

# MATHEMATISCHE

[www.mn-journal.org](http://www.mn-journal.org)

Founded in 1948 by Erhard Schmidt

Editors-in-Chief:

B. Andrews, Canberra

R. Denk, Konstanz

K. Hulek, Hannover

F. Klopp, Paris

NACHRICHTEN

WILEY

REPRINT

# On the automorphism group of the asymptotic pants complex of an infinite surface of genus zero

Louis Funar\* and Maxime Nguyen\*\*

Institut Fourier, UMR 5582, Laboratoire de Mathématiques, Université Grenoble Alpes, CS 40700, 38058 Grenoble cedex 9, France

Received 31 August 2013, revised 10 February 2016, accepted 10 February 2016

Published online 4 April 2016

**Key words** Universal mapping class group, pants complex, infinite type surfaces, group actions, braided Thompson group

**MSC (2010)** 20F34, 20F38, 57M07, 57N05

The braided Thompson group  $\mathcal{B}$  is an asymptotic mapping class group of a sphere punctured along the standard Cantor set, endowed with a rigid structure. Inspired from the case of finite type surfaces we consider a Hatcher–Thurston cell complex whose vertices are asymptotically trivial pants decompositions. We prove that the automorphism group  $\widehat{\mathcal{B}^{\frac{1}{2}}}$  of this complex is also an asymptotic mapping class group in a weaker sense. Moreover  $\mathcal{B}^{\frac{1}{2}}$  is obtained by  $\mathcal{B}$  by first adding new elements called half-twists and further completing it.

© 2016 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim

## 1 Definitions and statements

### 1.1 Motivation

Inspired by Royden’s theorem on the holomorphic automorphisms of Teichmüller spaces Ivanov proved in [14] (subsequently completed by Korkmaz and Luo [15], [19]) that the automorphism group of the complex of curves of most compact surfaces coincides with the extended mapping class group. This was the start-point of many results of similar nature, coming under the name of rigidity theorems. Margalit proved the rigidity (see [20]) of pants complexes and further work extended this to even stronger rigidity theorem (see e.g. [1], [11]–[13], [16], [19], [21] for a non-exhaustive list).

The study of such automorphisms groups in the pro-finite or pro-unipotent categories seems fundamental in Grothendieck’s program. For instance, although the pro-finite pants complexes are still rigid the automorphism group of the corresponding pro-finite curve complexes is a version of the Grothendieck–Teichmüller group ([18] and references there).

Simpler versions of this general question concern the solenoids, whose study was started in [2], and then infinite type surfaces corresponding to direct limits. The purpose of this article is to make progress in the second case using the formalism of asymptotically rigid homeomorphisms and braided Thompson groups developed in [9], [10]. A previous result in this direction is the rigidity theorem proved in [8] for an infinite type planar surface related to the Thompson group  $T$  (see [4]).

In this note we will consider an infinite surface obtained from the sphere by deleting the standard Cantor set from the equator. Since its mapping class group is a topological group, the authors of [9] introduced a smaller subgroup  $\mathcal{B}$  called asymptotically rigid mapping class group, which was proved to be finitely presented. As its name suggests, one restricts to mapping classes of those homeomorphisms which preserve an extra structure on the surface, but only outside of large enough compact sub-surfaces.

There were different but closely related versions of such asymptotically rigid mapping class groups considered independently by Brin ([3]) and Dehornoy ([6], [7]). All of them are usually designed by the generic term of

\* Corresponding author: e-mail: louis.funar@univ-grenoble-alpes.fr

\*\* e-mail: ngmaxime@gmail.com

braided Thompson groups, as they occur as extensions of some Thompson group (see [4]) by an inductive limit of mapping class groups, in particular by infinite braid groups.

For instance,  $\mathcal{B}$  is the extension of the Thompson group  $V$  by an inductive limit of pure mapping class groups of holed spheres corresponding to an exhaustion of the sphere punctured along a Cantor set. The extra structure in its definition is needed to make unique the extension of a given homeomorphism defined on a compact sub-surface to the whole surface. The action at infinity of such a homeomorphism is an avatar of the action of the Thompson group  $V$  on the Cantor set.

The novelty in the present setting is the appearance of some mild non-rigidity phenomenon of the corresponding Hatcher–Thurston asymptotic pants complex. Nevertheless the group of automorphisms is still an asymptotic mapping class groups of the surface, but now the extra structure preserved is weakened.

## 1.2 The surface $S_{0,\infty}$

Let  $\mathbb{D}^2$  be the (hyperbolic) disc and suppose that its boundary  $\partial\mathbb{D}^2$  is parametrized by the unit interval (with endpoints identified). Let  $\tau_*$  denote the (dyadic) Farey triangulation of  $\mathbb{D}^2$ . This triangulation is given by the family of bi-infinite geodesics representing the standard dyadic intervals, i.e. the family of geodesics  $I_a^n$  joining the points  $p = \frac{a}{2^n}$ ,  $q = \frac{a+1}{2^n}$  on  $\partial\mathbb{D}^2$ , where  $a, n$  are integers satisfying  $0 \leq a \leq 2^n - 1$ . Let  $T_3$  be the dual graph of  $\tau_*$ , which is an infinite (unrooted) trivalent tree. Let  $\Sigma$  be a closed  $\delta$ -neighborhood of  $T_3$ .

Let  $S_{0,\infty}$  be the infinite surface obtained from gluing two copies of  $\Sigma$  along its boundary. We assume in addition that the family of arcs coming from the two copies of  $\tau_*$  defines a collection of simple closed curves, denoted by  $E$ .

A *pants decomposition* of a surface is a maximal collection of distinct homotopically nontrivial simple closed curves on it which are pairwise disjoint and non-isotopic. The complementary regions (which are 3-holed spheres) are called *pair of pants*. The collection of simple closed curves  $E$  is a *pants decomposition* of  $S_{0,\infty}$ . Subsequently,  $E$  will be called the *standard pants decomposition* of  $S_{0,\infty}$ .

**Definition 1.1** A *prerigid structure* on a surface is a collection of disjoint properly embedded line segments, such that the complement of their union has two connected components.

A *rigid structure* is the data consisting of a pants decomposition and a prerigid structure such that:

1. The traces of the prerigid structure on each pair of pants are made of three connected components, called *seams*;
2. For each pair of boundary circles of a given pair of pants, there is exactly one seam joining the two circles.

We will often call *rigid surface* a surface endowed with a rigid structure.

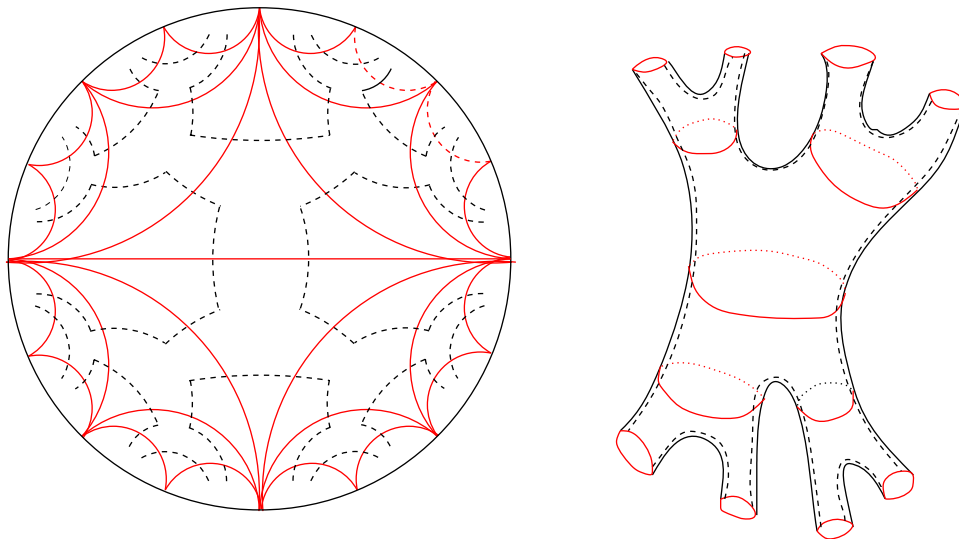
We arbitrarily fix a prerigid structure associated to the standard pants decomposition  $E$  to obtain a rigid structure which is called the *standard rigid structure* of  $S_{0,\infty}$ . The complement in  $S_{0,\infty}$  of the union of lines of the canonical prerigid structure has two components: we distinguish one of them as the *visible side* of  $S_{0,\infty}$ . Remark that the visible side of  $S_{0,\infty}$  is homeomorphic to the initial surface  $\Sigma$ .

## 1.3 The mapping class groups $\mathcal{B}$ , $\mathcal{B}^{\frac{1}{2}}$ and $\widehat{\mathcal{B}^{\frac{1}{2}}}$

**Definition 1.2** A compact sub-surface  $\Sigma_{0,n} \subset S_{0,\infty}$  (of genus zero with  $n$  boundary components) is *almost admissible* if its boundary is contained in the standard pants decomposition  $E$ . The *level* of a compact sub-surface is the number  $n$  of its boundary components.

Observe that any almost admissible sub-surface  $\Sigma_{0,n} \subset S_{0,\infty}$  inherits a (standard) rigid structure by restricting the standard rigid structure.

**Definition 1.3** Consider an almost admissible sub-surface  $\Sigma_{0,n} \subset S_{0,\infty}$  endowed with an arbitrary rigid structure. We say that the rigid sub-surface  $\Sigma_{0,n}$  is *quasi-admissible* if the seams of  $\Sigma_{0,n}$  have the same endpoints as the seams of the standard rigid structure  $E$  on  $S_{0,\infty} - \Sigma_{0,n}$ , so that we can glue together the seams of  $\Sigma_{0,n}$  with the seams of the standard rigid structure on  $S_{0,\infty} - \Sigma_{0,n}$  to obtain a prerigid structure on  $S_{0,\infty}$ . The visible



**Fig. 1** The surfaces  $(\Sigma, \tau_*)$  and  $(S_{0,\infty}, E)$ .

side of  $S_{0,\infty}$  induced from  $\Sigma_{0,n}$  is the one containing the visible side of  $\Sigma_{0,n}$ . If the trace of this visible side on  $S_{0,\infty} - \Sigma_{0,n}$  coincides with the trace of the visible side of the standard rigid structure  $E$  we say that the rigid sub-surface  $\Sigma_{0,n}$  is *admissible*.

**Definition 1.4** Let  $f$  be a homeomorphism of  $S_{0,\infty}$ , endowed with some rigid structure. One says that  $f$  is *almost-rigid* if it stabilizes the pants decomposition underlying the rigid structure. Further,  $f$  is *quasi-rigid* if it maps the pants decomposition into itself and the seams into the seams. If moreover  $f$  sends the visible side into the visible side, then  $f$  is said to be *rigid*.

Let  $\Sigma_{0,n} \subset S_{0,\infty}$  be a quasi-admissible rigid sub-surface and  $f$  a homeomorphism of  $S_{0,\infty}$ . When  $f(\Sigma_{0,n})$  is quasi-admissible, the image of the rigid structure of  $\Sigma_{0,n}$  by  $f$  is also a rigid structure on  $f(\Sigma_{0,n})$ , which is said to be *induced by  $f$* .

**Definition 1.5** The homeomorphism  $f$  of  $S_{0,\infty}$  is *asymptotically quasi-rigid* (resp. *asymptotically almost-rigid*) if there exists an almost-admissible sub-surface  $\Sigma_{0,n} \subset S_{0,\infty}$ , called a *support* of  $f$ , such that:

1.  $f(\Sigma_{0,n})$ , with the rigid structure induced by  $f$  from the standard rigid structure, is quasi-admissible;
2. the restriction of  $f : S_{0,\infty} - \Sigma_{0,n} \rightarrow S_{0,\infty} - f(\Sigma_{0,n})$  is quasi-rigid (resp. almost-rigid).

