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Centrally extended mapping class groups from quantum Teichmüller theory ☆

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Abstract

The central extension of the mapping class groups of punctured surfaces of finite type that arises in quantum Teichmüller theory is 12 times the Meyer class plus the Euler classes of the punctures. This is analogous to the result obtained in [12] for the Thompson groups.

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0. Introduction

The quantum theory of Teichmüller spaces of punctured surfaces of finite type, originally constructed in [6,16] and subsequently generalized to higher rank Lie groups and cluster algebras in [10,11], leads to one-parameter families of projective unitary representations of Ptolemy modular groupoids associated to ideal triangulations of punctured surfaces. We will call such representations (quantum) dilogarithmic representations, since the main ingredient in the theory

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is Faddeev's quantum dilogarithm function first introduced in the context of quantum integrable systems by L.D. Faddeev in [7].

These representations are infinite dimensional so that a priori it is not clear if they come from suitable $(2+1)$-dimensional topological quantum field theories (TQFT). Nonetheless, it is expected that in the singular limit, when the deformation parameter tends to a root of unity,[1] the "renormalized" theory corresponds to a finite dimensional TQFT first defined in [15,17] by using the cyclic representations of the Borel Hopf sub-algebra $BU_q(sl(2))$, and subsequently developed and generalized in [3]. One can get the same finite dimensional representations of Ptolemy modular groupoids directly from compact representations of quantum Teichmüller theory at roots of unity [5,1,16].

Projective representations of a group are well known to be equivalent to representations of central extensions of the same group by means of the following procedure. To a group G, a $\mathbb{C}$-vector space V and a group homomorphism $h : G \to PGL(V) \simeq GL(V)/\mathbb{C}^*$, where $\mathbb{C}^*$ is identified with a (normal) subgroup of $GL(V)$ through the embedding $z \mapsto z\,\mathrm{id}_V$, one can associate a central extension $\widetilde{G}$ of G by a subgroup A of $\mathbb{C}^*$ together with a representation $\widetilde{h} : \widetilde{G} \to GL(V)$ such that the following diagram is commutative and has exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & GL(V) & \longrightarrow & PGL(V) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow \scriptstyle{\widetilde{h}} & & \uparrow \scriptstyle{h} & & \\
1 & \longrightarrow & A & \longrightarrow & \widetilde{G} & \longrightarrow & G & \longrightarrow & 1
\end{array}$$

One such extension is the pull-back $\widetilde{\mathbf{G}}$ of the central extension $GL(V) \to PGL(V)$ under the homomorphism $G \to PGL(V)$, which is canonically defined. However it is possible to find also smaller extensions associated to proper subgroups $A \subset \mathbb{C}^*$. The central extension $\widetilde{G}$ associated to the smallest possible subgroup $A \subset \mathbb{C}^*$ for which there exists a linear representation as in the diagram above resolving the projective representation of G will be called the *minimal reduction* of $\widetilde{\mathbf{G}}$.

In this light, quantum Teichmüller theory gives rise to representations of certain central extensions of the surface mapping class groups which are the vertex groups of the Ptolemy modular groupoids. The main goal of this paper is to identify the isomorphism classes of those central extensions. Namely, by using the quantization approach of [16], we extend the analysis of the particular case of a once punctured genus three surface performed in [18] to arbitrary punctured surfaces of finite type.

Let a group G with a given presentation be identified as the quotient group F/R, where F is a free group and R, the normal subgroup generated by the relations. Then, a central extension of G can be obtained from a homomorphism $\bar{h} : F \to GL(V)$ with the property $\bar{h}(R) \subset \mathbb{C}^*$ so that it induces a homomorphism $h : G \to PGL(V)$. In this case, the homomorphism $\bar{h}$ will be called an *almost linear representation* of G, in order to distinguish it from a projective representation.

