condition for a space E to fiber over the S^{n-k} is that $\chi(E) = 0$ if $n - k$ is odd and $\chi(E) \equiv 0 \pmod 2$ if $n - k$ is even.

We can compute $\chi(M(\Gamma_1, \dots, \Gamma_p))$ using the local picture description of each singularity.

Consider first the case $k = 0$. A critical point associated with a vertex of Γ_i of valence $(d + 1)$ comes with a local model whose link has $(d + 1)$ components. As in the case of Lefschetz fibrations, we obtain the local model from a fibration over the punctured disk $D^n - \{0\}$ with fiber $D^n - \bigsqcup_{i=1}^{d} D_i^n$ by adjoining one singular fiber over 0 that is the cone over the boundary. This amounts to adjoin to the trivial fibration over D^n a number of d handles of index n, corresponding to crushing the vanishing cycle $\bigvee_d S_i^{n-1}$ to a point. This handlebody description can be turned into a cell-decomposition, and therefore each local model corresponds to a fibration with d cells of dimension n adjoined. Gluing together all local models by the patchwork explained in the introduction produces a block $X(\Gamma_j)$ obtained from a fibration over D^n with t_j cells of dimension n added, where $t_j = 2m_j - s_j$,

m_j being the total number of edges in the Γ_j, and s_j being the total number of vertices. Since each vertex has valence at least 2, we have $m_j - s_j \geq 0$. An alternative argument is to observe that $X(\Gamma_j)$ deformation retracts onto the singular fiber, which is obtained from the regular fiber by contracting the attaching $(n-1)$-spheres corresponding to the n-handles. This shows that the dimension of the cokernel of $H_n(\partial X(\Gamma_j)) \to H_n(X(\Gamma_j))$ equals t_j.

Therefore

$$\begin{aligned}\chi(M(\Gamma_1, \Gamma_2, \ldots, \Gamma_p)) &= \chi(S^n)\chi(\sharp_g S^1 \times S^{n-1}) + (-1)^n t \\ &= -g((1+(-1)^n)^2 + (-1)^n t,\end{aligned}$$

where t is the sum of all t_j. When n is odd, $\chi(M(\Gamma_1, \Gamma_2, \ldots, \Gamma_p)) = -t \neq 0$, and hence it cannot be a fibration over some n-manifold. When n is even, $\chi(M(\Gamma_1, \Gamma_2, \ldots, \Gamma_p)) \equiv -t \equiv s \not\equiv 0 \pmod 2$, and thus it cannot fiber over S^n.

Let now consider the case $k \geq 1$ by first analyzing the local picture. The link of each critical point is now connected. However, there exists a collection of disjoint embedded spheres S^{n-1} embedded in the local fiber $F_{\Pi^k \psi_L}$, which is diffeomorphic to an $(n+k)$-disk with d copies of $(n-1)$-handles attached to it. The singular fiber is then homeomorphic to a cone over the link. Therefore a regular neighborhood of the singular fiber is homeomorphic to the result of attaching d handles of index n to the regular neighborhood of a generic fiber. This description allows us to use the previous arguments for $k = 0$. We conclude as before.

Remark 3.1. When singular points arise from the fibered links as before, each critical point s contributes with $\chi(F_s) - 1$ to $\chi(M)$, where F_s is the local fiber around s. This holds more generally for all fibered links. On the other hand, if dimensions were of the form $(2n+1, k)$, then local fibers should verify $\chi(F_s) = 1$ according to [2; 14; 21]. This shows that the contribution of every critical point is trivial in odd dimensions, and hence the previous arguments cannot work.

3.3. *Proof of Theorem 2.2*

Every open book fibration $S^5 - N(K) \to S^2$ has a simply connected fiber F^3 (see, e.g., [4]). By Perelman's solution to the Poincaré conjecture F^3 is a disk with holes, and thus K is a disjoint union of spheres S^2. Therefore K is a generalized Hopf link L_Q for some matrix Q. Moreover, L_Q is fibered if and only if $Q = \widetilde{A}$, where A is unimodular according to [4]. Thus, for any smooth map $f : M^6 \to S^3$ with finitely many cone-like critical points, there are neighborhoods of the critical points to which the restriction of f is equivalent to some local model. Outside these neighborhoods the restriction of f should be a locally trivial fiber bundle. Therefore M^6 is diffeomorphic to some $M^6(\Gamma_1, \Gamma_2, \ldots, \Gamma_p)$, where Γ_i are bicolored decorated graphs, and f arises as before. We suppose that M^6 is not a fibration over S^3. Every graph Γ_i has at least one black vertex, as otherwise we could remove it. Each decorated graph Γ_i determines $f_{\tilde{\Gamma}_i} : X^6(\Gamma_i) \to D^3$, whose generic fiber is some closed 3-manifold F, which is independent on i.

Notice that the union V of singular fibers of f is a CW complex of dimension 3 embedded in M, so that $\pi_1(M^6 - V^3) \to \pi_1(M^6)$ is an isomorphism. The long exact sequence in homotopy associated with the fibration $f|_{M-V}$ implies that $\pi_1(F^3) \to \pi_1(M^6)$ is surjective, with free Abelian kernel. Let F_{ij} and $D_{ij} = D^3 - \bigsqcup_{s=1}^{n_{ij}} D_s^3$ denote the 3-manifolds with boundaries that occur as labels of the white vertices and black vertices, respectively, of the graph Γ_i. The key point is that local fibers D_{ij} are simply connected. Then the generic fiber F is obtained from the (graph) connected sum of F_{ij} and D_{ij}. The block $X^6(\Gamma_i) \setminus V$ is the union of fibered pieces $D_v^6 \setminus V$ associated with black vertices v and $F_{ij} \times (D^3 \setminus \{0\})$ associated with decorated white vertices. Moreover, we glue together two such adjacent pieces along the submanifold $N(L_{\widetilde{A(v)}}) \setminus L_{\widetilde{A(v)}}$, which is simply connected by transversality. Also, $\pi_1(D_v^6 \setminus V) = 1$. Then Van Kampen's theorem implies that the inclusion of F into $X^6(\Gamma_i) \setminus V$ induces an isomorphism at the level of fundamental groups, and hence $\pi_1(X^6(\Gamma_i) \setminus V)$ is isomorphic to $\pi_1(F) \cong *_j \pi_1(F_{ij}) * \mathbb{F}_r$, where r is the rank of $H_1(\Gamma_i)$. We obtain $M^6 \setminus V$ by first gluing together several blocks $X^6(\Gamma_i) \setminus V$ along neighborhoods of boundary fibers and second gluing to the result a trivial fibration $F \times D^3$ along the whole boundary $F \times S^2$. Further use of Van Kampen's theorem shows that the inclusion of F into M is also an isomorphism at the fundamental group level.

Every black vertex v of some Γ_i has associated a link of the form $L_{\widetilde{A}}$, where A is unimodular [4]. However, unimodular skew-symmetric matrices have to be of even size, so that every black vertex v has odd degree. Assume that $\pi_1(M)$ has no free factor, so that $r = 0$. Then each Γ_i should have only one black vertex, since otherwise the valence of a black vertex being odd it would be at least 3, and this would produce a free factor in $\pi_1(F)$. The local fiber associated with this black vertex is $D^3 \setminus \bigsqcup_{s=1}^{d} D_s^3$. Each F_{ij} must have one boundary component; if some F_{ij} had at least two boundary components, then gluing the local fiber $D^3 \setminus \bigsqcup_{s=1}^{d} D_s^3$ would produce a free factor in $\pi_1(F)$. Thus the generic fiber F of f is diffeomorphic to $\sharp_{s=1}^{d} F_s$.

Suppose now that $\pi_1(M) = 1$. Then F is simply connected and hence, by Perelman, is diffeomorphic to S^3. Moreover, each F_{ij} is diffeomorphic to a disk. The computation of the Euler characteristic from the previous section gives us

$$\chi(M^6(\Gamma_1, \Gamma_2, \ldots, \Gamma_p)) = -\sum_{i=1}^{p} d_i,$$

where $1 + d_i \geq 3$ is the degree of the black vertex of Γ_i. In particular, if $\pi_1(M) = 1$ and $\chi(M) \geq -1$, then $\varphi_c(M, S^3) = \infty$, as we supposed that M^6 does not fiber. This proves the claim.