If, moreover  $f(\Sigma_{0,n})$  is admissible and  $f : S_{0,\infty} - \Sigma_{0,n} \rightarrow S_{0,\infty} - f(\Sigma_{0,n})$  is rigid, then we call  $f$  *asymptotically rigid*.

**Definition 1.6** The *asymptotic mapping class groups*  $\mathcal{B}^{\frac{1}{2}}$ ,  $\widehat{\mathcal{B}}^{\frac{1}{2}}$  and  $\mathcal{B}$  of  $S_{0,\infty}$  denote the groups of isotopy classes of asymptotically quasi-rigid, asymptotically almost-rigid and orientation preserving asymptotically rigid homeomorphisms, respectively.

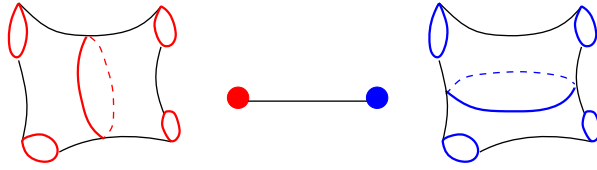
The group  $\mathcal{B}$  appeared in [9], where it was called the universal mapping class group of genus zero.

We will speak below of *asymptotically rigid* (resp. *quasi-rigid*) mapping classes, as being isotopy classes of asymptotically rigid (resp. quasi-rigid) homeomorphisms.

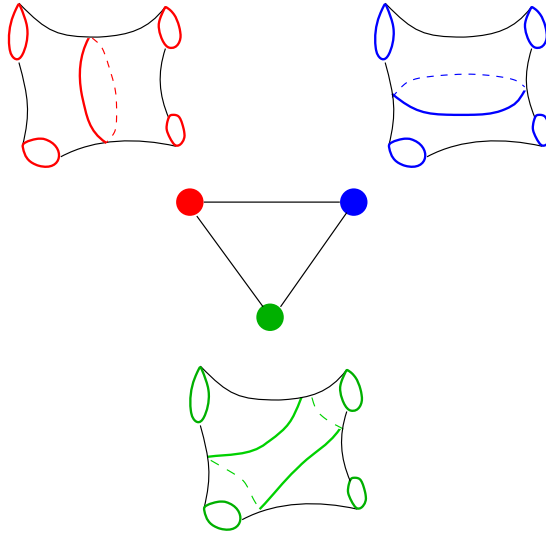
#### 1.4 The complex of pants decompositions of $S_{0,\infty}$

An *asymptotically trivial pants decomposition* of  $S_{0,\infty}$  is a pants decomposition which coincides with  $E$  outside an admissible sub-surface. We define a cellular complex whose vertex set is the set of asymptotically trivial pants decompositions.





**Fig. 2** Elementary move.



**Fig. 3** Triangular 2-cell in  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ .

**Definition 1.7** Let  $F$  and  $F'$  be two asymptotically trivial pants decompositions of  $S_{0,\infty}$ . Let  $c$  be a curve of  $F$ . We say that  $F$  and  $F'$  differ by an *elementary move* along the curve  $c$  if  $F'$  is obtained from  $F$  by replacing  $c$  by another curve which intersects  $c$  twice and does not intersect the other curves of  $F$ .

**Definition 1.8** Let  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  denote the *Hatcher–Thurston pants complex* of  $S_{0,\infty}$ , defined in ([9, Def. 5.1]) as follows:

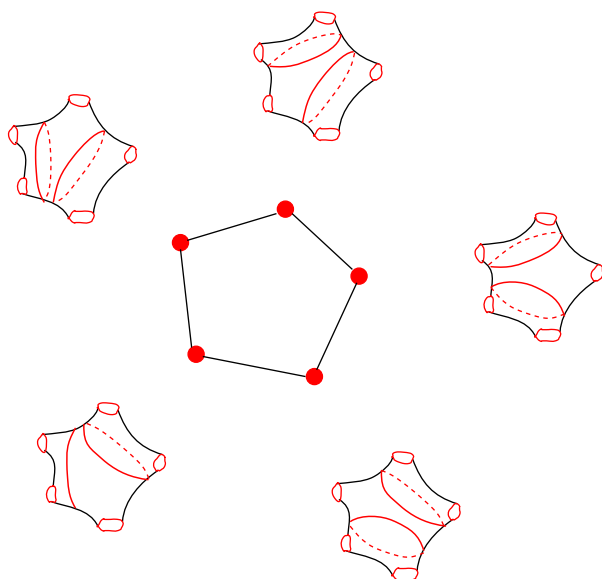
- The vertices are the asymptotically trivial pants decompositions of  $S_{0,\infty}$ ;
- The edges correspond to pairs of pants decomposition which differ by an elementary move;
- The 2-cells are introduced to fill triangular cycles (see Fig. 3), square cycles corresponding to commutativity of moves with disjoint supports and pentagonal cycles (see Fig. 4).

It is known (see [9, Prop. 5.4]) that  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  is connected and simply connected and  $\mathcal{B}$  acts cellularly on it, with one orbit of 0-cells, one orbit of 1-cells, one orbit of triangular 2-cells, one orbit of pentagonal 2-cells but infinitely many orbits of square 2-cells.

### 1.5 The main result

**Theorem 1.9** *The asymptotic mapping class group  $\widehat{\mathcal{B}}^{\frac{1}{2}}$  is isomorphic to the group of automorphisms of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ .*

Before to proceed with the proof we will give more details about the groups  $\mathcal{B}^{\frac{1}{2}}$  and  $\widehat{\mathcal{B}}^{\frac{1}{2}}$  in Section 2, by providing a finite generating set for  $\mathcal{B}^{\frac{1}{2}}$  and explaining in which sense  $\widehat{\mathcal{B}}^{\frac{1}{2}}$  is a completion of  $\mathcal{B}^{\frac{1}{2}}$ . We introduce the subgroups  $D$  and  $\widehat{D}$  of half-twists and give their structure, to be used later. In particular, one characterizes the group of half-twists as the common stabilizer of the standard pants decomposition  $E$  and a pair of pants on  $S_{0,\infty}$ .



**Fig. 4** Pentagonal 2-cell in  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ .

We prove in Section 3 that  $\widehat{\mathcal{B}}^{\frac{1}{2}}$  acts faithfully on the complex  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ , and thus it remains to show that every automorphism of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  is induced by an element of  $\widehat{\mathcal{B}}^{\frac{1}{2}}$ . From an explicit geometric characterization of the 2-cells of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  we deduce in Section 4 that two automorphisms of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  which coincide on the vertices adjacent to the base vertex  $E$  should be equal. One additional ingredient is the link graph  $\mathcal{L}(F)$  of a vertex  $F$  and its associated collapsed graph  $\mathcal{L}_{S_{0,\infty}}^A(F)$ , whose vertices are in one-to-one correspondence with the curves in the pants decomposition  $F$ . We then study the action of half-twists, on these links.

The final steps of the proof of our main result are given in Section 5. Given an automorphism of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  we construct an element of  $\widehat{\mathcal{B}}^{\frac{1}{2}}$  with the same action on the restricted link of  $E$  associated to a compact sub-surface. By induction we obtain a sequence of elements in  $\widehat{\mathcal{B}}^{\frac{1}{2}}$  which coincide with the given automorphism on sub-sets of  $\mathcal{L}(E)$  corresponding to bigger and bigger compact sub-surfaces. The associated infinite product is an element of  $\widehat{\mathcal{B}}^{\frac{1}{2}}$  whose action on  $\mathcal{L}(E)$ , and hence on all of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  is the prescribed one.

## 2 Preliminaries

### Thompson groups and $\mathcal{B}$

There is a natural projection  $\pi : S_{0,\infty} \rightarrow \Sigma$ , where  $\Sigma$  was defined in Section 1.2, such that the pullback of the arcs of  $\tau_*$  is the set of closed curves of  $E$ . Then we have a bijection between the set of standard dyadic intervals and the set of closed curves of  $E$ :

$$\{\text{curves of } E\} \leftrightarrow \{\text{arcs of } \tau_*\} \leftrightarrow \{\text{standard dyadic intervals}\}.$$

Recall that the Thompson group  $V$  is the group of right-continuous bijections of  $S^1$  that map images of dyadic rational numbers to images of dyadic rational numbers, that are differentiable except at finitely many images of dyadic rational numbers, and such that, on each maximal interval on which the function is differentiable, the function is linear with derivative a power of 2.

The Thompson group  $T$  is the group of piecewise linear homeomorphisms of  $S^1$  that map images of dyadic rational numbers to images of dyadic rational numbers, that are differentiable except at finitely many images of dyadic rational numbers and on intervals of differentiability on which the function is differentiable, the function is linear with derivative a power of 2.

For more details about Thompson groups, see [4]. Furthermore, we know (see [10]) that  $T$  can be viewed as an asymptotic mapping class group of the planar surface  $\Sigma$ .

Let  $K_\infty^*$  be the inductive limit  $\bigcup_{n=0}^\infty K^*(3 \cdot 2^n)$  where  $K^*(n)$  is the pure mapping class group of the  $n$ -holed sphere. We have the exact sequence ([9]):

$$1 \longrightarrow K_\infty^* \longrightarrow \mathcal{B} \longrightarrow V \longrightarrow 1.$$

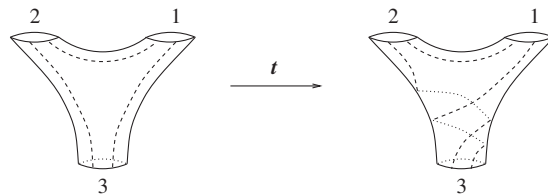
Moreover, we have the following result from [9, proof of Prop. 2.4]:

**Proposition 2.1** *The Thompson group  $T$  is the subgroup of elements of  $\mathcal{B}$  which preserve the visible side of  $S_{0,\infty}$ .*

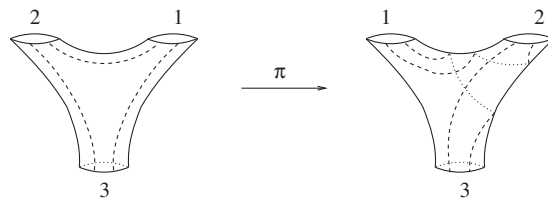
**Generators of  $\mathcal{B}$**

We fix an admissible pair of pants  $P$  of  $S_{0,\infty}$  to be called the *fundamental pair of pants*. The surface  $S_{0,\infty}$  retracts onto a tree which is the adjacency graph of the standard pants decomposition  $E$ . Choosing a fundamental pair of pants amounts to enhance this tree to a *rooted tree*  $\mathcal{T}$ . In particular, curves of  $E$  are in one-to-one correspondence with the mid-points of the edges of  $\mathcal{T}$ . Similarly, the *fundamental four-holed sphere* is the union of two adjacent admissible pairs of pants, one of them being the fundamental one.

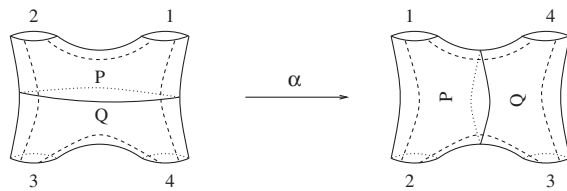
The group  $\mathcal{B}$  is generated by the twist  $t$  (see Fig. 5), the braid  $\pi$  (Fig. 6) and the lifts  $\alpha$  (see Fig. 7) and  $\beta$  (see Fig. 8) of the two generators of  $T$  which are usually denoted by the same letters. For more details see Section 3 of [9].