In quantum Teichmüller theory, central extensions of surface mapping class groups appear through almost linear representations. Specifically, let $\Gamma_{g,r}^s$ be the mapping class group of a surface $\Sigma_{g,r}^s$ of genus g with r boundary components and s punctures. These are mapping classes of homeomorphisms which fix the boundary point-wise and fix the set of punctures (not necessarily

[1] One should distinguish between two different limits, depending on whether $\frac{\log(q)}{2\pi i}$ tends to a positive or a negative rational number. In the case when this limit is a positive rational number, the limit of the representation is non-singular and so it stays infinite dimensional.

point-wise). Denoting $\Gamma_g^s = \Gamma_{g,0}^s$, the projective representations of Γ_g^s for $(2g-2+2s)s > 0$, constructed in [16,18], are almost linear representations corresponding to certain central extensions $\widetilde{\Gamma_g^s}$. By considering embeddings $\Sigma_{g,r}^s \subset \Sigma_{h,0}^t$, the central extensions $\widetilde{\Gamma_g^s}$ can be used to define central extensions for the mapping class groups $\Gamma_{g,r}^s$ for $s \geqslant 1$. According to [23], any embedding $\Sigma_{g,r}^s \subset \Sigma_{h,0}^t$, for which $\Sigma_h^t \setminus \Sigma_{g,r}^s$ contains no disk, punctured disk or cylinder components, induces an embedding of the corresponding mapping class groups. Using this fact, we can define the central extension $\widetilde{\Gamma_{g,r}^s}$ as the pull-back of the central extension $\widetilde{\Gamma_h^t}$ by the injective homomorphism $\Gamma_{g,r}^s \hookrightarrow \Gamma_h^t$ induced by an embedding of the corresponding surfaces. A priori, it is not clear whether such definition depends on a particular choice of the embedding, but our main result below shows that this is indeed the case.

Central extensions by an Abelian group A of a given group G are known to be classified, up to isomorphism, by elements of the 2-cohomology group $H^2(G; A)$. In the case of surface mapping class groups $\Gamma_{g,r}^s$, the latter was first computed by Harer in [14] for $g \geqslant 5$ and further completed by Korkmaz and Stipsicz in [20] for $g \geqslant 4$ (see also [19] for a survey). Specifically, we have

$$H^2\big(\Gamma_{g,r}^s\big) = \mathbb{Z}^{s+1}, \quad \text{if } g \geqslant 4,$$

where the generators are given by (one fourth of) the Meyer signature class χ (it is the only generator for the groups $H^2(\Gamma_g) \cong H^2(\Gamma_{g,1}) \simeq \mathbb{Z}$, see [22,14,20] for its definition) and s Euler classes e_i associated with s punctures. In the case when $g = 3$, the group $H^2(\Gamma_{3,r}^s)$ still contains the subgroup $\mathbb{Z}^{s+1}$ generated by the above mentioned classes, but it is not known whether there are other (2-torsion) classes. When $g = 2$ we will show that $H^2(\Gamma_{2,r}^s)$ contains the subgroup $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}^s$, whose torsion part is generated by χ and whose free part is generated by the Euler classes. The Universal Coefficients Theorem permits then to compute $H^2(G; A)$ for every Abelian group A.

Denote as above by $\widetilde{\boldsymbol{\Gamma}_{\mathbf{g,r}}^{\mathbf{s}}}$ the canonical central extension of $\Gamma_{g,r}^s$ by $\mathbb{C}^*$ which is obtained as the pull-back of the canonical central extension $GL(\mathcal{H}) \to PGL(\mathcal{H})$ under the quantum projective representation associated to a semi-symmetric T in the Hilbert space $\mathcal{H}$ (see the next section). Quantum representations depend on some parameter $\zeta \in \mathbb{C}^*$. Our main result is the following theorem.