3.4. Proof of Theorem 2.3

We first need the following:

LEMMA 3.2. *In dimensions $(2n, n)$, let the graph Γ_0 be a tree consisting of one black vertex decorated by the fibered link K_A that is adjacent to $d + 1 \geq 2$ white vertices decorated by disks D^n. The gluing maps correspond to the decomposition of S^n as the union of two smooth disks D^n along an equatorial sphere. Then $\partial X^{2n}(\Gamma_0)$ is diffeomorphic to $S^n \times S^{n-1}$, and the boundary fibration $\psi_{\Gamma_0} : \partial X^{2n}(\Gamma_0) \to S^{n-1}$ is trivial.*

Proof. We obtain $\partial X(\Gamma_0)$ by doing surgery on the link K_A, namely gluing to $S^{2n-1} \setminus N(K_A)$ the disjoint union of $(d + 1)$ solid tori $\bigsqcup_{j=0}^{d} D^n \times S^{n-1}$ such that the jth copy of the solid torus $D^n \times S^{n-1}$ is glued along $\partial N(K_j^{n-1})$ so that $\partial D^n \times \{pt\}$ correspond to the preferred longitude spheres. This is the same as doing surgery on $L_{\widetilde{A}}$ corresponding to the framings given by the longitude around S_0 and the meridian spheres along S_j, $j \geq 2$. Surgery along meridian spheres restores the sphere $S^{2n-1} \setminus N(S_0)$, whereas surgery along the longitude of S_0 yields $S^{n-1} \times S^n$. The fibration structure of ψ_{Γ_0} corresponds then to the projection onto S^{n-1}. □

LEMMA 3.3. *In dimensions $(2n, n - k)$ with $k \geq 1$, let the graph Γ_0 with a single black vertex v decorated by $L_{\Pi^k L}$, where L is an $(n-1)$-dimensional generalized Hopf link with $d+1 \geq 5$ components and a white vertex connected by an edge. The white vertex w is decorated by $F_w = \sharp\partial_{j=1}^{d} D^n \times S^k$. The gluing along ∂F_w corresponds to surgery of the core k-dimensional spheres, and hence the global fiber F is diffeomorphic to S^{n+k}. Then $\partial X(\Gamma_0)$ is diffeomorphic to $S^{n+k} \times S^{n-k-1}$, and the boundary fibration $\partial X(\Gamma_0) \to S^{n-k-1}$ is trivial.*

Proof. We have the decomposition

$$\partial X(\Gamma_0) = (S^{2n-1} \setminus ((\sharp_{j=1}^{d} S^{n-1} \times S^k) \times D^{n-k})) \cup (\sharp\partial_{j=1}^{d} D^n \times S^k) \times S^{n-k-1}$$

along $\partial(E_v)_k = (\sharp_{j=1}^{d} S^{n-1} \times S^k) \times S^{n-k-1}$. The result follows for $k = 0$ from Lemma 3.2. We use further induction on k. We add the subscript k to all objects defined so far. If the claim holds for k, then $(E_v)_k \subset S^{n+k} \times S^{n-k-1}$, and the projection $f_k : (E_v)_k \to S^{n-k-1}$ is the restriction of the second factor projection. Note that $(E_v)_{k+1} = (R\overline{\Pi})^{-1}(S^{n-k-1} \setminus (D^{n-k-1}(s) \cup D^{n-k-1}(n)))$, where $D^{n-k-1}(s)$ and $D^{n-k-1}(n)$ are two disk neighborhoods of the north and south poles n, s of S^{n-k-1}. Then the fibration $f_{k+1} : (E_v)_{k+1} \to S^{n-k-2}$ is the composition $(f_k)|_{(E_v)_{k+1}} : (E_v)_{k+1} \to S^{n-k-1} \setminus (D^{n-k-1}(s) \cup D^{n-k-1}(n))$ with the projection $R\overline{\Pi} : S^{n-k-1} \setminus (D^{n-k-1}(s) \cup D^{n-k-1}(n)) \to S^{n-k-2}$. Further, $(E_v)_{k+1}$ is a subfibration of the product fibration

$$S^{n+k} \times S^{n-k-1} \setminus (D^{n-k-1}(s) \cup D^{n-k-1}(n)) \to S^{n-k-2},$$

which itself is a subfibration of $S^{n+k+1} \times S^{n-k-2} \to S^{n-k-2}$.

It remains to observe that, during the process of filling, we adjoined the fibration $(E_v)_{k+1}$ along the boundary $((F_w)_{k+1} \times [0, 1]) \times S^{n-k-2}$, namely $(F_w)_{k+1} \times S^{n-k-2}$. This proves the induction claim. □

If $k = 0$, then we consider the manifold $M(\Gamma_0) = X(\Gamma_0) \bigcup_{\partial X(\Gamma_0)} S^n \times D^n$. First, $\pi_1(M(\Gamma_0)) = 0$, and further by Mayer–Vietoris $H_j(M(\Gamma_0)) = 0$ for $1 \le j \le 2n - 1$, $j \ne n$, and $H_n(M(\Gamma_0)) = \mathbb{Z}^{d+2}$. Thus $M(\Gamma_0)$ is $(n-1)$-connected.

Assume that $M(\Gamma_0)$ fiber over S^n with fiber F^n. Then the long exact sequence of the fibration shows that F^n must be $(n-2)$-connected. Further, the Wang sequence first yields

$$\to H_n(F) \to H_{2n-1}(M) \to H_{n-1}(F^n) \to H_{2n-2}(F^n) \to H_{2n-2}(M) \to,$$

and thus $H_{n-1}(F^n; \mathbb{Q}) = 0$ for $n \ge 3$, and second:

$$\to 0 = H_1(F) \to H_n(F) \to H_n(M) \to H_0(F^n) \to H_{n-1}(F^n) = 0,$$

and hence $H_0(F)$ has rank d, a contradiction, thereby proving the claim.

When $k \ge 1$, we consider $M(\Gamma_0) = X(\Gamma_0) \bigcup_{\partial X(\Gamma)} S^{n+k} \times D^{n-k}$. It follows that $\pi_1(M(\Gamma_0)) = 0$, and by Mayer–Vietoris $H_j(M(\Gamma_0)) = 0$ for $1 \le j \le 2n-1$, $j \notin \{n-k, n+k\}$, whereas $H_n(M(\Gamma_0)) = \mathbb{Z}^d$, $H_{n-k}(M(\Gamma_0)) = H_{n+k}(M(\Gamma_0)) = \mathbb{Z}$. Assume that $M(\Gamma_0)$ fibers over S^{n-k} with fiber F^{n+k}. Then the long exact sequence of the fibration shows that F is connected and simply connected. The Wang exact sequence

$$H_q(F) \to H_q(M(\Gamma_0)) \to H_{q-n+k}(F) \to H_{q-1}(F) \to H_{q-1}(M(\Gamma_0)) \to$$

for $q = 2n-1, 2n-2, \dots, n+k+2$ yields

$$H_{n+k-1}(F) = H_{n+k-2}(F) = \cdots = H_{2k+2}(F) = 0.$$

Further, for $q = n+k+1$, we obtain the exact sequence

$$\begin{aligned} 0 = H_{n+k+1}(M(\Gamma_0)) &\to H_{2k+1}(F) \to H_{n+k}(F) \to H_{n+k}(M(\Gamma_0)) \\ &\to H_{2k}(F) \to H_{n+k-1}(F) = 0, \end{aligned}$$

which implies that

$$\operatorname{rk} H_{2k+1}(F; \mathbb{Q}) = \operatorname{rk} H_{2k}(F; \mathbb{Q}) = u \in \{0, 1\}.$$

If $n \ge 2k+2$, then we can consider $q = n+k-1, \dots, n$ and derive

$$H_{2k-1}(F) = H_{2k-2}(F) = \cdots = H_n(F) = 0.$$

From the exactness of

$$0 = H_n(F) \to H_n(M(\Gamma_0)) \to H_k(F) \to H_{n-1}(F)$$

we obtain $\operatorname{rk} H_k(F; \mathbb{Q}) \ge d$. However, $H_n(F; \mathbb{Q}) = 0$ for $n \ge 2k$, and this contradicts the Poincaré duality for F^{n+k}.