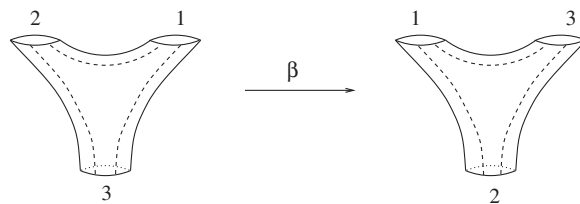
**Fig. 5** The action of a twist  $t$  on the fundamental pair of pants.



**Fig. 6** The action of a braid  $\pi$  on the fundamental pair of pants.



**Fig. 7** The action of  $\alpha$  on the fundamental four-holed sphere.



**Fig. 8** The action of a braid  $\beta$  on the fundamental pair of pants.

Recall that the subgroup of  $\mathcal{B}$  consisting of those mapping classes represented by rigid homeomorphisms is isomorphic to  $PSL_2(\mathbb{Z})$  (see [9, Rem. 2.1]). The action of  $PSL_2(\mathbb{Z})$  on the rooted tree  $\mathcal{T}$  is transitive. In particular, for any curve  $a$  of  $E$  the Dehn twist  $t_a$  along the curve  $a$  is a conjugate of  $t$  in  $\mathcal{B}$ .

Let  $\Sigma_{0,n}$  be a  $n$ -holed sphere embedded in  $S_{0,\infty}$  such that its boundary components lie in the standard decomposition  $E$  of  $S_{0,\infty}$ . Recall that the mapping class group  $\mathcal{M}(\Sigma_{0,n})$  is the group of isotopy classes of homeomorphisms of  $\Sigma_{0,n}$  preserving the orientation. The elements of  $\mathcal{M}(\Sigma_{0,n})$  are represented by homeomorphisms which can permute the boundary components of  $\Sigma_{0,n}$ . We can assume that they also preserve the trace of these boundary components on the visible side. Then there exists an embedding  $\mathcal{M}(\Sigma_{0,n}) \rightarrow \mathcal{B}$  obtained by extending rigidly a homeomorphism representing a mapping class of  $\mathcal{M}(\Sigma_{0,n})$ .

### 2.1 The extended mapping class group $\widehat{\mathcal{B}}^{\frac{1}{2}}$

We now define some mapping classes which are not represented by asymptotically rigid homeomorphisms. These elements will generate the extended mapping class group  $\mathcal{B}^{\frac{1}{2}}$ , whose completion is  $\widehat{\mathcal{B}}^{\frac{1}{2}}$ .

#### 2.1.1 The symmetry

**Definition 2.2** Let  $i_R$  be the isotopy class of the quasi-rigid homeomorphism acting as the symmetry on  $S_{0,\infty}$  which permute the visible side and the invisible side on each pair of pants of the standard decomposition  $E$  and preserves the seams.

#### 2.1.2 The half-twists

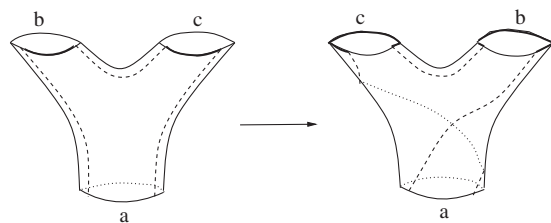
Let  $a$  be a curve of the standard decomposition  $E$ . Using the metric on the rooted tree  $\mathcal{T}$  we can speak about the distance between two curves of  $E$  and hence between subsets of  $E$ . There exists then a unique admissible pair of pants  $P_a$  which is different from the fundamental pair of pants  $P$ , such that  $a$  is the closest among the three boundary components of  $P_a$  to the fundamental pair of pants.

Let  $a, b, c$  be the bounding circles of  $P_a$ . By cutting along each of these three curves, we define three infinite connected components of  $S_{0,\infty}$  respectively denoted by  $S_a, S_b, S_c$  and one pair of pants  $P_a$ , such that  $S_a$  contains the fundamental pair of pants. Let  $D_a$  be a homeomorphism of  $P_a$  such that  $D_a$  fixes  $a$  and interchanges the other two curves, i.e.  $D_a(b) = c$  and  $D_a(c) = b$ , by means of a rotation of angle  $\pi$ . By definition the trace of  $b$  on the visible side is sent by  $D_a$  to the trace of  $c$  on the invisible side and the trace of  $c$  on the visible side is sent onto the trace of  $b$  on the invisible side. Then we quasi-rigidly extend this homeomorphism on  $S_{0,\infty}$ . This means that  $D_a$  acts on  $S_a$  as identity, sending the visible side of  $S_b$  on the invisible side of  $S_c$ , the invisible side of  $S_b$  on the visible side of  $S_c$ , the visible side of  $S_c$  on the invisible side of  $S_b$  and the invisible side of  $S_c$  on the visible side of  $S_b$ .

**Definition 2.3** We denote the isotopy class of the homeomorphism  $D_a$  by  $d_a$  and call it the *half-twist along a*.

We denote by  $supp(d_a)$  the sub-surface of  $S_{0,\infty}$  on which  $D_a$  acts non trivially up to homotopy, which is larger than its smallest support  $P_a$ . With the previous notations, we have  $supp(d_a) = P_a \cup S_b \cup S_c$ .

To emphasize the relation between  $\mathcal{B}^{\frac{1}{2}}$  and the Thompson-type groups note that each curve of the standard decomposition  $E$  is associated to a standard dyadic interval of the form  $I_k^n := [\frac{k}{2^n}, \frac{k+1}{2^n}]$  where  $k$  and  $n$  are integers



**Fig. 9** Action of a half-twist  $d_a$  on the pair of pants  $P_a$ .



such that  $0 \leq k \leq 2^n - 1$ . Moreover, the standard dyadic intervals  $I_{2^k}^{n+1}$  and  $I_{2^{k+1}}^{n+1}$  both define with  $I_k^n$  a pair of pants in  $E$ . Hence, each half-twist along a curve belonging in  $E$  is defined by a couple  $(k, n) \in \mathbb{N}^2$  such that  $0 \leq k \leq 2^n - 1$  and we can also use the notation  $d_a = d_{I_k^n}$ . For all  $n \leq m$  and  $j, k$  satisfying  $0 \leq j \leq 2^n - 1$  and  $0 \leq k \leq 2^m - 1$ , we have  $\text{supp}(d_{I_j^n}) \cap \text{supp}(d_{I_k^m}) = \emptyset$  or  $\text{supp}(d_{I_j^n}) \subset \text{supp}(d_{I_k^m})$ . Therefore, for all sequences  $(j_i, n_i, p_i)_{i \in \mathbb{N}}$  of elements of  $\mathbb{N} \times \mathbb{N} \times \mathbb{Z}$  such that for all  $i \in \mathbb{N}$ ,  $0 \leq j_i \leq 2^{n_i} - 1$  the sequence  $(n_i)_{i \in \mathbb{N}}$  is non decreasing, we can define the infinite product  $\prod_{i=0}^{\infty} (d_{I_{j_i}^{n_i}})^{p_i}$  as an element of the mapping class group of the surface  $S_{0,\infty}$ .

### 2.1.3 The group $\mathcal{B}^{\frac{1}{2}}$

**Proposition 2.4** *The group  $\mathcal{B}^{\frac{1}{2}}$  is generated by one half-twist around some curve of the standard decomposition  $E$ , the symmetry  $i_R$  and the elements  $\alpha$  and  $\beta$  of the group  $\mathcal{B}$ .*

*Convention.* By a slight abuse of language we speak about the image  $g(S)$ , where  $g$  is an isotopy class of a homeomorphism which is rigid on  $S_{0,\infty} - S$ , by letting it be the admissible sub-surface image of  $S$  by means of some homeomorphism representing  $g$ .

*Proof.* Let  $g \in \mathcal{B}^{\frac{1}{2}}$ . By possibly composing with the symmetry  $i_R$  we can assume that  $g$  is the mapping class of an orientation preserving homeomorphism of  $S_{0,\infty}$ . There exists an admissible standard sub-surface  $S \subset S_{0,\infty}$ , with  $g(S)$  quasi-admissible such that  $g$  is quasi-rigid outside  $S$ . There is no loss of generality in assuming that  $S$  contains the fundamental pair of pants  $P$ . Let  $a_j$ ,  $1 \leq j \leq n$ , be those boundary components of  $S$  such that the visible side induced from  $g(S)$  on the connected component of  $S_{0,\infty} - S$  containing  $a_j$  does not agree with the trace of the standard rigid structure  $E$ . Replace now  $S$  by  $S' = S \cup \bigcup_{j=1}^n P_{a_j}$ . Let  $a_{j,0}, a_{j,1}$  and  $a_j$  be the boundary components of  $P_{a_j}$ . Since  $g$  is quasi-rigid the element  $gd_{a_1}d_{a_2} \cdots d_{a_n}$  is also quasi-rigid. But now the visible sides of the connected components of  $S_{0,\infty} - S'$  containing either  $a_{j,0}$  or  $a_{j,1}$ , which are induced from  $gd_{a_1}d_{a_2} \cdots d_{a_n}(S')$  agree with the trace of the standard rigid structure, since the half-twist exchange the visible and invisible sides of the permuted boundary components. The remaining boundary components of  $S$  correspond to those of  $S'$ . It follows that  $gd_{a_1}d_{a_2} \cdots d_{a_n}(S')$  is admissible and hence the mapping class  $gd_{a_1}d_{a_2} \cdots d_{a_n}$  is rigid outside  $S'$  and hence asymptotically rigid. This means that  $gd_{a_1}d_{a_2} \cdots d_{a_n} \in \mathcal{B}$ . Therefore  $\mathcal{B}^{\frac{1}{2}}$  is generated by  $\mathcal{B}$  and the set of half-twists along curves in  $E$ .

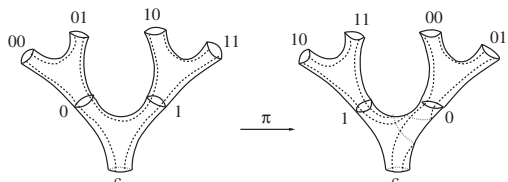
Further observe that the twist  $t_a$  is given by  $t_a = d_a^2$ . Note that a half-twist along any curve of  $S_{0,\infty}$  is conjugate to a half-twist along a curve of  $E$  by an element of  $\mathcal{B}$ , since  $\mathcal{B}$  acts transitively on the set of isotopy classes of simple closed curves of  $S_{0,\infty}$ .

Let now  $c$  be a curve of  $E$ . Set  $\pi_c$  for the conjugate of  $\pi$  which fixes  $c$  and acts in the same way on the pair of pants  $P_c$  as  $\pi$  does on the fundamental pair of pants. Denote by  $c_0, c_1$  the two other boundary components of  $P_c$ , so that  $c, c_0, c_1$  is a clockwise oriented triple, with respect to the local cyclic order structure induced by the planar visible part. We claim that:

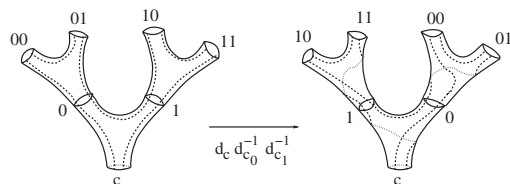
**Lemma 2.5** *We have  $\pi_c = d_c d_{c_0}^{-1} d_{c_1}^{-1}$ .*

We can obtain  $d_c, d_{c_0}$  and  $d_{c_1}$  as conjugates of a given half-twist by elements of  $T \subset \mathcal{B}$ . Assuming this lemma, it follows that  $\mathcal{B}^{\frac{1}{2}}$  is generated by  $T$  (which is generated by  $\alpha$  and  $\beta$ ),  $i_R$  and one half-twist, as claimed.  $\square$

*Proof of Lemma 2.5.* Let us denote by  $c_{i0}, c_{i1}$  the two other boundary components of  $P_{c_i}$ , with the same convention as above. To compare these two elements we have to consider a larger sub-surface of level 4, see the following picture:



The composition of half-twists  $d_c d_{c_0}^{-1} d_{c_1}^{-1}$  on a sub-surface of level 4 is viewed below:



Observe that the visible sides of the curves  $c_0$  and  $c_1$  are sent by  $\pi_c$  into the visible sides of the curves  $c_1$  and  $c_0$ , respectively. On the other hand, the visible sides of the curves  $c_0$  and  $c_1$  are sent by  $d_c d_{c_0}^{-1} d_{c_1}^{-1}$  into the invisible sides of the curves  $c_1$  and  $c_0$ , respectively.