Theorem 0.1. *The central extension $\widetilde{\boldsymbol{\Gamma}_{\mathbf{g,r}}^{\mathbf{s}}}$ can be reduced to a minimal central extension $\widetilde{\Gamma_{g,r}^s}$ of $\Gamma_{g,r}^s$ by a cyclic Abelian $A \subset \mathbb{C}^*$, where A is the subgroup of $\mathbb{C}^*$ generated by ζ^{-6}. Moreover its cohomology class is*

$$c_{\widetilde{\Gamma_{g,r}^s}} = 12\chi + \sum_{i=1}^{s} e_i \in H^2\big(\Gamma_{g,r}^s; A\big)$$

if $g \geqslant 2$ and $s \geqslant 4$. Here χ and e_i are one fourth of the Meyer signature class and respectively the i-th Euler class with A coefficients.

There is a geometric interpretation of this extension.

Corollary 0.2. *Let us consider the extension $\widehat{\Gamma_{g,r+s}}$ of class 12χ. Then there is an exact sequence*:

$$1 \to A^{s-1} \to \widehat{\Gamma_{g,r+s}} \to \widetilde{\Gamma_{g,r}^s} \to 1.$$

In some sense the quantum representations of punctured mapping class groups can be lifted to the mapping class groups of surfaces with boundary obtained by blowing up the punctures.

Corollary 0.3. *The cohomology class of the central extension* $\widetilde{\boldsymbol{\Gamma}^{\mathbf{s}}_{\mathbf{g,r}}}$ *is*

$$c_{\widetilde{\boldsymbol{\Gamma}^{\mathbf{s}}_{\mathbf{g,r}}}} = 12\chi + \sum_{i=1}^{s} e_i \in H^2\big(\Gamma^s_{g,r}; \mathbb{C}^*\big)$$

if $g \geqslant 3$ *and* $s \geqslant 4$. *The same formula holds also when* $g = 2$ *but the class* χ *vanishes in* $H^2(\Gamma^s_{g,r}; \mathbb{C}^*)$. *Here* χ *and* e_i *are one fourth of the Meyer signature class and respectively the* i*-th Euler class with* $\mathbb{C}^*$ *coefficients.*

Remark 0.1. The central extension arising from $SU(2)$-TQFT with p_1-structures was computed in [13,21] for Γ_g and it equals 12χ. It can be shown that their computations extend to the case of punctured surfaces and the associated class for $\Gamma^s_{g,r}$ is $12\chi + \sum_{i=1}^{s} e_i$. Our result shows that this extension coincides with the central extension arising from quantum Teichmüller theory. This is the counterpart for finite type surfaces and their mapping class groups of the result obtained in [12] for the Ptolemy–Thompson group.

The organization of the paper is as follows. In Section 1, we review the quantization of the Teichmüller space of a punctured surface and define the associated quantum representations of the decorated Ptolemy groupoid which correspond to linear representations of a central extension of the decorated Ptolemy groupoid. Then, in Section 2, we prove Theorem 0.1 by finding the pull-back of this central extension to the mapping class group of the surface. The key idea is to use a Grothendieck type principle. Namely, one can identify a central extension of the mapping class group of some surface, if one understands its restrictions to the mapping class groups of sub-surfaces of bounded topological types. The core of the proof consists in computing explicitly the lifts to the central extension of the decorated Ptolemy groupoid of the relations known to hold in the mapping class groups. When properly interpreted, these lifts yield the class of the mapping class group extension.

1. Quantum Teichmüller theory

1.1. The groupoid of decorated ideal triangulations

Let $\Sigma = \Sigma^s_{g,r}$ be an oriented closed surface of genus g with r boundary components and $s \geqslant 1$ punctures. When $r > 0$ we choose a set of points on each boundary, which will be called *boundary* punctures. When we need to single out the s punctures lying in the interior we will call them *interior* punctures. In this paper we will only consider the situation when each boundary component has exactly one boundary puncture, so that there is a total of $s + r$ punctures among which r are boundary punctures. The triangulations of $\Sigma^s_{g,r}$ whose vertices are the $s+r$ punctures will be called *ideal triangulations*. Then Σ is *large* if and only if $Ns > 0$, where $N = 4g - 4 + 2s + 3r$ is the number of triangles in an ideal triangulation.