If $n = 2k+1$, then from the exact sequence

$$H_{2k+1}(F) \to H_{2k+1}(M(\Gamma_0)) \to H_k(F) \to H_{2k}(F) \to H_{2k}(M(\Gamma_0))$$

we derive that both the kernel and the cokernel of the map $H_{2k+1}(M(\Gamma_0); \mathbb{Q}) \to H_k(F; \mathbb{Q}))$ have rank at most $\operatorname{rk}(H_{2k+1}(F; \mathbb{Q}) \le 1$. This implies that $\operatorname{rk} H_k(F; \mathbb{Q}) \ge d-2$. However, $\operatorname{rk} H_n(F; \mathbb{Q}) \le 1$ by the preceding, and this contradicts the Poincaré duality for F^{n+k}.

If $2k \geq n$, then let a be the smallest positive integer such that $a(n-k-1) \geq 2k-n$. Using induction and the Wang sequence, we obtain that, for all natural $m \leq a$,

$$H_{n+k-m(n-k-1)}(F) = \cdots = H_{2k+2-m(n-k-1)}(F) = 0,$$

$$\operatorname{rk} H_{2k+1-m(n-k-1)}(F;\mathbb{Q}) = \operatorname{rk} H_{2k-m(n-k-1)}(F;\mathbb{Q}) = u \in \{0,1\}.$$

By letting $q = n+k-1, \ldots, n \geq 2k - a(n-k)$ we derive again that $\operatorname{rk} H_n(F;\mathbb{Q}) \leq 1$, whereas $\operatorname{rk} H_k(F;\mathbb{Q}) \geq d-2$, a contradiction. This shows that $M(\Gamma_0)$ cannot fiber over S^{n-k}.

3.5. *Proof of Theorem 2.4*

LEMMA 3.4. *In dimensions $(2n+1, n)$, let the graph Γ_0 be a tree consisting of one black vertex decorated by the fibered link SK_A that is adjacent to $d+1 \geq 2$ white vertices, one of which is decorated by the disk D^{n+1}, and the remaining white vertices are decorated by $S^1 \times D^n$. The gluing maps correspond to the decomposition of S^{n+1} into the union of two disks D^{n+1} along an equatorial sphere. Then $\partial X^{2n+1}(\Gamma_0)$ is diffeomorphic to $S^{n+1} \times S^n$, and the boundary fibration $\psi_{\Gamma_0} : \partial X^{2n+1}(\Gamma_0) \to S^n$ is trivial.*

Proof. Let assume that the component S_0^{n-1} of $K_A = L_{A^*}$ is spun. We consider the link $L_{\widetilde{A}} = \bigsqcup_{j=0}^{d} S_j^{n-1} \subset S^{2n-1}$ as the boundary of a holed disk. The spun component S_0^n inherits a longitude by spinning the one of S_0^{n-1}, whereas the other components $S^1 \times S_j^{n-1}$ inherit well-defined meridians by taking their product with S^1. We obtain $\partial X(\Gamma_0)$ by doing surgery on the link SK_A, namely gluing to $S^{2n} \setminus N(SK_A)$ the disjoint union $D_0^n \times S^n \bigsqcup_{j=1}^{d} S^1 \times S^{n-1} \times D_j^n$ such that the jth copy of $S^1 \times S^{n-1} \times D_j^n$ is glued along $\partial N(K_j^n)$ and $D_0^n \times S^n$ is glued along $\partial N(K_0^n)$. Surgery along $\partial N(K_j^n)$ identifies ∂D_0^n with the longitude of S_0^n and $S^1 \times \partial D_j^n$ with the meridian of $S^1 \times S_j^{n-1}$. By completing the last surgeries we restore the sphere $S^{2n} \setminus N(S_0)$, whereas surgery along the longitude of S_0^n yields $S^{n+1} \times S^n$. The fibration structure of ψ_{Γ_0} then corresponds to the projection onto S^n. □

We consider the manifold $M(\Gamma_0) = X(\Gamma_0) \bigcup_{\partial X(\Gamma_0)} S^{n+1} \times D^n$. First, $\pi_1(M(\Gamma_0)) = 0$, and further by Mayer–Vietoris $H_j(M(\Gamma_0)) = 0$ for $1 \leq j \leq 2n$, $j \notin \{n, n+1\}$, $H_n(M(\Gamma_0)) = H_{n+1}(M(\Gamma_0)) = \mathbb{Z}^{d+1}$. Thus $M(\Gamma_0)$ is $(n-1)$-connected.

Assume that $M(\Gamma_0)$ fibers over S^n with fiber F^{n+1}. Then the long exact sequence of the fibration shows that F^{n+1} must be $(n-1)$-connected and the induced map $\pi_n(S^n) \to \pi_{n-1}(F)$ is surjective, so that $\pi_{n-1}(F) \cong H_{n-1}(F)$ has rank at most 1. Further, the Wang sequence first yields

$$\to H_{2n}(F) \to H_{2n}(M(\Gamma_0)) \to H_n(F) \to H_{2n-1}(F^{n+1}) \to H_{2n-1}(M(\Gamma_0)) \to,$$

and thus $H_n(F^{n+1};\mathbb{Q}) = 0$ for $n \geq 3$, and second:

$$\begin{aligned}&\to 0 = H_1(F) \to H_n(F) \to H_n(M(\Gamma_0)) \to H_0(F) \to H_{n-1}(F)\\&\to H_{n-1}(M(\Gamma_0)) = 0,\end{aligned}$$

and hence $H_0(F)$ has rank at least $(d+1)$, which contradicts the $(n-1)$-connectedness of F. This proves the claim.

LEMMA 3.5. *Consider the dimensions* $(2n+1, n-k)$, $k \geq 1$, *and the graph* Γ_0 *with a single black vertex* v *decorated by* $L_{\Pi^k SL}$, *where* L *is an* $(n-1)$*-dimensional generalized Hopf link with* $d+1 \geq 5$ *components and a white vertex connected by an edge. The white vertex* w *is decorated by* $F_w = (\sharp\partial_{j=1}^d D^n \times S^{k+1})\sharp\partial(\sharp\partial_{j=1}^d S^k \times D^{n+1})$. *The gluing along* ∂F_w *is the one arising in surgery of the* $(k+1)$ *and* k*-dimensional core spheres, and the global fiber* F *is then diffeomorphic to* S^{n+k+1}. *Then* $\partial X(\Gamma_0)$ *is diffeomorphic to* $S^{n+k+1} \times S^{n-k-1}$, *and the boundary fibration* $\psi_{\Gamma_0} : \partial X(\Gamma_0) \to S^{n-k-1}$ *is trivial.*

Proof. If L is fibered and $L = \partial F^n$, where the fiber $F^n = D^n \setminus \bigsqcup_{j=1}^d D_j^n$, then SL is fibered, and its associated fiber is $SF^{n+1} = D^{n+1} \setminus \bigsqcup_{j=1}^d S^1 \times D_j^n$. We obtain SF^{n+1} from $D^{n+1} \setminus \bigsqcup_{j=1}^d D_j^{n+1}$ by adjoining for each boundary component ∂D_j^{n+1} one $(n-1)$-handle along a trivially embedded $S^{n-2} \subset S^n$. Therefore SF^{n+1} is obtained from D^{n+1} by first adding d handles of index n and further d handles of index $(n-1)$, as before. The attaching spheres bound disjoint disks, and hence SF^{n+1} is the boundary connected sum of d copies of the corresponding result for $d = 1$, the later being $D^2 \times S^{n-1} \setminus D_0^{n+1}$. Further, $L_{\Pi^k SL}$ is fibered with fiber $SF_{(k)}^{n+k+1} = SF^{n+1} \times D^k$, which has the same description of handles addition along D^{n+k+1} as before. We obtain $SF_{(k)}^{n+k+1} = \sharp\partial_{j=1}^d D^{2+k} \times S^{n-1}\sharp\partial_{j=1}^d S^n \times D^{k+1}$. Note that $L_{\Pi^k SL} = \sharp_{j=1}^d S^{1+k} \times S^{n-1}\sharp_{j=1}^d S^n \times S^k$ for $k \geq 1$, and in particular it is connected.

The global fiber of $X(\Gamma_0)$ is the union of $SF_{(k)}^{n+k+1}$ and F_w. The gluing is the connected sum of gluings occurring in the two spheres decompositions $D^{2+k} \times S^{n-1} \cup S^{1+k} \times D^n = S^{n+k+1}$ and $S^n \times D^{k+1} \cup D^{n+1} \times S^k = S^{n+k+1}$, and thus the global fiber is S^{n+k+1}.

