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Abstract

The group of $\mathcal{C}^1$-diffeomorphisms of any sparse Cantor subset of a manifold is countable and discrete (possibly trivial). Thompson's groups come out of this construction when we consider central ternary Cantor subsets of an interval. Brin's higher-dimensional generalizations nV of Thompson's group V arise when we consider products of central ternary Cantor sets. We derive that the $\mathcal{C}^2$-smooth mapping class group of a sparse Cantor sphere pair is a discrete countable group and produce this way versions of the braided Thompson groups.

1. Introduction

Differentiable structures on Cantor sets have first been considered by Sullivan in [Sul88]. Our aim is to consider groups of diffeomorphisms of Cantor sets, mapping class groups of Cantor punctured spheres and their relations with Thompson-like groups. In particular, the usual Thompson groups (see [CFP96]) can be retrieved as diffeomorphism groups of Cantor subsets of suitable spaces (a line, a circle or a 2-sphere).

Let M be a compact manifold and $C \subset M$ be a Cantor set, namely a *compact totally disconnected subset without isolated points*. Any two Cantor sets are homeomorphic as topological spaces. But if M has dimension $m \geqslant 3$ there exist Cantor sets C_1 and C_2 embedded into M so that there is no ambient homeomorphism of M carrying C_1 into C_2. One says that C_1 and C_2 are not *topologically equivalent* Cantor set embeddings.

A Cantor subset of $\mathbb{R}^m$ is *tame* if there is a homeomorphism of $\mathbb{R}^m$ which sends it within a coordinates axis. All Cantor sets in $\mathbb{R}^m$, for $m \leqslant 2$ are tame, but there exist uncountably many *wild* (i.e. not tame) Cantor sets in $\mathbb{R}^m$, for every $m \geqslant 3$ (see [Bla51]).

One defines similarly *smooth equivalence* and *smoothly tame* Cantor sets. The analogous story for diffeomorphisms is already interesting for $m = 1$, as Cantor subsets of $\mathbb{R}$ might be differentiably non-equivalent. Our main concern is the image of the group of diffeomorphisms of M which preserve a Cantor set C into the automorphisms group of C. Under fairly general conditions we are able to prove that this is a countable group, thereby providing an interesting class of discrete groups. For Cantor sets obtained from a topological iterated function system the associated groups are non-trivial, while for many self-similar Cantor sets these are versions of Thompson's groups.

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A. General countability statements

1.1 Pure mapping class groups

DEFINITION 1. Let M be a compact orientable manifold and $C \subset M$ a Cantor subset. We denote by $\mathrm{Diff}^k(M, C)$ the group of diffeomorphisms of class $\mathcal{C}^k$ of M sending C to itself, by $\mathrm{Diff}^{k,+}(M, C)$ the subgroup of orientation preserving diffeomorphisms and by $P\mathrm{Diff}^{k,+}(M, C)$ the subgroup of *pure* orientation preserving diffeomorphisms, i.e. pointwise preserving C.

The $\mathcal{C}^k$-*mapping class group* $\mathcal{M}^{k,+}(M, C)$ is the group $\pi_0(\mathrm{Diff}^{k,+}(M, C))$ of $\mathcal{C}^k$-isotopy classes of orientation preserving diffeomorphisms rel C (i.e. which are the identity on C) of class $\mathcal{C}^k$. The *pure $\mathcal{C}^k$-mapping class group* $P\mathcal{M}^{k,+}(M, C)$ is the group $\pi_0(P\mathrm{Diff}^{k,+}(M, C))$ of $\mathcal{C}^k$-isotopy classes of pure orientation preserving $\mathcal{C}^k$-diffeomorphisms rel C.

In a similar vein but a different context, the group of homeomorphisms $\mathrm{Diff}^0(M, A)$ associated to a manifold M and a countable dense set $A \subset M$ was studied recently in [DM10]. The authors proved there that $\mathrm{Diff}^0(M, A)$ is either isomorphic to a countably infinite product of copies of $\mathbb{Q}$, when M is 1-dimensional, or the Erdös subgroup of l^2 elements, otherwise. In the present setting, when A is closed and the smoothness is at least $\mathcal{C}^1$ the situation is fundamentally different.

If we write $C = \bigcap_{j=1}^{\infty} C_j$, where each C_j is a compact submanifold of M and $C_{j+1} \subset \mathrm{int}(C_j)$ for all j, then the sequence $\{C_j\}$ is called a *defining sequence* for C. It is known that C is a tame Cantor set if and only if we can choose C_j to be finite unions of disjoint disks.

DEFINITION 2. The class of φ in $P\mathcal{M}^{k,+}(M, C)$ is *compactly supported* if there exists some defining sequence $\{C_j\}$ of C and some n for which the restriction of φ to C_n is $\mathcal{C}^k$-isotopic to identity rel C.

Note that the property of being compactly supported is independent of the choice of the defining sequence.

Our first result is as follows.

THEOREM 1. *Let C be a $\mathcal{C}^k$-tame Cantor set, namely a Cantor subset of a closed interval $\mathcal{C}^k$-embedded in a compact orientable manifold M of dimension at least 2. If $k \geqslant 2$, then all classes in the group $P\mathcal{M}^{k,+}(M, C)$ are compactly supported. In particular, the group $P\mathcal{M}^{k,+}(M, C)$ is countable.*

In contrast, the topological mapping class group $P\mathcal{M}^{0,+}(S^2, C)$ is uncountable. We might expect $P\mathcal{M}^{k,+}(M, C)$ to be countable for $k \geqslant 2$ even when C is only a $\mathcal{C}^0$-tame Cantor subset of M.

The following precises the second statement in Theorem 1.

COROLLARY 1. *Let C be a $\mathcal{C}^k$-tame Cantor subset of a compact orientable surface M and $\{C_j\}$ be a defining sequence for C consisting of finite unions of disjoint disks. If $k \geqslant 2$, then $P\mathcal{M}^{k,+}(M, C)$ coincides with the inductive limit $\lim_{j \to \infty} P\mathcal{M}^{k,+}(M - \mathrm{int}(C_j))$ of pure mapping class groups of compact subsurfaces.*

Note that, when N is a compact surface the isomorphism type of $P\mathcal{M}^{k,+}(N)$ is independent of k.

1.2 $\mathcal{C}^1$-diffeomorphism groups of Cantor sets

We now turn to the full mapping class groups. Several groups which arose recently in the literature could be thought to play the role of the mapping class groups for some infinite type surfaces, for instance the group $\mathcal{B}$ from [FK04] and its version BV, which was defined by Brin [Bri07] and Dehornoy [Deh06], independently. These two groups are braidings of the Thompson group V (see [CFP96]). Geometric constructions of the same sort permitted the authors of [FK08] to derive two braidings T^* and $T^\sharp$ of the Thompson group T.

Our next goal is to show that these groups are indeed smooth mapping class groups in the usual sense and that most (if not all) smooth mapping class groups of Cantor sets are related to some Thompson-like groups.

Assume for the moment that $C \subset M$ is smoothly tame. Set then $\mathfrak{diff}_M^k(C)$ and $\mathfrak{diff}_M^{k,+}(C)$ for the groups of induced transformations of C arising as restrictions of elements of $\mathrm{Diff}^k(M,C)$ and $\mathrm{Diff}^{k,+}(M,C)$, respectively. The $\mathcal{C}^k$ topology on $\mathrm{Diff}^k(M,C)$ induces a topology on $\mathfrak{diff}_M^k(C)$.

Notice now that we have the exact sequence:

$$1 \to P\mathcal{M}^{k,+}(M,C) \to \mathcal{M}^{k,+}(M,C) \to \mathfrak{diff}_M^{k,+}(C) \to 1. \tag{1}$$

By Theorem 1 the group $\mathcal{M}^{k,+}(M,C)$ is discrete countable if and only if $\mathfrak{diff}_M^{k,+}(C)$ does, when $k \geqslant 2$ and M is compact (or the interior of a compact manifold).

Classical Thompson groups can be realized as groups of dyadic piecewise linear homeomorphisms (or bijections) of an interval, circle or a Cantor set (see [CFP96, GS87]) or as groups of automorphisms at infinity of graphs (respecting or not the planarity), as in [Ner92]. Notice that the more involved construction from [GS87] provides embeddings of Thompson groups into the group of diffeomorphisms of the circle, admitting invariant minimal Cantor sets. In particular, Ghys and Sergiescu obtained embeddings as discrete subgroups of the group of diffeomorphisms (see [GS87, Theorem 2.3]).

In our setting we see that whenever it is discrete and countable the group $\mathcal{M}^{k,+}(M,C)$ is the braiding of $\mathfrak{diff}_M^{k,+}(C)$ according to Corollary 1, as in the cases studied in [Bri07, Deh06, FK04, FK08]. This raises the question whether $\mathfrak{diff}_M^{k,+}(C)$ is a Thompson-like group, in general. We were not able to solve this question in full generality and actually when C is a generic Cantor set of the interval we expect the group $\mathfrak{diff}_M^{k,+}(C)$ be trivial. To this purpose we introduce the following property of Cantor sets.

DEFINITION 3. The Cantor subset C of an interval is σ-*sparse* if, for any $a, b \in C$, there is a complementary interval $(\alpha, \beta) \subset (a,b) \cap \mathbb{R} \backslash C$ such that

$$\beta - \alpha \geqslant \sigma(b - a). \tag{2}$$

Moreover C is *sparse* if it is σ-sparse for some $\sigma > 0$.

Set $\mathfrak{diff}^k(C) = \mathfrak{diff}_\mathbb{R}^k(C)$, $\mathfrak{diff}^{k,+}(C) = \mathfrak{diff}_\mathbb{R}^{k,+}(C)$ for the sake of notational simplicity.

THEOREM 2. *If C is a sparse Cantor subset of $\mathbb{R}$, then the group $\mathfrak{diff}^1(C)$ is countable. If C is a sparse Cantor set in $S^1 = \mathbb{R}/\mathbb{Z}$, then $\mathfrak{diff}_{S^1}^1(C)$ is countable.*

Theorem 2 cannot be extended to all Cantor sets C, without additional assumptions, as we can see from the examples given in § 5.

We have the following more general version of the previous result.

THEOREM 3. *If C is a sparse Cantor subset of an interval $\mathcal{C}^1$-embedded into a compact orientable manifold M, then the group $\mathfrak{diff}^1_M(C)$ is countable and discrete.*

Although we only considered smoothly tame Cantor subsets above, there is a large supply of topologically tame Cantor subsets in any dimensions for which we can prove the countability.

THEOREM 4. *Let C_i be sparse Cantor sets in $\mathbb{R}$ and $C = C_1 \times C_2 \times \cdots \times C_n \subset \mathbb{R}^n$. Then the group $\mathfrak{diff}^1_{\mathbb{R}^n}(C)$ is countable.*

Observe that the Lebesgue measure of a sparse Cantor set is zero. In this direction, notice that Deroin, Kleptsyn and Navas recently proved that invariant Cantor sets of groups of real-analytic circle diffeomorphisms have zero Lebesgue measure (see [DKN13, Corollary 1.17]). The result cannot be extended to $\mathcal{C}^1$-diffeomorphisms, due to the Denjoy counterexamples, but it might hold more generally for all $\mathcal{C}^2$-diffeomorphisms of the circle according to a conjecture of Hector (see the discussion in [DKN13] and [DKN09, Conjecture 1.11]).

The key point is to show that the stabilizer of a point in this group is a finitely generated abelian group (see Lemma 7, Proposition 1). The discreteness of the stabilizers seems to be the counterpart to the following unpublished theorem of G. Hector (see [Nav06]): If the subgroup G of the group $\mathrm{Diff}^\omega(S^1)$ of analytic diffeomorphisms of the circle has an exceptional minimal set, then the stabilizer G_a of any point a of the circle in G is either trivial or $\mathbb{Z}$. As a corollary every subgroup of $\mathrm{Diff}^\omega(S^1)$ having a minimal Cantor set is countable. This is, of course, not true for subgroups of $\mathrm{Diff}^\infty(S^1)$. Nevertheless, the stabilizer G_a of a subgroup $G \subset \mathrm{Diff}^2(S^1)$ with an exceptional minimal set cannot contain two germs whose logarithm of their derivatives are rationally independent, according to a classical result of Sacksteder [Sac65, Theorem 2]. Sacksteder's result cannot be extended to $\mathcal{C}^1$-diffeomorphisms. The proof of our key result is related to Thurston's generalization of Reeb's stability theorem from [Thu74] and uses in an essential way the fact that the Cantor set is sparse while allowing only $\mathcal{C}^1$-smoothness of the diffeomorphisms.

Remark 1. Note that mapping class groups $\mathcal{M}^1(M, C)$ depend essentially on the ambient manifold M (see [AF17]). On the contrary, if $C \subset \mathrm{int}(D^n)$ is fixed, then for any embedding of the disk D^n in the interior of some orientable n-manifold M^n the groups $\mathfrak{diff}^1_M(C)$ are isomorphic. Moreover, these groups stabilize with respect to the standard embeddings $D^n \subset D^{n+k}$, for large enough k. One could however vary the groups $\mathfrak{diff}^1_M(C)$ by allowing C to intersect the boundary of M in different patterns (isolated points, Cantor subsets etc.).

B. Specific families of Cantor sets

1.3 Iterated functions systems

DEFINITION 4. A *contractive iterated function system* (abbreviated contractive *IFS*) is a finite family $\Phi = (\phi_0, \phi_1, \ldots, \phi_n)$ of contractive maps $\phi_j : \mathbb{R}^d \to \mathbb{R}^d$. Recall that a map ϕ is *contractive* if its Lipschitz constant is smaller than unity, namely:

$$\sup_{x,y \in \mathbb{R}^d} \frac{d(\phi(x), \phi(y))}{d(x, y)} < 1.$$

According to Hutchinson (see [Hut81]) there exists a unique non-empty compact $C = C_\Phi \subset \mathbb{R}^d$, called the *attractor* of the contractive IFS Φ, such that $C = \bigcup_{j=0}^n \phi_j(C)$.

Example 1. The central Cantor set C_λ, with $\lambda > 2$, is the attractor of the IFS $\Phi = (\phi_0, \phi_1)$ on $\mathbb{R}$ given by

$$\phi_0(x) = \frac{1}{\lambda} x, \quad \phi_1(x) = \frac{1}{\lambda} x + \frac{\lambda - 1}{\lambda}.$$

Although the IFS makes sense also when $1 < \lambda \leqslant 2$, in this case the attractor is not a Cantor set but the whole interval $[0, 1]$.

Consider now the following type of IFS of topological nature.

DEFINITION 5. Let U be an orientable manifold (possibly non-compact) and $\phi_j : U \to U$ be finitely many orientation preserving homeomorphisms on their image. We say that $\Phi = (\phi_0, \phi_1, \ldots, \phi_n)$ has a *strict attractive basin* M if M is a compact orientable submanifold $M \subset U$ with the following properties:

(i) $\phi_j(M) \subset \text{int}(M)$, for all $j \in \{0, 1, \ldots, n\}$;
(ii) $\phi_i(M) \cap \phi_j(M) = \emptyset$, for any $j \neq i \in \{0, 1, \ldots, n\}$.

We say that the pair (Φ, M) is an *invertible* IFS if M is a strict attractive basin for Φ. If moreover, ϕ_j are $\mathcal{C}^k$-diffeomorphisms on their image, then we say that the IFS is of class $\mathcal{C}^k$.

The existence of an attractive basin is a topological version of uniform contractivity of ϕ_j. There exists then a unique invariant non-empty compact $C_\Phi \subset M$ with the property that $C_\Phi = \bigcup_{i=0}^n \phi_i(C_\Phi)$.

THEOREM 5. *Consider a $\mathcal{C}^1$ contractive invertible IFS (Φ, M), $\Phi = (\phi_0, \phi_1, \ldots, \phi_n)$, whose strict attractive basin M is diffeomorphic to a d-dimensional ball. Then, the group $\mathfrak{diff}_M^1(C_\Phi)$ contains the Thompson group F_{n+1}, when M is of dimension $d = 1$ and the Thompson group V_{n+1}, when $d \geqslant 2$, respectively.*

In particular, the groups $\mathfrak{diff}_M^1(C_\Phi)$ are (highly) non-trivial.

For a clear introduction to the classical Thompson groups F, T, V we refer to [CFP96]. The generalized versions F_n, T_n, V_n were considered by Higman ([Hig74] and further extended and studied by Brown and Stein (see [Ste92]), Bieri and Strebel (see [BS16]) and Laget [Lag04]. We will recall their definitions in §2.

The result of the theorem does not hold when the attractive basin M is not a ball. For instance, when M is a 3-dimensional solid torus, by taking non-trivial (linked) embeddings $\bigcup_{i=0}^n \phi_i(M) \subset M$ we can provide examples of wild Cantor sets, some of them being topologically rigid, in which case the group $\mathfrak{diff}_M^1(C)$ is trivial (see [Shi74, Wri86]).

1.4 Self-similar Cantor subsets of the line

The second part of this paper is devoted to concrete examples of groups arising from these constructions, for particular choices of Cantor sets. We will be concerned in this section with self-similar Cantor sets, namely attractors of IFS which consist only of similitudes. The typical example is the central ternary Cantor set $C_\lambda \subset [0, 1]$ of parameter $\lambda > 2$ from Example 1.

Let $\Phi = (\phi_0, \phi_1, \ldots, \phi_n)$ be an IFS of affine transformations of $[0, 1]$, given by

$$\phi_j(x) = \lambda_j x + a_j,$$

where

$$0 = a_0 < \lambda_0 < a_1 < \lambda_1 + a_1 < a_2 < \cdots < \lambda_{n-1} + a_{n-1} < a_n < \lambda_n + a_n = 1.$$

The last condition means that the segments $\phi_j([0,1])$ are mutually disjoint, so that the attractor $C = C_\Phi$ is a sparse Cantor subset of $[0,1]$. The positive reals $g_j = a_{j+1} - \lambda_j - a_j$ are the initial *gaps* as they represent the distance between consecutive intervals $\phi_j([0,1])$ and $\phi_{j+1}([0,1])$. The image of $[0,1]$ by the elements of the monoid generated by Φ are called *standard intervals*.

We consider the groups F_C and T_C defined as follows. Let $PL(\mathbb{R}, C)$ and $PL(S^1, C)$ be the groups of orientation preserving piecewise linear homeomorphisms of $\mathbb{R}$ and $S^1 = \mathbb{R}/\mathbb{Z}$ respectively, keeping invariant C, i.e. of those homeomorphisms φ for which there exists a finite covering of C by standard disjoint intervals $\{I_j\}$, $\mathbf{k}_j \in \mathbb{Z}^{n+1}$ and $a_j, b_j \in C$, such that

$$\varphi(x) = b_j + \Lambda_{\mathbf{k}_j}(x - a_j) \quad \text{for any } x \in I_j, \tag{3}$$

where $\Lambda_{\mathbf{k}} = \prod_{i=0}^{n} \lambda_i^{k_i}$, for each multi-index $\mathbf{k} = (k_0, k_1, \ldots, k_n) \in \mathbb{Z}^{n+1}$. Eventually F_C and T_C are the images of $PL(\mathbb{R}, C)$ and $PL(S^1, C)$, respectively, in the group of homeomorphisms of C. Similarly we have the group of piecewise affine exchanges $PE(C)$ which are (not necessarily orientation preserving) left continuous bijections of S^1 preserving C, i.e. of those (not necessarily continuous) maps φ for which there exists a finite covering of C by standard disjoint intervals $\{I_j\}$, $\mathbf{k}_j \in \mathbb{Z}^{n+1}$ and $a_j, b_j \in C$, such that

$$\varphi(x) = b_j \pm \Lambda_{\mathbf{k}_j}(x - a_j) \quad \text{for any } x \in I_j. \tag{4}$$

We denote by $V_C^\pm$ its image into the group of homeomorphisms of C. We denote by $V_C \subset V_C^\pm$ the subgroup obtained by requiring the restrictions of φ to each standard interval I_j be orientation-preserving, as in (3).

Definition 6. The self-similar Cantor set $C \subset [0,1]$ satisfies the genericity condition (C) if

(i) either all homothety ratios λ_i are equal and all initial generation gaps g_α are equal;

(ii) or the factors λ_i and the gaps g_α are incommensurable, in the following sense:

    (a) $\Lambda_{\mathbf{k}} g_\alpha = g_\beta$ implies that $\mathbf{k} = 0$ and $\alpha = \beta$;

    (b) there exists no permutation σ different from identity and $\mathbf{k}, \mathbf{k}_\alpha \in \mathbb{Z}_+^{n+1}$ such that for all β we have:
$$\frac{g_{\sigma(\beta)}}{g_\beta} = \Lambda_{-\mathbf{k}_\beta + (1/n) \sum_{\alpha=1}^{n} \mathbf{k}_\alpha}.$$

Theorem 6. *Let $C \subset [0,1]$ be a self-similar Cantor set satisfying the genericity condition (C). Then for every $\varphi \in \mathfrak{diff}^{1,+}(C)$ we can find a covering of C by a finite collection of disjoint standard intervals $\{I_j\}$, whose images are also standard intervals, integers $\mathbf{k}_j \in \mathbb{Z}^{n+1}$ and $a_j, b_j \in C$, such that the restriction of the map φ has the form*

$$\varphi(x) = b_j + \Lambda_{\mathbf{k}_j}(x - a_j) \quad \text{for any } x \in I_j \cap C. \tag{5}$$

In particular, $\mathfrak{diff}^{1,+}(C)$ is isomorphic to F_C, $\mathfrak{diff}_{S^1}^{1,+}(C)$ is isomorphic to T_C and $\mathfrak{diff}_{S^2}^{1,+}(C)$ is isomorphic to $V_C^\pm$. Moreover, these are isomorphic to the Thompson groups F_{n+1}, T_{n+1} and the signed Thompson group $V_{n+1}^\pm$, respectively.