Nevertheless, there exists an isotopy of  $S_{0,\infty}$  which is identity outside the sub-surface  $\Sigma_{0,4}$  sending the rigid structure induced by  $\pi_c$  into the one induced by  $d_c d_{c_0}^{-1} d_{c_1}^{-1}$ : twist along each  $c_0$  and  $c_1$  counterclockwise (with respect to the boundary orientation) until the endpoints of the seams are switched, thereby exchanging the visible and the invisible part of these circles. This proves that these two elements of  $\widehat{\mathcal{B}}^{\frac{1}{2}}$  coincide.  $\square$

### 2.1.4 The group $\widehat{\mathcal{B}}^{\frac{1}{2}}$

We will show now that  $\widehat{\mathcal{B}}^{\frac{1}{2}}$  is obtained from  $\mathcal{B}^{\frac{1}{2}}$  by a process of passing to limit.

**Proposition 2.6** Every  $g \in \widehat{\mathcal{B}}^{\frac{1}{2}}$  can be written as an infinite product:

$$g = f \cdot \prod_{i=1}^{\infty} d_{c_i}^{p_i}$$

where:

1.  $f \in \mathcal{B}^{\frac{1}{2}}$  acts quasi-rigidly outside some admissible sub-surface  $S \subset S_{0,\infty}$ ;
2.  $c_i$  is a sequence of curves belonging to  $E \cap (S_{0,\infty} - S)$  which goes to infinity, i.e. which eventually leaves any compact sub-surface of  $S_{0,\infty}$  and  $p_i \in \mathbb{Z}$ .

Note that both the left hand side and the infinite product in the right hand side are well-defined elements in the mapping class group of (all homeomorphisms of)  $S_{0,\infty}$ .

*Proof.* There exists a quasi-admissible sub-surface  $S \subset S_{0,\infty}$ , with  $g(S)$  quasi-admissible, such that  $g$  preserves globally the pants decomposition  $E \cap (S_{0,\infty} - S)$  of  $S_{0,\infty} - S$ , namely:  $g(E \cap (S_{0,\infty} - S)) = E \cap (S_{0,\infty} - g(S))$ . There is no loss of generality in assuming that  $S$  contains the fundamental pair of pants. Denote by  $f$  the element of  $\mathcal{B}^{\frac{1}{2}}$  which acts as  $g$  on  $S$  and is extended quasi-rigidly on  $S_{0,\infty} - S$ . Note that  $f$  might not belong to  $\mathcal{B}$  since the boundary circles do not necessarily inherit the right decomposition into visible and invisible part from  $g$ . It follows that  $f^{-1}g$  stabilizes every connected component of  $S_{0,\infty} - S$ .

Let us stick for the moment to the restriction to one connected component. Let  $U$  be a connected component of  $S_{0,\infty} - S$  and  $y \in \widehat{\mathcal{B}}^{\frac{1}{2}}$  be an element of its stabilizer. The unique boundary circle  $c$  of  $U$  has to be sent into itself by  $y$ , and its visible part is sent into its visible part. Let  $c_0, c_1$  be the two other curves which occur along with  $c$  as the boundary of the pair of pants  $P_c$  in  $E \cap U$ . Since  $y$  is the mapping class of a homeomorphism, the curves  $y(c_0), y(c_1)$  and  $y(c) = c$  belong to  $E \cap U$  and bound a pair of pants of  $E$ . This means that  $y(P_c) = P_c$  and thus  $\{y(c_0), y(c_1)\} = \{c_0, c_1\}$ . Up to composition with  $d_c$  we can therefore assume that  $y(c_i) = c_i$ , for  $i = 0, 1$ .

If the visible parts of  $c_i$  and  $y(c_i)$  agree then, up to composition with some powers of  $t_c, t_{c_0}$  and  $t_{c_1}$  we can assume that the restriction of  $y$  to  $P_c$  is identity, namely it sends the seams into seams and the visible part into the visible part.

If, moreover the visible part of  $c_i$  and  $y(c_i)$  disagree then we compose  $y$  with  $d_{c_i}$  and we reduce ourselves to the previous situation. In order to see this, recall that  $P_{c_i}$  denotes the pair of pants of  $E \cap U$  which is adjacent to  $P_c$  along  $c_i$ . Then the composition of  $y$  and those  $d_{c_i}$  needed above is the mapping class of a homeomorphism of

$P_c \cup P_{c_1} \cup P_{c_2}$  which is isotopic rel boundary to one whose restriction to  $P_c$  sends the visible part of  $c_i$  into the visible part of  $c_i$  (see the figures used in the proof of Lemma 2.5).

Note that  $d_{c_0}$  and  $d_{c_1}$  commute with each other. Therefore, in both cases, by composing  $y$  with some element of the form  $t_c^{-n} d_{c_0}^{-n_0} d_{c_1}^{-n_1}$  we can make the restriction of  $y$  to  $P_c$  to be identity.

Now, for any finite sequence  $I$  with entries from  $\{0, 1\}$ , we denote by  $c_{I0}$  and  $c_{I1}$  the two curves at unit distance from the curve  $c_I$  (using the metric induced from the tree  $\mathcal{T}$ ) which are farther from  $c$  than  $c_I$ . Let  $\|I\|$  denote the number of entries of  $I$ . By recurrence on  $\|I\|$ , for any  $k$  there exists  $n_i \in \mathbb{Z}$ , for  $i \leq k$ , such that the restriction of  $y t_c^{-n} \prod_{\|I\|=1}^k d_{c_I}^{-n_i}$  to the admissible sub-surface bounded by  $c$  and all the curves  $c_I$ , where  $\|I\| = k$ , is identity. It follows that we can write in  $\widehat{\mathcal{B}}^{\frac{1}{2}}$ :

$$y = t_c^n \prod_{\|I\|=1}^{\infty} d_{c_I}^{n_I}.$$

Notice that  $d_{c_I}$  and  $d_{c_J}$  commute when  $\|I\| = \|J\|$  and more generally, if  $\text{supp}(d_{c_I}) \cap \text{supp}(d_{c_J}) = \emptyset$ .

We can now resume the proof of the statement. Set  $U_1, U_2, \dots, U_n$  for the connected components of  $S_{0,\infty} - S$ , whose boundary circles are denoted by  $a_j$ . From above we can write:

$$f^{-1}g|_{U_j} = t_{a_j}^{n_j} \prod_{\|I\|=1}^{\infty} d_{a_{j,I}}^{n_{j,I}}, \quad \text{for } 1 \leq j \leq n.$$

Since  $c_j$  are disjoint and  $S$  contains the fundamental pair of pants, we have  $\text{supp}(d_{c_j}) \cap \text{supp}(d_{c_k}) = \emptyset$ , for  $j \neq k$ . This implies that:  $\text{supp}(d_{c_{j,I}}) \cap \text{supp}(d_{c_{k,J}}) = \emptyset$ , for  $j \neq k$ , because  $\text{supp}(d_{c_{j,I}}) \subset \text{supp}(d_{c_j})$ . We obtain therefore the identity:

$$g = f \prod_{j=1}^n t_{a_j}^{n_j} \prod_{\|I\|=1}^{\infty} \prod_{j=1}^n d_{a_{j,I}}^{n_{j,I}}$$

which proves the claim. □

### 2.1.5 The sub-groups $D$ and $\widehat{D}$

The sub-group of standard half-twists  $D \subset \widehat{\mathcal{B}}^{\frac{1}{2}}$  is the subgroup generated by the half-twists  $d_a$ , where  $a$  runs over the set of curves in  $E$ . Note that the extended supports  $\text{supp}(d_a)$  are all contained in the complement of the fundamental pair of pants, so that whenever  $a, b \in E$  then  $\text{supp}(d_a)$  and  $\text{supp}(d_b)$  are either disjoint or else one of them is contained in the other. In particular, although one might think that it is reasonable to call the element  $\alpha^2 d_a \alpha^2$  (where  $\alpha$  is the element of  $\mathcal{B}$  from picture 7) also a standard half-twist, it does not belong to the sub-group of standard half-twists  $D$ .

Let  $\widehat{D} \subset \widehat{\mathcal{B}}^{\frac{1}{2}}$  denote the sub-group consisting of possibly infinite products

$$d = d_{a_1} d_{a_2} \cdots d_{a_n} \cdots$$

of the half-twists  $d_{a_i}$ , where  $a_i$  is a sequence of curves from  $E$  which goes to infinity, i.e. which eventually leaves every compact sub-surface of  $S_{0,\infty}$ .

We wish to emphasize the fact that the sub-groups  $D$  and  $\widehat{D}$  depend on the choice of a fundamental pair of pants, up to conjugacy, although their isomorphism type does not.

Let  $\text{Perm}_{2^n}^3$  denote the group of those automorphisms of a rooted trivalent tree of level  $n + 1$  which fix the neighbours of the root, or equivalently, the automorphism group of three copies of the rooted binary tree of level  $n$ . The action of  $\text{Perm}_{2^n}^3$  on the set of  $3 \cdot 2^n$  boundary leaves is faithful. Denote by  $\text{Perm}_{\infty}^3$  the inductive limit  $\lim_{n \rightarrow \infty} \text{Perm}_{2^n}^3$ , where  $\text{Perm}_{2^n}^3 \rightarrow \text{Perm}_{2^{n+1}}^3$  is induced by the embedding of the corresponding rooted trees.

**Proposition 2.7** *The abelian sub-group  $D[2] \subset D$  generated by the twists  $t_a = d_a^2$  along the curves  $a$  in  $E$  is a normal subgroup of  $D$  which fits into the exact sequence:*

$$1 \longrightarrow D[2] \longrightarrow D \longrightarrow \text{Perm}_{\infty}^3 \longrightarrow 1$$

**P r o o f .** The homomorphism  $D \rightarrow Perm_n^3$  corresponds to the action of  $D$  on the trivalent sub-tree  $\mathcal{T}_n$  of  $\mathcal{T}$  whose vertices are those curves in  $E$  at distance at most  $n$  from the fundamental pair of pants  $P$ . Each element of  $Perm_n^3$  is a product of transpositions. Here by transposition we mean the transformation exchanging two branches having a common vertex by means of a planar symmetry, so that the cyclic order of their leaves is reversed. Every transposition is the image of the half-twist along the curve in  $E$  corresponding to the vertex, so that  $D \rightarrow Perm_n^3$  is surjective for every  $n$  and hence also for  $n = \infty$ .