The triviality of the S^{n+k+1}-fibration $\partial X(\Gamma_0) \to S^{n-k}$ follows by induction on k as before. □

Let now $k \geq 1$ and consider $M(\Gamma_0) = X(\Gamma_0) \bigcup_{\partial X(\Gamma)} S^{n+k+1} \times D^{n-k}$. It follows that $\pi_1(M(\Gamma_0)) = 0$ and by Mayer–Vietoris $H_j(M(\Gamma_0)) = 0$ for $1 \leq j \leq 2n-1$, $j \notin \{n-k, n, n+1, n+k+1\}$, whereas $H_n(M(\Gamma_0)) = H_{n+1}(M(\Gamma_0)) = \mathbb{Z}^d$, $H_{n-k}(M(\Gamma_0)) = H_{n+k+1}(M(\Gamma_0)) = \mathbb{Z}$. Thus $M(\Gamma_0)$ is $(n-k-1)$-connected. Assume that $M(\Gamma_0)$ fibers over S^{n-k} with fiber F^{n+k+1}. Then the long exact sequence of the fibration shows that F is connected and simply connected. The Wang exact sequence

$$\to H_q(F) \to H_q(M(\Gamma_0)) \to H_{q-n+k}(F) \to H_{q-1}(F) \to H_{q-1}(M(\Gamma_0)) \to$$

for $q = 2n, 2n-1, \ldots, n+k+3$ yields inductively:

$$H_{n+k}(F) = H_{n+k-1}(F) = \cdots = H_{2k+3}(F) = 0.$$

Further, by taking $q = n+k+2$ we find that

$$\begin{aligned}0 = H_{n+k+2}(M(\Gamma_0)) &\to H_{2k+2}(F) \to H_{n+k+1}(F) \to H_{n+k+1}(M(\Gamma_0))\\ &\to H_{2k+1}(F) \to H_{n+k}(F) = 0.\end{aligned}$$

Therefore

$$\operatorname{rk} H_{2k+2}(F;\mathbb{Q}) = \operatorname{rk} H_{2k+1}(F;\mathbb{Q}) = u \in \{0,1\}.$$

If $n+1 \geq 2k+3$, then we can consider $q = n+k, n+k-1, \ldots, n+1$ and derive

$$H_{2k}(F) = H_{2k-1}(F) = \cdots = H_{n+1}(F) = 0.$$

From the exactness of

$$0 = H_{n+1}(F) \to H_{n+1}(M(\Gamma_0)) = \mathbb{Z}^d \to H_{k+1}(F) \to H_n(F) \to H_n(M(\Gamma_0))$$

we obtain that $\operatorname{rk} H_{k+1}(F;\mathbb{Q}) \geq d$. However, $\operatorname{rk} H_n(F;\mathbb{Q}) \leq 1$ for $n \geq 2k+2$, which contradicts the Poincaré duality for F^{n+k+1}.

If $n = 2k+1$, then from the exact sequence

$$H_{2k+2}(F) \to H_{2k+2}(M(\Gamma_0)) \to H_{k+1}(F) \to H_{2k+1}(F) \to H_{2k+1}(M(\Gamma_0))$$

we derive that both the kernel and the cokernel of the map $H_{2k+2}(M(\Gamma_0);\mathbb{Q}) \to H_{k+1}(F;\mathbb{Q}))$ have rank at most $\operatorname{rk} H_{2k+2}(F;\mathbb{Q}) \leq 1$. This implies that $\operatorname{rk} H_{k+1}(F;\mathbb{Q}) \geq d-2$. However, $\operatorname{rk} H_n(F;\mathbb{Q}) \leq 1$ by the preceding, and this contradicts the Poincaré duality for F^{n+k+1}.

If $2k \geq n$, then let a be the smallest positive integer such that $a(n-k-1) \geq 2k+1-n$. By using induction and the Wang sequence we obtain, for all natural $m \leq a$,

$$H_{n+k-m(n-k-1)}(F) = \cdots = H_{2k+3-m(n-k-1)}(F) = 0,$$
$$\operatorname{rk} H_{2k+2-m(n-k-1)}(F;\mathbb{Q}) = \operatorname{rk} H_{2k+1-m(n-k-1)}(F;\mathbb{Q}) = u \in \{0,1\}.$$

By letting $q = n+k, \ldots, n+1 \geq 2k+1-a(n-k-1)$ we derive as before that $\operatorname{rk} H_n(F;\mathbb{Q}) \leq 1$, whereas $\operatorname{rk} H_{k+1}(F;\mathbb{Q}) \geq d-2$, a contradiction. This shows that $M(\Gamma_0)$ cannot fiber over S^{n-k}.

3.6. *Proof of Proposition 2.1*

Let $F : X \to Y$ be a differentiable map. We denote by $d_x F : T_x(X) \to T_{F(x)}(Y)$ its differential at $x \in X$. If $(G, \cdot)$ is a Lie group, then the left and right translations by $g \in G$ are the maps $L_g : G \to G$, $L_g(z) = gz$ and $R_g : G \to G$, $R_g(z) = zg$, respectively. Smooth maps $A : M \to G$ and $B : N \to G$ have a well-defined multiplication by setting $A \odot B : M \times N \longrightarrow G$ and $(A \odot B)(z,w) = A(z)B(w)$.

LEMMA 3.6. *Let M^m and N^n be smooth manifolds, and let $(G,\cdot)$ be a Lie group of dimension* $\dim G \leq \min(m,n)$. *For any smooth maps $A : M \to G$ and $B : N \to G$, we have*

$$C(A \odot B) \subseteq C(A) \times C(B). \tag{3}$$

Proof. We first need the following easy formula (for the particular case $M = N = G$ and $A = B = id_G$, see [19, p. 42]):

$$[d_{(x,y)}(A \odot B)](u, v) = (d_{B(y)}L_{A(x)})(d_y B(v)) + (d_{A(x)}R_{B(y)})(d_x A(u)) \quad (4)$$

for all $(u, v) \in T_x(M) \times T_y(N) \cong T_{(x,y)}(M \times N)$. This implies that the image of $d(A \odot B)]_{(x,y)}$ is the subspace

$$(d_{B(y)}L_{A(x)})(d_y B(T_y(N))) + (d_{A(x)}R_{B(y)})(d_x A(T_x(M))).$$

If $(x, y) \in (M \times N) \setminus (C(A) \times C(B))$, then either x is a regular point for A, or y is a regular point for B. By symmetry we may assume that x is a regular point of A and hence, by our assumptions on the dimensions, $(d_x A)(T_x(M)) = T_{B(y)}(G)$. Then, by the previous formula the range of $d(A \odot B)_{(x,y)}$ contains

$$(d_{A(x)}R_{B(y)})(d_x A)(T_x(M)) = (d_{A(x)}R_{B(y)})(T_{A(x)}(G)).$$

Since $R_{B(y)}$ is a diffeomorphism of G, the last vector space is the same as $T_{A(x)B(y)}(G) = T_{(A\odot B)(x,y)}(G)$. This shows that (x, y) is a regular point of $A \odot B$. $\square$

Therefore, if M^m and N^n and smooth manifolds and $(G, \cdot)$ is a Lie group of dimension $\dim G \leq \min(m, n)$ such that $\varphi(M, G)$ and $\varphi(N, G)$ are finite, then for any closed subgroup $H \subset G$, we have that $\varphi(M \times N, G/H)$ is finite and

$$\varphi(M \times N, G/H) \leq \varphi(M, G)\varphi(N, G). \quad (5)$$

The right-hand side inequalities follow from this inequality and the facts that $\varphi(S^4, S^3) = 2$ (see [1]) $\varphi(\sharp_s S^2 \times S^2, S^3) = 2s + 2$ (see [10; 11]).

The left-hand side inequalities follow from the fact that the manifolds considered do not fiber over S^3, by the same argument as in the proof of Proposition 2.1. In fact, we first have $\chi(S^4 \times \cdots \times S^4) = 2^m$. Further, the Euler characteristic is almost additive, namely $\chi(M\sharp N) = \chi(M) + \chi(N) - (1 + (-1)^n)$ for closed n-manifolds M and N. Therefore we can compute:

$$\chi((\sharp_{r_1} S^2 \times S^2) \times (\sharp_{r_2} S^2 \times S^2) \times \cdots \times (\sharp_{r_m} S^2 \times S^2)) = 2^m(r_1 + 1)\cdots(r_m + 1).$$

REMARK 3.2. If $f : M^m \longrightarrow S^{n+1}$ $(m \geq n + 1 \geq 3)$ is a smooth map with r critical points, then we can construct a map F with rs critical points by using fiber connected sums (see [11], proof of Prop. 3.1). The target manifolds are of the form $\sharp_g(S^1 \times S^n)$. Thus there are examples with finite $\varphi(M^{4m}, \sharp_g(S^1 \times S^2))$.