The main points in the statement of the theorem are the finiteness of the covering and the fact that the intervals are standard.

Remark 2. If the self-similar Cantor set does not satisfy the genericity condition (C), then the same proof provides for every $\varphi \in \mathfrak{diff}^{1,+}(C)$ a covering of C by a finite collection of disjoint intervals $\{I_j\}$, integers $\mathbf{k}_j \in \mathbb{Z}^{n+1}$ and $a_j, b_j \in C$, such that the restriction of the map φ is given by the formula (39), a slight generalization of (5) above. However, it is not clear that one could assume that the images of $\{I_j\}$ are standard intervals and, in particular, that $\mathfrak{diff}^{1,+}(C)$ is isomorphic to some Thompson group.

We derive easily now the interpretation for the Thompson groups and their braided versions as follows.

Corollary 2.

(i) *Let C be the image of the standard ternary Cantor subset into the equatorial circle of the sphere S^2 and $k \geqslant 2$.*

    (a) *The smooth mapping class group $\mathcal{M}^{k,+}(D_+^2, C)$ is the Thompson group T, where D_+^2 denotes the upper hemisphere.*

    (b) *The smooth mapping class group $\mathcal{M}^{k,+}(S^2, C)$ is the group of half-twists $\mathcal{B}^{1/2}$ from [FN16].*

(ii) *Let C be the standard ternary Cantor subset of an interval contained in the interior of a 2-disk D^2 and $k \geqslant 2$. Then $\mathcal{M}^{k,+}(D^2, C)$ is the group of half-twists of the punctured disk (see [AF17]).*

Remark 3. The group of half twists $\mathcal{B}^{1/2}$ is an extension of the signed Thompson group $V^{\pm}$ by the compactly supported pure mapping class group of $S^2 - C$. It is similar to the braided Thompson group $\mathcal{B}$ from [FK04] (see § 2.5), which is an extension of V by the same pure mapping class group, in particular it is also finitely presented (see [AF17]).

Remark 4. The central ternary Cantor sets C_λ are pairwise non-diffeomorphic, i.e. there is no $\mathcal{C}^1$ diffeomorphism of $\mathbb{R}$ sending C_λ into $C_{\lambda'}$ for $\lambda \neq \lambda'$. Indeed, if it were such a diffeomorphism then the Hausdorff dimensions of the two Cantor sets would agree, while the Hausdorff dimension of C_λ is $\log 2/\log \lambda$ (see [Fal03], Theorem 1.14). Nevertheless, the groups $\mathfrak{diff}^{1,+}(C_\lambda)$ are all isomorphic, for $\lambda > 2$, according to Theorem 6.

We notice that a weaker version of our Theorem 6 concerning the form of $\mathcal{C}^1$-diffeomorphisms of the central Cantor sets C_λ, was already obtained in [BMPV97, Proposition 1].

A case which attracted considerable interest is that of bi-Lipschitz homeomorphisms of Cantor sets (see [CP88, FM92] and the recent [RRY12, XX13]). In particular, the results of Falconer and Marsh [FM92] imply that every bi-Lipschitz homeomorphism of a Cantor set is given by a pair of possibly infinite coverings of the Cantor set by disjoint intervals and affine homeomorphisms between the corresponding intervals. Notice that any countable subgroup of $\mathrm{Diff}^0(S^1)$ (or $\mathrm{Diff}^0([0, 1])$ can be conjugated (by a homeomorphism) into the corresponding group of bi-Lipschitz homeomorphisms (see [DKN07, Theorem D]).

1.5 Self-similar Cantor dusts

The next step is to go to higher dimensions. Examples of Blankenship (see [Bla51]) show that there exist wild Cantor sets in $\mathbb{R}^n$, for every $n \geqslant 3$. A Cantor set C is tame if and only if for every $\varepsilon > 0$ there exist finitely many disjoint piecewise linear cells of diameter smaller than ε

whose interiors cover C. In particular, products of tame Cantor sets are tame. More generally, the product of a Cantor subset of $\mathbb{R}^n$ with any compact 0-dimensional subset $Z \subset \mathbb{R}^m$ is a tame Cantor subset of $\mathbb{R}^{m+n}$ (see [Mcm64, Corollary 2]).

In order to emphasize the role of the embedding we will consider now the simplest Cantor subsets, which although tame they are not smoothly tame. Let $C_\lambda^n \subset \mathbb{R}^n$ be the Cartesian product of n copies of C_λ, where $n \geqslant 2$ and $\lambda > 2$, which is itself a Cantor set.

THEOREM 7. *Let $\varphi \in \mathfrak{diff}_{\mathbb{R}^n}^{1,+}(C_\lambda^n)$, where $\lambda > 2$. Then there is a covering of C_λ^n by a finite collection of disjoint standard parallelepipeds $\{I_j\}$, integers $k_{j,i} \in \mathbb{Z}$ and $a_{j,i}, b_{j,i} \in C_\lambda$, such that*

$$\varphi(x) = (b_{j,i} + \lambda^{k_j}(x_i - a_{j,i}))_{i=1,n} \circ S_j \quad \text{for any } x \in I_j \cap C_\lambda^n, \tag{6}$$

where S_j is an orientation preserving symmetry of the cube. In particular, $\mathfrak{diff}_{\mathbb{R}^n}^{1,+}(C_\lambda^n)$ is isomorphic to the n-dimensional Brin group nV^{sym} decorated by the group D_n of positive symmetries of the cube (see § 2).

Notice that in a series of papers (see [Bri04, Bri10, BL10, HM12]) by Brin, Bleak and Lanoue, Hennig and Matucci the authors proved that the higher-dimensional Thompson groups nV defined by Brin are pairwise non-isomorphic finitely presented simple groups (see also [Rub96, Rub89]).

Remark 5. Note that the group $\mathfrak{diff}_{[0,1]^n}^1(C_\lambda^n)$ is a proper subgroup of $\mathfrak{diff}_{\mathbb{R}^n}^1(C_\lambda^n)$.

2. Definition of Thompson-like groups

The standard reference for the classical Thompson groups is [CFP96]. For the sake of completeness we provide here the basic definitions from several different perspectives, which lead naturally to the generalizations considered by Brown and Stein and further to the high dimensional Brin groups.

2.1 Groups of piecewise affine homeomorphisms/bijections

Thompson's group F is the group of piecewise dyadic affine homeomorphisms of the interval $[0,1]$. Namely, for each $f \in F$, there exist two dyadic subdivisions of $[0,1]$, $a_0 = 0 < a_1 < \cdots < a_n = 1$ and $b_0 = 0 < b_1 \cdots < b_n$, with $n \in \mathbb{N}^*$, such that $a_{i+1} - a_i$ and $b_{i+1} - b_i$ belong to $\{n/2^k, \ n, k \in \mathbb{N}\}$, so that the restriction of f to $[a_i, a_{i+1}]$ is the unique increasing affine map onto $[b_i, b_{i+1}]$.

Therefore, an element of F is completely determined by the data of two dyadic subdivisions of $[0,1]$ having the same cardinality.

Let us identify the circle to the quotient space $[0,1]/0 \sim 1$. *Thompson's group T* is the group of piecewise dyadic affine orientation preserving homeomorphisms of the circle. In other words, for each $g \in T$, there exist two dyadic subdivisions of $[0,1]$, $a_0 = 0 < a_1 < \cdots < a_n = 1$ and $b_0 = 0 < b_1 \cdots < b_n$, with $n \in \mathbb{N}^*$, and $i_0 \in \{1, \ldots, n\}$, such that, for each $i \in \{0, \ldots, n-1\}$, the restriction of g to $[a_i, a_{i+1}]$ is the unique increasing map onto $[b_{i+i_0}, b_{i+i_0+1}]$. The indices must be understood modulo n.

Therefore, an element of T is completely determined by the data of two dyadic subdivisions of $[0,1]$ having the same cardinality, say $n \in \mathbb{N}^*$, plus an integer $i_0 \bmod n$.

Finally, *Thompson's group V* is the group of bijections of $[0,1[$, which are right-continuous at each point, piecewise non-decreasing and dyadic affine. In other words, for each $h \in V$, there exist two dyadic subdivisions of $[0,1]$, $a_0 = 0 < a_1 < \cdots < a_n = 1$ and $b_0 = 0 < b_1 \cdots < b_n$, with $n \in \mathbb{N}^*$, and a permutation $\sigma \in \mathfrak{S}_n$, such that, for each $i \in \{1, \ldots, n\}$, the restriction of h to

$[a_{i-1}, a_i[$ is the unique non-decreasing affine map onto $[b_{\sigma(i)-1}, b_{\sigma(i)}[$. It follows that an element h of V is completely determined by the data of two dyadic subdivisions of $[0, 1]$ having the same cardinality, say $n \in \mathbb{N}^*$, plus a permutation $\sigma \in \mathfrak{S}_n$. Denoting $I_i = [a_{i-1}, a_i]$ and $J_i = [b_{i-1}, b_i]$, these data can be summarized into a triple $((J_i)_{1 \leqslant i \leqslant n}, (I_i)_{1 \leqslant i \leqslant n}, \sigma \in \mathfrak{S}_n)$.

The *signed Thompson group* $V^\pm$ is the group of right-continuous bijections of the unit circle that map the set of dyadic rationals to itself, are differentiable except at finitely many points, and such that, on every interval of differentiability, they are affine maps whose derivatives are (positive or negative) powers of 2.

We have obvious inclusions $F \subset T \subset V \subset V^\pm$. R.J. Thompson proved in 1965 that F, T and V are finitely presented groups and that T and V are simple (cf. [CFP96]). The group $V^\pm$ is also finitely presented and simple (see [AF17]). The group F is not perfect, as $F/[F, F]$ is isomorphic to $\mathbb{Z}^2$, but $F' = [F, F]$ is simple. However, F' is not finitely generated (this is related to the fact that an element f of F lies in F' if and only if its support is included in $]0, 1[$).

2.2 Groups of diagrams of finite binary trees

A *finite binary rooted planar tree* is a finite planar tree having a unique 2-valent vertex, called the *root*, a set of monovalent vertices called the *leaves*, and whose other vertices are 3-valent. The planarity of the tree provides a canonical labelling of its leaves, in the following way. Assuming that the plane is oriented, the leaves are labelled from 1 to n, from left to right, the root being at the top and the leaves at the bottom.

There exists a bijection between the set of dyadic subdivisions of $[0, 1]$ and the set of finite binary rooted planar trees. Indeed, given such a tree, one may label its vertices by dyadic intervals in the following way. First, the root is labelled by $[0, 1]$. Suppose that a vertex is labelled by $I = [k/2^n, (k+1)/2^n]$, then its two descendant vertices are labelled by the two halves I: $[k/2^n, (2k+1)/2^{n+1}]$ for the left one and $[(2k+1)/(2^{n+1}), (k+1)/2^n]$ for the right one. Finally, the dyadic subdivision associated to the tree is the sequence of intervals which label its leaves.

Thus, an element h of V is represented by a triple (τ_1, τ_0, σ), where τ_0 and τ_1 have the same number of leaves $n \in \mathbb{N}^*$, and $\sigma \in \mathfrak{S}_n$. Such a triple will be called a *symbol* for h. It is convenient to interpret the permutation σ as the bijection φ_σ which maps the ith leaf of the source tree τ_0 to the $\sigma(i)$th leaf of the target tree τ_1. When h belongs to F, the permutation σ is identity and the symbol reduces to a pair of trees (τ_1, τ_0).

Now, two symbols are equivalent if they represent the same element of V and one denotes by $[\tau_1, \tau_0, \sigma]$ the equivalence class. The composition law of piecewise dyadic affine bijections is pushed out on the set of equivalence classes of symbols in the following way. In order to define $[\tau_1', \tau_0', \sigma'] \cdot [\tau_1, \tau_0, \sigma]$, one may suppose, at the price of refining both symbols, that the tree τ_1 coincides with the tree τ_0'. Then the product of the two symbols is

$$[\tau_1', \tau_1, \sigma'] \cdot [\tau_1, \tau_0, \sigma] = [\tau_1', \tau_0, \sigma' \circ \sigma].$$

It follows that V is isomorphic to the group of equivalence classes of symbols endowed with this internal law. Now, every element of $V^\pm$ is encoded by an *enhanced symbol* $(T, T', \sigma, \varepsilon)$, where T, T' are admissible trees, $\sigma : \partial T \to \partial T'$ is a bijection and $\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^{\partial T'}$, up to equivalence.

2.3 Partial automorphisms of trees

The beginning of the article [GS91] formalizes a change of point of view, consisting in considering, not the finite binary trees, but their complements in the infinite binary tree.

Let $\mathcal{T}_2$ be the infinite binary rooted planar tree (all its vertices other than the root are 3-valent). Each finite binary rooted planar tree τ can be embedded in a unique way into $\mathcal{T}_2$, assuming that the embedding maps the root of τ onto the root of $\mathcal{T}_2$, and respects the orientation. Therefore, τ may be identified with a subtree of $\mathcal{T}_2$, whose root coincides with that of $\mathcal{T}_2$.

DEFINITION 7 (Cf. [KS05]). A *partial isomorphism* of $\mathcal{T}_2$ consists of the data of two finite binary rooted subtrees τ_0 and τ_1 of $\mathcal{T}_2$ having the same number of leaves $n \in \mathbb{N}^*$, and an isomorphism $q : \mathcal{T}_2 \backslash \tau_0 \to \mathcal{T}_2 \backslash \tau_1$. The complements of τ_0 and τ_1 have n components, each one isomorphic to $\mathcal{T}_2$, which are enumerated from 1 to n according to the labeling of the leaves of the trees τ_0 and τ_1. Thus, $\mathcal{T}_2 \backslash \tau_0 = T_0^1 \cup \cdots \cup T_0^n$ and $\mathcal{T}_2 \backslash \tau_1 = T_1^1 \cup \cdots \cup T_1^n$ where the T_j^i are the connected components. Equivalently, the partial isomorphism of $\mathcal{T}_2$ is given by a permutation $\sigma \in \mathfrak{S}_n$ and, for $i = 1, \ldots, n$, an isomorphism $q_i : T_0^i \to T_1^{\sigma(i)}$.

Two partial automorphisms q and r can be composed if and only if the target of r coincides with the source of r. One gets the partial automorphism $q \circ r$. The composition provides a structure of inverse monoid on the set of partial automorphisms.

Let $\partial \mathcal{T}_2$ be the boundary of $\mathcal{T}_2$ (also called the set of 'ends' of $\mathcal{T}_2$) endowed with its usual topology, for which it is a Cantor set. Although a partial automorphism does not act (globally) on the tree, it does act on its boundary. One has therefore a morphism from the monoid of partial isomorphism into the homeomorphisms of $\partial \mathcal{T}_2$, whose image N is the *spheromorphisms group of Neretin* (see [Ner92]).

Thompson's group V can be viewed as the subgroup of N which is the image of those partial automorphisms which respect the local orientation of the edges.

2.4 Generalizations following Brown and Stein, Bieri and Strebel

Brown considered in [Bro87] similar groups $F_{n,r} \subset T_{n,r} \subset V_{n,r}$ (extending the previous work of Higman), which were defined as in the last two constructions above but using instead of binary trees forests of r copies of n-ary trees so that F, T, V correspond to $n = 2$ and $r = 1$. The isomorphism type of $V_{n,r}$ and $T_{n,r}$ only depends on $r \pmod{n}$ while $F_{n,r}$ depends only on n. We drop the subscript r when $r = 1$. These groups are finitely presented and of type FP_∞ according to [BG84] for the case of F and T and then [Bro87, Theorem 4.17] for its extension to all other groups from this family. Moreover, Higman has proved (see [Hig74]) that $V_{n,r}$ has a simple subgroup of index g.c.d$(2, n-1)$, and this was extended by Brown who showed that F_n have simple commutator and $T_{n,r}$ have simple double commutator groups (see [Bro87] for more details and refinements).

One can obtain these groups also by considering n-adic piecewise affine homeomorphisms (or bijections) of $[0, r]$ (with identified endpoints for $T_{n,r}$) i.e. having singularities in $\mathbb{Z}[1/n]$ and derivatives in $\{n^a, a \in \mathbb{Z}\}$. This point of view was taken further by Bieri, Strebel and Stein in [BS16, Ste92]. Specifically, given a multiplicative subgroup $P \subset \mathbb{R}$, a $\mathbb{Z}[P]$-submodule $A \subset \mathbb{R}$ satisfying $P \cdot A = A$, and a positive $r \in A$, one can consider the group $F_{A,P,r}$ of those PL homeomorphisms of $[0, r]$ with finite singular set in A and all slopes in P. There are similar families $T_{A,P,r}$ and $V_{A,P,r}$. Brown and Stein proved that $F_{\mathbb{Z}[1/(n_1 n_2 \cdots n_k)],\langle n_1, n_2, \ldots, n_k \rangle, r}$ is finitely presented of FP_∞ type. Furthermore $F_{A,P,r}$ and $V_{A,P,r}$ have simple commutator subgroups, while $T_{A,P,r}$ have simple second commutator subgroups.

The signed version $V_{n,r}^{\pm}$ of $V_{n,r}$ is defined as above, by allowing both orientation preserving and orientation reversing piecewise affine homeomorphisms.

2.5 Mapping class groups of infinite surfaces and braided Thompson groups

Let $\mathscr{S}_{0,\infty}$ be the oriented surface of genus zero, which is the following inductive limit of compact oriented genus zero surfaces with boundary $\mathscr{S}_n$. Starting with a cylinder $\mathscr{S}_1$, one gets $\mathscr{S}_{n+1}$ from $\mathscr{S}_n$ by gluing a pair of pants (i.e. a three-holed sphere) along each boundary circle of $\mathscr{S}_n$. This construction yields, for each $n \geqslant 1$, an embedding $\mathscr{S}_n \hookrightarrow \mathscr{S}_{n+1}$, with an orientation on $\mathscr{S}_{n+1}$ compatible with that of $\mathscr{S}_n$. The resulting inductive limit (in the topological category) of the $\mathscr{S}_n$ is the surface $\mathscr{S}_{0,\infty} = \varinjlim_n \mathscr{S}_n$.

By the above construction, the surface $\mathscr{S}_{0,\infty}$ is the union of a cylinder and of countably many pairs of pants. This topological decomposition of $\mathscr{S}_{0,\infty}$ will be called the *canonical pair of pants decomposition*.

The set of isotopy classes of orientation-preserving homeomorphisms of $\mathscr{S}_{0,\infty}$ is an *uncountable* group. By restricting to a certain type of homeomorphism (called asymptotically rigid), we shall obtain countable subgroups (see [FK04, FK08]).

Any connected and compact subsurface of $\mathscr{S}_{0,\infty}$ which is the union of the cylinder and finitely many pairs of pants of the canonical decomposition will be called an *admissible subsurface* of $\mathscr{S}_{0,\infty}$. The *type* of such a subsurface S is the number of connected components in its boundary.

A *rigid structure* on $\mathscr{S}_{0,\infty}$ is given by a pants decomposition together with a set of disjoint proper arcs joining distinct ends such that each pair of pants intersects essentially only three arcs which join different boundary circles. One component of complement of the union of arcs is called the *visible side*. We fix a rigid structure on the surface underlying the canonical pants decomposition.

DEFINITION 8 (Following [KS05, FK04]). A homeomorphism φ of $\mathscr{S}_{0,\infty}$ is *asymptotically rigid* if there exist two admissible subsurfaces S_0 and S_1 having the same type, such that $\varphi(S_0) = S_1$ and whose restriction $\mathscr{S}_{0,\infty} \backslash S_0 \to \mathscr{S}_{0,\infty} \backslash S_1$ is rigid, meaning that it maps each pair of pants (of the canonical pants decomposition) onto a pair of pants and the visible side onto the visible side. If we drop the last requirement we say that the homeomorphism φ is *asymptotically quasi-rigid*. The *asymptotically rigid* and *quasi-rigid mapping class groups* of $\mathscr{S}_{0,\infty}$ are the groups of isotopy classes of asymptotically rigid and quasi-rigid homeomorphisms, respectively.

The *asymptotically rigid mapping class group* $\mathcal{B}$ and the *quasi-rigid mapping class group* $\mathcal{B}^{1/2}$ of $\mathscr{S}_{0,\infty}$ are finitely presented groups (see [FK04, AF17]) which fit into the exact sequences:

$$1 \to P\mathcal{M}(\mathscr{S}_{0,\infty}) \to \mathcal{B} \to V \to 1,$$
$$1 \to P\mathcal{M}(\mathscr{S}_{0,\infty}) \to \mathcal{B}^{1/2} \to V^{\pm} \to 1.$$

Some very similar versions of the same group (using a Cantor disk instead of a Cantor sphere or a more combinatorial framework) were obtained independently by Brin [Bri07] and Dehornoy [Deh06]. We will call any version of them *braided Thompson groups*.

2.6 Brin's groups nV and their decorated versions

A rather different direction was taken in the seminal paper [Bri04] of Brin, where the author constructed a family of countable groups nV acting as homeomorphisms of the product of n-copies of the standard triadic Cantor, generalizing the group V which occurs for $n = 1$.

Let $I^n \subset \mathbb{R}^n$ denote the unit cube. A *numbered pattern* is a finite dyadic partition of I^n into parallelepipeds along with a numbering. A dyadic partition is obtained from the cube by dividing at each step of the process one parallelepiped into two equal halves by a cutting hyperplane parallel to one of the coordinates' hyperplane.

One definition of nV is the group of piecewise affine (not continuous!) transformations associated to pairs of numbered patterns. Given the numbered patterns $P = (L_1, L_2, \ldots, L_n)$ and $Q = (R_1, R_2, \ldots, R_n)$, we set $\varphi_{P,Q}$ for the unique piecewise affine transformation of the cube sending affinely each L_i into R_i and preserving the coordinates' hyperplanes. Thus nV is the group of piecewise affine transformations of the form $\varphi_{P,Q}$, with P, Q running over the set of all possible dyadic partitions.