We show further that  $D[2]$  is a normal subgroup of  $D$ . By direct inspection we find that:

$$d_c d_a = \begin{cases} d_a d_c, & \text{if } \text{supp}(d_a) \cap \text{supp}(d_c) = \emptyset, \\ d_{d_c(a)} d_c, & \text{if } \text{supp}(d_a) \subset \text{supp}(d_c), \\ d_a d_{d_a^{-1}(c)}, & \text{if } \text{supp}(d_c) \subset \text{supp}(d_a). \end{cases}$$

We derive that:

$$d_c d_a^2 d_c^{-1} = \begin{cases} d_a^2, & \text{if } \text{supp}(d_a) \cap \text{supp}(d_c) = \emptyset, \\ d_{d_c(a)}^2, & \text{if } \text{supp}(d_a) \subset \text{supp}(d_c), \\ d_a^2, & \text{if } \text{supp}(d_c) \subset \text{supp}(d_a), \end{cases}$$

and our claim follows.

Let now  $g \in \ker(D \rightarrow Perm_n^3)$  be a product of half-twists along curves of  $E \cap \Sigma_{0,3,2^n}$ , where  $\Sigma_{0,3,2^n}$  is the surface whose boundary circles are at distance  $n$  from the fundamental pair of pants  $P$ . Since  $g$  is a homeomorphism preserving  $E$  and fixing the boundary of  $P$ ,  $g$  should send  $\Sigma_{0,3,2^k}$  into itself, for every  $k$ . By induction on  $k$ , the action of  $g$  on the set of boundary components of  $\Sigma_{0,3,2^k}$  must be trivial, for any  $k \geq 0$ . The induction step follows from the fact that  $\Sigma_{3,2^{k+1}} - \Sigma_{3,2^k}$  is a union of disjoint pair of pants. Then the restriction of  $g$  to each such pair of pants either sends the new level  $k + 1$  boundary components into themselves, or else it permutes them. But a non-trivial permutation of them induces a non-trivial element in  $Perm_\infty^3$ . This proves that  $g$  keeps fixed any curve of  $E$  and hence all pair of pants in  $\Sigma_{0,3,2^n}$ . Since the mapping class group of a pair of pants is the abelian group generated by the three boundary Dehn twists, it follows that  $g$  belongs to the normal subgroup of  $D$  generated by the Dehn twists along curves of  $E$ , i.e. to  $D[2]$ .  $\square$

**Proposition 2.8** Set  $\widehat{D[2]} \subset \widehat{\mathcal{B}^{\frac{1}{2}}}$  for the sub-group consisting of possibly infinite products

$$d = t_{a_1} t_{a_2} \cdots t_{a_n} \cdots$$

of Dehn twists  $t_{a_i}$ , where  $a_i$  is a sequence of curves from  $E$  which goes to infinity. Then  $\widehat{D[2]}$  is a normal subgroup of  $\widehat{D}$  which fits into the exact sequence:

$$1 \longrightarrow \widehat{D[2]} \longrightarrow \widehat{D} \longrightarrow Perm_\infty^3 \longrightarrow 1$$

where  $Perm_\infty^3$  is as above.

**P r o o f .** Note first that any element  $d \in \widehat{D}$  can be written as an infinite product of the form:

$$d = d_{a_1}^{p_1} d_{a_2}^{p_2} \cdots d_{a_n}^{p_n} \cdots$$

of powers of half-twists  $d_{a_i}$  along curves from  $E$ , where  $p_j \in \mathbb{Z}$ , so that there exist  $k_0 \leq k_1 \leq \cdots \leq k_n \leq \cdots$  with the properties:

1.  $a_{k_n}, a_{k_{n+1}}, \dots, a_{k_{n+1}-1}$  are curves of level  $n$ , for each  $n$ ;
2. there are no more than  $3 \cdot 2^n$  curves of level  $n$ , i.e.  $k_{n+1} - k_n \leq 3 \cdot 2^n$ , for all  $n$ .

Then the proof given above for Proposition 2.7 works without essential modifications. We skip the details.  $\square$

**Proposition 2.9** Let  $g \in \widehat{\mathcal{B}^{\frac{1}{2}}}$  be an element such that  $g(E) = E$  and  $g$  fixes the fundamental pair of pants  $P$ , namely  $g$  is the mapping class of a homeomorphism whose restriction to  $P$  is identity. Then  $g \in \widehat{D}$ . In particular, if additionally  $g \in \mathcal{B}^{\frac{1}{2}}$ , then  $g \in D$ .

**Proof.** The arguments are similar to those used in the second part of Proposition 2.7. Since  $g$  is a homeomorphism preserving  $E$  and fixing the boundary of  $P$ ,  $g$  sends  $\Sigma_{0,3 \cdot 2^k}$  into itself, for every  $k$ . By induction on  $k$ , the action of  $g$  on the set of boundary components of  $\Sigma_{0,3 \cdot 2^k}$  is the same as that of a product  $d_k$  of half-twists along curves of level  $\leq k - 1$ , for any  $k \geq 1$ . The induction step follows from the fact that  $\Sigma_{3 \cdot 2^{k+1}} - \Sigma_{3 \cdot 2^k}$  is a union of disjoint pair of pants. Then the restriction of  $g$  to each such pair of pants either sends the new level  $k + 1$  boundary components of into themselves, or else permutes them.

Now, the element  $d = d_1 d_2 \cdots d_k \cdots$  belongs to  $\widehat{D}$  and the action of  $d^{-1}g$  on the set of curves of  $E$  is trivial. Therefore, as above,  $d^{-1}g \in \widehat{D}[2]$ , so that  $g \in \widehat{D}$ , as claimed.

The second assertion follows from Proposition 2.7.  $\square$

### 3 The complex of decompositions of $S_{0,\infty}$

#### 3.1 Geometric interpretation of 2-cells

We say that two curves of a pants decomposition  $F$  are *adjacent* if they bound the same pair of pants in  $F$ .

**Proposition 3.1** *Let  $F_1, F_2, F_3$  be three vertices of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  such that for all  $i \in 1, 2$ ,  $F_i$  and  $F_{i+1}$  are joined by an edge in  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ . Then there exists a unique 2-cell in  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  containing  $F_1, F_2, F_3$  as vertices.*

**Proof.** There are three possible cases:

- If  $d(F_1, F_3) = 1$ , then  $F_1, F_2, F_3$  are vertices of a triangular 2-cell;
- If  $d(F_1, F_3) = 2$ , there are two possible cases. Let  $m$  and  $m'$  be the elementary moves along the curves  $c$  and  $c'$ , which are represented by the edges  $(F_1, F_2)$  and  $(F_2, F_3)$ , respectively;
  - If  $c$  and  $c'$  are not adjacent in the decomposition  $F_2$ , the associated elementary moves have disjoint supports and they commute. Hence,  $F_1, F_2, F_3$  belong to a unique squared 2-cell;
  - If  $c$  and  $c'$  are adjacent, the associated elementary moves do not commute. Let  $F_0$  be the decomposition obtained from  $F_1$  by applying the elementary move  $m'$  and  $F_4$  the decomposition obtained from  $F_3$  by applying  $m$ . Then  $d(F_0, F_4) = 1$ .  $\square$

#### 3.2 Action of the mapping class group of the complex

**Lemma 3.2** *The extended mapping class group  $\widehat{\mathcal{B}}^{\frac{1}{2}}$  acts by automorphisms on the complex  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ . This action is transitive on the set of vertices of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ . Moreover, the natural map  $\Psi : \widehat{\mathcal{B}}^{\frac{1}{2}} \rightarrow \text{Aut}(\mathcal{C}_{\mathcal{P}}(S_{0,\infty}))$  is injective.*

**Proof.** The first part of this result is a weak version of Proposition 5.4 of [9]. We prove the injectivity part as follows. We take an element  $g \in \widehat{\mathcal{B}}^{\frac{1}{2}}$  and assume  $g$  is non trivial. We want to show that we can find a vertex  $F$  of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  such that  $g \cdot F \neq F$ .

We first consider the special case where  $g \in \mathcal{B}^{\frac{1}{2}}$ . Let  $\Sigma_{0,n}$  be a quasi-admissible sub-surface of  $S_{0,\infty}$  such that  $n \geq 5$  and  $g$  preserves the trace of pants decompositions and seams on  $S_{0,\infty} - \Sigma_{0,n}$ .

If  $\Sigma_{0,n} \neq g(\Sigma_{0,n})$ , there is a curve  $c$  of the standard decomposition  $E$  such that  $c$  is a boundary component of  $\Sigma_{0,n}$ ,  $g(c)$  is a boundary component of  $g(\Sigma_{0,n})$ ,  $g(c)$  is still in  $E$  but  $g(c)$  is different from  $c$ . Let  $F$  be a decomposition obtained from  $E$  by an elementary move on  $c$ . Then  $g \cdot F \neq F$ .

If  $\Sigma_{0,n} = g(\Sigma_{0,n})$ , then the restriction of  $g$  to  $\Sigma_{0,n}$  is a non-trivial element of the mapping class of  $\Sigma_{0,n}$ . The injectivity part of the main theorem of [20] says that there exists a pants decomposition  $F_n$  of  $\Sigma_{0,n}$  such that  $g \cdot F_n \neq F_n$ . Let  $F$  be the decomposition of  $S_{0,\infty}$  which coincide with  $E$  outside  $\Sigma_{0,n}$  and with  $F_n$  inside  $S_{0,\infty}$ . Then  $g \cdot F \neq F$ .

Now, we turn to the general case  $g \in \widehat{\mathcal{B}}^{\frac{1}{2}}$ . We can write, following Proposition 2.6:

$$g = f \cdot \prod_{i=0}^{\infty} d_{c_i}^{p_i}$$



where  $f \in \mathcal{B}^{\frac{1}{2}}$  and the sequence of curves  $c_i$  belong to  $E$  and eventually leave every compact sub-surface. If  $f$  is non trivial, the previous case leads to the conclusion. If  $f$  is trivial, then there exists  $i \in \mathbb{N}$  such that  $p_i \neq 0$ . Let  $F$  be a decomposition obtained from  $E$  by an elementary move applied on the curve  $c_i$ . Then  $g \cdot F \neq F$ .  $\square$

#### 4 The automorphism group of $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$

In this section, we introduce a graph whose vertices represent the neighbours of the vertex  $E$  in  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ . We will prove that the automorphism group of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  acts naturally on this graph. We first consider the following general result.

**Lemma 4.1** *Let  $\phi$  and  $\phi'$  be two automorphisms of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  such that for any decomposition  $F$  joined by an edge to  $E$  in  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ , we have  $\phi(F) = \phi'(F)$ . Then  $\phi = \phi'$ .*

*Proof.* The proof is based on the geometric characterization of the 2-cells of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ . We denote by  $d$  the combinatorial distance on the 1-skeleton of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ . We prove that  $\phi(F) = \phi'(F)$  for any vertex  $F$  of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ , by induction on the distance between  $E$  and  $F$ . We have:

1. First,  $\phi(E) = \phi'(E)$ . Indeed, let  $E, F$  and  $F'$  be the vertices of a triangular 2-cell. There exists precisely one more triangular 2-cell sharing two vertices with the former cell, say of vertices  $E, F$  and  $F''$ . By direct inspection the three vertices  $F, F'$  and  $F''$  uniquely determine the fourth vertex of this configuration, namely any vertex  $H$  such that both  $H, F, F'$  and  $H, F, F''$  are triangular 2-cells of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  coincides with  $E$ . Therefore any automorphism of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  which fixes  $F, F'$  and  $F''$  should also fix  $E$ .
2. For any  $F$  at distance one from  $E$ ,  $\phi(F) = \phi'(F)$ ;
3. By the induction hypothesis, assume that  $\phi$  and  $\phi'$  coincide on the ball of center  $E$  and radius  $n$  in  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ , with  $n \geq 1$ . Let  $G$  be any vertex at distance  $n + 1$  of  $E$ . We consider a path  $p = H_0 = E, H_1, \dots, H_{n-1}, H_n, G$  of length  $n + 1$  joining  $E$  to  $G$ . The vertices  $H_{n-1}, H_n, G$  define a unique 2-cell of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ . This 2-cell contains at least one fourth vertex  $G'$  at distance less than or equal to  $n$  from  $E$ . Indeed otherwise this 2-cell were triangular, so  $G$  would be at distance less or equal to  $n$  from  $E$ , contradicting our assumptions. Hence,  $\phi(G)$  is necessary the vertex of the unique 2-cell of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  defined by  $\phi(G')$ ,  $\phi(H_{n-1})$ ,  $\phi(H_n)$  which is joined by an edge to  $\phi(H_n)$  and different from  $\phi(H_{n-1})$ . By uniqueness,  $\phi(G) = \phi'(G)$ .  $\square$

##### 4.1 The link of a decomposition

For any vertex  $F$  of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ , we define the *link*  $\mathcal{L}(F)$  to be a graph whose set of vertices consists of those vertices in  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  which are adjacent to  $F$ . Then two vertices are connected within  $\mathcal{L}(F)$  by an (A)-edge when they lie together with  $F$  in the same pentagonal 2-cell of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ , and by a (B)-edge when they belong to the same triangular 2-cell in  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ . By the geometric interpretation of 2-cells in  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ , this bi-colored link graph is well defined.