Definition 1.1. A *decorated ideal triangulation* of Σ is an ideal triangulation τ up to isotopy fixing the boundary, where all triangles are provided with a marked corner, and a bijective ordering map

$$\bar{\tau} : \{1, \dots, N\} \ni j \mapsto \bar{\tau}_j \in T(\tau)$$

is fixed. Here $T(\tau)$ is the set of all triangles of τ.

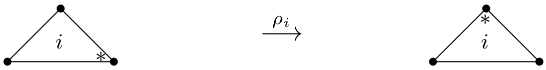


Fig. 1. The transformation ρ_i.

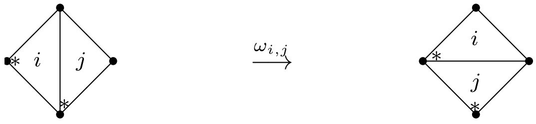


Fig. 2. The transformation $\omega_{i,j}$.

Graphically, the marked corner of a triangle is indicated by an asterisk and the corresponding number is put inside the triangle. The set of all decorated ideal triangulations of Σ is denoted Δ_Σ.

Recall that if a group G freely acts on a set X, then there is an associated groupoid defined as follows. The objects are the G-orbits in X, while morphisms are G-orbits in $X \times X$ with respect to the diagonal action. Denote by $[x]$ the object represented by an element $x \in X$ and by $[x, y]$ the morphism represented by a pair of elements $(x, y) \in X \times X$. Two morphisms $[x, y]$ and $[u, v]$, are composable if and only if $[y] = [u]$ and their composition is $[x, y][u, v] = [x, gv]$, where $g \in G$ is the unique element sending u to y. The inverse and the identity morphisms are given respectively by $[x, y]^{-1} = [y, x]$ and $\mathrm{id}_{[x]} = [x, x]$. In what follows, products of the form $[x_1, x_2][x_2, x_3] \cdots [x_{n-1}, x_n]$ will be shortened as $[x_1, x_2, x_3, \ldots, x_{n-1}, x_n]$.

The mapping class group $\Gamma_{g,r}^s$ of Σ acts freely on Δ_Σ. In this case, we let $\mathcal{G}_\Sigma$ denote the corresponding groupoid, called the *groupoid of decorated ideal triangulations*, or *decorated Ptolemy groupoid*. This groupoid first considered in [16] is an enhanced version of the usual Ptolemy groupoid introduced and studied by Penner in [24] (see also [25]), which arises in the Fock–Goncharov quantization [9–11] of the Teichmüller space. There is a presentation for $\mathcal{G}_\Sigma$ with three types of generators and four types of relations.

The generators are of the form $[\tau, \tau^\sigma]$, $[\tau, \rho_i\tau]$, and $[\tau, \omega_{i,j}\tau]$, where τ^σ is obtained from τ by replacing the ordering map $\bar{\tau}$ by the map $\bar{\tau} \circ \sigma$, where $\sigma \in S_N$ is a permutation of the set $\{1, \ldots, N\}$, $\rho_i\tau$ is obtained from τ by changing the marked corner of triangle $\bar{\tau}_i$ as in Fig. 1, and $\omega_{i,j}\tau$ is obtained from τ by applying the flip transformation in the quadrilateral composed of triangles $\bar{\tau}_i$ and $\bar{\tau}_j$ as in Fig. 2.