Another description is as a group of homeomorphisms of the product C^n of the standard triadic Cantor set C. Parallelepipeds in a dyadic partition correspond to closed and open (clopen) subsets of C^n. Every dyadic cutting hyperplane H subdividing some parallelepiped R into two halves determines a parallel shadow (open) parallelepiped in R whose width is one third of the width of R in the direction orthogonal to H. Notice then that the complement of the union of all shadow parallelepipeds is C^n. Every pattern $P = (R_1, R_2, \ldots, R_n)$ determines a numbered collection of parallelepipeds $X_P = (X(R_1), X(R_2), \ldots, X(R_n))$ whose complementary is the set of shadow parallelepipeds of those cutting hyperplanes used to build P. Then $A(R_i) = X(R_i) \cap C^n$ form a clopen partition of C^n. For a pair of patterns P, Q we define the homeomorphism $h_{P,Q}$ of C^n as the unique homeomorphism which sends affinely $A(L_i)$ into $A(R_i)$ and preserves the orientation in each coordinate. This amounts to say that $h_{P,Q}$ is the restriction to C^n of the piecewise affine transformation sending affinely $X(L_i)$ into $X(R_i)$ and preserving the coordinates' hyperplanes.

The groups nV are simple (see e.g. [Bri04, Bri10]) and finitely presented (see [HM12]). The stabilizer at some $a \in C^n$ of the (germs of) homeomorphisms in nV is isomorphic to $\mathbb{Z}^{r(a)}$, where $r(a)$ is the number of rational coordinates of a. This implies that the groups nV are pairwise non-isomorphic (see [BL10] for details).

We could of course extend this construction to arbitrary products of central Cantor sets C_λ in the spirit of Brown and Stein, Bieri and Strebel as above.

As in the case of groups $V_{n,r}$ there exists a decorated version nV^{sym} of nV by allowing piecewise affine transformations $h_{P,Q}$ to be arbitrary affine isomorphisms between $A(L_i)$ into $A(R_i)$, not necessarily preserving each coordinate's hyperplane. We will say that nV^{sym} is the *n-dimensional Brin group decorated by D_n*, where D_n denotes the group of orientation preserving symmetries of the cube. Its elements correspond to numbered patterns $P = (L_1, L_2, \ldots, L_n)$ and $Q = (R_1, R_2, \ldots, R_n)$, along with a n-tuple $\Sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ of orientation preserving symmetries of the n-cube. The map $\varphi_{P,Q,\Sigma}$ consists of the unique piecewise affine transformation sending $\sigma_i(L_i)$ into R_i. Recall that D_n is the group of orthogonal $n \times n$ matrices with integer entries and unit determinant. One defines in the same way the *n-dimensional Brin group $nV^{\pm\mathrm{sym}}$ decorated by O_n*, where O_n denotes the hyperoctahedral group of all symmetries of the cube, which is the higher-dimensional generalization of $V^{\pm}$.

3. Proof of general countability statements

3.1 Proof of Theorem 1

We parameterize the interval E containing the Cantor set C by the $\mathcal{C}^k$-curve $\gamma : [0,1] \to M$ and denote by $A \subset [0,1]$ the preimage of C, which is still a Cantor set. We may assume that $\{0,1\} \subset A$. For the sake of simplicity we suppose that the interval E lies in the interior of M. The proof works in general, with only minor modifications. Let $\varphi \in \mathrm{Diff}^{k,+}(M, C)$ and denote by $\xi(t) = \varphi \circ \gamma(t)$. Consider a $\mathcal{C}^k$-coordinates chart $U \subset M$ containing E, such that U is identified with an open disk, while γ is now linear and parameterized by arc length, namely that $\|\dot{\gamma}\| = 1$ and $\ddot{\gamma} = 0$. The norm $\|\cdot\|$ is associated to the standard scalar product $\langle\,,\rangle$ on U induced from $\mathbb{R}^n$.

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The strategy of the proof is as follows. We define a subset $I_\varepsilon \subset [0,1]$ consisting of finitely many intervals which contains A. At first one straightens out ξ in the complementary of I_ε. To this purpose we modify φ by composing with a convenient compactly supported diffeomorphism. Further we show that there is an isotopy rel A which straightens out the remaining arcs of ξ. Eventually, one proves that a diffeomorphism preserving the orientation of the surface which fixes the arc E is, up to isotopy, supported outside a disk neighborhood of E. This will show that φ has a compactly supported class.

Assume for the moment that A is just an infinite set without isolated points. The set of those t for which $\gamma(t) = \xi(t)$ is a closed subset of $[0,1]$ containing A and hence its closure $\overline{A}$. Now let $t_0 \in \overline{A}$. Then, since γ and ξ are differentiable at t_0 we have

$$\dot{\gamma}(t_0) = \lim_{t \in A, t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} = \lim_{t \in A, t \to t_0} \frac{\xi(t) - \xi(t_0)}{t - t_0} = \dot{\xi}(t_0). \tag{7}$$

If φ is twice differentiable then the same argument shows that

$$\ddot{\gamma}(t_0) = \ddot{\xi}(t_0). \tag{8}$$

Since φ is of class $\mathcal{C}^2$, for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that whenever $s_1, s_2 \in A$, with $|s_1 - s_2| < \delta(\varepsilon)$ we have

$$1 - \varepsilon < \langle \dot{\gamma}(t), \dot{\xi}(t) \rangle \leqslant 1 \quad \text{for all } t \in [s_1, s_2], \tag{9}$$
$$|\ddot{\xi}(t)| < \varepsilon \quad \text{for all } t \in [s_1, s_2]. \tag{10}$$

We assume now that $A = \bigcap_{j=1}^\infty A_j$ is the infinite nested intersection of the closed finite unions of intervals $A_j \supset A_{j+1} \supset \cdots$.

We denote $I_\varepsilon = \bigcup_{s_1, s_2 \in A; |s_1 - s_2| \leqslant (\delta(\varepsilon)/2)} [s_1, s_2] \subset [0,1]$. We choose $\varepsilon > 0$ small enough such that the image of $\xi|_{I_\varepsilon}$ is contained within the coordinates disk U.

Set further $\gamma_s(t) = (1-s)\gamma(t) + s\,\xi(t)$, for $t \in [s_1, s_2] \subset I_\varepsilon$ and $s \in [0,1]$.

LEMMA 1. *Fix $\varepsilon < 1$ as above. Let $s_1, s_2 \in A$, such that $|s_1 - s_2| \leqslant \delta(\varepsilon)/2$. Then $\gamma_s|_{[s_1, s_2]}$ provides a $\mathcal{C}^k$-isotopy between the restrictions $\gamma|_{[s_1, s_2]}$ and $\xi|_{[s_1, s_2]}$ to the interval $[s_1, s_2]$. In particular, the image $\xi(I_\varepsilon)$ is contained within the union of orthogonal strips $(\gamma(I_\varepsilon) \times \mathbb{R}) \cap U$.*

Proof. We have to prove that for any $s \in [0,1]$ the curve $\gamma_s|_{[s_1, s_2]}$ is simple. This follows immediately from the fact that whenever $\varepsilon < 1$ we have

$$\langle \dot{\gamma}_s(t), \dot{\gamma}(t) \rangle \geqslant 1 - s + s\langle \dot{\xi}(t), \dot{\gamma}(t) \rangle \geqslant 1 - \varepsilon s > 0 \tag{11}$$

for any $t \in [0,1]$, $s \in [0,1]$. Further, note that the curve $\gamma_s(t)$, for $s \in [0,1]$ and fixed $t \in I_\varepsilon$ is a segment joining $\xi(t)$ with its orthogonal projection onto $\gamma(I_\varepsilon)$. $\qquad\square$

We set

$$\eta(t) = \begin{cases} \xi(t) & \text{if } t \in I_\varepsilon, \\ \gamma(t) & \text{if } t \notin I_\varepsilon. \end{cases} \tag{12}$$

LEMMA 2. *There exists a compactly supported diffeomorphism $\psi \in P\mathrm{Diff}^{k,+}(M, C)$ such that $\psi(\xi)$ and η are isotopic rel A.*

Proof. Lemma 1 shows that the image of η is a simple curve, as $\xi(I_\varepsilon)$ is contained within the union of orthogonal strips $(\gamma(I_\varepsilon) \times \mathbb{R}) \cap U$, and thus it cannot intersect $\gamma([0,1] \backslash I_\varepsilon)$.

Note that $A \subset I_\varepsilon$, since A has no isolated points. The endpoints of a maximal complementary interval should belong to A, by maximality. In particular, its length should be greater than $\delta(\varepsilon)/2$, and hence there are only finitely many maximal complementary intervals say $J_1, J_2, \ldots, J_p$. Then $\xi(J_i)$ are pairwise disjoint smooth arcs whose interiors are $\xi(\text{int}(J_i)) \subset M - C$, each such arc joining two distinct points of C. Moreover, as $\xi(t) = \gamma(t)$, for $t \in \bigcup_{i=1}^p \partial J_i$, we can straighten out the half-arcs of ξ around these points. Namely, there exists a small neighborhood $N(I_\varepsilon)$ of I_ε within $[0,1]$ such that after perturbing ξ by an isotopy supported in $N(I_\varepsilon) \cap (\bigcup_{i=1}^p J_i)$ we have $\xi(t) = \gamma(t)$, for $t \in N(I_\varepsilon) \cap (\bigcup_{i=1}^p J_i)$.

Now the arcs $\xi(I_\varepsilon)$ are disjoint both from $\xi(J_i \backslash N(I_\varepsilon))$ and $\gamma(J_i \backslash N(I_\varepsilon))$. There exists then a small enough open neighborhood V of $\xi(I_\varepsilon)$ within U which is disjoint from both $\xi(J_i \backslash N(I_\varepsilon))$ and $\gamma(J_i \backslash N(I_\varepsilon))$. Therefore there exists an orientation preserving diffeomorphism ψ supported on $M - V$, thus compactly supported, such that $\psi(\xi(J_i \backslash N(I_\varepsilon))) = \gamma(J_i \backslash N(I_\varepsilon))$, and hence $\psi(\xi(J_i)) = \eta(J_i)$. Thus $\psi(\xi)$ and η are isotopic rel A, as claimed. $\qquad \square$

LEMMA 3. *The curves γ and η are isotopic rel A.*

Proof. We will prove that the family

$$\eta_s(t) = \begin{cases} \gamma_s(t) & \text{if } t \in I_\varepsilon, \\ \gamma(t) & \text{if } t \notin I_\varepsilon, \end{cases} \tag{13}$$

is the desired isotopy. From Lemma 1 it suffices to show that there are not intersections between the segments of curves $\gamma_s|_{[s_1,s_2]}$ and $\gamma_s|_{[s_3,s_4]}$, when $s_i \in A$ and $[s_1, s_2], [s_3, s_4] \subset I_\varepsilon$ are disjoint.

Let $p = \gamma_s|_{[s_1,s_2]}(t_0)$ be a point on the first curve segment. We want to estimate the angle β of the Euclidean triangle with vertices $p, \gamma(s_1), \gamma(s_2)$ at $\gamma_s(s_1)$. We can write then

$$\langle \gamma_s(t_0) - \gamma_s(s_1), \dot{\gamma}(0) \rangle = \int_0^{t_0 - s_1} \langle \dot{\gamma}_s(s_1 + x), \quad \dot{\gamma}(0) \rangle \, dx = \int_0^{t_0 - s_1} 1 - s + s \langle \dot{\xi}(s_1 + x), \dot{\gamma}(0) \rangle \, dx. \tag{14}$$

Then (9) implies

$$\| \gamma_s(t_0) - \gamma_s(s_1) \| \cos(\beta) = \langle \gamma_s(t_0) - \gamma_s(s_1), \quad \dot{\gamma}(0) \rangle \geqslant (t_0 - s_1)(1 - s\varepsilon). \tag{15}$$

On the other hand from (10) we derive

$$\| \dot{\xi}(x) - \dot{\xi}(s_1) \| \leqslant \varepsilon(x - s_1), \tag{16}$$

and then

$$\| \gamma_s(t_0) - \gamma_s(s_1) \| \leqslant \int_0^{t_0 - s_1} \| \dot{\gamma}_s(x) \| \, dx \leqslant \int_0^{t_0 - s_1} (s \| \dot{\xi}(x) \| + (1 - s)) \, dx \leqslant t_0 - s_1 + \frac{\varepsilon}{2}(t_0 - s_1)^2. \tag{17}$$

From (15) we obtain

$$\cos(\beta) \geqslant \frac{1 - s\varepsilon}{1 + (\varepsilon/2)(t_0 + s_1)} \geqslant \frac{1 - \varepsilon}{1 + \varepsilon}. \tag{18}$$

If we choose $\varepsilon \leqslant \frac{1}{3}$ then $\beta \in [-\pi/3, \pi/3]$.

Assume now the contrary of our claim, namely that there exists some intersection point p between $\gamma_s|_{[s_1,s_2]}$ and $\gamma_s|_{[s_3,s_4]}$. Up to a symmetry of indices we can assume that the Euclidean triangle with vertices at $p, \gamma_s(s_1)$ and $\gamma_s(s_2)$ has the angle β at $\gamma_s(s_1)$ within the interval $[\pi/2, \pi)$. This contradicts our estimates (18) for β. $\qquad \square$

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The last ingredient of the proof of Theorem 1 is as follows.

LEMMA 4. *Assume that there exists an isotopy of class $\mathcal{C}^k$ between γ and $\eta = \psi(\varphi(\gamma))$ rel A. Then φ is $\mathcal{C}^k$-isotopic to a compactly supported diffeomorphism from $P\mathrm{Diff}^{k,+}(M, C)$.*

Proof. We can assume that the disk D of diameter E is contained in U.

If the dimension of M is 2, the endpoints of E separate the circle ∂D into two arcs, say F^+ and F^-. The circular order of the three arcs F^+, E, F^- around an endpoint of E is preserved by $\psi \circ \varphi \in \mathrm{Diff}^{k,+}(M, C)$. Note that E is fixed by $\psi \circ \varphi$. Thus there exists a $\mathcal{C}^k$-isotopy which is identity on E, sends $\psi(\varphi(F^+))$ to F^+ and $\psi(\varphi(F^-))$ to F^-. Therefore $\psi \circ \varphi$ is isotopic to a diffeomorphism supported on the complement of D and hence its class is compactly supported. The claim follows now, because ψ is equally compactly supported.

If the dimension of M is at least 3, there exists an isotopy of M sending $\psi(\varphi(\partial D))$ to ∂D, which is identity on E, because $\psi \circ \varphi \in P\mathrm{Diff}^{k,+}(M, C)$ and we conclude as above. $\qquad\square$

3.2 Sparse sets and proofs of Theorems 2, 3 and 4

3.2.1 *Preliminaries.* Let $\mathcal{N}_\varepsilon(a)$ denote the ε-neighborhood $|x - a| < \varepsilon$ of a in $\mathbb{R}$, $\mathcal{N}_\varepsilon^\pm(a)$ the punctured right and left semi-neighborhoods of a, i.e., $a < x < a + \varepsilon$ and $a - \varepsilon < x < a$, respectively.

We say that $a \in C$ is a *left point* of C if there is a left semi-neighborhood $\mathcal{N}^-(a)$ such that $\mathcal{N}^-(a) \cap C = \emptyset$. In the same way we define *right points*.

For $a \in C$ denote by Diff_a^k the stabilizer of a in $\mathrm{Diff}^k(\mathbb{R}, C)$, and by $\mathfrak{diff}_a^k$ the group of k-germs of elements of the stabilizer of a in $\mathfrak{diff}^k(C)$. The superscript $+$ in $\mathrm{Diff}_a^{k,+}$ and $\mathfrak{diff}_a^{k,+}$ means that we only consider those diffeomorphisms that preserve the orientation of the interval, i.e. increasing.

Let φ be a diffeomorphism with $\varphi(a) = a$. We say that φ is *N-flat* at a if

$$\varphi(x) - x = o((x - a)^N) \quad \text{as } x \to a. \tag{19}$$

LEMMA 5. *Assume that C is a σ-sparse subset of $\mathbb{R}$. Let $\varphi \in \mathrm{Diff}_a^1$ be 1-flat at $a \in C$. Then $\varphi|_C$ is identity in a small neighborhood of a.*

Proof. Observe first that $\varphi \in \mathrm{Diff}_a^{1,+}$, since $\varphi'(a) = 1$ and hence φ must be increasing. We can assume without loss of generality that a is not a right point of C. Suppose that φ is non-trivial on $\mathcal{N}_\delta^+(a) \cap C$ for any $\delta > 0$.

We first claim that fixed points of φ accumulate from the right to a. Otherwise, there exists some δ such that $\varphi(x) - x$ keeps a constant sign for all $x \in \mathcal{N}_\delta^+(a)$. Assume that this sign is positive and choose $b \in \mathcal{N}_\delta^+(a) \cap C$. Let $(\alpha, \beta) \subset (a, b)$ be a maximal complementary interval of length at least $\sigma(b - a)$. By maximality $\alpha \in C$. Since $\varphi(\alpha) \in C$ and $\varphi(\alpha) > \alpha$ we have $\varphi(\alpha) \geqslant \beta$, so that

$$\frac{\varphi(\alpha) - a}{\alpha - a} = \frac{\varphi(\alpha) - \alpha}{\alpha - a} + 1 \geqslant \frac{\beta - \alpha}{\alpha - a} + 1 \geqslant \frac{\sigma(b - a)}{\alpha - a} + 1 \geqslant 1 + \sigma. \tag{20}$$

By the mean value theorem there exists $\xi \in (a, \alpha)$ such that:

$$\varphi'(\xi) = \frac{\varphi(\alpha) - a}{\alpha - a} \geqslant 1 + \sigma.$$

But this inequality contradicts the 1-flatness condition for small δ, as taking the limit when $\delta \to 0$ we would obtain $\varphi'(a) \geqslant 1 + \sigma$.

When the sign of $\varphi(x) - x$ is negative we reach the same conclusion by considering $\varphi(\beta) - \beta$. This proves the claim.

Therefore there is a decreasing sequence u_k accumulating at a, such that $\varphi(u_k) = u_k$. As $\varphi|_{C \cap \mathcal{N}_\delta^+(a)}$ is not identity for any $\delta > 0$ there exists a decreasing sequence $v_k \in C$ accumulating on a, such that all $\varphi(v_k) - v_k$ are of the same sign, say positive. Therefore, up to passing to a subsequence, we obtain a sequence of disjoint intervals (α_j, β_j) such that $\beta_{j+1} \leqslant \alpha_j$, $\varphi(\alpha_j) = \alpha_j$, $\varphi(\beta_j) = \beta_j$, and $v_j \in (\alpha_j, \beta_j)$.

Since φ is monotone, it has to be monotone increasing, by above. Thus $\varphi^k(v_j) \in [\alpha_j, \beta_j]$, for any $k \in \mathbb{Z}$, where φ^k denotes the kth iterate of φ. The bi-infinite sequence $\varphi^k(v_j)$ is increasing and so

$$\alpha_j \leqslant \lim_{k \to -\infty} \varphi^k(v_j) < \lim_{k \to \infty} \varphi^k(v_j) \leqslant \beta_j. \tag{21}$$

Now $\lim_{k \to -\infty} \varphi^k(v_j)$ and $\lim_{k \to \infty} \varphi^k(v_j)$ are fixed points of φ and we can assume, without loss of generality that our choice of intervals is such that $\alpha_j = \lim_{k \to -\infty} \varphi^k(v_j)$, $\lim_{k \to \infty} \varphi^k(v_j) = \beta_j$. In particular $\alpha_j, \beta_j \in C$.

As C is σ-sparse there is a complementary interval $(\gamma_j, \delta_j) \subset (\alpha_j, \beta_j)$ of length at least $\sigma(\beta_j - \alpha_j)$. The interval (γ_j, δ_j) cannot contain any point $\varphi^k(v_j)$ and thus there exists some $k_j \in \mathbb{Z}$ such that

$$\varphi^{k_j}(v_j) \leqslant \gamma_j < \delta_j \leqslant \varphi^{k_j+1}(v_j). \tag{22}$$

Denote $\varphi^{k_j}(v_j) = \eta_j$. We have then

$$\frac{\varphi(\eta_j) - \varphi(\alpha_j)}{\eta_j - \alpha_j} - 1 = \frac{\varphi(\eta_j) - \eta_j}{\eta_j - \alpha_j} \geqslant \frac{\sigma(\beta_j - \alpha_j)}{\eta_j - \alpha_j} \geqslant \sigma. \tag{23}$$

By the mean value theorem there exists $\xi_j \in (\alpha_j, \eta_j)$ such that

$$\frac{\varphi(\eta_j) - \varphi(\alpha_j)}{\eta_j - \alpha_j} = \varphi'(\xi_j), \tag{24}$$

and thus such that $\varphi'(\xi_j) \geqslant 1 + \sigma$. As φ' is continuous at a, by letting j go to infinity we derive $\varphi'(a) \geqslant 1 + \sigma$ which contradicts the 1-flatness. $\qquad\square$

LEMMA 6. *If C is σ-sparse and $\varphi \in \mathrm{Diff}_a^1$ is not 1-flat then*

$$|\varphi'(a) - 1| \geqslant \sigma. \tag{25}$$

Proof. Let $\varphi \in \mathrm{Diff}_a^1$ not 1-flat, so that $\varphi'(a) \neq 1$. Let us further suppose that $\varphi'(a) > 1$, the other situation being similar. For any $\delta > 0$ we can choose $b \in \mathcal{N}_\delta^+(a) \cap C$. There is then a maximal complementary interval $(\alpha, \beta) \subset (a, b)$ of length at least $\sigma(b - a)$. By maximality $\alpha \in C$.