4. Signatures

4.1. Signature Definition

To compute the signature $\sigma(X(\Gamma))$, we need a description of the cup product $\cup$. Recall that, for a $2n$-manifold with boundary M, we have $H^n(M, \partial M) \cong H_n(M)$, whereas $H^n(M, \partial M) \cong H_n(M, \partial M)^*$ by the universal coefficients theorem. The signature of M is the one of the $(-1)^n$-symmetric bilinear form $\phi_M : H^n(M, \partial M) \times H^n(M, \partial M) \to \mathbb{R}$ given by

$$\phi_M(x, y) = \langle x \cup y, [M]\rangle.$$

The adjoint of this bilinear form is the homomorphism $\phi_M : H_n(M) \to H_n(M)^*$, which can be identified (see, e.g., [26]) with the inclusion induced morphism in the long exact sequence

$$H_n(\partial M) \to H_n(M) \overset{\phi_M}{\to} H_n(M, \partial M) \to H_{n-1}(\partial M).$$

It follows that $\ker \phi_M$ is precisely the image of $H_n(\partial M)$ into $H_n(M)$.

Our purpose is an explicit description of the bilinear form and its kernel in the case of $X(\Gamma)$. Unless explicitly stated, we consider here $k = 0$, the last subsection being concerned with the modifications to the present arguments for $k \geq 1$.

4.2. Notation

Assume that we have a graph Γ with vertices decorated by generalized fibered $(n-1)$-links L_v such that (S^{2n-1}, L_v) are Neuwirth–Stallings pairs. We assume that $n \geq 3$.

On one hand, L_v are links of the form K_{A_v} for some unimodular integer matrices A_v. The linking matrices in the canonical framing A_v^* of K_{A_v} are given by Lemma 3.1. For simplicity, we assume that only white vertices, which are labeled by disks, can occur, and their effect is to cap off the boundary components. In particular, we can realize trees Γ whose leaves are white vertices of this kind.

If v is a vertex of Γ, then we denote by Γ_v the set of edges issued from v and by $E(\Gamma)$ the set of all edges of Γ. The link L_v has $d(v)$ components indexed by the edges in Γ_v.

Let $E_v = S^{2n-1} - N(L_v)$ denote the link complement endowed with its canonical boundary trivialization. Thus $\partial E_v = \bigsqcup_{e \in \Gamma_v} (S^{n-1} \times S^{n-1})_e$, boundary components being indexed by the edges e in Γ_v.

4.3. Homology of $X(\Gamma)$

We have the map $f_\Gamma : X(\Gamma) \to D^n$ with one critical value and one singular fiber $V^n(\Gamma) = f_\Gamma^{-1}(0)$. The retraction $D^n \to \{0\}$ lifts to a deformation retraction $X(\Gamma) \to V^n(\Gamma)$, so that

$$H_*(X(\Gamma)) \cong H_*(V(\Gamma)).$$

On the other hand, the singular fiber $V^n(\Gamma)$ is obtained from the regular fiber F^n by crushing vanishing cycles to points. Vanishing cycles on the local fiber F_v correspond to the attaching spheres described before. Specifically, these are $(d(v) - 1)$ embedded $(n-1)$-spheres carrying the homology of $F^n = S^n \setminus \bigsqcup_{i=1}^{d(v)} D_i^n$. The contribution of a white vertex v to $V_n(\Gamma)$ is just the fiber F_v, which is a disk.

We can also obtain $V^n(\Gamma)$ by gluing along the pattern Γ the local singular fibers V_v that are cones along the boundary spheres in ∂F_v. Thus each edge e of Γ gives raise to a topological sphere $S_e^n \subset V_v$ obtained by suspending the sphere $S_e^{n-1} \subset \partial V_v \cong \partial F_v$ associated with the edge e at two points corresponding to the two vertices of e. Gluing together all the spheres S_e^n by identifying the cone points corresponding to the same vertex of Γ, we obtain $V^n(\Gamma)$. It follows that

$$H_*(V^n(\Gamma)) \cong \begin{cases} \bigoplus_{e\in E(\Gamma)} H_*(S^n_e) & \text{for } * \neq 1, \\ H_1(\Gamma) & \text{for } * = 1. \end{cases}$$

In particular, we have the following:

LEMMA 4.1. *There is a preferred basis $\{\beta_e, e \in E(\Gamma)\}$ of $H_n(V(\Gamma))$ given by the n-cycles S^n_e.*

Note that the links L_v are naturally oriented, as they bound the local fiber F_v. This induces a well-defined orientation of the n-cycle representing β_e.

Recall that F is diffeomorphic to $\sharp_1^g S^1 \times S^{n-1}$, where g is the rank of $H_1(\Gamma)$, and thus $H_2(F) \cong H_{n-2}(F) = 0$ if $n \neq 3$.

Now the boundary $E = \partial X(\Gamma)$ is endowed with a fibration over $S^{n-1} = \partial D^n$ with fiber F. The Wang sequence in homology with rational coefficients first gives us

$$H_{n+1}(F) \to H_{n+1}(E) \to H_2(F),$$

so that $H_{n+1}(E) = 0$ and by duality $H_{n-2}(E) = 0$. Further, the Wang sequence reads

$$0 \to H_n(F) \to H_n(E) \to H_1(F) \to H_{n-1}(F) \to H_{n-1}(E) \to H_0(F) \to 0.$$

4.4. The Cup Product Bilinear Form

Let A^*_v denote the $d(v) \times d(v)$ linking matrix in the canonical framing of the link L_v. We define the matrix A^*_Γ indexed by the set of edges $E(\Gamma)$:

$$A^*(\Gamma)_{ef} = \begin{cases} (A^*_v)_{ef} & \text{if } e \cap f = v, \\ (A^*_v)_{ef} + (A^*_w)_{ef} & \text{if } e \cap f = \{v, w\}, \\ 0 & \text{if } e \cap f = \emptyset. \end{cases}$$

LEMMA 4.2. *The cup product bilinear form $\phi_{X(\Gamma)}$ is expressed by the matrix $A^*(\Gamma)$ in the basis $\{\beta_e, e \in E(\Gamma)\}$ of $H_n(V(\Gamma))$.*

Proof. The inclusions $F \to E$, $E \to X(\Gamma)$ induce a morphism $H_n(F) \to H_n(X(\Gamma))$ whose image lies in the kernel of $\varphi_{X(\Gamma)}$. If we identify $H_n(X(\Gamma))$ to $H_n(V(\Gamma))$, then this has a simple description. Specifically, the fundamental class $[F]$ of F is sent into $\sum_{e\in E(\Gamma)} \beta_e$. By the previous discussion this element belongs to the kernel of $\phi_{X(\Gamma)}$.

Consider now two cycles β_{e_1} and β_{e_2} in $H_n(X(\Gamma))$. If $e_1 \cap e_2 = \emptyset$, then the intersection of these two cycles is trivial. Therefore

$$\phi_{X(\Gamma)}(\beta_{e_1}, \beta_{e_2}) = 0 \quad \text{if } e_1 \cap e_2 = \emptyset. \tag{6}$$

Recall that $V^n_v \subset D^{2n}_v$ is a cone over $L_v = \bigsqcup_{e\in\Gamma_v} S^{n-1}_e$. Let $e_i = vw_i$ with distinct w_i. There are two n-cycles in D^{2n}_v that bound $S^{n-1}_{e_1}$ and $S^{n-1}_{e_2}$; after putting them in general position, their algebraic intersection number is $\mathrm{lk}(S^{n-1}_{e_1}, S^{n-1}_{e_2})$ (see, e.g., [22], 5.D, Ex. 9, p. 134 for $n = 2$). Moreover, $S^{n-1}_{e_1}$ and $S^{n-1}_{e_2}$ also bound disjoint n-cycles in $D^{2n}_{w_1}$ and $D^{2n}_{w_2}$, respectively. Therefore we can perturb β_{e_1} and β_{e_2} to have algebraic intersection number $\mathrm{lk}(S^{n-1}_{e_1}, S^{n-1}_{e_2})$. Since this algebraic intersection number is an invariant of their homology classes, we derive:

$$\phi_{X(\Gamma)}(\beta_{e_1}, \beta_{e_2}) = \mathrm{lk}(S^{n-1}_{e_1}, S^{n-1}_{e_2}) = (A^*_v)_{e_1 e_2}. \tag{7}$$

Note that A^*_v is the $d(v) \times d(v)$ linking matrix in the canonical framing of the link $L_v \subset \partial D_v$.