There are two sets of relations satisfied by these generators. The first set is as follows:

$$[\tau, \tau^\alpha, (\tau^\alpha)^\beta] = [\tau, \tau^{\alpha\beta}], \quad \alpha, \beta \in S_N, \tag{1}$$

$$[\tau, \rho_i\tau, \rho_i\rho_i\tau, \rho_i\rho_i\rho_i\tau] = \mathrm{id}_{[\tau]}, \tag{2}$$

$$[\tau, \omega_{ij}\tau, \omega_{ik}\omega_{ij}\tau, \omega_{jk}\omega_{ik}\omega_{ij}\tau] = [\tau, \omega_{jk}\tau, \omega_{ij}\omega_{jk}\tau], \tag{3}$$

$$[\tau, \omega_{ij}\tau, \rho_i\omega_{ij}\tau, \omega_{ji}\rho_i\omega_{ij}\tau] = [\tau, \tau^{(ij)}, \rho_j\tau^{(ij)}, \rho_i\rho_j\tau^{(ij)}], \tag{4}$$

where the first two relations are evident, while the other two are shown graphically in Figs. 3, 4.

The following commutation relations fulfill the second set of relations:

$$[\tau, \rho_i\tau, (\rho_i\tau)^\sigma] = [\tau, \tau^\sigma, \rho_{\sigma^{-1}(i)}\tau^\sigma], \tag{5}$$

$$[\tau, \omega_{ij}\tau, (\omega_{ij}\tau)^\sigma] = [\tau, \tau^\sigma, \omega_{\sigma^{-1}(i)\sigma^{-1}(i)}\tau^\sigma], \tag{6}$$

$$[\tau, \rho_j\tau, \rho_i\rho_j\tau] = [\tau, \rho_i\tau, \rho_j\rho_i\tau], \tag{7}$$

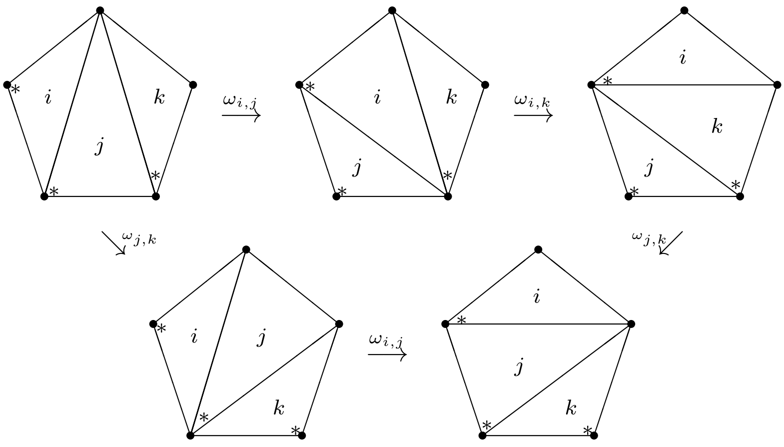


Fig. 3. The Pentagon relation (3).

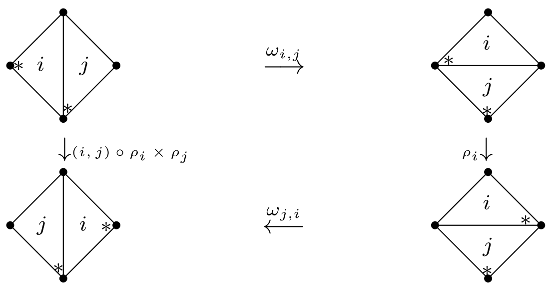


Fig. 4. The Inversion relation (4).

$$[\tau, \rho_i\tau, \omega_{jk}\rho_i\tau] = [\tau, \omega_{jk}\tau, \rho_i\omega_{jk}\tau], \quad i \notin \{j,k\}, \tag{8}$$

$$[\tau, \omega_{ij}\tau, \omega_{kl}\omega_{ij}\tau] = [\tau, \omega_{kl}\tau, \omega_{ij}\omega_{kl}\tau], \quad \{i,j\} \cap \{k,l\} = \emptyset. \tag{9}$$

Consider now an embedding of $\Sigma_{g,r}^s$ into $\Sigma_{h,v}^t$ sending all punctures (both interior and boundary) to punctures. Of course boundary punctures are sent into interior punctures unless the respective boundary circle is also a boundary of the larger surface.