We claim that for small enough δ we have $\varphi(\alpha) > \alpha$. Assume the contrary. By the mean value theorem there exists $\xi \in (a, \alpha) \subset (a, b)$ such that

$$\varphi'(\xi) = 1 + \frac{\varphi(\alpha) - \alpha}{\alpha - a} \leqslant 1 \tag{26}$$

and letting δ go to 0 we would obtain $\varphi'(a) \leqslant 1$, contradicting our assumptions. Thus $\varphi(\alpha) > \alpha$, and hence $\varphi(\alpha) \geqslant \beta$. As above, the mean value theorem provides us $\xi \in (a, \alpha)$ so that

$$\varphi'(\xi) = 1 + \frac{\varphi(\alpha) - \alpha}{\alpha - a} \geqslant 1 + \sigma. \tag{27}$$

Letting δ go to zero we obtain $\varphi'(a) \geqslant 1 + \sigma$. When $\varphi'(a) < 1$ we can use similar methods or pass to φ^{-1} in order to obtain $\varphi'(a) \leqslant 1 - \sigma$. $\qquad\square$

Lemma 7. *Let C sparse and $a \in C$. Then one of the following holds:*

(i) *either for any $\varphi \in \mathrm{Diff}_a^{1,+}$, the restriction $\varphi|_C$ is identity in a small neighborhood of a, so that $\mathfrak{diff}_a^{1,+} = 1$;*

(ii) *or else, there is $\psi_a \in \mathrm{Diff}_a^{1,+}$ such that for any $\varphi \in \mathrm{Diff}_a^{1,+}$ the restriction of φ to a small neighborhood $\mathcal{N}_\delta(a) \cap C$ coincides with the iterate $\psi_a^k|_C$ for some $k \in \mathbb{Z}\setminus\{0\}$. Moreover, any such ψ_a is of the form*

$$\psi_a(x) = a + p(x - a) + o(x - a) \quad \text{as } x \to a, \tag{28}$$

where $|p - 1| \geqslant \sigma$. Thus $\mathfrak{diff}_a^{1,+} = \mathbb{Z}$.

Proof. If the first alternative doesn't hold, by Lemma 5 we can assume that there exists some $\varphi \in \mathrm{Diff}_a^{1,+}$ which is not 1-flat.

The map $\chi : \mathrm{Diff}_a^1 \to \mathbb{R}^*$ given by $\chi(\varphi) = \varphi'(a)$ is easily seen to be a group homomorphism. By Lemma 6 the subgroup $\chi(\mathrm{Diff}_a^{1,+})$ of $\mathbb{R}_+^*$ is discrete and non-trivial and thus it is isomorphic to $\mathbb{Z}$. Let $\psi_a \in \mathrm{Diff}_a^{1,+}$ be a germ whose image $\chi(\psi_a)$ is a generator of $\chi(\mathrm{Diff}_a^{1,+})$. Then ψ_a is not 1-flat and thus, by Lemma 6, it satisfies (28).

If $\varphi \in \mathrm{Diff}_a^{1,+}$, then we can write $\varphi = \psi_a^k \theta$, for some $k \in \mathbb{Z}\setminus\{0\}$ and $\theta \in \ker \chi$. But the kernel of χ consists of those $\theta \in \mathrm{Diff}_a^{1,+}$ which are 1-flat. By Lemma 5 the restriction of θ to some neighborhood $\mathcal{N}_\delta(a) \cap C$ is identity. This proves that $\varphi = \psi_a^k$ in a neighborhood $\mathcal{N}_\delta(a) \cap C$, as claimed. $\square$

Remark 6. If $C = C_\lambda$ is the ternary central Cantor set in $\mathbb{R}$, then $\mathfrak{diff}_a^{1,+}(C_\lambda)$ is not always $\mathbb{Z}$. An element a of C_λ is called λ-rational if it has an eventually periodic development

$$a = \sum_{i=1}^{\infty} a_i \lambda^i,$$

where $a_i \in \{0, \lambda - 1\}$. Therefore $\mathfrak{diff}_a^{1,+}(C_\lambda)$ is $\mathbb{Z}$ if and only if a is λ-rational and trivial, otherwise.

Remark 7. Since the subgroup $\chi(\mathrm{Diff}_a^1) \subset \mathbb{R}^*$ is discrete there exists $\lambda \geqslant 1$ such that $\chi(\mathrm{Diff}_a^1)$ is of the form $\langle \lambda \rangle$, $\langle -\lambda \rangle$ or $\langle \pm \lambda \rangle$. Here $\langle x \rangle$ denotes the subgroup of $\mathbb{R}^*$ generated by x. In particular $\mathfrak{diff}_a^1$ is isomorphic to either 1, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}$, or else $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

However, if a is a left (or right) point of C then there is no decreasing homeomorphism of $(\mathbb{R}, C)$ fixing a. Thus $\mathrm{Diff}_a^1 = \mathrm{Diff}_a^{1,+}$, and the result of Lemma 28 holds more generally for Diff_a^1.

3.2.2 *Proof of Theorem 2.* We need to show that the identity is an isolated point of the group $\mathfrak{diff}^1(C)$, if C is σ-sparse. To this purpose consider an element $\mathfrak{diff}^1(C)$ having a representative $\psi \in \mathrm{Diff}^1(\mathbb{R}, C)$ such that

$$1 - \sigma < \psi'(x) < 1 + \sigma \quad \text{for any } x \in C. \tag{29}$$

There is no loss of generality in assuming that $\psi \in \mathrm{Diff}^{1,+}(\mathbb{R}, C)$, i.e. that ψ is monotone increasing. The minimal element $\min C$ of C should therefore be fixed by any element of $\mathrm{Diff}^{1,+}(\mathbb{R}, C)$, in particular by ψ. By Lemma 6, $\psi \in \mathrm{Diff}_{\min C}^1(\mathbb{R}, C)$ must be 1-flat at $\min C$.

Consider the set

$$U = \{x \in C; \psi(z) = z \text{ for any } z \in C \cap (-\infty, x]\}. \tag{30}$$

The set U is non-empty, as $\min C \in U$. Let $\xi = \sup U$.

Assume first that ξ is not a right point of C. Since ψ is continuous, $\xi \in U$ so that $\psi \in \mathrm{Diff}^1_\xi$. From Lemma 6 $\psi'(\xi) = 1$ and ψ is 1-flat at ξ. According to Lemma 5 there is some $\delta > 0$ such that the restriction $\psi|_{C \cap \mathcal{N}^+_\delta(\xi)}$ is identity, which contradicts the maximality of ξ.

If ξ is a right point of C, then there is some maximal complementary interval $(\xi, \eta) \subset \mathbb{R} \setminus C$. Since $\psi|_{C \cap [\min C, \xi]}$ is identity it follows that $\psi(C \cap [\xi, \infty)) \subset C \cap [\xi, \infty)$. As η is the minimal element of $C \cap (\xi, \infty)$ it should be a fixed point of $\psi|_{[\xi, \infty)}$ and so $\eta \in U$. This contradicts the maximality of ξ. Hence ψ is identity on C.

Remark 8. The same arguments show that if $C \subset [0,1]$ is a sparse Cantor set and $\mathfrak{diff}^{1,+}_0(C) = 1$, then $\mathfrak{diff}^{1,+}(C) = 1$.

For the second claim of the theorem let V_δ be the set of those elements in $\mathfrak{diff}^1_{S^1}(C)$ having a representative $\psi \in \mathrm{Diff}^1(S^1, C)$ such that

$$1 - \delta < \psi'(x) < 1 + \delta \quad \text{for any } x \in C. \tag{31}$$

Here elements of $\mathrm{Diff}^1(S^1)$ are identified with real periodic functions on $\mathbb{R}$. We choose $\delta < \min(\sigma, 0.3)$. It is enough to prove that V_δ is finite.

Consider a complementary interval $J \subset S^1 - C$ of maximal possible length, say $|J|$. Consider its right end η, with respect to the cyclic orientation. If $\psi \in V_\delta$ is such that $\psi(\eta) = \eta$, then the arguments from the proof of Theorem 2 show that $\psi(x) = x$ when $x \in C$.

We claim that the set of intervals of the form $\psi(J)$, for $\psi \in V_\delta$ is finite. Each $\psi(J)$ is a maximal complementary interval, because if it were contained in a larger interval J', then $\psi^{-1}(J)$ would be a complementary interval strictly larger than J. This shows that any two such intervals $\psi(J)$ and $\varphi(J)$ are either disjoint or they coincide, for otherwise their union would contradict their maximality. Further, each $\psi(J)$ has length at least $(1 - \delta)|J|$. This shows that the set of intervals is a finite set $\{J_1, J_2, \ldots, J_k\}$.

Assume that $\psi(J) = \varphi(J)$ and both ψ and φ preserve the orientation of the circle. If the right end of J is η, with respect to the cyclic orientation, then $\varphi \circ \psi^{-1}$ sends J to J and hence fixes η. Then the arguments from the proof of Theorem 2 show that $\varphi \circ \psi^{-1}(x) = x$ when $x \in C$. It follows that there are at most $2k$ elements in V_δ, finishing the proof of the first part.

3.2.3 *Proof of Theorem 3.* Let C be a Cantor set contained within a $\mathcal{C}^1$-embedded simple closed curve L on the orientable manifold M. For the sake of simplicity we will suppose from now on that M is a surface, but the proof goes on without essential modifications in higher dimensions. Let φ be a diffeomorphism of M sending C into C. Fix a parameterization of a collar N such that (N, L) is identified with $(L \times [-1,1], L \times \{0\})$. Denote by $\pi : N \to L$ the projection on the first factor and by $h : N \to [-1,1]$ the projection on the second factor.

There exists an open neighborhood U of C in L so that $\varphi(U) \subset N$. In particular, the closure $\overline{U}$ is a finite union of closed intervals. The map $\varphi : \overline{U} \to N = L \times [-1,1]$ has the property $h \circ \varphi(a) = 0$, for each $a \in C$. Therefore the differential $D_a(h \circ \varphi) = 0$, for each $a \in C$. Since φ is a diffeomorphism $D_a(\pi \circ \varphi) \neq 0$, for every $a \in C$.

For each $a \in C$ consider an open interval neighborhood U_a within L, so that $D_x(\pi \circ \varphi) \neq 0$ and $\| D_x(h \circ \varphi) \| < 1$, for every $x \in \overline{U_a}$. We obtain an open covering $\{U_a; a \in C\}$ of C. As C is compact there exists a finite subcovering by intervals $\{U_1, U_2, \ldots, U_n\}$. Without loss of generality one can suppose that $U_j \subset U$, for all j. We consider such a covering having the minimal number of elements. This implies that $\overline{U_j}$ are disjoint intervals.

For every j the map $\pi|_{\varphi(\overline{U_j})} : \varphi(\overline{U_j}) \to \pi(\varphi(\overline{U_j})) \subset L$ is a diffeomorphism on its image, since $\varphi(\overline{U_j})$ is connected and $D_x(\pi \circ \varphi) \neq 0$, for any $x \in \overline{U_j}$.

Consider a slightly smaller closed interval $I_j \subset U_j$ such that $I_j \cap C = U_j \cap C$.

Let μ be a positive smooth function on $\bigsqcup_{j=1}^n \overline{U_j}$ such $\mu(t)$ equals 1 near the boundary points and vanishes on $\bigsqcup_{j=1}^n I_j$. Define $\phi_s : \bigsqcup_{j=1}^n \overline{U_j} \to N$ by

$$\phi_s(x) = (\pi \circ \varphi(x), (s\mu(x) + 1 - s) \cdot h \circ \varphi(x)). \tag{32}$$

Then $\phi_0(x) = \varphi(x)$ and for each $s \in [0, 1]$ we have $\phi_s(x) = \varphi(x)$, for x near the boundary points of $\bigsqcup_{j=1}^n \overline{U_j}$. Furthermore $\phi_1(x) = \pi \circ \varphi(x) \in L$, when $x \in \bigsqcup_{j=1}^n I_j$. One should also notice that $\phi_s(x) = \varphi(x)$, for each $x \in C$ and $s \in [0, 1]$.

Now denote $J_j = \pi \circ \varphi(I_j)$. It is clear that $C = \varphi(C) \subset \bigcup_{j=1}^n J_j$. We claim that we can assume that J_j are disjoint. Indeed, since φ is bijective we have $\varphi(I_j \cap C) \cap \varphi(I_k \cap C) = \emptyset$, for any $j \neq k$. Since $\varphi(I_j \cap C)$ are closed subsets of L there exists $\varepsilon > 0$ so that $d(\varphi(I_j \cap C), \varphi(I_k \cap C)) \geqslant \varepsilon$, for $j \neq k$, where d is a metric on L. Since $\phi_1(I_j \cap C) = \varphi(I_j \cap C)$, we have $d(\phi_1(I_j \cap C), \phi_1(I_k \cap C)) \geqslant \varepsilon$, for $j \neq k$. Thus there exist some open neighborhoods J_j' of $\phi_1(I_j \cap C)$ within L so that $d(J_j', J_k') \geqslant \frac{1}{2}\varepsilon$, for all $j \neq k$. As ϕ_1 is a diffeomorphism there exist open neighborhoods I_j' of $I_j \cap C$ with the property that $\phi_1(I_j') \subset J_j'$, for all j. Now I_j' and J_j' are finite unions of open intervals. We can replace them by closed intervals with the same intersection with C. This produces two new families of disjoint closed intervals related by ϕ_1, as the initial situation. This proves the claim.

We obtained that there exist two coverings $\{I_1, I_2, \ldots, I_n\}$ and $\{J_1, J_2, \ldots, J_n\}$ of C by disjoint closed intervals and a diffeomorphism $\phi_1 : \bigsqcup_{j=1}^n I_j \to \bigsqcup_{j=1}^n J_j$ such that $\phi_1(x) = \varphi(x)$, for any $x \in C$.

Notice that the sign of $D_a(\pi \circ \varphi)$ might not be the same for all intervals.

Every partition of C induced by a covering $\{I_1, I_2, \ldots, I_n\}$ as above is determined by the choice of complementary intervals, namely the $n - 1$ connected components of $L \setminus \bigcup_{j=1}^n I_j$. It follows that there are only countably many finite partitions of C of the type considered here. Next, the set of those elements of $\mathfrak{diff}_M^1(C)$ which arise from partitions induced by the coverings $\{I_1, I_2, \ldots, I_n\}$ and $\{J_1, J_2, \ldots, J_n\}$ of C is acted upon transitively by the stabilizer of one partition. The stabilizer of one partition embeds into the product of $\mathfrak{diff}_{I_j}^1(C \cap I_j)$. Theorem 2 then implies that $\mathfrak{diff}_M^1(C)$ is countable.

3.2.4 *Proof of Theorem 4.* Before proceeding, we need some preparatory material. Let $A \subset \mathbb{R}^n$ be a set without isolated points. Let $T_p\mathbb{R}^n$ denote the tangent space at p on $\mathbb{R}^n$ and $UT_p\mathbb{R}^n \subset T_p\mathbb{R}^n$ the sphere of unit vectors. For any $p \in A$ one defines the *unit tangent spread* $UT_pA \subset UT_p\mathbb{R}^n$ at p as the set of vectors $v \in UT_p\mathbb{R}^n$ for which there exists a sequence of points $a_i \in A$ with $\lim_{i \to \infty} a_i = p$ and

$$\lim_{i \to \infty} \frac{a_i - p}{\| a_i - p \|} = v.$$

Vectors in UT_pA will also be called *(unit) tangent vectors* at p to A. We also set $T_pA = \mathbb{R}_+ \cdot UT_pA \subset T_p\mathbb{R}^n$ for the *tangent spread* at p.

A differentiable map $\varphi : (\mathbb{R}^n, A) \to (\mathbb{R}^n, B)$ induces a tangent map $T_p\varphi : T_pA \to T_{\varphi(p)}B$. Specifically, let $D_p\varphi : T_p\mathbb{R}^n \to T_{\varphi(p)}\mathbb{R}^n$ be the differential of φ; then we have

$$T_p\varphi = U(D_p\varphi),$$

where for a linear map $L : V \to W$ between vector spaces we denoted by $U(L) : U(V) \to U(W)$ the map induced on the unit spheres, namely

$$U(L)v = \frac{L(v)}{\|L(v)\|}.$$

As the unit tangent spread UT_pA is a subset of the unit sphere, it inherits the spherical geometry and metric. In particular, it makes sense to consider the convex hull $\mathrm{Hull}(UT_pA) \subset UT_p\mathbb{R}^n$ in the sphere.

Although tangent spreads to product Cantor sets might depend on the particular factors, their convex hulls have a simple description. Let $C = C_1 \times C_2 \times \cdots \times C_n \subset \mathbb{R}^n$ be a product of Cantor sets $C_i \subset \mathbb{R}$. The usual cubical complex underlying the n-dimensional cube $[0,1]^n$ will be denoted by $\square^n$. Then denote by $Lk(p)$ the spherical link of $p \in \square^n$. If p belongs to a k-dimensional cube but not to a $(k+1)$-dimensional cube of $\square^n$ then $Lk(p)$ is isometric to the link $\mathcal{L}_{k,n}$ of the origin in $\mathbb{R}^k \times \mathbb{R}_+^{n-k}$. Thus there are precisely $n+1$ different isometry types of links of points.

Now a direct inspection shows that for each $p \in C$ there exists some k so that the convex hull $\mathrm{Hull}(UT_pA)$ is isometric to $\mathcal{L}_{k,n}$.

When the diffeomorphism $\varphi : (\mathbb{R}^n, C) \to (\mathbb{R}^n, C)$ is also *conformal*, then the tangent maps are isometries between the unit tangent spreads, because the spherical distance is given by angles between the corresponding vectors. However this is not true for general diffeomorphisms.

Nevertheless the spherical links $\mathcal{L}_{k,n}$ are quite particular. There exist $n+k$ vectors along the coordinates' axes which are extremal points of UT_pC, such that their convex hull is $\mathrm{Hull}(UT_pC)$, so isometric to $\mathcal{L}_{k,n}$. These are vectors of the form $e_i, -e_i, e_j$, where e_i correspond to the coordinates' axes in $\mathbb{R}^k$ and e_j to those in $\mathbb{R}^{n-k}$. Now, any diffeomorphism $\varphi : (\mathbb{R}^n, C) \to (\mathbb{R}^n, C)$ should send a unit tangent spread of type $\mathcal{L}_{k,n}$ into one of the same type, since $\mathcal{L}_{k,n}$ is not affinely equivalent to $\mathcal{L}_{k',n}$, for $k \neq k'$. Moreover, the extremal vectors are sent into extremal vectors of the same type.

Further, let $\varphi \in \mathrm{Diff}^1(\mathbb{R}^n, C)$ such that $\|D_a\varphi - \mathbf{1}\| \leqslant \varepsilon$ for all $a \in C$. Assume now that the unit tangent spread UT_aC is isometric to $\mathcal{L}_{0,n}$, namely it is of *corner* type. In this case $U(D_a\varphi)$ should permute the n coordinate vectors, which are the extremal vectors of $\mathcal{L}_{0,n}$. Therefore either $U(D_a\varphi) = \mathbf{1}$, or else

$$\|U(D_a\varphi) - \mathbf{1}\| \geqslant \sqrt{2},$$

which yields

$$\|D_a\varphi - \mathbf{1}\| \geqslant \sqrt{2}.$$

In other words, taking $\varepsilon < \sqrt{2}$ any diffeomorphism φ as above should satisfy $U(D_a\varphi) = \mathbf{1}$. Now, if φ is of class $\mathcal{C}^1$ then $U(D_a\varphi)$ is continuous. Since the set of corner points is dense in C we derive $U(D_a\varphi) = \mathbf{1}$, for any $a \in C$. This is the same as saying that for any $a \in C$ the linear map $D_a\varphi$ is represented by a diagonal matrix, with respect to the standard coordinate system of $\mathbb{R}^n$.

PROPOSITION 1. *Let $a \in C$ be a corner point. The map $\chi : \mathfrak{diff}^1_{\mathbb{R}^n,a}(C) \to (\mathbb{R}^*)^n$, which associates to the germ φ the eigenvalues of $D_a\varphi$ is an isomorphism onto a discrete subgroup of $(\mathbb{R}^*)^n$.*

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be the standard coordinate functions on $\mathbb{R}^n$ and $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denote the projection onto the hyperplane $H_j = \{x_j = 0\}$. For the sake of simplicity we assume that $a = (0, 0, \ldots, 0)$, and that the convex hull of the unit tangent spread is the union of the sets $H_j^+ = H_j \cap \{x_i \geqslant 0, i = 1, \ldots, n\}$. We will use induction on n. The claim was proved in Lemma 7 for $n = 1$. Assume it holds for all dimensions at most $n - 1$.

Let $\varphi \in \mathrm{Diff}^1(\mathbb{R}^n, C)$ such that $\varphi(a) = a$. Assume that $\|D_x\varphi - \mathbf{1}\| < \frac{1}{2}\sigma < \frac{1}{2}$ for all x in a neighborhood V of a in $\mathbb{R}^n$. We will prove that $\varphi|_C$ is a trivial germ at a. This shows that the image of χ is a discrete subgroup of $(\mathbb{R}^*)^n$ and the kernel of χ is trivial.

Consider the maps $\varphi_j : H_j \to H_j$ given by $\varphi_j(x) = \pi_j \circ \varphi(x)$. The determinant of $D_a\varphi_j$ is the product of all eigenvalues of $D_a\varphi$ but the jth eigenvalue, and hence it is non-zero. Moreover, we have $\|D_a\varphi_j - \mathbf{1}\| < \frac{1}{2}\sigma$.

We claim the following.

LEMMA 8. *The map $\varphi_j : H_j \cap V \to H_j$ is injective.*

Proof. Assume the contrary, namely that there exist two points $p, q \in H_j \cap V$ such that $\pi_j(\varphi(p)) = \pi_j(\varphi(q))$. Consider the first non-trivial case $n = 2$, when H_j^+ are half-lines issued from a. The mean value theorem and the previous equality prove that there exists some $\xi \in H_j^+ \cap V$ between p and q so that $(\pi_j \circ \varphi)'(\xi) = 0$. This amounts to the fact that the image of $D_\xi\varphi$ is contained in the kernel of $D_{\varphi(\xi)}\pi_j$, namely that

$$\langle D_\xi\varphi(v_j), v_j \rangle = 0,$$

where v_j is a unit tangent vector to H_j^+ at ξ. We derive $\|D_\xi\varphi_j - \mathbf{1}\| \geqslant 1$, contradicting our assumptions.