Further, denote by  $\mathcal{L}_{S_{0,\infty}}^A(F)$  the graph obtained from  $\mathcal{L}(F)$  by collapsing each (B)-edge (along with its endpoints) to a vertex.

Let  $\Sigma_{0,n}$  be an  $n$ -holed sphere and  $F_n$  a pants decomposition of  $\Sigma_{0,n}$ , viewed as vertex of the complex  $\mathcal{C}_{\mathcal{P}}(\Sigma_{0,n})$ . Then one defines the *restricted link*  $\mathcal{L}_{\Sigma_{0,\infty}}(F_n)$  as above, but using only the neighbours in  $\mathcal{C}_{\mathcal{P}}(\Sigma_{0,n})$ .

Now, for every vertex  $F$  of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  and any curve  $c$  in  $F$ , we denote by  $V_c(F)$  the subset of those vertices of  $\mathcal{L}(F)$  corresponding to the decompositions obtained by a single elementary move on  $c$ . Thus, the set of vertices of  $\mathcal{L}(F)$  is the disjoint union of all  $V_c(F)$ , where  $c$  runs through the set of curves of  $F$ .

We obtain directly from the definitions the following lemma:

**Lemma 4.2** *Let  $\phi$  be any automorphism of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ . Then  $\phi$  induces an isomorphism  $\phi_{*,F} : \mathcal{L}(E) \rightarrow \mathcal{L}(F)$  where  $F = \phi(E)$ . Moreover,  $\phi_{*,F}$  preserves the (A)- and (B)-edge types.*

In particular, this lemma says that any link  $\mathcal{L}(F)$  is isomorphic to  $\mathcal{L}(E)$ .

## 4.2 Structure of $\mathcal{L}(E)$

We give a description of the graph  $\mathcal{L}(E)$  based on the geometric characterization of 2-cells in  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ .

### (A)-edges

Given two curves  $c$  and  $c'$  in  $E$  and any two distinct vertices  $a \in V_c(E)$  and  $b \in V_{c'}(E)$ , there exists an (A)-edge joining the vertices  $a$  and  $b$  in  $\mathcal{L}(E)$  if and only if  $c$  and  $c'$  are *adjacent* in  $E$ , that is  $c$  and  $c'$  bound the same pair of pants in  $E$ .

### (B)-edges

Given two curves  $c$  and  $c'$  in  $E$  and two distinct vertices  $a \in V_c(E)$  and  $b \in V_{c'}(E)$ , if there exists a (B)-edge joining the vertices  $a$  and  $b$  in  $\mathcal{L}(E)$  then  $c = c'$ .

### Structure of $V_c(E)$

Let  $c$  be any curve in  $E$ . We apply an elementary move on  $c$  such that the resulting curve  $c_0$  has only one component in the visible side of  $S_{0,\infty}$ . Denote by  $F_{c_0}$  the decomposition obtained from  $E$  by this elementary move.

Any other curve intersecting  $c$  twice and which is disjoint from the other curves in  $E$  can be obtained from  $c_0$  by applying  $k$  half-twists along  $c$ , where  $k$  is some non-zero integer. We denote by  $c_k$  the curve obtained from  $k$  half-twists along  $c$  on  $c_0$  and  $F_{c_k}$  the resulting decomposition of  $S_{0,\infty}$ . Then we have a canonical identification:

$$V_c(E) = \{F_{c_k}, k \in \mathbb{Z}\}.$$

Furthermore,  $F_{c_k}$  and  $F_{c_m}$  are joined by an edge (which is necessarily of B-type) if and only if  $|k - m| = 1$ .

We remark that a vertex in  $V_c(E)$  represents a homotopy class of a non-trivial curve in  $\Sigma_{0,4}$  and that two vertices in  $V_c(E)$  are joined by an edge if the associated curves on  $\Sigma_{0,4}$  have geometric intersection equal to 2. This graph has been considered before, in relation with the curve complex of  $\Sigma_{0,4}$ . It was already known to Max Dehn that this graph is isomorphic to the Farey graph  $\tau_*$  (see [5] and [19, Section 3.2]).

Note that  $V_c(F)$  has a similar structure, for any  $F$ . By choosing  $g \in \mathcal{B}^{\frac{1}{2}}$  such that  $F = g(E)$ , we derive an identification of  $V_c(F)$  with  $V_{g^{-1}(c)}(E)$ . This identification is not unique, as we can alter  $g$  by an element of  $D$ .

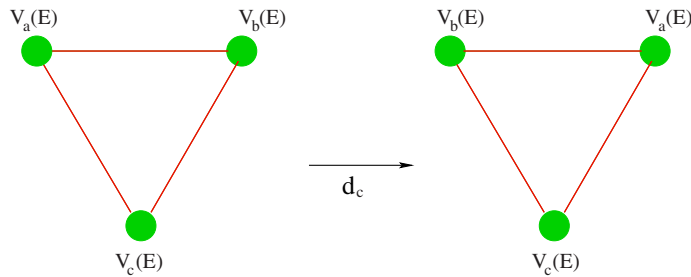
## 4.3 Structure of $\mathcal{L}_{S_{0,\infty}}^A(E)$

The image of  $V_c(E)$  by the collapsing map  $\mathcal{L}(E) \rightarrow \mathcal{L}_{S_{0,\infty}}^A(E)$  is a single point. Thus it will make sense to speak about the vertex  $V_c(E)$  of  $\mathcal{L}_{S_{0,\infty}}^A(E)$ . This provides a one-to-one correspondence between the set of vertices of  $\mathcal{L}_{S_{0,\infty}}^A(E)$  and the set of curves in  $E$ . Moreover, two vertices of  $\mathcal{L}_{S_{0,\infty}}^A(E)$  are connected by an edge if and only if they represent two adjacent curves in  $E$ . Thus  $\mathcal{L}_{S_{0,\infty}}^A(E)$  consists of triangles indexed by the pair of pants occurring in  $E$ . We will often consider the dual of  $\mathcal{L}_{S_{0,\infty}}^A(E)$ , whose vertices correspond to pair of pants in  $E$  and edges to curves in  $E$ , which is therefore combinatorially isomorphic to the binary tree  $\mathcal{T}$ .

## 4.4 Action of half-twists on the links

Let  $d_c$  be a half-twist whose support is a pair of pants bounded by the curves  $a, b, c$  of  $E$ , such that  $d$  permutes  $a$  and  $b$ . Then  $d_c$  acts on  $\mathcal{L}_{S_{0,\infty}}^A(E)$  by reversing the cyclic order of the vertices of the triangle associated to the support of  $d_c$  in  $\mathcal{L}_{S_{0,\infty}}^A(E)$  and fixing the vertex  $V_c(E)$ . Moreover  $d_c$  acts on  $\mathcal{L}(E)$  and preserves the type of edges in  $\mathcal{L}(E)$ . Hence,  $d_c$  acts on the union of the sets  $V_{c'}(E)$  over the curves  $c'$  of  $E$ . We distinguish three cases in the description of the action on  $V_{c'}(E)$ , according to whether  $c'$  is adjacent to one of the curves  $a, b, c$  or not:

1. Let  $c'$  be a curve from  $E$  contained in  $\text{supp}(d_c)$  and different from  $c$ . The half-twist  $d_c$  sends  $c'$  into a different curve  $c''$  of  $E$ . The induced map  $V_{c'}(E) \rightarrow V_{c''}(E)$  is given by  $F_{c'_k} \mapsto F_{c''_k}$ , for  $k \in \mathbb{Z}$ .



**Fig. 10** Action of the half-twist  $d$  on the link  $\mathcal{L}_{S_{0,\infty}}^A(E)$ .

2. Further  $V_c(E)$  is stable by  $d_c$  and  $d_c$  acts on this set by the map  $F_{c_k} \mapsto F_{c_{k+1}}, k \in \mathbb{Z}$ .
3. If  $c'$  is a curve belonging to the component of  $S_{0,\infty} \setminus \{c\}$  which does not contain  $a$  and  $b$ , then  $d_c$  fixes  $V_{c'}(E)$  point-wise.

#### 4.5 Action of $T$ on the links

Let  $t$  be an element of the Thompson group  $T$ . Then  $t$  induces an isomorphism  $t_{*,F} : \mathcal{L}(E) \rightarrow \mathcal{L}(F)$ , where  $F = t(E)$ . If  $c$  is a curve of  $E$ , there exists a curve  $c'$  of  $F$  such that  $t_{*,F}(V_c(E)) = V_{c'}(F)$ . The mapping class  $t$  preserves the visible face of  $S_{0,\infty}$  from Proposition 2.1. For any  $k \in \mathbb{Z}$ , we have then  $t_{*,F}(F_{c_k}) = F_{c'_k}$ .

#### 4.6 Characterization of automorphisms of $\mathcal{L}(E)$

We deduce some conditions for an isomorphism  $\mathcal{L}(E) \rightarrow \mathcal{L}(E)$  to be induced by an element of  $\widehat{\mathcal{B}}^{\frac{1}{2}}$ .

**Lemma 4.3** *Let  $f \in \widehat{\mathcal{B}}^{\frac{1}{2}}$  such that  $\phi(E) = E$ , where  $\phi = \Psi(f) \in \text{Aut}(\mathcal{C}_{\mathcal{P}}(S_{0,\infty}))$ . Let  $c$  be a curve in  $E$  and  $c'$  be the curve of  $E$  such that  $\phi(V_c(E)) = V_{c'}(E)$ . Then there exists  $p \in \mathbb{Z}$ , such that for all  $k \in \mathbb{Z}$ , we have  $\phi(F_{c_k}) = F_{c'_{\varepsilon(f)k+p}}$ , where  $\varepsilon(f) = 1$ , if  $f$  preserves the orientation and  $\varepsilon(f) = -1$ , otherwise.*

**Proof.** Assume we choose a set of identifications  $\alpha_{c,F} : V_c(F) \rightarrow \mathbb{Z}$ , for all  $F$  and  $c \in F$ , which coincides with the canonical identification above for  $V_c(E)$ . Set  $\text{Perm}(\mathbb{Z})$  for the group of bijections of  $\mathbb{Z}$ . We have a map  $\mu_{c,F} : \widehat{\mathcal{B}}^{\frac{1}{2}} \rightarrow \text{Perm}(\mathbb{Z})$  defined by  $\mu_{c,F}(g)(k) = \alpha_{g(c),g(F)} \circ g_* \circ \alpha_{c,F}^{-1}(k)$ , where  $k \in \mathbb{Z}, c \in F$  and  $g_* : V_c(F) \rightarrow V_{g(c)}(g(F))$  is the map induced by  $g \in \widehat{\mathcal{B}}^{\frac{1}{2}}$ . This map  $\widehat{\mu}$  is a twisted group homomorphism, in the sense that  $\mu_{c,g \circ f}(g \circ f) = \mu_{f(c),f(F)}(g) \circ \mu_{c,F}(f)$ , if  $f, g \in \widehat{\mathcal{B}}^{\frac{1}{2}}$ . We set  $\mu_c(f) = \mu_{c,E}(f)$ , when  $c \in E$  and  $f(E) = E$ . Note that  $\mu_c(f)$  is independent of the choice of  $\alpha_{c,F}$  extending the canonical identification, if  $f(E) = E$ , and we have  $\phi(F_{c_k}) = F_{c'_{\mu_c(f)(k)}}$ , where  $\phi = \Psi(f)$ .