Lemma 1.1. *Assume that each component of $\Sigma_{h,v}^t \setminus \mathrm{int}(\Sigma_{g,r}^s)$ is large. Then there is a natural embedding of $\mathcal{G}_{\Sigma_{g,r}^s}$ into $\mathcal{G}_{\Sigma_{h,v}^t}$.*

Proof. Let τ_{ext} be a fixed decorated triangulation of $\Sigma_{h,v}^t \setminus \mathrm{int}(\Sigma_{g,r}^s)$. If τ is a decorated triangulation of $\Sigma_{g,r}^s$ we denote by $\tau \cup \tau_{ext}$ the result of gluing the two triangulations along their corresponding boundary circles with the induced decoration. The isotopy class of the resulting triangulation is unique up to the action of Dehn twists along boundary components of $\Sigma_{g,r}^s$. This induces an injective map between the set of objects of the two groupoids. Then, the map which associates to the class $[\tau_1, \tau_2]$ of decorated triangulations of $\Sigma_{g,r}^s$ the class $[\tau_1 \cup \tau_{ext}, \tau_2 \cup \tau_{ext}]$ is well-defined. Since the restriction of a homeomorphism of $\Sigma_{h,v}^t$ preserving the isotopy class of the decorated triangulation τ_{ext} to $\Sigma_{h,v}^t \setminus \mathrm{int}(\Sigma_{g,r}^s)$ is isotopic to identity by Alexander's trick, the map defined above is injective. □

Remark 1.1. When $r > 0$ the construction of the decorated Ptolemy groupoid $\mathcal{G}_{\Sigma^s_{g,r}}$ depends on the choice of the set of boundary punctures, which might have more than r elements, in general.

1.2. Hilbert spaces of square integrable functions associated to triangulations

In what follows, we work with Hilbert spaces

$$\mathcal{H} \equiv L^2(\mathbb{R}), \qquad \mathcal{H}^{\otimes n} \equiv L^2\big(\mathbb{R}^n\big).$$

Any two self-adjoint operators p and q, acting in $\mathcal{H}$ and satisfying the Heisenberg commutation relation

$$\mathsf{pq} - \mathsf{qp} = (2\pi \mathsf{i})^{-1} \operatorname{id}_{\mathcal{H}}, \tag{10}$$

can be realized as differentiation and multiplication operators. Such "coordinate" realization in Dirac's bra–ket notation has the form

$$\langle x|\mathsf{p} = \frac{1}{2\pi \mathsf{i}} \frac{\partial}{\partial x} \langle x|, \qquad \langle x|\mathsf{q} = x\langle x|. \tag{11}$$

Formally, the set of "vectors" $\{|x\rangle\}_{x\in\mathbb{R}}$ forms a generalized basis of $\mathcal{H}$ with the following orthogonality and completeness properties:

$$\langle x|y\rangle = \delta(x-y), \qquad \int_{\mathbb{R}} |x\rangle\, dx\, \langle x| = \operatorname{id}_{\mathcal{H}}.$$

For any $1 \leqslant i \leqslant m$ we shall use the following notation

$$\iota_i : \operatorname{End}\mathcal{H} \ni \mathsf{a} \mapsto \mathsf{a}_i = \underbrace{1\otimes\cdots\otimes 1}_{i-1 \text{ times}} \otimes \mathsf{a} \otimes 1 \otimes \cdots \otimes 1 \in \operatorname{End}\mathcal{H}^{\otimes m}.$$

Besides that, if $\mathsf{u} \in \operatorname{End}\mathcal{H}^{\otimes k}$ for some $1 \leqslant k \leqslant m$ and $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, m\}$, then we shall write

$$\mathsf{u}_{i_1 i_2 \dots i_2} \equiv \iota_{i_1} \otimes \iota_{i_2} \otimes \cdots \otimes \iota_{i_k}(\mathsf{u}).$$

The symmetric group S_m naturally acts in $\mathcal{H}^{\otimes m}$:

$$P_\sigma(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_m) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(i)} \otimes \cdots \otimes x_{\sigma^{-1}(m)}, \qquad \sigma \in S_m. \tag{12}$$

1.3. Semi-symmetric T-matrices

We define now the algebraic structure needed for constructing representations of the decorated Ptolemy groupoid $\mathcal{G}_\Sigma$.