In the general case $n > 2$ we will use a trick to reduce ourselves to $n = 2$, because we lack a multidimensional mean value theorem. Let P be a generic affine 2-dimensional half-plane whose boundary line passes through p and q. We can find arbitrarily small $\mathcal{C}^1$-isotopy deformations ψ of φ_j so that $\psi(H_j)$ is transversal to P and $\|D_a\psi - \mathbf{1}\| < \sigma$. It follows that $\psi(H_j) \cap P$ is a 1-dimensional manifold Z with a boundary containing both p and q. Now either there exist two distinct points of the boundary ∂Z joined by an arc within Z, or else there is an arc of Z issued from p which returns to p, contradicting the transversality of the intersection $\psi(H_j) \cap P$. In any case the mean value argument above shows that a point $\psi(\xi)$ of Z should exist for which the tangent vector v is orthogonal to H_j. We can write $v = D_\xi\psi(w)$, for some tangent vector $w \in H_j$ at ξ. It follows that

$$\langle D_\xi\psi(w), w \rangle = 0,$$

which implies $\|D_\xi\psi - \mathbf{1}\| \geqslant 1$, contradicting our assumptions. $\square$

It follows that $\varphi_j : H_j \cap V \to H_j$ is an injective map of maximal rank in a neighborhood V of a, and hence a diffeomorphism on its image. The projection π_j sends C into $C \cap H_j$, so that

$$\varphi_j(C \cap H_j \cap V) \subset C \cap \varphi(H_j \cap V) \subset C \cap H_j.$$

Our aim is to use the induction hypothesis for φ_j. In order to do that we need to show that the class of φ_j defines indeed an element of $\mathfrak{diff}^1_{\mathbb{R}^{n-1}, a}(C)$, where we identified H_j with $\mathbb{R}^{n-1}$.

We assume from now on that the neighborhood V is a parallelepiped, all whose vertices being corner points. Its boundary ∂V will then consist of the union of the faces $V_j = \partial V \cap H_j^+$ with their respective parallel faces V_j'. The parallelepiped V is surrounded by gaps, whose smaller width is some $\delta > 0$. Let V^δ be the δ-neighborhood of V. If φ is Lipschitz with Lipschitz constant $1 + \varepsilon$ and

$$(1 + \varepsilon)l_i < \delta + l_i$$

where l_i are the edge lengths of V then the image $\varphi(V)$ is contained in V^δ, so that $\varphi_j(V_j) \subset V^\delta \cap H_j$.

Further $\varphi_j(\partial V_j)$ bounds $\varphi_j(V_j)$ and thus there are no points of $C \cap H_j$ accumulating on $\varphi_j(V_j)$, as their unit tangent spread cannot be of the type $\mathcal{L}_{n-1,n-1}$. Thus $C - \varphi_j(V_j)$ is a closed subset of C and hence its distance to $\varphi_j(V_j)$ is strictly positive. There exists then an open set $U \subset V^\delta$ which contains $\varphi_j(V_j)$ such that $U \cap (C - \varphi_j(V_j)) = \emptyset$. It follows that there exists an extension of φ_j to a diffeomorphism Φ_j of (H_j, C) which is identity outside U, and hence on $(V_\delta \cap H_j) \cup (C - \varphi(V_j))$.

It only remains to check that $\Phi_j^{-1}(C)$ is also contained in C, as needed for $\Phi_j \in \mathrm{Diff}^1$ $(\mathbb{R}^{n-1}, C)$. This follows as shown below.

LEMMA 9. *The map φ_j has the property*

$$\varphi_j(C \cap V_j) = C \cap \varphi(V_j).$$

Proof. Assume that there exists some point p in $\varphi(V_j) \cap C$ which does not belong to V_j. Then the line issued from p which is orthogonal to V_j intersects $\varphi(V_j)$ only once, from Lemma 8. On the other hand there are points of C on this line, as C is a product and $p \notin V_j$. By Jordan's theorem there exist points of C which belong to different components of $R^n - \varphi(\partial V)$ which contradicts the fact that φ is surjective on C.

Thus $\varphi(C \cap V_j) \subset C \cap V_j$. The same argument for φ^{-1} yields the opposite inclusion and hence $\varphi(C \cap V_j) = C \cap V_j$. Our claim follows. $\qquad\square$

Lemma 9 tells us that φ_j defines a germ in $\mathfrak{diff}^1_{H_j,a}(C \cap H_j)$, namely both φ_j and φ_j^{-1} sends $C \cap H_j$ into itself. By the induction hypothesis $\varphi_j|_{C \cap H_j}$ must be identity in a neighborhood of a within H_j.

Notice that this implies already that $D_a\varphi = \mathbf{1}$, and hence establishing the first claim of Proposition 1.

For the second claim we consider the distance $d(C - V, V) = \mu > 0$, as V is surrounded by gaps. We suppose further that

$$\|D_x\varphi - \mathbf{1}\| < \min\left(\frac{\sigma}{2}, \frac{\mu}{1+\sigma}\right).$$

We know that $\varphi(y, 0) = (y, u(y))$, for $y \in C \cap V \cap H_n$ and some function $u \geqslant 0$. The next step is to show that $u|_{C \cap V \cap H_n} = 0$.

Assume that there exists some $x \in C \cap V \cap H_n$ so that $u(x) > 0$. Observe that $u(x) \in C_n$, since $\varphi(C) \subset C$. Since points of C_n which are not endpoints are dense in C_n there should exist $x \in C$ for which $u(x)$ is not an endpoint of C_n. Set $z = (x, u(x)) \in C$.

Then for each $\nu > 0$ there exist points $z_+, z_- \in C$ with $\pi_n(z_+) = \pi_n(z_-) = x$, so that the distances $d(z_+, z), d(z_-, z) < \nu$.

Observe that the segment z_+z_- intersects just once $\varphi(H_n^+)$, namely at z. One might expect to use Jordan's theorem in order to derive that $z_+ \in C$ and $z_- \in C$ could not belong to the same connected component of $\varphi(\partial V)$. This is not exactly true, as the segment z_+z_- could possibly intersect other sheets like $\varphi(H_j^+)$ which are part of $\varphi(\partial V)$.

Set r for the distance between $x \in H_n^+$ and the union of the other $2n - 1$ faces of ∂V. By the induction hypothesis we can assume that $r > 0$. Choose now ν so that $\nu < \min((1 - \sigma)r/2, \mu(1-\sigma)/2)$.

Suppose that there exist $x_+, x_- \in C \cap V$ such that $\varphi(x_+) = z_+$ and $\varphi(x_-) = z_-$. By Jordan's theorem the segment z_+z_- intersects at least once $\varphi(\partial V - H_n^+)$, say in a point $\tilde{z} = \varphi(\tilde{x})$.

We have then $d(x, \tilde{x}) \geqslant r$, while

$$d(\varphi(x), \varphi(\tilde{x})) \leqslant d(z_+, z_-) \leqslant 2\nu.$$

On the other hand the $\mathcal{C}^1$-diffeomorphism φ^{-1} is Lipschitz with Lipschitz constant bounded by $\sup_{x \in V} \|D_x \varphi^{-1}\|$. Now, by standard functional calculus we have:

$$\|D_x \varphi^{-1}\| \leqslant \sum_{k=0}^{\infty} \|\mathbf{1} - D_x \varphi\|^k < \frac{1}{1 - \sigma}.$$

Therefore the Lipschitz constant of φ^{-1} is bounded above by $1/(1 - \sigma)$ so that

$$d(x, \tilde{x}) \leqslant \frac{1}{1 - \sigma} d(\varphi(x), \varphi(\tilde{x})) \leqslant \frac{2\nu}{1 - \sigma}.$$

This contradicts our choice of ν.

Furthermore if one of x_+, x_-, say x_+ belongs to $C - V$ then we have $d(x, x_+) \geqslant \mu$ while

$$d(\varphi(x), \varphi(x_+)) \leqslant \nu$$

and the argument above still leads to a contradiction.

This shows that φ cannot be surjective on C. On the other hand a diffeomorphism of $\mathbb{R}^n$ which preserves C restricts to a bijection on C. If it were not surjective then its inverse would send points of C outside.

In particular $u(x)|_{C \cap H_n^+} = 0$ and so $\varphi|_{C \cap H_n^+}$ is identity. The same proof shows that $\varphi|_{C \cap H_j^+}$ is identity, for all j.

By using the same argument when a runs over the points of $V \cap C \cap \bigcup_{j=1}^n H_j^+$ we derive that $\varphi|_{C \cap V}$ is identity, as claimed. □

End of the proof of theorem 4. The proof is by induction on n. For $n = 1$ this was already proved above. Let V denote now the smallest parallelepiped containing C, in order to match previous notations and constructions. Suppose that $\varphi \in \text{Diff}^1(\mathbb{R}^n, C)$ is such that $\|D_x \varphi - \mathbf{1}\| < \varepsilon$, for all $x \in V^\delta$. Then $\varphi(\partial V)$ surrounds C and the proof of Lemma 9 gives us $\varphi(C \cap \partial V) = C \cap \partial V$. Moreover, each φ_j preserves the associated face V_j. By the induction hypothesis φ_j is the identity. It follows that $\varphi|_{C \cap \partial V}$ is the identity. We can therefore use Proposition 1 to derive that around every corner point of $C \cap \partial V$ the map $\varphi|_C$ is identity. The same argument works for all corner points of V.

Remark 9. If $C = C_\lambda^n$, then $\mathfrak{diff}_a^{1,+}(C_\lambda)$ is isomorphic to $\mathbb{Z}^{r(a)}$, where $r(a)$ is the number of coordinates of a which are λ-rational (compare with [BL10]).

4. Diffeomorphism groups of specific Cantor sets

4.1 Proof of Theorem 5

Observe first that C_Φ is a Cantor set. Indeed the contractivity assumption implies that an infinite intersection $\lim_{p \to \infty} \phi_{i_1} \phi_{i_2} \cdots \phi_{i_p}(M)$ cannot contain but a single point. Two such points which are distinct are separated by some smoothly embedded sphere, which is the image of ∂M by an element of the semigroup generated by Φ, so that the set C_Φ is totally disconnected. The perfectness follows the same way.

We will draw a rooted $(n+2)$-valent tree $\mathcal{T}$ with edges directed downwards. When $M = [0,1]$ there is an extra structure on $\mathcal{T}$, as all edges issued from a vertex are enumerated from left to right.

There is a one-to-one correspondence between the points of the boundary at infinity of the tree and the points of the Cantor set $C = C_\Phi$ associated to the invertible IFS (Φ, M). To each point $\xi \in C$ we can assign an infinite sequence $I = i_1 i_2 \cdots i_p \cdots$, so that $\xi = \xi(I)$ where we denoted:

$$\xi(I) = \bigcap_{p=1}^{\infty} \phi_{i_1} \phi_{i_2} \cdots \phi_{i_p}(M).$$

The vertices of the tree are endowed with a compatible labeling by means of finite multi-indices I, where the root has associated the empty index and the vertex v_I is the one reached after traveling along the edges labeled $i_1, i_2, \ldots, i_p$. We also put

$$\phi_I(x) = \phi_{i_1} \phi_{i_2} \cdots \phi_{i_p}(x)$$

for finite I. This extends obviously to the case of infinite multi-indices I.

We further need to introduce a special class of germs, as follows.

DEFINITION 9. The standard germ associated with the finite multi-indices I and J is the diffeomorphism $\phi_{I/J} : \phi_I(M) \to \phi_J(M)$ given by

$$\phi_{I/J}(\phi_I(x)) = \phi_J(x). \tag{33}$$

Standard germs preserve the Cantor set C as germs, namely $\phi_{I/J}(C \cap \phi_I(M)) \subset C \cap \phi_J(M)$. In fact if S is an infinite multi-index then

$$\phi_{I/J}(\xi(IS)) = \xi(JS).$$

Graphically we can realize this map as a partial isomorphism of the tree $\mathcal{T}$ which maps the subtree hanging at v_I onto the subtree hanging at v_J.

Consider a pair (t_1, t_2) of finite labeled subtrees of the same degree of $\mathcal{T}$ both containing the root, and whose leaves are enumerated $v_{I_1}, v_{I_2}, \ldots, v_{I_p}$ and $v_{J_1}, v_{J_2}, \ldots, v_{J_p}$.

LEMMA 10. Assume that ϕ_j are orientation preserving diffeomorphisms of M. Then the map

$$\phi(x) = \phi_{I_k/J_k}(x) \quad \text{if } x \in \phi_{I_k}(C) \tag{34}$$

defines an element $\phi_{(t_1, t_2)} \in \mathfrak{diff}^{1,+}(C)$.

Proof. We know that $C = \bigcup_{i=0}^{n} \phi_i(C)$, since C is the attractor of Φ. By recurrence on the number of leaves we show that

$$C = \bigcup_{i=0}^{n} \phi_{I_i}(C)$$

for any finite subtree t of $\mathcal{T}$ containing the root and having leaves v_{I_i}, $i = 0, n$. Now ϕ is a smooth orientation preserving map defined on $\bigcup_{i=0}^{n} \phi_{I_i}(M)$, and so its domain of definition contains C.

When the dimension $d = 1$, the complementary $M \backslash \bigcup_{i=0}^{n} \phi_{I_i}(M)$ is the union of finitely many intervals, which we call gaps and there exists by orientability assumption an extension of ϕ to a diffeomorphism of $M = [0,1]$ sending gaps into gaps.

When the dimension $d > 1$, the complementary gap $M \setminus \bigcup_{i=0}^{n} \phi_{I_i}(M)$ is now connected and diffeomorphic to the standard disk with $(n+1)$ holes. Moreover, the restriction of ϕ to every sphere $\partial \phi_{I_i}(M)$ is isotopic to identity since it is orientation preserving and it admits an extension to the ball. Taking a suitable smoothing at the singular vertex of the conical extension of $\phi|_{\bigcup_{i=0}^{n} \phi_{I_i}(\partial M)}$ we obtain an extension of ϕ to a diffeomorphism of the ball M, possibly non-trivial on ∂M.

This extension preserves C invariant as gaps are disjoint from C and therefore defines an element $\phi_{(t_1,t_2)} \in \mathfrak{diff}^{1,+}(C)$. $\qquad \square$

End of the proof of Theorem 5. Let us stabilize the pair of trees (t_1, t_2) to a pair (t_1', t_2'), where t_j' is obtained from t_j by adding the first descendants at vertex v_{I_s}, for $j = 1$ and v_{J_s}, when $j = 2$. The new vertices come with a compatible labeling. Moreover, an orientation preserving diffeomorphism of C induces a monotone map of the boundary of the tree, when $d = 1$.

By direct inspection using the explicit form of ϕ we find that:

$$\phi_{(t_1,t_2)} = \phi_{(t_1',t_2')}.$$

Thus the map which associates to the pair (t_1, t_2) of labeled trees the element $\phi_{(t_1,t_2)}$ factors through a map $F_{n+1} \to \mathfrak{diff}^{1,+}(C)$, for $d = 1$, and $V_{n+1} \to \mathfrak{diff}^{1,+}(C)$, for $d = 2$, respectively. This is easily seen to be a homomorphism. When $I \neq J$ the map $\varphi_{I/J}|_C$ is not identity since $\varphi_I(M) \cap \varphi_J(M) = \emptyset$. This proves that the homomorphism defined above is injective, thereby ending the proof of Theorem 5.

Remark 10. There is a more general setting in which we allow basins to have boundary fixed points. We say that the compact submanifold M is an *attractive basin* for $\Phi = (\phi_0, \phi_1, \ldots, \phi_n)$ if, for all $j \in \{0, 1, \ldots, n\}$ we have

  (i) $\phi_j(\text{int}(M)) \subset \text{int}(M)$;
 (ii) $\text{int}(\phi_j^{-1}(\phi_j(\partial M) \cap \partial M)) \supset \text{int}(\phi_j(\partial M) \cap \partial M)$;
(iii) $\phi_i(M) \cap \phi_j(M) = \emptyset$, for any $i \neq j \in \{0, 1, \ldots, n\}$;
(iv) $\text{int}(\phi_j(\partial M) \cap \partial M) \subset \text{int}(\phi_j^{-1}(\phi_j(\partial M) \cap \partial M))$.

Using similar arguments one can show that $\mathfrak{diff}^{1}(C_\Phi)$ contains F_{n+1} whenever Φ has an attractive basin.

Remark 11. If the Cantor set C is invertible, namely there exists an orientation reversing diffeomorphism ϕ of M preserving C, then we can replace the homeomorphisms ϕ_j which reverse the orientation by $\phi \circ \phi_j$. However, there exist non-invertible Cantor subsets, for instance the union of two copies $C_\lambda \cup (1 + C_\mu)$, for $\lambda \neq \mu$.

4.2 Proof of Theorem 6 for $C = C_\lambda$

Our strategy is to give first a detailed proof of Theorem 6 in the case when $C = C_\lambda$ and then to explain the necessary changes needed to achieve the general case in the next section.

We first need the following lemma.

LEMMA 11. *If a is a left (or right) point of C_λ, then $\chi(\text{Diff}_a^1)$ is the subgroup $\langle \lambda \rangle \subset \mathbb{R}^*$.*

Proof. Recall from Remark 7 that $\text{Diff}_a^1(C_\lambda) = \text{Diff}_a^{1,+}(C_\lambda)$. The set $L(C_\lambda)$ of left points of C_λ is affinely locally homogeneous, namely for any two left points a and b there exists an affine

germ sending a neighborhood of a in C_λ into a neighborhood of b in C_λ. Therefore it suffices to analyze $\mathrm{Diff}_0^{1,+}(C_\lambda)$. Moreover, 0 is the minimal element of C_λ and therefore it should be fixed by any element of $\mathrm{Diff}^{1,+}(C_\lambda)$.

Elements of $L(C_\lambda)$ can be described explicitly, as

$$L(C_\lambda) = \bigcup_{n=1}^\infty \left\{ x \in [0,1]; x = \sum_{j=1}^n a_j \lambda^{-j}, \text{ where } a_j \in \{0, \lambda - 1\} \right\}. \tag{35}$$

Therefore there exists δ such that the multiplication by $\lambda \in \mathbb{R}^*$ sends $C_\lambda \cap \mathcal{N}_\delta(0)$ into C_λ. This easily implies that $\chi(\mathrm{Diff}_a^{1,+})$ contains the subgroup $\langle \lambda \rangle$.

For the reverse inclusion we need a sharpening of Lemma 6. Note first that the set of lengths of gaps in C_λ is $\{(\lambda - 2)\lambda^{-n}, n \in \mathbb{Z}_+ \setminus \{0\}\}$. In particular, the quotients of the lengths of any two gaps belong to $\langle \lambda \rangle$.

Let $\alpha > 1$ be a minimal element occurring in $\chi(\mathrm{Diff}_0^{1,+}(C_\lambda)) \subset \mathbb{R}_+^*$. By Lemma 7 there exists $k \in \mathbb{Z}_+$ such that $\lambda^{-1} = \alpha^k$. Let $\varphi \in \mathrm{Diff}_0^{1,+}$ be such that $\varphi'(0) = \lambda^{-1/k}$.

For every gap I, the image $J = \varphi(I)$ is another gap and the ratio $|J|/|I|$ is an element in $\langle \lambda \rangle$, hence of the form $\lambda^{i(I)}$, for some integer $i(I)$. Further, there is a point $x_I \in I$ for which $\varphi'(x_I) = \lambda^{i(I)}$. Letting I_n be a sequence of gaps converging to the origin, we have that $\varphi'(x_{I_n})$ converges to $\varphi'(0) = \lambda^{-1/k}$. The sequence of integers $i(I_n)$ hence converges to $-1/k$, which forces $k = 1$. $\qquad\square$

We next observe that for each left point a of C_λ there exists a small neighborhood U_a of a such that the affine map $\psi_a = a + \lambda(x - a)$ sends $U_a \cap C_\lambda$ into C_λ, defining therefore a germ in $\mathfrak{diff}_a^{1,+}$. Then Lemmas 11 and 7 imply together that $\mathfrak{diff}_a^{1,+}$ is generated by $\psi_a = a + \lambda(x - a)$.

Let a and b be two left points of C_λ. Denote by $D(a,b)$ the set of germs at a of classes of local diffeomorphisms φ of $(\mathbb{R}, C_\lambda)$ such that $\varphi(a) = b$. Then $D(a,b)$ is acted upon transitively by $\mathfrak{diff}_a^{1,+}$. Using an argument similar to the one from above concerning stabilizers, $D(a,b)$ consists of germs of maps of the form $\psi_{a,b,k} = b + \lambda^k(x - a)$, with $k \in \mathbb{Z}$.

Now let $\varphi \in \mathrm{Diff}^{1,+}(\mathbb{R}, C_\lambda)$ such that $\varphi(a) = b$. From above there exists $\delta > 0$ such that $\varphi|_{C_\lambda \cap \mathcal{N}_\delta(a)}$ coincides with $\psi_{a,b,k}|_{C_\lambda \cap \mathcal{N}_\delta(a)}$ and hence $\varphi'(a) \in \langle \lambda \rangle$. Therefore, for any left point $a \in C_\lambda$ we have $\varphi'(a) \in \langle \lambda \rangle$. Now, left points of C_λ are dense in C_λ, φ' is continuous and $\langle \lambda \rangle$ has no other accumulation points in $\mathbb{R}^*$. It follows that $\varphi'(a) \in \langle \lambda \rangle$, for any $a \in C_\lambda$ and any $\varphi \in \mathrm{Diff}^{1,+}(\mathbb{R}, C_\lambda)$.