From Section 4.5, we have  $\mu_{c,E}(t) = id_{\mathbb{Z}}$  for any  $t \in T$ . From Section 4.4 if  $d$  is a half-twist then  $\mu_c(d)$  is either the identity  $id_{\mathbb{Z}}$ , or the translation  $\mu_c(d)(k) = k + 1$ . Also  $\mu_c(i_R)(k) = -k$ . Since any element of  $\widehat{\mathcal{B}}^{\frac{1}{2}}$  is a product of an element of  $T$  along with infinitely many half-twists accumulating at infinity,  $\mu_c(f)$  has the required form when  $f(E) = E$ . □

**Lemma 4.4** *Let  $f \in \widehat{\mathcal{B}}^{\frac{1}{2}}$  such that  $\phi(E) = E$ , where  $\phi = \Psi(f) \in \text{Aut}(\mathcal{C}_{\mathcal{P}}(S_{0,\infty}))$ . Let  $P'$  and  $P''$  be two almost admissible pairs of pants which are bounded by the curves  $a, a', a''$  and  $a, b, b'$ , respectively. We assume that:*

1.  $a$  is closer to the fundamental pair of pants than  $a'$  and  $a''$ ;
2. the vertices  $V_a(E), V_{a'}(E), V_{a''}(E)$  and  $V_b(E)$  of  $\mathcal{L}_{S_{0,\infty}}^A(E)$  are point-wise fixed by  $\phi$ .

Then  $\mu_a(f)(k) = \varepsilon(f)k + p$ , where  $p$  is even.

**Proof.** We reduce ourselves to the case when  $f$  preserves the orientation, by composing if needed with the symmetry  $i_R$ . Choose an orientation preserving rigid mapping class  $h$  such that  $h(P'') = P$  and  $h(a)$  is a boundary curve of the fundamental pair of pants  $P$ , whereas  $h(a')$  and  $h(a'')$  are not. Then  $\tilde{f} = h \circ f \circ h^{-1}$  still fulfills  $\tilde{f}(E) = E$ . Since two vertices of the triangle  $V_a(E), V_b(E), V_{b'}(E)$  of  $\mathcal{L}_{S_{0,\infty}}^A(E)$  are fixed by  $\phi$ , the third one will also be fixed, i.e.  $\phi(V_{b'}(E)) = V_{b'}(E)$ . This implies that  $f(b) = b$  and  $f(b') = b'$ , so that  $\tilde{f}$  preserves each boundary component of  $P$ . One can compose  $\tilde{f}$  with a product  $g$  of Dehn twists along closed curves in  $P$  and possibly half-twists along  $h(b)$  and  $h(b')$  such that  $\tilde{f} \circ g$  is identity on  $P$ . Hence  $\tilde{f} \circ g \in \widehat{D}$ , by Proposition 2.9. Now, the triangle with vertices  $V_{h(a)}(E), V_{h(a')}(E)$  and  $V_{h(a'')}(E)$  is pointwise fixed by  $\Psi(\tilde{f} \circ g)$ . It follows that in the expression of  $\tilde{f} \circ g$ , and hence of  $\tilde{f}$ , as a product of half-twists along curves in  $E$  going to infinity the half-twist along  $h(a)$  occurs with an even exponent. The claim is a consequence of the fact that  $\mu_{h(a)}(h \circ f \circ h^{-1}) = \mu_a(h) \circ \mu_a(f) \circ (\mu_a(h))^{-1}$ .  $\square$

## 5 Compact sub-surfaces

Observe first, that for all  $n \geq 5$  and any admissible sub-surface  $\Sigma_{0,n}$  of  $S_{0,\infty}$  the inclusion  $\Sigma_{0,n} \subset S_{0,\infty}$  induces an embedding of  $\mathcal{C}_{\mathcal{P}}(\Sigma_{0,n})$  into  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ . In this section, we will show that, for any automorphism  $\phi$  of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  there exists  $n \geq 5$ , admissible sub-surfaces  $\Sigma_{0,n}$  and  $\Sigma'_{0,n}$  of  $S_{0,\infty}$  and an induced isomorphism between the pants complexes

$$\phi_n : \mathcal{C}_{\mathcal{P}}(\Sigma_{0,n}) \longrightarrow \mathcal{C}_{\mathcal{P}}(\Sigma'_{0,n}),$$

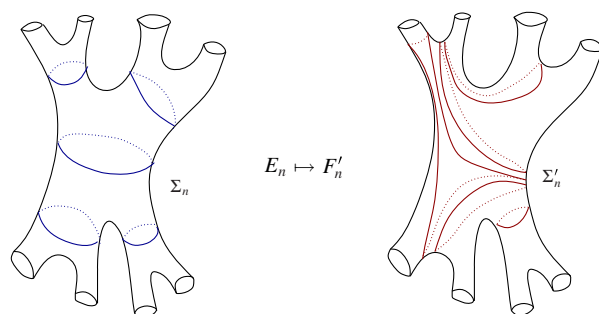
in the sense that the restriction of  $\phi$  to the sub-complex  $\mathcal{C}_{\mathcal{P}}(\Sigma_{0,n}) \subset \mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  coincides with  $\phi_n$ .

### 5.1 Definition of the induced isomorphism

Let  $\phi$  be an automorphism of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$  and set  $F = \phi(E)$ . There exists an admissible sub-surface  $\Sigma'_{0,n}$  of level  $n \geq 5$  such that  $F$  coincides with  $E$  outside  $\Sigma'_{0,n}$ . The decomposition  $F$  induces a pants decomposition  $F'_n$  of  $\Sigma'_{0,n}$ .

#### Definition of $\Sigma_{0,n}$ :

Consider the set  $W'$  of vertices of  $\mathcal{L}_{S_{0,\infty}}^A(F)$  of the form  $V_{c'}(F)$ , for all curves  $c' \in F'_n$ . The set  $W'$  spans a connected tree in the dual tree of  $\mathcal{L}_{S_{0,\infty}}^A(F)$ , which is isomorphic to the dual tree of  $F'_n$ . The automorphism  $\phi^{-1}$  induces an isomorphism  $\phi_{*,F}^{-1}$  from  $\mathcal{L}_{S_{0,\infty}}^A(F)$  to  $\mathcal{L}_{S_{0,\infty}}^A(E)$ . Therefore  $W = \phi_{*,F}^{-1}(W')$  is a set of vertices of  $\mathcal{L}_{S_{0,\infty}}^A(E)$  of the form  $V_c(E)$ , where  $c$  belongs to some finite subset  $E_n$  of curves of  $E$ . Since  $W$  also spans a connected subtree of the dual of  $\mathcal{L}_{S_{0,\infty}}^A(F)$ , the curves of  $E_n$  form a pants decomposition of a connected admissible sub-surface  $\Sigma_{0,n}$  of  $S_{0,\infty}$ .



**Fig. 11** The isomorphism  $\mathcal{C}_{\mathcal{P}}(\Sigma_{0,n}) \rightarrow \mathcal{C}_{\mathcal{P}}(\Sigma'_{0,n})$ .

**The isomorphism  $\phi_n : \mathcal{C}_{\mathcal{P}}(\Sigma_{0,n}) \rightarrow \mathcal{C}_{\mathcal{P}}(\Sigma'_{0,n})$ :**

Let  $G$  be a decomposition adjacent to the canonical decomposition  $E$  in  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ . Assume that this decomposition is issued from an elementary move on a curve belonging to the interior of the sub-surface  $\Sigma_{0,n}$ . Then  $\Sigma_{0,n}$  contains a support of  $G$ . Moreover, by construction of  $\Sigma_{0,n}$ ,  $\phi(G)$  is issued from an elementary move in  $F$  on a curve of  $\Sigma'_{0,n}$  and induces a pants decomposition of  $\Sigma'_{0,n}$ .

By induction on the number of moves, the image by  $\phi$  of any decomposition  $G$  of  $S_{0,\infty}$  obtained by applying finitely many elementary moves on curves lying in the interior of  $\Sigma_{0,n}$  is a decomposition of  $S_{0,\infty}$  whose support is included in  $\Sigma'_{0,n}$ . In other words,  $\phi(G) \cap \Sigma'_{0,n}$  is a well-defined vertex of  $\mathcal{C}_{\mathcal{P}}(\Sigma'_{0,n})$ . It remains to note that the pants decomposition of  $\Sigma'_{0,n}$  is obtained by finitely many elementary moves on the decomposition of  $\Sigma_{0,n}$ . This correspondence provides a cellular isomorphism  $\phi_n : \mathcal{C}_{\mathcal{P}}(\Sigma_{0,n}) \rightarrow \mathcal{C}_{\mathcal{P}}(\Sigma'_{0,n})$ .

**Lemma 5.1** *Let  $\phi$  be an element of  $\text{Aut}(\mathcal{C}_{\mathcal{P}}(S_{0,\infty}))$  and  $\Sigma_{0,n}$ , with  $n \geq 5$ , be a sub-surface as above. There exists an element  $g \in \mathcal{B}^{\frac{1}{2}}$  such that  $\phi$  and  $\Psi(g) \in \text{Aut}(\mathcal{C}_{\mathcal{P}}(S_{0,\infty}))$  coincide on the restriction of the link  $\mathcal{L}(E)$  to vertices of  $\mathcal{C}_{\mathcal{P}}(\Sigma_{0,n})$ .*

**Proof.** Let  $E'_n$  denote the trace of the canonical decomposition on  $\Sigma'_{0,n}$ . The dual trees of  $E_n$  and of  $E'_n$  are not necessarily isomorphic, in general. Let  $\partial E_n$  and  $\partial F'_n$  denote the set of those curves in  $E_n$  and  $F'_n$  lying in the boundary of  $\Sigma_{0,n}$  and  $\Sigma'_{0,n}$ , respectively. The map  $\phi_*$  at the level of  $\mathcal{L}^A_{S_{0,\infty}}(E)$  induced an isomorphism between the dual trees of  $E_n$  and  $F'_n$ , sending therefore leaves onto leaves and thus  $\partial E_n$  onto  $\partial F'_n$ . We can choose then a homotopy class of a homeomorphism  $f$  sending  $\Sigma_{0,n}$  on  $\Sigma'_{0,n}$  such that the restriction of  $f$  to  $\partial E_n$  acts combinatorially as  $\phi_*$ . This homeomorphism induces an isomorphism between the pants complexes  $\bar{f} : \mathcal{C}_{\mathcal{P}}(\Sigma_{0,n}) \rightarrow \mathcal{C}_{\mathcal{P}}(\Sigma'_{0,n})$ . By left-composing  $\phi_n$  by  $\bar{f}^{-1}$ , we define an automorphism  $\phi_{n,f}$  of  $\mathcal{C}_{\mathcal{P}}(\Sigma_{0,n})$ .