Definition 1.2. A *semi-symmetric T-matrix* consists of two operators $\mathsf{A} \in \operatorname{End}(\mathcal{H})$ and $T \in \operatorname{End}(\mathcal{H}^{\otimes 2})$ satisfying the equations:

$$\mathsf{A}^3 = 1, \tag{13}$$

$$T_{12}T_{13}T_{23} = T_{23}T_{12}, \tag{14}$$

$$T_{12}\mathsf{A}_1 T_{21} = \zeta \mathsf{A}_1\mathsf{A}_2 P_{(12)}, \tag{15}$$

where $\zeta \in \mathbb{C}^*$ and the permutation operator $P_{(12)}$ is defined by Eq. (12), for σ denoting the transposition (12).

Examples of semi-symmetric T-matrices could be obtained as follows. Fix some self-conjugate operators p, q satisfying the Heisenberg commutation relation (10). Choose a parameter b satisfying the condition:

$$\big(1-|b|\big)\operatorname{Im} b = 0,$$

and define then two unitary operators by the following formulas:

$$\mathsf{A} \equiv e^{-\mathrm{i}\pi/3} e^{\mathrm{i}3\pi \mathsf{q}^2} e^{\mathrm{i}\pi(\mathsf{p}+\mathsf{q})^2} \in \operatorname{End}(\mathcal{H}), \tag{16}$$

$$T \equiv e^{\mathrm{i}2\pi \mathsf{p}_1\mathsf{q}_2} \varphi_b(\mathsf{q}_1+\mathsf{p}_2-\mathsf{q}_2) \in \operatorname{End}\big(\mathcal{H}^{\otimes 2}\big). \tag{17}$$

They satisfy the defining relations for a semi-symmetric T-matrix, where

$$\zeta = e^{\mathrm{i}\pi c_b^2/3}, \qquad c_b = \frac{\mathrm{i}}{2}\big(b+b^{-1}\big), \tag{18}$$

and φ_b is Faddeev's quantum dilogarithm defined on $\{z \in \mathbb{C};\ |\operatorname{Im}(z)| < |\operatorname{Im}(c_b)|\}$ by means of

$$\varphi_b(z) = \exp\left(-\frac{1}{4}\int_{-\infty}^{\infty} \frac{\exp(-2izx)\,dx}{\sinh(xb)\sinh(x/b)x}\right). \tag{19}$$

Faddeev's quantum dilogarithm is closely related to the double gamma and double sine functions [2,26] and was used by Baxter [4] and Faddeev (see [7,8]). Its main feature is the following functional equation (see [7,8]):

$$\varphi_b(q)\varphi_b(p) = \varphi_b(p)\varphi_b(p+q)\varphi_b(q)$$

whenever $pq - qp = \frac{1}{2\pi i}\mathbf{1}$.

Remark that the operator A is characterized (up to a normalization factor) by the equations:

$$\mathsf{A}\mathsf{q}\mathsf{A}^{-1} = \mathsf{p}-\mathsf{q}, \qquad \mathsf{A}\mathsf{p}\mathsf{A}^{-1} = -\mathsf{q}.$$

Note that Eqs. (13)–(15) correspond to relations (2)–(4).