For a given $\varphi \in \mathrm{Diff}^{1,+}(\mathbb{R}, C_\lambda)$ its derivative φ' is continuous on the whole interval $[0,1]$ and hence is bounded. Moreover, the same argument for φ^{-1} shows that φ' is also bounded from below away from 0, so that $\varphi'|_{C_\lambda}$ can only take finitely many values of the form λ^n, $n \in \mathbb{Z}$.

The following is a key ingredient in the description of the group $\mathfrak{diff}^{1,+}(C_\lambda)$.

LEMMA 12. *Let $\varphi \in \mathfrak{diff}^{1,+}(C_\lambda)$. There is a covering of C_λ by a finite collection of disjoint closed intervals I_k, such that $\varphi|_{C_\lambda \cap I_k}$ is the restriction of an affine function to $I_k \cap C_\lambda$. Specifically,*

$$\varphi(x) = \varphi(c_k) + \lambda^{j_k}(x - c_k) \quad \text{for } x \in I_k \cap C_\lambda, \tag{36}$$

where c_k is a left point of $C_\lambda \cap I_k$.

Proof. For $c \in C_\lambda$ there is some $m \in \mathbb{Z}$ such that $\varphi'(c) = \lambda^m$. We want to prove that there exists an open neighborhood U of c such that

$$\varphi(x) = \varphi(c) + \lambda^{j_k}(x - c) \quad \text{for } x \in U \cap C_\lambda. \tag{37}$$

Then such neighborhoods will cover C_λ and we can extract a finite subcovering by clopen (closed and open) subsets with the same property.

This claim is true for any left (and by similar arguments for right) end points c of C_λ. It is then sufficient to prove that whenever we have a sequence of left points $a_n \to a_\infty$ contained in a closed interval $U \subset [0,1]$ and a $\mathcal{C}^1$-diffeomorphism $\varphi : U \to \varphi(U) \subset [0,1]$ with $\varphi(C \cap U) \subset C$, there exists a neighborhood U_{a_∞} of a_∞ and an affine function ψ such that for large enough n the following holds:
$$\varphi(x) = \psi(x) \quad \text{for } x \in C_\lambda \cap U_{a_\infty}.$$

Around each left point a_n there are affine maps $\psi_{a_n,k_n} : U_{a_n,k_n} \to [0,1]$ defining germs in $D(a_n, c_n)$, where $c_n = \varphi(a_n)$, such that c_n converge to $c_\infty = \varphi(a_\infty)$ and
$$\varphi(x) = \psi_{a_n,k_n}(x) \quad \text{for } x \in C_\lambda \cap U_{a_n,k_n}.$$

We can further assume that $U_{a_n,k_n} \cap C_\lambda$ are clopen sets and we can take $U_{a_n,k_n} = [a_n, b_n]$ where b_n are right points of C_λ, and the sequence a_n is monotone, say increasing.

There is no loss of generality to assume that $\psi'_{a_n,k_n}|_{C \cap U_{a_n,k_n}}$ is independent on n, say it equals λ^m, namely $k_n = m$. Replacing φ by its inverse φ^{-1} we can also assume that $m \leqslant 0$. Since C_λ is invariant by the homothety of factor λ and center 0, we can further reduce the problem to the case where $m = 0$. We have then $\varphi'(a_\infty) = 1$, by continuity.

Choose n large enough so that $|\varphi'(x) - 1| < \varepsilon$, for any $x \in [a_n, a_\infty]$, where the exact value of ε will be chosen later. Now consider the maximal interval of the form $[a_n, b]$ to which we can extend $\psi_{a_n,0}$ to an affine function which coincides with φ on $C \cap [a_n, b]$.

If $b = a_\infty$, then the Lemma follows. Otherwise, it is no loss of generality in assuming that $b = b_n$ and thus b is a right point of C_λ. Then b_n is adjacent to some gap (b_n, d). Since d is a left point of C_λ and $\varphi'(d) = 1$, we can suppose that $d = a_{n+1}$.

Since φ preserves $C \cap U$, it should send the gap (b_n, a_{n+1}) into some gap contained into $[\varphi(a_n), \varphi(b_{n+1})]$. Recall from above that the ratios of lengths of gaps of C_λ is the discrete subset $\langle \lambda \rangle \subset \mathbb{R}^*$. When $|\varphi'(x) - 1| < \varepsilon$, we derive that the ratio of the lengths of the gaps $\varphi(b_n, a_{n+1})$ and (b_n, a_{n+1}) is bounded by $1 + \varepsilon$. By taking $\varepsilon < 1 - \lambda$ we see that the only possibility is that the lengths of these two gaps coincide, namely that
$$\varphi(a_{n+1}) = \varphi(b_n) + a_{n+1} - b_n.$$

This implies that there is a smooth extension of $\psi_{a_n,0}$ to an affine function on $[a_n, b_{n+1}]$ which coincides with φ on points of C_λ, contradicting the maximality of $b = b_n$. This proves that $b = a_\infty$, proving the claim.

When a_∞ is not a right point we also have an affine extension of φ to a right neighborhood of a_∞, by the same argument. $\qquad\square$

Consider the rooted binary tree $\mathcal{T}$ embedded in the plane so that its ends abut on the interval $[0,1]$. We label each edge e by $l(e) \in \{0, \lambda - 1\}$, such that the leftmost edge is always labeled 0. Let v be a vertex of $\mathcal{T}$ and $e_1, e_2, \ldots, e_n$ the sequence of edges representing the geodesic which joins the root to v. To the vertex v one associates then the number
$$r(v) = \sum_{j=1}^{n} l(e_j)\lambda^{-j}. \tag{38}$$

Denote by $D(v)$ the set of all descendants of the vertex v. If I is a closed interval in $[0,1]$ we claim that $L(C_\lambda) \cap I$ coincides with the set $r(D(v_I))$, for some unique vertex $v_I \in \mathcal{T}$. Furthermore, if

$I_1, I_2, \ldots, I_k$ is a set of disjoint standard intervals covering C_λ then $v_{I_1}, v_{I_2}, \ldots, v_{I_k}$ are the leaves of a finite binary subtree $T(I_1, I_2, \ldots, I_k)$ of $\mathcal{T}$ containing the root. In particular, if $J_1, J_2, \ldots, J_k$ is another covering of C_λ by standard intervals then we have two finite trees $T(I_1, I_2, \ldots, I_k)$ and $T(J_1, J_2, \ldots, J_k)$. Further, we also have affine bijections $\varphi_j : I_j \to J_j$ which are of the form $\varphi_j(x) = b_j + \lambda^{k_j}(x - a_j)$, where $a_j, b_j \in L(C_\lambda)$. It is clear that $\varphi_j(I_j \cap L(C_\lambda)) = J_j \cap L(C_\lambda)$. The explicit form of $\varphi_j|_{I_j \cap L(C_\lambda)}$ actually can be interpreted in terms of $r(v_{I_j})$, as follows. Let $\mathcal{D}(v)$ be the planar rooted subtree of $\mathcal{T}$ of vertices $D(v)$ and root v. There is a natural identification $\iota_{v,w}$ of the planar binary rooted trees $D(v)$ and $D(w)$, for any $v, w \in \mathcal{T}$. When we further identify $L(C_\lambda) \cap I_j$ with the set $r(D(v_{I_j}))$ the induced action of φ_j on $w \in D(v_{I_j})$ coincides with $\iota_{v_{I_j}, v_{J_j}}$.

Consider now the operation of replacing an interval I_j by two disjoint intervals I_j' and I_j'' whose union is disjoint from the other intervals I_k. Correspondingly we replace J_j by the couple $\{J_j', J_j''\} = \{\varphi_j(I_j'), \varphi_j(I_j'')\}$ and φ_j by its restrictions to these smaller intervals. This operation does not change the element in $\mathfrak{diff}^1(C_\lambda)$. The immediate consequence of the description of φ_j is that the pairs of trees $T(I_1, \ldots, I_j', I_j'', \ldots, I_k)$ and $T(J_1, \ldots, J_j', J_j'', \ldots, J_k)$ are both obtained from $T(I_1, I_2, \ldots, I_k)$ and $T(J_1, J_2, \ldots, J_k)$ by adding one caret at the jth leaf. This proves that this pair of trees is a well-defined element of the standard Thompson group F. It is rather clear that the map defined this way $\mathfrak{diff}^{1,+}(C_\lambda) \to F$ is an isomorphism.

In a similar way we define an isomorphism $\mathfrak{diff}_{S^1}^{1,+}(C_\lambda) \to T$, when we work with the infinite unrooted binary tree $\mathcal{T}$ embedded in the plane so that its ends abut to S^1.

In the case of $\mathfrak{diff}_{S^2}^1(C_\lambda)$ we use the proof of Theorem 3 and the infinite unrooted binary tree $\mathcal{T}$ without any planar structure. The only difference is that the restrictions $\varphi|_{I_j}$ are not having anymore a coherent orientation. Some of them might be orientation preserving while the others may not. This explains the isomorphism between $\mathfrak{diff}_{S^2}^1(C_\lambda)$ and the signed Thompson group $V^\pm$. This ends the proof of Theorem 6 in the case of $C = C_\lambda$.

4.3 Proof of the general case of Theorem 6

The only missing ingredient is the result generalizing Lemma 12 to the more general self-similar sets considered here, as follows.

LEMMA 13. *Let $\varphi \in \mathrm{Diff}^{1,+}(\mathbb{R}, C)$, where $C = C_\Phi$ is a self-similar Cantor set satisfying the genericity condition (C). Then there is a covering of C by a finite collection of disjoint intervals I_k, such that $\varphi|_{C \cap I_k}$ is the restriction of an affine function to $I_k \cap C$.*

The proof of this lemma for incommensurable parameters will occupy § 4.3.1. In the case when gaps and homothety factors are respectively equal the proof given above extends word by word.

Now, any φ in the group $\mathfrak{diff}^{1,+}(C)$ corresponds to a pair of coverings of C by intervals $(I_1, I_2, \ldots, I_k)$ and $(J_1, J_2, \ldots, J_k)$ so that φ sends affinely I_j into J_j, for all j. These intervals could be chosen to be of the form $[a, b]$, where a is a left point of C and b is a right point of C. We call them *clopen* intervals.

Particular examples of clopen intervals are the images of $[0, 1]$ by the semigroup generated by Φ, which will be called *standard (clopen)* intervals. Each standard clopen interval corresponds to a finite geodesic path descending from the root in the (regular rooted) tree of valence $n + 2$ associated to Φ. Thus standard intervals are associated to vertices of the $(n+2)$-valent tree, and one says that they belong to the kth *generation of standard intervals* if the associated vertex is at distance k from the root. The complementary intervals to the union of all kth generation of standard intervals will be the kth *generation of gaps*. Moreover, given a standard interval I of

the kth generation, the gaps of the $(k + 1)$th generation lying in I will also be called the first generation of gaps in I. Notice that, conversely, every gap is a first generation gap for some uniquely determined standard interval, to be called its *antecedent* standard interval.

Note that the intervals obtained in the previous lemma were not necessarily standard intervals. It then remains to prove the following enhancement of Lemma 13.

LEMMA 14. *We assume that $C = C_\Phi$, where Φ verifies the genericity condition (C) from Definition 6. Then any $\varphi \in \mathfrak{diff}^{1,+}(C)$ corresponds to a pair of coverings of C by standard intervals $(I_1, I_2, \ldots, I_k)$ and $(J_1, J_2, \ldots, J_k)$ so that φ sends affinely I_j into J_j, for all j.*

Proof. Every clopen interval is the disjoint union of finitely many standard intervals and open gaps. We can therefore suppose that all I_j are standard intervals.

Note that for any two standard intervals $I, J \subset [0, 1]$ there exists an affine bijection $(I, I \cap C) \to (J, J \cap C)$, because this holds when $I = [0, 1]$.

We now claim that, conversely, if there exists an affine bijection $\varphi : (I, I \cap C) \to (J, J \cap C)$ and I is standard then its image J is also a standard interval. This will prove Lemma 14.

Consider a maximal standard interval $I' \subset J$. Composing φ with the affine map in $\mathfrak{diff}^{1,+}(C)$ sending bijectively $(I', I' \cap C)$ onto $(I, I \cap C)$ we can assume that $I = I'$. In particular, the homothety factor μ of the affine map $\varphi : I \to J$ is at least 1.

Assume first that all homothety ratios are equal to λ and all initial gaps have the same length g, as in Definition 6(i). Observe that all gaps will have sizes of the form $\lambda^m g$, for some $m \in \mathbb{Z}_+$. Moreover, if I is a standard interval of the kth generation, then the set of largest gaps in I consists of n equidistant gaps of size $\lambda^k g$. Their image by the affine map φ is the set of largest gaps in J, so that the latter are also n equidistant equal gaps in J, necessarily of size $\lambda^{n+k} g$, for some $n \in \mathbb{Z}_-$. In particular the homothety factor is $\mu = \lambda^n$.

We now consider the antecedent standard intervals associated to the largest gaps of J. If such a gap had size $\lambda^{n+k} g$, its antecedent interval would have size λ^{n+k}. If two of the largest gaps in J have distinct antecedent intervals, then they would be separated by another gap of size $\lambda^{n-1+k} g$, contradicting their maximality in J. Therefore all but possibly the leftmost and rightmost intervals of the complement of these n gaps in J are standard.

Now, the interval between two consecutive gaps in I is a standard interval of length λ^{k+1}, whose image by the affine map φ has length λ^{n+k+1}. This shows that the leftmost and the rightmost intervals also should be standard intervals, as they have the same size as the remaining $(n-1)$ standard intervals between consecutive image gaps. This proves that J is a standard interval.

Consider now the case when homothety factors and gap lengths are incommensurable, as in Definition 6(ii). The set of gaps of the same generation is totally ordered from the leftmost gap towards the right. The sequence of lengths of $(k+1)$th generation gaps within a standard interval of the kth generation is of the form $(\Lambda_{\mathbf{k}} g_1, \Lambda_{\mathbf{k}} g_2, \ldots, \Lambda_{\mathbf{k}} g_n)$, for some $\mathbf{k}$. Consider now a gap of the first generation, say $\Lambda_{\mathbf{k}} g_\alpha$, of I. Its image by an affine map should be a gap of J. It follows that there exists some $\sigma(\alpha) \in \{1, 2, \ldots, n\}$ and $\mathbf{k}_\alpha \in \mathbb{Z}_+^{n+1}$, so that:

$$\mu \Lambda_{\mathbf{k}} g_\alpha = \Lambda_{\mathbf{k}_\alpha} g_{\sigma(\alpha)},$$

where μ is the homothety factor of the map φ. Conversely, any gap of $I \subset J$ is the image by φ of some gap of I, and hence there exists some $\tau(\alpha) \in \{1, 2, \ldots, n\}$ and $\mathbf{l}_\alpha \in \mathbb{Z}_+^{n+1}$, so that:

$$\frac{1}{\mu} \Lambda_{\mathbf{k}} g_\alpha = \Lambda_{\mathbf{l}_\alpha} g_{\tau(\alpha)}.$$

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Getting rid of μ in the two equalities above we obtain the following identities, for all α, β:

$$\Lambda_{\mathbf{k}_\alpha + \mathbf{l}_\beta - 2\mathbf{k}} \quad g_{\sigma(\alpha)} g_{\tau(\beta)} = g_\alpha g_\beta.$$

By taking $\beta = \sigma(\alpha)$ we derive:

$$\Lambda_{\mathbf{k}_\alpha + \mathbf{l}_{\sigma(\alpha)} - 2\mathbf{k}} \, g_{\tau(\sigma(\alpha))} = g_\alpha.$$

If g_α and λ_j satisfy the genericity condition (C) the last equality implies $\tau(\sigma(\alpha)) = \alpha$ and $\mathbf{k}_\alpha + \mathbf{l}_{\sigma(\alpha)} = 2\mathbf{k}$, for every α. A symmetric argument yields $\sigma(\tau(\alpha)) = \alpha$, so that σ and τ are bijections inverse to each other. Furthermore we derive:

$$\mu^n = \prod_{\alpha=1}^n \Lambda_{\mathbf{k}_\alpha - \mathbf{k}} \frac{g_{\sigma(\alpha)}}{g_\alpha} = \Lambda_{\sum_{\alpha=1}^n (\mathbf{k}_\alpha - \mathbf{k})},$$

so that

$$\mu = \Lambda_{-\mathbf{k} + (1/n) \sum_{\alpha=1}^n \mathbf{k}_\alpha}.$$

Therefore, for each β we have:

$$\frac{g_{\sigma(\beta)}}{g_\beta} = \Lambda_{-\mathbf{k}_\beta + (1/n) \sum_{\alpha=1}^n \mathbf{k}_\alpha}.$$

Then our assumptions of genericity imply that σ must be identity. It turns out that all $\mathbf{k}_\alpha$ are equal to some $\overline{\mathbf{k}}$ and hence $\mu = \Lambda_{\overline{\mathbf{k}} - \mathbf{k}}$.

Observe that there exists a standard interval J' and an affine bijection $\psi : (I, I \cap C) \to (J', J' \cap C)$ with homothety factor $\Lambda_{\overline{\mathbf{k}} - \mathbf{k}}$. Therefore $\varphi \circ \psi^{-1} : J' \to J$ is a translation. Moreover, as σ was identity the sequence of first generation gaps in J' is sent by $\varphi \circ \psi^{-1}$ into the sequence of first generation gaps of some standard interval J''. By induction, the ordered sequence of the kth generation of gaps in J' is sent by $\varphi \circ \psi^{-1}$ into the sequence of the kth generation gaps of J''. Since J and J'' have the same length it follows that $J = J''$ and hence J is a standard interval, as claimed. $\qquad\square$

Now, it is immediate that $\langle \lambda_0 \rangle \subset \chi(\mathrm{Diff}_0^{1,+})$, and by Lemma 7 there exists some $N \in \mathbb{Z}_+$ so that $\chi(\mathrm{Diff}_0^{1,+}) = \langle \lambda_1^{1/N} \rangle$. Since $L(C)$ is affinely locally homogeneous this holds for any left point a of C.

Then, the general form of an affine germ locally preserving C around a left point $c_k \in C \cap I_k$ is

$$\varphi(x) = \varphi(c_k) + \Lambda_{\mathbf{j}_k, N}(x - c_k) \quad \text{for } x \in I_k \cap C, \tag{39}$$

where, for each multi-index $\mathbf{k} = (k_0, k_1, \ldots, k_n)$ we put

$$\Lambda_{\mathbf{k}, N} = \lambda_0^{k_0/N} \prod_{i=1}^n \lambda_i^{k_i}. \tag{40}$$

Furthermore, we can modify any germ in $\mathrm{Diff}_0^{1,+}$ by using homotheties of ratios λ_0^k, $k \in \mathbb{Z}$ in order to obtain a diffeomorphism $\varphi : [0, 1] \to [0, r]$ sending C into C. By Lemma 13 we can assume that φ is an affine map, and by Lemma 14 $[0, r]$ must be a standard interval. It follows that the homothety factor of φ is a power of λ_0. This implies that $N = 1$.

Pairs of coverings by standard clopen intervals of C correspond to pairs of finite rooted subtrees. Subdividing the covering by standard subintervals is then equivalent to stabilizing the trees. This provides isomorphisms with the Thompson groups F_{n+1}, T_{n+1} and the signed Thompson group $V_{n+1}^\pm$, respectively, ending the proof of Theorem 6.

4.3.1 *Proof of Lemma 13 for incommensurable parameters.* We will use a much weaker restriction than the total incommensurability, see the conditions used below.

Recall from §4.1 that the rooted $(n + 2)$-valent tree associated to the IFS has the edges issued from a vertex labeled by integers from 0 to n (from left to right). Then left points of C correspond to sequences which eventually end in 0, namely of the form

$$L(i_1 \cdots i_p) = i_1 i_2 \cdots i_p 000000 \cdots ,$$

while right points correspond to sequences which eventually end in n:

$$R(i_1 \cdots i_p) = i_1 \cdots i_p nnnnnn \cdots .$$

Consider two finite multi-indices $I = i_1 \cdots i_p$ and $J = j_1 \cdots j_q$ and set $a = L(i_1 \cdots i_p)$, $b = R(i_1 \cdots i_p)$, $\alpha = L(j_1 \cdots j_q)$, $\beta = R(j_1 \cdots j_q)$. Following Definition 9 the *standard germ* $\psi_{I,J}$ is the affine map $\psi_{I,J} : [a, b] \to [\alpha, \beta]$ given by the formula:

$$\psi_{I,J}(x) = a + \left(\frac{\prod_{m=1}^{q} \lambda_{j_m}}{\prod_{k=1}^{p} \lambda_{i_k}^{-1}} \right)(x - a).$$

Each multi-index I determines a vertex v_I of the tree, which is the endpoint of the geodesic issued from the root which travels along the edges labeled $i_1, i_2, \ldots, i_p$. Then, at the level of trees a standard germ corresponds to a combinatorial map sending the subtree hanging at the vertex v_I onto the subtree issued from the vertex v_J, as in the figure below:
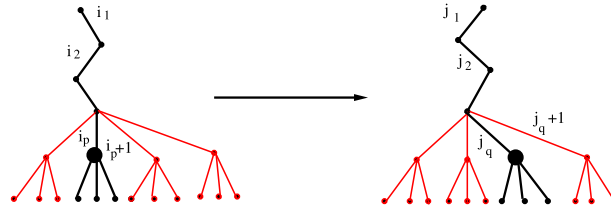
An *extension* of the standard germ $\psi : [a, b] \to [\alpha, \beta]$ is a standard germ defined on $[c, d] \supset [a, b]$ whose restriction to $[a, b]$ coincides with ψ such that $[c, d]$ corresponds to a vertex $v_{I'}$ of the tree whose multi-index I' is a prefix of I, namely $I' = i_1 i_2 \cdots i_r$ with $r \leqslant p$. Note that a non-trivial extension of ψ exists only if $i_p = j_q$.