Moreover, if both edges $e \neq f$ have the same endpoints $v \neq w$, then a similar argument shows that

$$\phi_{X(\Gamma)}(\beta_e, \beta_f) = (A^*_v)_{ef} + (A^*_w)_{ef}. \tag{8}$$

Suppose further that $e_1 = e_2$. If n is odd, then the antisymmetry of the bilinear form yields

$$\phi_{X(\Gamma)}(\beta_e, \beta_e) = 0. \tag{9}$$

If n is even, then as $\sum_{e \in E(\Gamma)} \beta_e$ lies in the kernel of $\varphi_{X(\Gamma)}$, we derive:

$$\phi_{X(\Gamma)}\Bigg(\sum_{e \in E(\Gamma)} \beta_e, \beta_{e_0}\Bigg) = \sum_{e \cap e_0 \neq \emptyset} \phi_{X(\Gamma)}(\beta_e, \beta_{e_0}) = 0.$$

Writing $e_0 = vw$, we derive

$$\begin{aligned} \phi_{X(\Gamma)}(\beta_{e_0}, \beta_{e_0}) &= -\sum_{e \in \Gamma_v \setminus \{e_0\}} (A^*_v)_{ee_0} - \sum_{e \in \Gamma_w \setminus \{e_0\}} (A^*_w)_{e_0 e} \\ &= (A^*_v)_{e_0 e_0} + (A^*_w)_{e_0 e_0}. \end{aligned} \tag{10}$$

□

4.5. *Geometric Interpretation of the Kernel of* $\phi_{X(\Gamma)}$

By induction on the number of boundary components we find:

$$H_i(E_v) = \begin{cases} 0 & \text{if } i \notin \{0, n-1\}, \\ \mathbb{Q}^{d(v)} & \text{if } i = n-1. \end{cases}$$

This makes sense also when v is a white vertex and hence $d(v) = 1$.

We can represent classes in $H_{n-1}(E_v)$ by means of *meridian* $(n-1)$-spheres on ∂E_v, which are represented as $\{p\} \times \partial D^n \subset L_v \times D^n \subset S^{2n-1}$ after its identification with a regular neighborhood $N(L_v)$ of L_v in S^{2n-1} given by the trivialization. We have then a preferred basis of $H_{n-1}(E_v) = \mathbb{Q}\langle \mu_e, e \in \Gamma_v \rangle$.

To compute $H_*(E)$, we will use a refined version of the Mayer–Vietoris exact sequence. If we have an open covering U_i of E such that $U_i \cap U_j \cap U_k = \emptyset$ for distinct i, j, k, then the following sequence is exact:

$$\to \bigoplus_{i<j} H_k(U_i \cap U_j) \to \bigoplus_i H_k(U_i) \to H_k(E) \to \bigoplus_{i<j} H_{k-1}(U_i \cap U_j) \to .$$

By taking U_i to be small neighborhoods of E_v, we derive that

$$H_n(E) = \ker\Bigg(\bigoplus_{e\in E(\Gamma)} H_{n-1}((S^{n-1}\times S^{n-1})_e) \to \bigoplus_v H_{n-1}(E_v)\Bigg), \tag{11}$$

where the map $H_{n-1}((S^{n-1}\times S^{n-1})_e) \to \bigoplus_v H_{n-1}(E_v)$ is the morphism induced by inclusion if v is a vertex of e and zero otherwise.

Let us give explicit cycles for classes in $H_n(E)$. An $(n-1)$-cycle $(z_e) \in \bigoplus_{e\in\Gamma_v} H_{n-1}((S^{n-1}\times S^{n-1})_e)$ is called *bounding* if its image by the inclusion-induced morphism in $H_{n-1}(E_v)$ vanishes. Thus $H_n(E)$ is identified with the space of cycles $(z_e)_{e\in E(\Gamma)}$ that restrict to bounding $(n-1)$-cycles on every Γ_v. There exists an n-cycle Z_v in E_v such that $\partial Z_v = \sum_{e\in\Gamma_v} z_e$ in E_v. Therefore the union $Z = \bigcup_v Z_v$ is an n-cycle in E representing the class $(z_e)_{e\in E(\Gamma)}$.

Recall that $(S^{n-1}\times S^{n-1})_e$ is endowed with a canonical trivialization issued from the open book structure of L_v, namely it is foliated by the $(n-1)$-spheres arising as intersections between ∂E_v and the local fibers. This provides a family of isotopic $(n-1)$ spheres to be called *preferred longitudinal* spheres, in the homology class of the canonical framing. Now

$$H_{n-1}(\partial E_v) = \mathbb{Q}\langle \lambda_e, \mu_e, e\in\Gamma_v\rangle$$

has a basis consisting in classes of the form λ_e represented by the preferred longitudinal $(n-1)$-sphere in $(S^{n-1}\times S^{n-1})_e$ and the classes of meridian spheres μ_e. We want to describe the map

$$i_{e,v} : H_{n-1}((S^{n-1}\times S^{n-1})_e) \to \bigoplus_v H_{n-1}(E_v)$$

in the basis defined. By the definition of the meridian classes

$$i_{e,v}(\mu_e) = \mu_e. \tag{12}$$

Recall that by Hurewicz there exists an isomorphism (for $n \geq 3$)

$$\pi_{n-1}(E_v) \to H_{n-1}(E_v;\mathbb{Z}) \to \mathbb{Z}^{\Gamma_v}.$$

The image of the class of an embedded sphere S^{n-1} in E_v is given by the vector $(\mathrm{lk}(S^{n-1}, S_e^{n-1}))_{e\in\Gamma_v}$. Note that the image of μ_e is the vector $(\delta_{ef})_{f\in\Gamma_v}$. Further, the preferred longitudinal spheres λ_e and λ_f are isotopic in E_v to S_e^{n-1} and S_f^{n-1}, respectively, so that the linking number between the corresponding embedded spheres is

$$\mathrm{lk}(\lambda_e, S_f^{n-1}) = (A_v^*)_{ef}.$$

Since the union of all preferred longitudinal spheres λ_e for $e\in\Gamma_v$ bounds a copy of the local fiber F_v, we have $\sum_{e\in\Gamma_f} i_{e,v}(\lambda_e) = 0$, and hence

$$\sum_{e\in\Gamma_v} \mathrm{lk}(\lambda_e, S_f^{n-1}) = 0.$$

This yields

$$\mathrm{lk}(\lambda_e, S_e^{n-1}) = -\sum_{f\neq e, f\in\Gamma_v} (A_v^*)_{ef} = (A_v^*)_{ee}.$$

This proves that the image of λ_e by the Hurewicz isomorphism is the vector $((A_v^*)_{ef})_{f\in\Gamma_v}$. Therefore

$$i_{e,v}(\lambda_e) = \sum_{f\in\Gamma_v} (A_v^*)_{ef}\mu_f. \tag{13}$$

This identifies $\ker i_{e,v}$ with the kernel of the linear map expressed by the matrix $(\mathbf{1}|A_v^*)$ consisting of two square blocks in the basis. Note that A_v^* is not of maximal rank.