Now, Margalit's rigidity theorem ([20]) states that there exists an element  $F_{n,f}$  of the extended mapping class group of  $\Sigma_{0,n}$  whose action on  $\mathcal{C}_{\mathcal{P}}(\Sigma_{0,n})$  is the automorphism  $\phi_{n,f}$ .

We now consider the map  $f \circ F_{n,f} : \Sigma_{0,n} \rightarrow \Sigma'_{0,n}$  and extend it rigidly to the surface  $S_{0,\infty}$ . Let  $g_{n,f}$  be the asymptotically rigid mapping class of  $S_{0,\infty}$  resulting from this extension. By construction,  $\Psi(g_{n,f})$  coincides with  $\phi$  on the restricted link  $\mathcal{L}_{S_{0,\infty}}(E_n)$ . □

**5.2 Extension of the isomorphisms of the link**

We consider an extension of the map  $\mu_c : \widehat{\mathcal{B}}^{\frac{1}{2}} \rightarrow \text{Perm}(\mathbb{Z})$ , defined in the proof of Lemma 4.3, from  $\widehat{\mathcal{B}}^{\frac{1}{2}}$  to  $\text{Aut}(\mathcal{C}_{\mathcal{P}}(S_{0,\infty}))$ . As the most general case will not be needed later, we restrict ourselves to the subgroup  $\text{Aut}(\mathcal{C}_{\mathcal{P}}(S_{0,\infty}))_E$  of those  $\phi \in \text{Aut}(\mathcal{C}_{\mathcal{P}}(S_{0,\infty}))$  satisfying  $\phi(E) = E$ . For every  $c \in E$  let  $c' \in E$  be the curve with the property that  $\phi(V_c(E)) = V_{c'}(E)$ . Then  $\mu_c$  is defined by  $\phi(F_{c_k}) = F_{c'_{\mu_c(\phi)(k)}}$ . It is clear that  $\mu_c : \text{Aut}(\mathcal{C}_{\mathcal{P}}(S_{0,\infty}))_E \rightarrow \text{Perm}(\mathbb{Z})$  is a twisted homomorphism, namely  $\mu_c(\phi_1 \circ \phi_2) = \mu_{\phi_2(c)}(\phi_1) \circ \mu_c(\phi_2)$ .

Lemma 5.1 allows us to generalize some local characterizations described in section 4.6 to all automorphisms of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ . More precisely, we have the following result:

**Lemma 5.2** *Consider  $\phi \in \text{Aut}(\mathcal{C}_{\mathcal{P}}(S_{0,\infty}))$  such that  $\phi(E) = E$ . Let  $P'$  and  $P''$  be two almost admissible pairs of pants which are bounded by the curves  $a, a', a''$  and  $a, b, b'$ , respectively. We assume that:*

1.  *$a$  is closer to the fundamental pair of pants than  $a'$  and  $a''$ ;*
2. *the vertices  $V_a(E), V_{a'}(E), V_{a''}(E)$  and  $V_b(E)$  of  $\mathcal{L}^A_{S_{0,\infty}}(E)$  are point-wise fixed by  $\phi$ .*

*Then  $\mu_a(\phi)(k) = \varepsilon(\phi)k + p$ , where  $p$  is even and  $\varepsilon(\phi) \in \{-1, 1\}$ .*

**Proof.** We choose a large enough admissible sub-surface  $\Sigma'_{0,n}$  as in Section 5.1 so that the restricted link  $\mathcal{L}_{S_{0,\infty}}(E_n)$  contains  $V_a(E), V_{a'}(E), V_{a''}(E), V_b(E)$  and  $V_{b'}(E)$ . From Lemma 5.1, there is an element  $f \in \widehat{\mathcal{B}}^{\frac{1}{2}}$  such that the automorphisms  $\Psi(f)$  and  $\phi$  coincide on the restricted link  $\mathcal{L}_{S_{0,\infty}}(E_n)$ . Thus, the properties valid for  $\Psi(f)$  which were given in Lemma 4.4 are also valid for  $\phi$ . □



### 5.3 Proof of the theorem

Let  $\phi$  be an automorphism of  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ . We shall construct an element of  $\widehat{\mathcal{B}}^{\frac{1}{2}}$  which acts as  $\phi$  on  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ .

According to Lemma 5.1, there is an element  $g$  of  $\mathcal{B}^{\frac{1}{2}}$  which is quasi-rigid outside a quasi-admissible sub-surface  $\Sigma_{0,n}$ ,  $n \geq 5$ , of  $S_{0,\infty}$ , such that  $\phi(E)$  coincide with  $E$  outside  $g(\Sigma_{0,n})$ . Moreover, the automorphisms  $\phi$  and  $\Psi(g)$  coincide on the restricted link  $\mathcal{L}_{S_{0,\infty}}(E_n) = \mathcal{L}(E) \cap \mathcal{C}_{\mathcal{P}}(\Sigma_{0,n})$  on  $\Sigma_{0,n}$ . We can assume that  $\Sigma_{0,n}$  contains the fundamental pair of pants.

We now consider the automorphism  $\phi \circ \Psi(g^{-1})$ . Let  $a$  be a boundary component of  $\Sigma_{0,n}$  (so that  $a$  is a curve of  $E$ ) and  $P_a$  be the pair of pants of  $S_{0,\infty}$  with boundary in  $E$  such that  $P_a \cap \Sigma_{0,n} = a$ . Then  $P_a$  has three boundary components  $a, a', a''$  defining a triangle  $V_a(E), V_{a'}(E), V_{a''}(E)$  in  $\mathcal{L}_{S_{0,\infty}}^A(E)$ . Both the triangle and its vertex  $V_a(E)$  are invariant by  $\phi \circ \Psi(g^{-1})$ . By possibly composing  $g$  with  $d_a$  we can assume that this triangle in  $\mathcal{L}_{S_{0,\infty}}^A(E)$  is point-wise invariant, while  $\phi \circ \Psi(g^{-1})$  still acts as identity on  $\mathcal{L}_{S_{0,\infty}}(E_n)$ , because  $\Psi(d_a)$  is identity on  $\mathcal{L}_{S_{0,\infty}}(E_n)$ . Observe that  $\varepsilon(\phi \circ \Psi(g^{-1})) = 1$ , as this automorphism is identity on  $V_b(E_n) \subset \mathcal{C}_{\mathcal{P}}(\Sigma_{0,n})$ , when  $b$  is a curve of  $E \cap \Sigma_{0,n}$  not homotopic to a boundary component and adjacent to  $a$ .

According to Lemma 5.2, there exists an even  $p \in \mathbb{Z}$  such that for any  $k \in \mathbb{Z}$ ,  $\mu_a(\phi \circ \Psi(g^{-1}))(k) = k + p$ , because the vertices  $V_a(E), V_{a'}(E), V_{a''}(E)$  and  $V_b(E)$  are point-wise invariant by  $\phi \circ \Psi(g^{-1})$ . Let then  $g' = d_a^p \circ g$ . Due to the description of the action of a half-twist on the link, we know that  $\phi$  and  $\Psi(g')$  coincide on  $\mathcal{L}(E) \cap \mathcal{C}_{\mathcal{P}}(\Sigma_{0,n} \cup P_a)$ .

By induction, there exists an infinite product  $d$  of half-twists along curves in  $E$  accumulating to infinity such that the automorphisms  $\Psi(d \circ g)$  and  $\phi$  coincide on all vertices adjacent to  $E$  in  $\mathcal{C}_{\mathcal{P}}(S_{0,\infty})$ . Now  $d \circ g \in \widehat{\mathcal{B}}^{\frac{1}{2}}$  and these two automorphisms are the same by Lemma 4.1. Thus  $\Psi$  is onto.

**Acknowledgements** The authors are indebted to Javier Aramayona, Ariadna Fossas, Pierre Lochak, Athanase Papadopoulos and Vlad Sergiescu for useful discussions and to the referee for pointing out several inaccuracies and for his/her suggestions which helped improving the presentation. They were supported by the ANR 2011 BS 01 020 01 ModGroup. Part of this work was done while the first author visited Erwin Schrödinger Institute which he wants to thank warmly for hospitality and support.

### References

- [1] J. Aramayona, Simplicial embeddings between pants graphs, *Geom. Dedicata* **144**, 115–128 (2010).
- [2] S. Bonnot, R. Penner, and D. Saric, A presentation for the baseleaf preserving mapping class group of the punctured solenoid, *Algebr. Geom. Topol.* **7**, 1171–1199 (2007).
- [3] M. G. Brin, The algebra of strand splitting. I. A braided version of Thompson’s group V, *J. Group Theory* **10**(6), 757–788 (2007).
- [4] J. W. Cannon, W. J. Floyd, and W. R. Parry, Introductory notes in Richard Thompson’s groups, *Enseign. Math.* **42**, 215–256 (1996).
- [5] M. Dehn, *Papers on Group Theory and Topology*. J. Stillwell (eds.), (Springer Verlag, Berlin-New York, 1987).
- [6] P. Dehornoy, Geometric presentations for Thompson’s groups, *J. Pure Appl. Algebra* **203**(1–3), 1–44 (2005).
- [7] P. Dehornoy, The group of parenthesized braids, *Adv. Math.* **205**(2), 354–409 (2006).
- [8] A. Fossas and M. Nguyen, Thompson’s group T is the orientation-preserving automorphism group of a cellular complex, *Publ. Mat.* **56**, 305–326 (2012).
- [9] L. Funar and C. Kapoudjian, On a universal mapping class group of genus zero, *Geom. Funct. Anal.* **14**, 965–1012 (2004).
- [10] L. Funar and C. Kapoudjian, The braided Ptolemy–Thompson group is finitely presented, *Geom. Topol.* **12**, 475–530 (2008).
- [11] E. Irmak, Complexes of nonseparating curves and mapping class groups, *Michigan Math. J.* **54**, 81–110 (2006).
- [12] E. Irmak and M. Korkmaz, Automorphisms of the Hatcher–Thurston complex, *Israel J. Math.* **162**, 183–196 (2007).
- [13] E. Irmak and J. McCarthy, Injective simplicial maps of the arc complex, *Turkish J. Math.* **34**, 339–354 (2010).
- [14] N. Ivanov, Automorphism of complexes of curves and of Teichmüller spaces, *Int. Math. Res. Not. IMRN* **14**, 651–666 (1997).
- [15] M. Korkmaz, Automorphisms of complexes of curves on punctured spheres and on punctured tori, *Topology Appl.* **95**, 85–111 (1997).
- [16] M. Korkmaz and A. Papadopoulos, On the arc and curve complex of a surface, *Math. Proc. Cambridge Philos. Soc.* **148**(3), 473–483 (2010).

- [17] P. Lochak and L. Schneps, The universal Ptolemy-Teichmüller groupoid, in: *Geometric Galois Actions, Vol. 2, Lecture Notes Series Vol. 243* (Cambridge Univ. Press, Cambridge, 1997), pp. 325–347.
- [18] P. Lochak, Results and conjectures in profinite Teichmüller theory, in: *Galois–Teichmüller Theory and Arithmetic Geometry, Advanced Studies in Pure Mathematics Vol. 63* (Math. Soc. Japan, Tokyo, 2012), pp. 263–335.
- [19] F. Luo, Automorphisms of the complex of curves, *Topology* **39**, 283–298 (2000).
- [20] D. Margalit, The automorphism group of the pants complex, *Duke Math. J.* **121**, 457–479 (2004).
- [21] P. Schmutz-Schaller, Mapping class groups of hyperbolic surfaces and automorphism groups of graphs, *Compositio Math.* **122**, 243–260 (2000).