A *multi-germ* is a finite collection of standard germs $\psi_j : [a_j, b_j] \to [\alpha_j, \beta_j]$ such that:

$$a_1 < b_1 < a_2 < b_2 < c < \cdots < a_k < b_k, \quad \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_k < \beta_k$$

and $[b_j, a_{j+1}]$ and $[\beta_j, \alpha_{j+1}]$ are gaps of C, for all j.

Eventually an *extension of a multi-germ* $\{\psi_j\}_{j=1,k}$ is a multi-germ $\{\theta_j\}_{j=1,m}$ such that every standard germ ψ_j is extended by some θ_i. Notice that several elements of the multi-germ $\{\psi_j\}_{j=1,k}$ might have the same extension θ_i.

LEMMA 15. *Let $\{\psi_j\}_{j=1,m}$ be a multi-germ with the property that there exist constants $\mu, \nu > 0$ satisfying*

$$\frac{\mu}{\nu} > \frac{1}{\max(\lambda_1, \lambda_2, \ldots, \lambda_n)},$$

such that

$$\mu \leqslant \psi_j'(x) \leqslant \nu \quad \textit{for every } x. \tag{41}$$

If one standard germ ψ_i, for some $i \in \{1, 2, \ldots, m\}$, admits an extension χ, then there exists an extension of the multi-germ $\{\psi_j\}_{j=1,m}$ containing the standard germ χ.

Moreover, if a diffeomorphism $\varphi \in \mathrm{Diff}^1(\mathbb{R}, C)$ whose derivative φ' verifies the condition for derivative (41) coincides with the multi-germ $\{\psi_j\}_{j=1,m}$ on $[a_1, b_m]$, then it coincides with χ on its domain of definition.

Proof. The standard germ ψ_j is of the form $\psi_j = \psi_{I,J}$, with $i_p = j_q = k$. We want to construct an increasing function extending the standard germ $\psi_{I,J}$ which satisfies the condition (41) for the derivative. Such a function will be called a *continuation* of ψ_j. Moving one step upward on the tree (i.e. the ancestor vertices) we arrive at the vertices $v_{I'}$ and $v_{J'}$, where $I = I'k$, $J = J'k$.

Consider first $k < n$ and seek for a continuation on the right side of the interval on which $\psi_{I,J}$ is defined. Therefore the continuation must have the form drawn below, where points marked by squares correspond to each other:
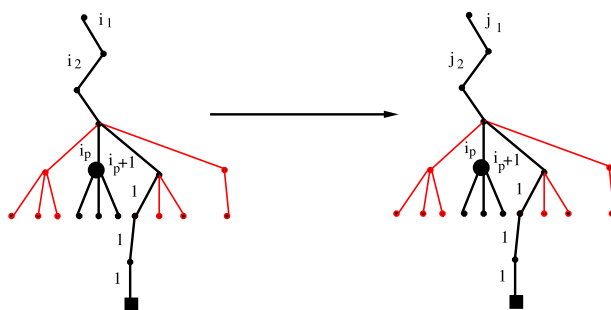

Since the ratio of the derivatives is uniformly bounded, the vertices corresponding to squares should be on the same level, namely at equal distance from the vertices $v_{I'}$ and $v_{J'}$, respectively. Consider the highest possible level of such squares for which the extended map is compatible with the standard germ ψ_{j+1}. We claim that this continuation has the following form, namely that squares sit on the vertices $v_{I'k+1}$ and $v_{J'k+1}$:
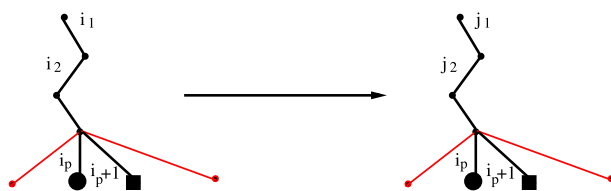

Assume the contrary holds, namely that the squares sit on lower levels, as in the figure below:
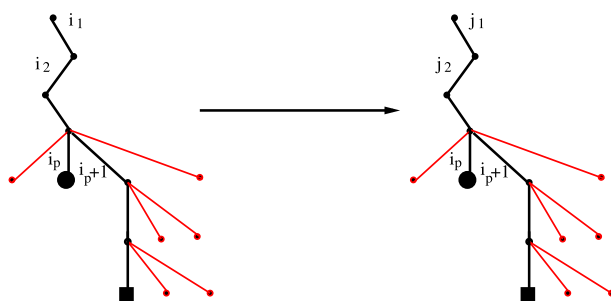
Consider further continuation to the right of this increasing function. We label points on the next branch issued from the ancestor of square vertices by triangles and further by hexagons etc. Consider further the highest levels for which continuation is compatible. Then the picture

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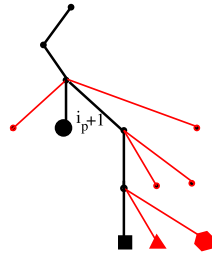
is impossible, since the ancestor of the square vertex also should have been labeled by a square. Therefore we must continue along an infinite path down to a boundary point of the tree, as in the following figure:
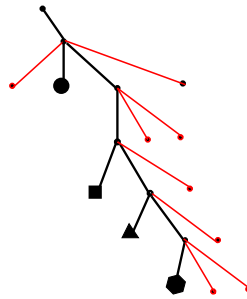
The boundary point corresponds to an infinite multi-index I. Then $\xi = \xi(I) \in [0,1]$ cannot be a right point of the Cantor set, since this would give a continuation to a whole subtree issued from $v_{I'}$, contradicting the form of our path.

Now our continuation coincides with the multi-germ $\{\psi_j\}_{j=1,m}$ for values $x \in [a_j, \xi]$. Since ξ is not a right point, they coincide in a right semi-neighborhood of ξ and this contradicts the choice of our infinite path.

We summarize the discussion above as follows. Let $k_r < n$; then the only possible right continuation (which satisfies the condition (41)) of $\psi_{Ik_1\cdots k_r, Jk_1\cdots k_r}$ is by the germ $\psi_{Ik_1\cdots k_{r-1}k_r+1, Jk_1\cdots k_{r-1}k_r+1}$. A similar argument shows that whenever $k_r > 0$ the only possible left continuation (which satisfies the condition (41)) of $\psi_{Ik_1\cdots k_r, Jk_1\cdots k_r}$ is by the germ $\psi_{Ik_1\cdots k_{r-1}k_r-1, Jk_1\cdots k_{r-1}k_r-1}$.

Repeating the same argument, we get the desired statement. □

LEMMA 16. *There exists $\varepsilon > 0$ with the following property. Consider a standard germ $\psi_{I,J}$ with $i_p \neq j_q$ and $j_q \neq n \neq i_p$.*

Then any continuation of $\psi_{I,J}$ to a standard germ θ sending $L(i_1 i_2 \cdots i_{p-1} i_p + 1)$ to $L(j_1 j_2 \cdots j_{q-1} j_q + 1)$ which is defined in a right semi-neighborhood of $L(i_1 i_2 \cdots i_{p-1} i_p + 1)$ is either an extension of the standard germ $\psi_{I,J}$, or else it verifies:

$$\left| \frac{\psi'_{I,J}}{\theta'} - 1 \right| > \varepsilon.$$

Notice that θ is locally affine and hence we don't need to specify the point (of the corresponding domain of definition) in which we consider the derivative.

Proof. The ratio of the derivatives of the standard germs $\psi_{I,J}$ and $\theta = \psi_{i_1 i_2 \cdots i_{p-1} i_p+1,\, j_1 j_2 \cdots j_{q-1} j_q+1}$ is given by

$$\frac{\psi'}{\theta'} = \frac{\lambda_{i_p}^{-1} \lambda_{i_q}}{\lambda_{i_p+1}^{-1} \lambda_{i_q+1}} \lambda_1^m,$$

where $m \in \mathbb{Z}$. This is a discrete subset of $\mathbb{R}^*$ and hence the claim. $\qquad \square$

We can apply the same arguments when $i_p \neq 0 \neq j_q$. Specifically, we have the following lemma.

LEMMA 17. *Let $n \geqslant 2$. Then there exists $\varepsilon > 0$ such that any multi-germ $\{\psi_j\}_{j=1,m}$ with the property:*
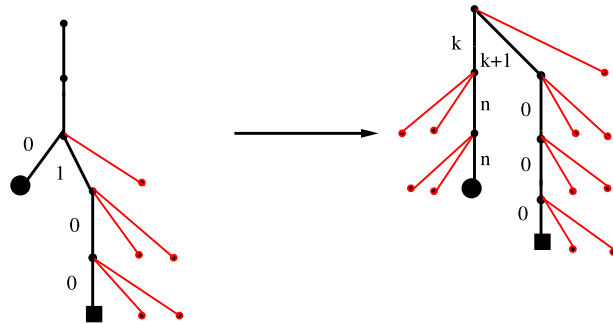
$$\left| \frac{\psi_i'}{\psi_j'} - 1 \right| < \varepsilon$$

admits an extension containing with at most two elements.

Proof. It remains to examine the standard germs $\psi_{I,J}$ in following two cases:

$$(I, J) \in \{(i_1 \cdots i_{p-1} 0,\; J = j_1 \cdots j_{q-1} n),\; (i_1 \cdots i_{p-1} n,\; j_1 \cdots j_{q-1} 0)\}.$$

The corresponding picture depends on the number s of occurrences of n in the tail of $j_1 \cdots j_{q-1} n$ and the positions of the the square vertices (having r and m respectively ancestors labeled 0) as below:

The ratio of derivatives is

$$\frac{\lambda_n^s \lambda_k \lambda_{k+1}^{-1} \lambda_0^{-r}}{\lambda_0 \lambda_1^{-1} \lambda_0^{-m}} = \frac{\lambda_k \lambda_1}{\lambda_{k+1}} \cdot \frac{\lambda_n^s}{\lambda_0^{r-m+1}}.$$

Letting s and $\mu = r - m + 1$ be large enough we can insure that $\lambda_n^s / \lambda_0^\mu$ is arbitrarily close to $\lambda_{k+1}/\lambda_k \lambda_1$. In this case $\mu > 0$, so that we can automatically extend the new standard germ obtained this way and get the figure below, where the position of the squared vertex is the highest possible:


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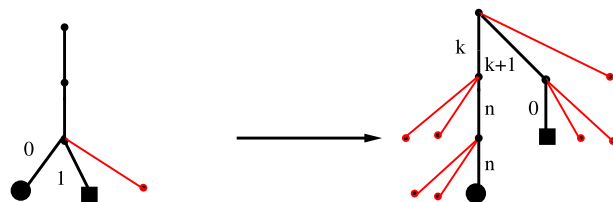
Now, as $n \geqslant 2$ we cannot find a non-trivial extension of the two standard germs corresponding to the labeled vertices. This means that there is an extension with at most two elements, thereby proving our statement. □

LEMMA 18. *Let $n = 1$. Then there exists $\varepsilon > 0$ such that any multi-germ $\{\psi_j\}_{j=1,m}$ verifying the condition:*
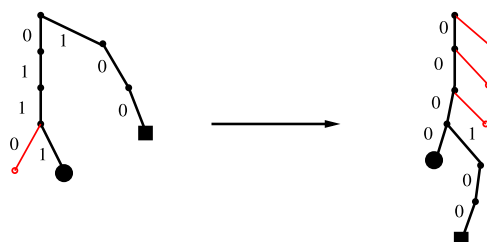
$$\left| \frac{\psi_i'}{\psi_j'} - 1 \right| < 1 + \varepsilon$$

admits an extension containing at most 4 elements.

*Proof.* The only possible situation is that pictured below:
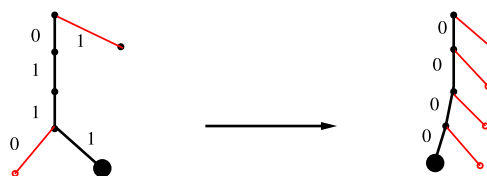

Consider a right continuation as follows:

In the left-hand side picture we have $r + 1$ ancestors of the fat dotted vertex which are labeled 1 and s ancestors of the square vertex labeled 0, while in the right-hand picture there are v ancestors of the square vertex labeled 0. Then the ratio of derivatives of the two standard germs is:

$$\frac{\lambda_1^{-r-1} \lambda_1 \lambda_0^s}{\lambda_1 \lambda_0^v} = \frac{\lambda_0^{s-v}}{\lambda_1^{r+1}}.$$

We can approximate arbitrarily close 1 by $\lambda_0^{s-v}/\lambda_1^{r+1}$, but then $s - v$ must be large, and in particular positive. This implies that we can automatically extend this to a standard germ as follows:
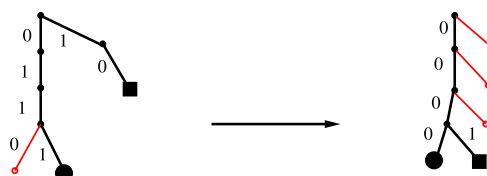
or, after removing non-essential information:
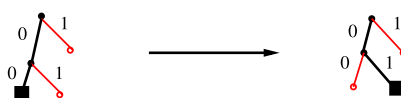
And we now see that a right continuation is impossible. Thus we get our claim. □

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4.3.2 *Cantor sets with commensurable parameters.* The genericity condition (C) could be extended to include also the case when all homothety factors λ_i are commensurable. We skip the details and present instead an example of an asymmetric Cantor set AC which is the attractor of the IFS:

$$\phi_0(x) = \tfrac{1}{4}x, \quad \phi_1(x) = \tfrac{1}{2}x + \tfrac{1}{2}.$$

For each finite multi-index $I = i_1 i_2 \cdots i_k$, with $i_j \in \{0, 1\}$ we set $l_\emptyset = 0$ and define by induction:

$$l_{0I} = \tfrac{1}{4}l_I, \quad l_{1I} = \tfrac{1}{2}l_I + \tfrac{1}{2}.$$

Then $L(AC) = \{l_I; I$ finite and admissible$\}$, where $I = i_1 i_2 \cdots i_k$ is admissible if it is either empty or else $i_k = 1$. If we set $l_I = \lim_{k \to \infty} l_{i_1 i_2 \ldots i_k}$ for infinite I, then AC is the union of $L(AC)$ and the set of l_I, with infinite I. Further, $L(AC)$ is identified with the set of those l_I for which I is infinite and eventually 0. It follows as in the case of C_λ that $\chi(\mathrm{Diff}_0^1(AC)) = \langle 4 \rangle$. Further, for $a, b \in L(AC)$, we obtain

$$D(a, b) = \left\{ \psi_{a,b,k} = b + \frac{1}{2^{n(a,b)}} 4^{-k}(x - a), k \in \mathbb{Z} \right\}, \tag{42}$$

where $n(a, b) \in \{0, 1\}$ is the parity of the length of the geodesic joining a to b in the reduced binary tree associated to the IFS.

The previous arguments show that any element φ of $\mathfrak{diff}^{1,+}(AC)$ determines a *finite* covering of AC by intervals I_j on which $\varphi|_{I_j}$ is of the form ψ_{a_j, b_j, k_j}, for some $a_j \in L(AC)$. Moreover $\mathfrak{diff}^{1,+}(AC)$ is isomorphic to the Thompson group F.

4.4 Proof of Theorem 7

Let $\mathrm{Diff}_a^1(\mathbb{R}^n, C)$ denote the stabilizer of $a \in C$ in the group $\mathrm{Diff}^1(\mathbb{R}^n, C)$. We verify immediately that the map $\chi : \mathrm{Diff}_a^1(\mathbb{R}^n, C) \to \mathrm{GL}(n, \mathbb{R})$, given by $\chi(\varphi) = D_a \varphi$ is a homomorphism. In the case when C is a product we can describe explicitly $\chi(\mathrm{Diff}_a^1(\mathbb{R}^n, C))$. For the sake of simplicity we restrict ourselves to the case $n = 2$. Consider $C = C_{\lambda_1} \times C_{\lambda_2}$. We say that $a = (a_1, a_2) \in C$ is an *end* point of C if both a_i are endpoints of C_{λ_i}.

LEMMA 19. *Suppose that $\lambda_i > 2$ and a is an end point of C.*

(i) *If $\lambda_1 \neq \lambda_2$ then*

$$\chi(\mathrm{Diff}_a^1(\mathbb{R}^2, C)) = \langle \lambda_1 \rangle \oplus \langle \lambda_2 \rangle. \tag{43}$$

(ii) *If $\lambda_1 = \lambda_2 = \lambda$ then*

$$\chi(\mathrm{Diff}_a^{1,+}(\mathbb{R}^2, C)) = \langle \lambda \rangle \oplus \langle \lambda \rangle. \tag{44}$$

Proof. From the first part of the proof of Theorem 4 we infer that whenever C is a product and $a \in C$ is fixed by φ, its differential $D_a \varphi$ must send both horizontal and vertical vectors into horizontal or vertical vectors.

Moreover, when λ_i are distinct, the horizontality/verticality of the segment should be preserved. Otherwise φ would induce a germ of $\mathcal{C}^1$-diffeomorphism $\phi : (\mathbb{R}, C_{\lambda_1}) \to (\mathbb{R}, C_{\lambda_2})$. By Remark 4 we need $\lambda_1 = \lambda_2$.

Therefore φ restricts to germs of diffeomorphisms $\phi_i \in \mathrm{Diff}_{a_i}^1(\mathbb{R}, C_{\lambda_i})$. By Lemma 11 and Remark 7 $\chi(\phi_i) = \langle \lambda_i \rangle$. This proves the first item.

On the other hand when $\lambda_1 = \lambda_2$ we can locally identify (C_{λ_1}, a_1) and (C_{λ_2}, a_2) by an affine germ. The linear map $R_a = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$ which exchanges the two orthogonal axes meeting at

$a \in C$ belongs to $\mathrm{Diff}_a^1(\mathbb{R}^2, C)$. We can compose φ with R_a, if needed, in order to have $D_a\varphi$ diagonal. Thus $\chi(\mathrm{Diff}_a^1(\mathbb{R}^2, C)) = \langle\langle\lambda\rangle \oplus \langle\lambda\rangle, R_a\rangle$. Taking into account that $\det(R_a) = -1$, so R_a is orientation reversing, we obtain the claim.

Observe that $\mathfrak{diff}_{a,\mathbb{R}^2}^1(C)$ is either isomorphic to $\mathbb{Z}^2$, when λ_i are distinct, or an extension of $\mathbb{Z}^2$ by $\mathbb{Z}/2\mathbb{Z}$, otherwise. $\qquad\square$

Consider now that $\lambda_1 = \lambda_2$. Now let a and b be two end points of C. Denote by $D(a,b)$ the set of germs at a of classes in $\mathfrak{diff}_{\mathbb{R}^2}^1(C)$ having representatives $\varphi \in \mathrm{Diff}^1(\mathbb{R}^2, C)$ such that $\varphi(a) = b$. This set is acted upon transitively by $\mathfrak{diff}_{a,\mathbb{R}^2}^{1,+}(C)$, so that $D(a,b)$ consists of maps of the form

$$\psi_{a,b,\mathbf{k},n}(x) = (b_{j,i} + \lambda^{k_j}(x_i - a_{j,i}))_{i=1,2} \circ S_a^{n_b} \quad \text{for any } x \in I_j \cap C. \tag{45}$$

where $S_a^{n_b}$ is an element of the group D_2 of orientation preserving symmetries of the square, namely S_a is a rotation of order 4 fixing a and $n_b \in \{0, 1, 2, 3\}$.

Now the set of endpoints of C is kept invariant by any $\varphi \in \mathrm{Diff}^1(\mathbb{R}^2, C)$. Therefore, for any endpoint $a \in C$ there exists some k_i, n depending on a such that $D_a\varphi = (\lambda^{k_1} \oplus \lambda^{k_2}) \circ S_a^n$. The set of possible values of $D_a\varphi$ is then a discrete subset of $\mathrm{GL}(2, \mathbb{R})$. Since endpoints of C are dense in C and $D\varphi$ is continuous we have $D_a\varphi$ is of the form $(\lambda^{k_1} \oplus \lambda^{k_2}) \circ S_a^n$, for any $a \in C$ and any $\varphi \in \mathrm{Diff}^{1,+}(\mathbb{R}, C)$.

For a given φ both the norm $\|D\varphi\|$ and the determinant $\det(D\varphi)$ of its differential are continuous on $[0,1] \times [0,1]$ and hence they are bounded. Moreover, the same argument for φ^{-1} shows that these quantities are also bounded from below away from 0, so that $D\varphi|_C$ can only take finitely many values. The next point is the analogue of Lemma 12 to this situation.

LEMMA 20. *There is a covering of C by a finite collection of disjoint standard rectangles I_k whose images are standard rectangles such that $\varphi|_{C \cap I_k}$ is the restriction of an affine function and more precisely we have*

$$\varphi(x) = (\lambda^{j_{k,1}} \oplus \lambda^{j_{k,2}}) \circ S_{b_k}^{m_k}(x - (\alpha_1, \alpha_2)) + \varphi(\alpha_1, \alpha_2) \quad \text{for } x \in I_k \cap C, \tag{46}$$

where α_i are left points of C_i.

Proof. We can choose both I_k and their images to be standard rectangles, as in the case of central Cantor sets C_λ.

Let $c \in C$. Then $D_c\varphi = (\lambda^{j_{k,1}} \oplus \lambda^{j_{k,2}}) \circ S_{b_k}^{m_k}$, which we denote by A for simplicity in the proof. We have to prove that there exists a neighborhood U of c such that

$$\varphi(x) = A(x - (\alpha_1, \alpha_2)) + \varphi(\alpha_1, \alpha_2) \quad \text{for } x \in U \cap C. \tag{47}$$

Such neighborhoods will cover C and we can extract a finite subcovering by clopen sets to get the statement.