The description of the map $H_n(E) \to H_n(V(\Gamma))$ is as follows. The retraction r respects the decomposition of $E = \bigcup_{v\in\Gamma} E_v$ and $V(\Gamma) = \bigcup_{v\in\Gamma} V_v$, and it induces a commutative diagram:

$$\begin{tikzcd}
H_n(E) \arrow[r] \arrow[d] & H_n(V(\Gamma)) \arrow[d] \\
\bigoplus_{v\in\Gamma} H_{n-1}(\partial E_v) \arrow[r] \arrow[d] & \bigoplus_{v\in\Gamma} H_{n-1}(\partial V_v) \arrow[d] \\
\bigoplus_{v\in\Gamma} H_{n-1}(E_v) \arrow[r] & \bigoplus_{v\in\Gamma} H_{n-1}(V_v)
\end{tikzcd}$$

Suppose that the class in $H_n(E)$ is given by the vector $(z_e)_{e\in E(\Gamma)}$. Then the retraction $r : E \to V(\Gamma)$ acts at the level of ∂E_v as the parallel transport in the trivial boundary fibration toward ∂V_v. This means that the image of $z_e = n_e\lambda_e + m_e\mu_e$ in $H_{n-1}(\partial V_v)$ is $n_e\lambda_e$. Therefore we obtain

$$\ker\phi_{X(\Gamma)} = \Bigg\{ \sum_{e\in E(\Gamma)} n_e\beta_e;\ \exists m_e \in \mathbb{Z},\ \text{such that } (n_e\lambda_e + m_e\mu_e)_e \in \ker i_{e,v}, \\ \forall e \in E(\Gamma), \forall v \in e \Bigg\}. \tag{14}$$

Now $(n_e\lambda_e + m_e\mu_e)_e \in \ker i_{e,v}$ if and only if

$$m_f = -\sum_{e\in\Gamma_v} n_e(A_v^*)_{ef}.$$

Let $e = vw$. When computing $i_{e,w}$, we have to note a change in orientation since E_v and E_w induce different orientations on their common boundary. We find that $(n_e\lambda_e + m_e\mu_e)_e \in \ker i_{e,w}$ if and only if

$$m_f = \sum_{e\in\Gamma_w} n_e(A_w^*)_{ef}.$$

Therefore

$$\ker\phi_{X(\Gamma)} = \Bigg\{ \sum_{e\in E(\Gamma)} n_e\beta_e;\ \sum_{e\in\Gamma_v} n_e(A_v^*)_{ef} + \sum_{e\in\Gamma_w} n_e(A_w^*)_{ef} = 0, \forall f \in \Gamma_v\cap\Gamma_w \Bigg\}.$$

This coincides indeed with the left kernel of the linear map given by $A^*(\Gamma)$.

When Γ is a tree, $H_1(F)=0$, and the inclusion-induced map $H_n(F)\to H_n(E)$ is an isomorphism. In this case the map $H_n(E)\to H_n(X)$ can be identified with the inclusion-induced map $H_n(F)\to H_n(X)$. After the identification $H_n(X)\xrightarrow{\cong} H_n(V^n(\Gamma))$, the previous map is the same as that induced by the retraction $H_n(F)\to H_n(V^n(\Gamma))$. In this case the kernel is one-dimensional by the previous Wang sequence, and hence

$$\ker\phi_{X(\Gamma)}=\mathbb{Q}\Big\langle \sum_{e\in E(\Gamma)}\beta_e\Big\rangle. \tag{15}$$

4.6. *Signature Computation When* $1\le k\le n-2$

For simplicity, we denote $L_{\Pi^k\psi_L}$ as $L_{\Pi^k L}$.

We only consider the following cases:

(1) the graph Γ consists of two black vertices v, w and an edge. The decoration is given by links of the form $L_v=L_{\Pi^k L_1}$ and $L_w=L_{\Pi^k L_2}$, obtained by the procedure of Section 2.5 from the $(n-1)$-dimensional generalized Hopf links L_1 and L_2 in S^{2n-1};

(2) the graph Γ_0 consists of one black vertex v and a white vertex w connected by an edge. The white vertex w is decorated by $F_w=\sharp_{\partial d-1}D^n\times S^k$. The gluing along ∂F_v is the identity map of $\sharp_{\partial d-1}S^{n-1}\times S^k$, and the global fiber F is then diffeomorphic to S^{n+k}.

We define the matrices A^*_Γ and $A^*_{\Gamma_0}$ indexed by the set of edges $\{1,2,\ldots,d\}$ (and not by the edges of the corresponding graphs):

$$(A^*_\Gamma)_{ef}=(A^*_v)_{ef}+(A^*_w)_{ef},\qquad (A^*_{\Gamma_0})_{ef}=(A^*_v)_{ef}.$$

LEMMA 4.3. *The cup product bilinear forms $\phi_{X(\Gamma)}$ and $\phi_{X(\Gamma_0)}$ are expressed by the matrices A^*_Γ and $A^*_{\Gamma_0}$, respectively, in their basis $\{\beta_e, e\in\{1,2,\ldots,d\}\}$.*

Proof. If $E_v=S^{2n-1}\setminus N(L_v)$, then $\partial X(\Gamma)$, which is also denoted $E=E_v\cup E_w$, is obtained by gluing together the link complements in a way that respects the trivialization on the boundary.

The generic fiber F of the map $X(\Gamma)\to D^{n-k}$ is the union of the local fibers $F_v\cup F_w$ along the link $L_v\cong L_w\cong \sharp_{d-1}S^{n-1}\times S^k$. Each local fiber F_v or F_w is diffeomorphic to $\sharp_{\partial d-1}S^{n-1}\times D^{k+1}$. The homeomorphism between F_w and $(S^n\setminus\bigsqcup_{i=1}^d D^n_i)\times[0,1]^k$ also provides an embedding of the $(n-1)$-dimensional link $L_1=\bigsqcup_{e=1}^d S^{n-1}_e\subset F_v$. Here the subscript $e\in\{1,2,\ldots,d\}$ corresponds to the numbering of the spheres in the link as in the previous section. Their homology classes $\{\beta_e, 1\le e\le d\}$ generate $H_{n-1}(F)$, and according to Mayer–Vietoris, we have:

$$H_*(F;\mathbb{Q})=\begin{cases}\mathbb{Q}^{d-1} & \text{if } *\in\{k+1,n-1\},\\ \mathbb{Q} & \text{if } *\in\{0,n+k\},\\ 0 & \text{otherwise,}\end{cases}$$

where, by notation abuse for $(n, k) = (3, 1)$, the set $\{k + 1, n - 1\}$ reduces to a singleton $\{2\}$. Therefore $H_{n-1}(F; \mathbb{Q})$ is identified with the quotient

$$H_{n-1}(F; \mathbb{Q}) = \mathbb{Q}\langle \beta_e, 1 \le e \le d\rangle \Big/ \left(\sum_{e=1}^{d} \beta_e = 0 \right).$$

As in the case $k = 0$, the block $X(\Gamma)$ retracts onto the singular fiber $V(\Gamma)$, which is the suspension $\Sigma(\sharp_{d-1} S^{n-1} \times S^k)$ of the link L_v, and therefore:

$$H_*(X(\Gamma); \mathbb{Q}) = \begin{cases} \mathbb{Q}^{d-1} & \text{if } * \in \{k+1, n\}, \\ \mathbb{Q} & \text{if } * \in \{0, n+k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Similar computations also provide:

$$H_*(X(\Gamma_0); \mathbb{Q}) = \begin{cases} \mathbb{Q}^{d-1} & \text{if } * = n, \\ \mathbb{Q} & \text{if } * = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that, for $k = 0$, the homology of $H_n(\Sigma(\sharp_{d-1} S^{n-1} \times S^k); \mathbb{Q}) = \mathbb{Q}^d$ according with the previous section.