This claim is true for endpoints $a = (\alpha_1, \alpha_2)$ of C. It is then sufficient to prove that whenever we have a sequence of endpoints $a_n \to a_\infty$ contained in a closed rectangle $U \subset [0,1]$ and a $\mathcal{C}^1$-diffeomorphism $\varphi : U \to \varphi(U) \subset [0,1]$ with $\varphi(C \cap U) \subset C$, there exists a neighborhood U_{a_∞} of a_∞ and an affine function ψ such that for large enough n the following holds:

$$\varphi(x) = \psi(x) \quad \text{for } x \in C_\lambda \cap U_{a_\infty}.$$

Around each left point a_n there are affine maps $\psi_{a_n} : U_{a_n, k_n} \to [0,1]$ defining germs in $D(a_n, c_n)$, where $c_n = \varphi(a_n)$, such that c_n converge to $c_\infty = \varphi(a_\infty)$ and

$$\varphi(x) = \psi_{a_n}(x) \quad \text{for } x \in C_\lambda \cap U_{a_n, k_n}.$$

We can further assume that $U_{a_n} \cap C_\lambda$ are clopen sets and we can take U_{a_n} to be standard rectangles $[\alpha_{n,1}, \beta_{n,1}] \times [\alpha_{n,2}, \beta_{n,2}]$ where $\beta_{n,i}$ are right points of C_i.

There is no loss of generality to assume that $D\psi_{a_n}|_{C \cap U_{a_n}}$ is independent on n, say it equals $(\lambda^{m_1} \oplus \lambda^{m_2})S^j$. Replacing φ by its inverse φ^{-1} we can also assume that $m_1 \leqslant 0$. Since C_1 is invariant by the homothety of factor λ and center 0, we can further reduce the problem to the case where $m_1 = 0$. We can further assume that $m_2 \leqslant 0$ by the same trick and finally get rid of the second diagonal component of the differential. Then, by continuity, we have $D_{a_\infty}\varphi = S^j$.

Choose n large enough so that $\|D_x\varphi(x) - \mathbf{1}\| < \varepsilon$, for any x in a square centered at a_∞ and containing all U_{a_n}, with n large enough, where the exact value of ε will be chosen later.

Now consider the maximal standard rectangle of the form $U' = [\alpha_{n,1}, \beta_1] \times [\alpha_{n,2}, \beta_2]$ to which we can extend ψ_{a_n} to an affine function which coincides with φ on $C \cap U'$.

The endpoint (β_1, β_2) belongs to the closure of three maximal rectangular gaps: the rectangle Q which is opposite to U' is a product of two gaps, while the other two Q_v and Q_h are products of gaps with one (vertical or horizontal) side of U'. Since $D_x\varphi$ is close to identity the image of the rectangular gaps are closed to rectangular gaps of approximatively the same sizes. Now, the images by φ of the vertices of Q are points of C forming a rectangle, which is itself the product of two gaps. Thus the sizes of this rectangle belong to the set $\{(\lambda - 2)\lambda^{-n}; n \in \mathbb{Z}_+\} \times \{(\lambda - 2)\lambda^{-n}; n \in \mathbb{Z}_+\}$. Since the ratios of two different lengths form a discrete set and $D_x\varphi$ is close to identity, the four points in the image form a rectangle congruent to Q. A similar argument holds now for the rectangles Q_v and Q_h. This implies that ψ_{a_n} can be extended to an affine function on a strictly larger rectangle, contradicting our assumptions. This proves the claim. $\qquad\square$

This description shows that $\mathfrak{diff}^{1,+}_{\mathbb{R}^2}(C)$ is isomorphic to $2V^{\mathrm{sym}}$, namely the Brin's group decorated by D_2 (see [Bri04] for the non-decorated case). Here D_2 is the group of the orientation preserving symmetries of the cube, namely the group of orthogonal matrices with integral entries and unit determinant.

Remark 12. We can obtain smaller decoration by choosing self-similar Cantor sets with fewer symmetries. For instance, $\mathfrak{diff}^{1,+}_{\mathbb{R}^n}(C)$ is isomorphic to Brin's group decorated by the positive isometry group of a rectangular parallelepiped with edges of different sizes, if $C = C_{\lambda_1} \times C_{\lambda_2} \times \cdots C_{\lambda_n}$, where λ_i are pairwise distinct but commensurable, namely there exists $\alpha \in \mathbb{R}^*_+$ and $k_i \in \mathbb{Z}$ such that $\lambda_i = \alpha^{k_i}$, for all i. We expect a similar result when λ_i are incommensurable. Moreover, by replacing each C_{λ_i} by some non-invertible self-similar Cantor set, the corresponding group $\mathfrak{diff}^{1,+}_{\mathbb{R}^n}(C)$ is isomorphic to Brin's group nV.

Remark 13. Following the arguments in the proof of Theorem 3 one shows that $\mathfrak{diff}^{1,+}_{\mathbb{R}^{n+k}}(C)$, for $k \geqslant 1$, is Brin's group $nV^{\pm\mathrm{sym}}$ decorated by the hyperoctahedral group O_n, namely the group of symmetries of the cube (possibly reversing the orientation).

5. Examples and counterexamples

5.1 Non-sparse Cantor sets with uncountable diffeomorphism group

Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a C^∞-function satisfying the following conditions:

$$h(x) = 0 \quad \text{for } 0 \leqslant x \leqslant 1, x \geqslant 2,$$
$$h(x) > 0 \quad \text{for } 1 < x < 2,$$
$$h'(x) > -1.$$

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Since the maps $g_j : [0,1] \to [0,1]$ given by

$$g_j(x) = x + 2^{-2^j} h(2^j x), \quad j \in \mathbb{Z}_+^*, \tag{48}$$

are strictly increasing, they are smooth diffeomorphisms of the interval. The support of g_j is $[2^{-j}, 2^{-j+1}]$ and hence the diffeomorphisms g_j pairwise commute. Their derivatives are of the form:

$$g_j'(x) = 1 + 2^{j-2^j} h'(2^j x),$$

and respectively

$$g_j^{(k)}(x) = 2^{kj-2^j} h^{(k)}(2^j x) \quad \text{for } k \geqslant 2.$$

Consider the group R consisting of bounded infinite sequences $\mathbf{m} = m_1, m_2, \ldots$ of integers, endowed with the term-wise addition.

There is a map $\Theta : R \to \mathrm{Diff}^0([0,1])$ given by

$$\Theta(\mathbf{m}) = \lim_{n \to \infty} g_1^{m_1} \circ g_2^{m_2} \circ \cdots \circ g_n^{m_n}, \tag{49}$$

where g_j^m is the m-fold composition of g_j. The order in the previous definition does not matter, as the maps commute. The limit map $\Theta(m)$ is immediately seen to be a homeomorphism of $[0,1]$ which is a diffeomorphism outside 0.

Let us first consider only those $\mathbf{m}$ where $m_j \in \{0,1\}$. Then we can compute first

$$\lim_{x \to 0} \Theta(\mathbf{m})'(x) = 1,$$

and then

$$\lim_{x \to 0} \Theta(\mathbf{m})^{(k)}(x) = 1 \quad \text{for } k \geqslant 2.$$

Therefore $\Theta(\mathbf{m})$ is a $\mathcal{C}^\infty$ diffeomorphism of $[0,1]$.

Moreover any element of R can be represented as a product of $\Theta(\mathbf{m})$, with $\mathbf{m}$ of having only 0 or 1 entries. This implies that $\Theta(R) \subset \mathrm{Diff}^\infty([0,1])$. Furthermore it is clear that Θ is injective, by looking at the factor corresponding to the first place where two sequences disagree. This implies that Θ provides a faithful $\mathcal{C}^\infty$ action of R by $\mathcal{C}^\infty$ diffeomorphisms on $[0,1]$.

The dynamics of each g_j on its support $[2^{-j}, 2^{-j+1}]$ is of type north-south with repelling and attracting fixed points on the boundary. Pick up some $a_j \in (2^{-j}, 2^{-j+1})$, so that $b_j = g_j(a_j) > a_j$. Then the intervals $g_j^n((a_j, b_j))$ are all pairwise disjoint. If $C_j^0 \subset [a_j, b_j]$ is some Cantor set, then the closure of its orbit, namely $C_j = \bigcup_{k=-\infty}^\infty g_j^k(C_j^0) \cup \{2^{-j}, 2^{-j+1}\}$ is a g_j-invariant Cantor subset of $[2^{-j}, 2^{-j+1}]$. Moreover, for any $n \neq 0$ the restriction $g_j^n|_{C_j}$ cannot be identity, since g_j^n is strictly increasing.

Then their union $C = \bigcup_{j=1}^\infty C_j$ is a Cantor subset of $[0,1]$ and for $\mathbf{m}$ not identically 0 we also have $\Theta(\mathbf{m})|_C$ is not identity. This shows that the diffeomorphism group $\mathfrak{diff}^\infty(C)$ contains R. In particular, $\mathfrak{diff}^\infty(C)$ is uncountable.

This technique of construction is close to that involved in other classical constructions in the field, as in [Tsu95, Nav08, Jor12]. In particular, such a group arises as the group of fragmentations of elements of the Grigorchuk–Maki group of intermediate growth acting by $\mathcal{C}^1$-diffeomorphisms of the interval from [Nav08].

5.2 Non-sparse Cantor set with trivial diffeomorphism group

Let X be obtained by removing a sequence of intervals, as follows. At the first step we remove from $[0, 1]$ the central interval of length $1/4$. At the mth step we have 2^m intervals which we label, starting from the leftmost to the rightmost as $I_1^{(m)}, I_2^{(m)}, \ldots, I_{2^m}^{(m)}$. We remove then from $I_j^{(m)}$ the central interval of length $2^{-2^{2^{m-1}-1+j}}$. The result of this procedure is a Cantor set X which is not sparse.

Let $\varphi \in \mathrm{Diff}^1(\mathbb{R}, X)$. Suppose first that there exists a sequence I_n of gaps approaching 0 from the right side with the property that for every n we have $J_n = \varphi(I_n) \neq I_n$. Then there exists points $x_{I_n} \in I_n$ such that $\varphi'(x_{I_n}) = |J_n|/|I_n|$. The sequence x_{I_n} converges to 0. Now lengths of gaps belong to the discrete set $\{2^{-2^n}, n \in \mathbb{Z}_+\}$ and there are not two gaps of the same length. Therefore any infinite sequence of lengths $|J_n|/|I_n| \neq 1$ has a subsequence which either converges to 0 or is unbounded. This implies that $\varphi'(0)$ is either 0 or infinite, which contradicts the fact that φ was a diffeomorphism.

It follows that for any sequence I_n as above and large enough n we have $\varphi(I_n) = I_n$. In particular $\varphi(0) = 0$. This holds for any left point of X and hence $\mathfrak{diff}^1(X) = 1$.

Now let $I, J \subset [0, 1]$ be closed intervals intersecting X intersecting along Cantor sets. The arguments above also show that there exists a diffeomorphism $\varphi : (I, X \cap I) \to (J, X \cap J)$ if and only if $I = J$ and $\varphi|_{X \cap I}$ is the identity. From the proof of Theorem 3 (see § 3.2.3) we deduce that $\mathfrak{diff}_M^1(X) = 1$ for any manifold M containing the $\mathcal{C}^1$-interval $[0, 1]$.

5.3 Sparse Cantor set with trivial diffeomorphism group

Start from the interval $I^{(0)} = [0, 1]$ by removing a central gap $J_1^{(1)}$ of size $(1 - \varepsilon)$. By recurrence at the nth step we have 2^n intervals $I_j^{(n)}$, $j = 1, \ldots, 2^n$, numbered from the left to the right. To go further we remove a central gap $J_j^{(n+1)}$ from $I_j^{(n)}$ of length $|J_j^{(n+1)}| = (1 - \varepsilon^n)|I_j^{(n)}|$. The set so obtained is obviously a sparse Cantor set C_0.

Let $a \in C_0$. Let b_n be the right endpoint of the interval $I_{j_n}^{(n)}$ to which a belongs. Then set (x_n, y_n) for the gap $J_{j_n}^{(n+1)} \subset I_{j_n}^{(n)}$. There is no loss of generality of assuming that $a < x_n < y_n < b_n$. Given $\varphi \in \mathrm{Diff}_a^1(\mathbb{R}, C_0)$, with $\varphi'(a) \neq 1$, there are infinitely many n for which the gap $J_{j_n}^{(n)}$ is not fixed by φ. It follows that either $\varphi(y_n) < x_n$, or $\varphi(x_n) > y_n$, for infinitely many n. By symmetry we can assume that the second alternative holds. Then

$$\frac{\varphi(x_n) - x_n}{x_n - a} \geqslant \frac{|y_n - x_n|}{|x_n - a|} \geqslant \frac{(1 - \varepsilon^n)|b_n - a|}{|x_n - a|} \geqslant \frac{(1 - \varepsilon^n)|b_n - a|}{\varepsilon^n|b_n - a|} = \frac{1 - \varepsilon^n}{\varepsilon^n}. \qquad (50)$$

Letting $n \to \infty$ we obtain that $\varphi'(a) = \infty$, contradiction. This proves that $\mathfrak{diff}_a^{1,+}(C_0) = 1$.

From Remark 8 we have $\mathfrak{diff}^{1,+}(C_0) = 1$. Moreover, the proof of Theorem 3 implies that $\mathfrak{diff}_M^{1,+}(C_0) = 1$ for any manifold M containing the $\mathcal{C}^1$-interval $[0, 1]$.

Another potential example. In order to convert the non-sparse example above X into a sparse Cantor set with the same properties, we have to mix ordinary gaps and very small gaps. Start as above from the interval $I^{(0)} = [0, 1]$ by removing a central gap $LG^{(1)}$ of size $\frac{1}{3}$ and two very small gaps each one centered within an interval component of $I^{(0)} - LG^{(1)}$, namely $SG_1^{(1)}$ and $SG_2^{(1)}$ of lengths 2^{-2^α} and $2^{-2^{\alpha+1}}$, respectively. Here α is chosen so that

$$\tfrac{1}{3} - 2^{-2^\alpha} > \tfrac{1}{6}.$$

We obtain at the next stage four intervals $I_1^{(1)}, I_2^{(1)}, I_3^{(1)}, I_4^{(1)}$, labeled from the left to the right.

By recurrence at the nth step we have 4^n intervals $I_j^{(n)}$, $j = 1, \ldots, 4^n$. To go further we remove first a central gap $LG_j^{(n+1)}$ from $I_j^{(n)}$ of length $|LG_j^{(n+1)}| = \frac{1}{3}|I_j^{(n)}|$. Further we remove two very small gaps each one centered within an interval component of $I_j^{(n)} - LG_j^{(n)}$, namely $SG_{2j+1}^{(n)}$ and $SG_{2j+2}^{(n)}$ of lengths $2^{-2^{\alpha+j+4^n}}$ and $2^{-2^{\alpha+j+1+4^n}}$. Letting n go to ∞ we obtain a $\frac{1}{3}$-sparse Cantor set MC. We believe that $\mathfrak{diff}^{1,+}(MC) = 1$.

5.4 Split Cantor sets

Two Cantor sets $C_i \subset \mathbb{R}^n$ are *locally smoothly non-equivalent* if for any $p_i \subset C_i$ there is no $\mathcal{C}^1$-diffeomorphism germ $(\mathbb{R}^n, C_1, p_1) \to (\mathbb{R}^n, C_2, p_2)$.

A Cantor set in $C \subset \mathbb{R}^n$ is said to be *smoothly split* if we can write $C = C_1 \cup C_2$ as a union of two Cantor sets with C_1 and C_2 locally smoothly non-equivalent and contained in disjoint intervals.

We have the following proposition.

PROPOSITION 2. *Let $n \geqslant 1$ and $C \subset \mathbb{R}^n$ be a Cantor set which is smoothly split as $C_1 \cup C_2$. Then $\mathfrak{diff}^{1,+}(C) = \mathfrak{diff}^{1,+}(C_1) \times \mathfrak{diff}^{1,+}(C_2)$.*

Proof. In this situation, C_i are contained into disjoint intervals U_i. Then diffeomorphisms preserving C should also send each C_i into itself. Furthermore all elements from $\mathfrak{diff}^{1,+}(C_1) \times \mathfrak{diff}^{1,+}(C_2)$ can be realized as classes of pairs of commuting diffeomorphisms supported in U_i. □

According to Remark 4 the central Cantor sets C_λ are pairwise locally smoothly non-equivalent. In particular the union $C_\lambda \cup 2 + C_\mu$ of two distinct Cantor sets, one of which is translated by 2 is a split Cantor set. It follows that

$$\mathfrak{diff}^{1,+}(C_\lambda \cup 2 + C_\mu) = \mathfrak{diff}^{1,+}(C_\lambda) \times \mathfrak{diff}^{1,+}(C_\mu) \cong F \times F,$$

for distinct λ and μ. It is not clear what would be $\mathfrak{diff}^{1,+}(C_\lambda \cup C_\mu)$ or $\mathfrak{diff}^{1,+}(C_\lambda + C_\mu)$ (for those parameters for which the sum is still a Cantor set).

5.5 Questions

The countability of $\mathfrak{diff}_M^1(C)$ relies on the description of the group of germs of the stabilizer G_a of a point $a \in C$ for a finitely generated group $G \subset \mathrm{Diff}^1(M, C)$: it is *cyclic* when $\dim M = 1$ and C is sparse and (a finite extension of) a subgroup of $\mathbb{Z}^n$ if $\dim M = n$ and C is a product of n sparse Cantor subsets of the line. We think that a similar result holds for any sparse enough Cantor subset of M in higher dimensions, not necessarily a product. One should note that if the action of a group $G \subset \mathrm{Diff}^2(S^1)$ admits a Markov partition (see [DKN09, DKN13]) and an exceptional minimal set C then C must indeed be sparse.

It seems presently unknown whether for any $\mathcal{C}^1$-locally discrete group G of $\mathrm{Diff}^2(S^1)$ with an exceptional minimal set every point stabilizer should have a cyclic group of germs. Recall that a group $G \subset \mathrm{Diff}^2(S^1)$ is $\mathcal{C}^1$-locally discrete if the restriction of the identity to any interval intersecting its minimal set is isolated in the $\mathcal{C}^1$-topology among the restrictions of elements of G. In particular groups $G \subset \mathrm{Diff}^\omega(S^1)$ with an exceptional minimal set are $\mathcal{C}^1$-locally discrete, as well as Fuchsian groups. One believes that the action of every $\mathcal{C}^1$-locally discrete subgroup of $\mathrm{Diff}^\omega(S^1)$ has a Markov partition (see [AFKM+15, Main Conjecture]).

A well-known conjecture of Dippolito (see [Dip78, pp. 448–449]) states that for a finitely generated group $G \subset \mathrm{Diff}^2(S^1)$ with a minimal exceptional C such that the groups of germs

of stabilizers are cyclic the Radon–Nikodym derivative of every element of G (with respect to an invariant measure) should be locally constant. This suggests that our results describing the elements of $\mathfrak{diff}_M^1(C)$ for Cantor sets associated to generic affine IFS might be extended to more general Cantor sets. For instance, when $C = \Lambda\Gamma \subset S^1$ is the limit set of a second kind Fuchsian group Γ then $\mathfrak{diff}_M^{1,+}(C)$ should also be isomorphic to one of the Greenberg generalizations T_Γ or $V_\Gamma^\pm$ of the Thompson groups associated to Γ (see [Gre91]). More generally, if $\Gamma \subset \mathrm{Diff}^1(M)$ is a finitely generated group having an exceptional minimal set C, the group $\mathfrak{diff}_M^1(C)$ is closely related to the group of piecewise-Γ homeomorphisms of (M, C). Note that an exceptional minimal set of a Denjoy $\mathcal{C}^1$-diffeomorphism (i.e. without periodic points and whose derivative has bounded variation) of the circle is not generic in our sense, as the spectrum of ratios of lengths of gaps contains 1 in its closure (see [Mcd81, Por09]). On the other hand Triestino conjectured that any finitely generated $\mathcal{C}^1$-locally discrete subgroup of $\mathrm{Diff}^2(S^1)$ is $\mathcal{C}^1$-semi-conjugate to a subgroup of a generalized subgroup T_Γ, where now $\Gamma \subset \mathrm{Diff}^\omega(S^1)$ is Gromov-hyperbolic and $\mathcal{C}^1$-locally discrete. Similar questions arise for $\mathcal{C}^1$-locally discrete subgroups of $\mathrm{Diff}^2(M, C)$ in relation to the generalized groups $V_\Gamma^\pm$, associated to $\Gamma \subset \mathrm{Diff}^\omega(M, C)$.

Every couple of Cantor sets which are attractors of IFS arising from $\mathcal{C}^1$-diffeomorphisms of the line have arbitrarily small perturbations in the $\mathcal{C}^1$-topology which makes them disjoint and generically their arithmetic difference is still a Cantor set, according to a remarkable result of Moreira [Mor11]. We don't know how the group $\mathfrak{diff}_M^1(C)$ varies under a $\mathcal{C}^1$-perturbation of the IFS and in particular whether it might be larger than the Thompson-type group associated to the IFS.

The validity of some version of the Tits alternative for diffeomorphism groups has its counterpart both for $\mathfrak{diff}_M^1(C)$ and the smooth mapping class groups: does any finitely generated subgroup contains a free subgroup on two generators or else it has a finite orbit on the Cantor set C? This was recently settled in the affirmative by Hurtado and Militon (see [HM15]) for $\mathcal{M}^1(M, C_\lambda)$ where C_λ is the standard ternary Cantor set. Whether a strong Tits alternative could hold for the groups associated to some Cantor sets comprises the question on finding the solvable subgroups of $\mathfrak{diff}_M^1(C)$, which started to be investigated in [BMNR17]. Recall that the Thompson group F does not contain any free non-abelian group though is not virtually solvable. Notice that, by slightly extending [CJN14], every finitely-generated torsion-free nilpotent group can be made act on the interval with an invariant Cantor set, but we do not know whether there exists a non-virtually-abelian, nilpotent group of $\mathcal{C}^1$-diffeomorphisms of a generic Cantor set.

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