The boundary fibration $\partial E_v \to S^{n-k-1}$, which is the restriction of the corresponding fibration of E_v, extends over D^{n-k}, and hence $\partial E_v = (\sharp_{d-1} S^{n-1} \times S^k) \times S^{n-k-1}$. Denote by $\lambda_e \in H_{n-1}(\partial E_v)$ the preferred longitudinal classes of the cycles $S_e^{n-1} \times \{pt\} \times \{pt\}$ obtained by pushing the cycles S_e^{n-1} along a direction of the local fiber. We also defines the meridian classes $\mu_e \in H_{n-1}(\partial E_v)$ as being the classes of the cycles $\{pt\} \times S_e^k \times S^{n-k-1}$, where S_e^k is a k-cycle linking once S_e^{n-1} and trivially the others S_f^{n-1} for $f \ne e$ corresponding to the boundary of the fiber disk D^k of Π^k. It is immediate that $H_{n-1}(\partial E_v) = \mathbb{Q}^{2(d-1)}$ and a specific basis is deduced from the generators system

$$H_{n-1}(\partial E_v) = \mathbb{Q}\langle \mu_e, \lambda_e; 1 \le e \le d\rangle \Big/ \left(\sum_{e=1}^{d} \mu_e = \sum_{e=1}^{d} \lambda_e = 0 \right).$$

We denote by the same symbols μ_e and λ_e the images of these classes in the homology of E_v. Using Mayer–Vietoris, we deduce that

$$H_n(E_v) = \begin{cases} \mathbb{Q}^{d-1} & \text{if } k = n-2, \\ 0 & \text{otherwise,} \end{cases}$$

and a description in the nontrivial case $k = n - 2$ is provided by the quotient

$$H_n(E_v) = \mathbb{Q}\langle \lambda_e \times S^1, 1 \le e \le d\rangle \Big/ \left(\sum_{e=1}^{d} \lambda_e \times S^1 = 0 \right).$$

Further, we have

$$H_{n-1}(E_v) = \mathbb{Q}\langle \mu_e, 1 \le e \le d\rangle \Big/ \left(\sum_{e=1}^{d} \mu_e = 0 \right).$$

Again by Mayer–Vietoris we obtain that the boundary map induces an isomorphism for $k \neq n-2$:

$$H_n(E) = \ker(H_{n-1}(\partial E_v) \to H_{n-1}(E_v) \oplus H_{n-1}(E_w)).$$

However, this also holds when $k = n-2$. Indeed, the map $H_n(E_v) \to H_n(V(\Gamma))$ factors through $H_n(V_v) = 0$. To understand $H_n(E)$, we need to describe the map $i_v : H_{n-1}(\partial E_v) \to H_{n-1}(E_v)$. It is clear that

$$i_v(\mu_e) = \mu_e.$$

The inclusion map $E_v \subset S^{2n-1} \setminus L_1$ induces a homomorphism $H_{n-1}(E_v) \to H_{n-1}(S^{2n-1} \setminus L_1) \cong \mathbb{Q}^d$. Its image is the subspace $\mathbb{Q}^{d-1}$ of vectors whose sum vanishes. By the computations from the previous section we have

$$i_v(\lambda_e) = \sum_{f=1}^{d} (A_v^*)_{ef} \mu_f.$$

Note that the right-hand is well-defined in $H_{n-1}(E_v)$.

The description of the map $H_n(E) \to H_n(V(\Gamma))$ is similar to the case $k = 0$. The retraction r respects the decomposition of $E = E_v \cup E_w$ and $V(\Gamma) = V_v \cup V_w$ and induces a commutative diagram

$$\begin{tikzcd}
H_n(E) \arrow[r] \arrow[d] & H_n(V(\Gamma)) \arrow[d] \\
H_{n-1}(\partial E_v) \oplus H_{n-1}(\partial E_w) \arrow[r] \arrow[d] & H_{n-1}(\partial V_v) \oplus H_{n-1}(\partial V_w) \arrow[d] \\
H_{n-1}(E_v) \oplus H_{n-1}(E_w) \arrow[r] & H_{n-1}(V_v) \oplus H_{n-1}(V_w)
\end{tikzcd}$$

Suppose that the class in $H_n(E)$ is given by the vector $(z_v, z_w) \in H_{n-1}(E_v) \oplus H_{n-1}(E_w)$. Then the retraction $r : E \to V(\Gamma)$ acts at the level of ∂E_v as the parallel transport in the trivial boundary fibration toward ∂V_v. This means that the image of $z_v = \sum_{e=1}^{d} n_e \lambda_e + m_e \mu_e$ in $H_{n-1}(\partial V_v)$ is $\sum_{e=1}^{d} n_e \lambda_e$.

Further, the arguments of Section 4.4 carry over without essential changes. □

4.7. *Indefinite Bilinear Forms*

Since A has zero diagonal, the associated bilinear form is indefinite. Now, the classification of indefinite symmetric unimodular bilinear forms over $\mathbb{Z}$ is known up to equivalence. Recall that bilinear forms associated with matrices A and B are equivalent if there exists an invertible integral matrix M such that $A = MBM^{\perp}$, where $M^{\perp}$ denote its transpose. Then, any indefinite unimodular symmetric A is equivalent to $pE_8 \oplus qH$ for some $p, q \in \mathbb{Z}_+$, $q \geq 1$ (see [20], II.5.3). Here E_8 denotes the Cartan matrix for the unimodular E_8 lattice, and H is the metabolic matrix:

$$E_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\{e_{i,s}, i = 1, 8\}$ and $\{f_{1,t}, f_{2,t}\}$ be bases for the inner product space associated with the sth factor E_8 and the tth factor H, respectively. The unimodular change of basis

$$e'_{i,s} = e_{i,s} + f_{1,1} - f_{2,1}, \quad f'_{j,t} = f_{j,t}, 1 \le i \le 8, 1 \le s \le p, 1 \le j \le 2, 1 \le t \le q,$$

shows that $pE_8 \oplus qH$ is equivalent to $pE'_8 \oplus qH$, where

$$E'_8 = \begin{pmatrix} 0 & -1 & -2 & -2 & -2 & -2 & -2 & -2 \\ -1 & 0 & -1 & -2 & -2 & -2 & -2 & -2 \\ -2 & 1 & 0 & -1 & -2 & -2 & -2 & -2 \\ -2 & -2 & -1 & 0 & -1 & -2 & -2 & -2 \\ -2 & -2 & -2 & -1 & 0 & -1 & -2 & -1 \\ -2 & -2 & -2 & -2 & -1 & 0 & -1 & -2 \\ -2 & -2 & -2 & -2 & -2 & -1 & 0 & -2 \\ -2 & -2 & -2 & -2 & -1 & -2 & -2 & 0 \end{pmatrix}.$$

Therefore unimodular symmetric matrices with zeros on the diagonal are equivalent to $pE'_8 \oplus qH$, $q \ge 1$.

4.8. *Proof of Theorem 2.5*

If Γ is a tree and $k = 0$, then the generic global fiber is $F = S^n$. Assume that K_A is fibered, for example, $A = \bigoplus_{\theta_{2n-1}} A_0$ for some unimodular symmetric A_0. The matrix associated with $\phi_{X(\Gamma)}$ is the matrix A^*that has the nonsingular minor A^{-1}. Since A is symmetric and unimodular $AA^{-1}A^{\perp} = A$, so that A^{-1} is equivalent over $\mathbb{Z}$ to A. We know that A is equivalent to $pE'_8 \oplus qH$ for some $q \ge 1$ and $p \ge 0$. We derive that

$$\sigma(X(\Gamma)) = 8p.$$

Then it suffices to consider $A_0 = p_0E'_8 \oplus q_0H$ with $p_0, q_0 \ge 1$ to obtain blocks $X(\Gamma)$ of nonzero signature.

Novikov's additivity of the signature shows that the resulting manifold $M^{2n}(\Gamma)$ has signature $8p \ne 0$. In particular, $\varphi(M^{2n}(\Gamma), S^n) = 1$ for even n, thereby obtaining another proof of Theorem 2.3.

If $k \ge 1$, then we consider either of the graphs Γ or Γ_0 from Section 4.6. The matrix associated with $\phi_{X(\Gamma)}$ is A^*_{Γ}. Then, as before, we take $A_v = A_w$ equivalent to $pE'_8 \oplus qH$ to obtain blocks $X(\Gamma)$ and $X(\Gamma_0)$ of signature $8p \ne 0$.

Then we can consider $M(\Gamma) = X(\Gamma) \cup S^{n+k} \times D^{n-k}$. By Novikov's additivity of signatures $\sigma(M(\Gamma)) = \sigma(X(\Gamma))$. By taking $A = pE_8' \oplus qH$ with $p, q \geq 1$ we obtain $\sigma(M(\Gamma)) = 8pm \neq 0$. Note that if $n - k$ is even, then $M(\Gamma)$ cannot fiber over S^{n-k} by the signature criterion.

REMARK 4.1. Observe that gluing several such blocks $X(\Gamma_i)$ is only possible when the boundary fibrations $\partial X(\Gamma_i)$ are cobounding. The examples obtained in the case where n is odd are doubles of such blocks, namely obtained by gluing $X(\Gamma)$ and $\overline{X(\Gamma)}$. Doubles of oriented manifolds are bounding, and therefore their signatures vanish.

REMARK 4.2. All examples obtained by this procedure have signature divisible by $8\theta_{2n-1}$. We can drop the factor θ_{2n-1} if we work instead of the smooth category in the topological category.

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