Let us introduce now some notation which will be useful in the sequel. For any operator $\mathsf{a} \in \operatorname{End}\mathcal{H}$ we set:

$$\mathsf{a}_{\hat{k}} \equiv \mathsf{A}_k \mathsf{a}_k \mathsf{A}_k^{-1}, \qquad \mathsf{a}_{\check{k}} \equiv \mathsf{A}_k^{-1} \mathsf{a}_k \mathsf{A}_k. \tag{20}$$

It is evident that

$$\mathsf{a}_{\check{\hat{k}}} = \mathsf{a}_{\hat{\check{k}}} = \mathsf{a}_k, \qquad \mathsf{a}_{\hat{\hat{k}}} = \mathsf{a}_{\check{k}}, \qquad \mathsf{a}_{\check{\check{k}}} = \mathsf{a}_{\hat{k}},$$

where the last two equations follow from Eq. (13). In particular, we have

$$\mathsf{p}_{\hat{k}} = -\mathsf{q}_k, \qquad \mathsf{q}_{\hat{k}} = \mathsf{p}_k - \mathsf{q}_k, \tag{21}$$

$$\mathsf{p}_{\check{k}} = \mathsf{q}_k - \mathsf{p}_k, \qquad \mathsf{q}_{\check{k}} = -\mathsf{p}_k. \tag{22}$$

Besides that, it will be also useful to use the notation

$$P_{(kl\ldots m\hat{k})} \equiv \mathsf{A}_k P_{(kl\ldots m)}, \qquad P_{(kl\ldots m\check{k})} \equiv \mathsf{A}_k^{-1} P_{(kl\ldots m)}, \tag{23}$$

where $(kl\ldots m)$ is the cyclic permutation

$$(kl\ldots m): k \mapsto l \mapsto \cdots \mapsto m \mapsto k.$$

Eq. (15) in this notation takes a rather compact form

$$T_{12}T_{2\hat{1}} = \zeta P_{(12\hat{1})}. \tag{24}$$

Remark 1.2. Notice that the Pentagon relation (14) can be applied whenever any of the indices $k \in \{1, 2\}$ arising among subscripts is replaced everywhere by either $\hat{k}$ or else $\check{k}$.

Remark 1.3. A T-matrix has the following symmetry property: $T_{12} = T_{\hat{2}\check{1}}$. This can be obtained using twice relation (24):

$$T_{12} = T_{12}T_{2\hat{1}}T_{2\hat{1}}^{-1} = \zeta P_{(12\hat{1})}T_{2\hat{1}}^{-1} = T_{\hat{1}\hat{2}}^{-1}\zeta P_{(12\hat{1})} = T_{\hat{1}\hat{2}}^{-1}\zeta P_{(\hat{1}\hat{2}\check{1})} = T_{\hat{2}\check{1}}. \tag{25}$$

1.4. The quantum Teichmüller space

The quantization of the Teichmüller space of a punctured surface Σ with boundary induced by a semi-symmetric T-matrix is defined by means of a *quantum functor*:

$$\mathsf{F} : \mathcal{G}_\Sigma \to \operatorname{End}(\mathcal{H}^{\otimes N}).$$

Its meaning is that we have an operator-valued function:

$$\mathsf{F} : \Delta_\Sigma \times \Delta_\Sigma \to \operatorname{End}(\mathcal{H}^{\otimes N}),$$

satisfying the following equations:

$$\mathsf{F}(\tau, \tau) = \mathrm{id}_{\mathcal{H}^{\otimes N}}, \qquad \mathsf{F}(\tau, \tau')\mathsf{F}(\tau', \tau'')\mathsf{F}(\tau'', \tau) \in \mathbb{C} \setminus \{0\}, \quad \forall \tau, \tau', \tau'' \in \Delta_\Sigma, \tag{26}$$

$$\mathsf{F}(f(\tau), f(\tau')) = \mathsf{F}(\tau, \tau'), \quad \forall f \in \mathcal{M}_\Sigma, \tag{27}$$

$$\mathsf{F}(\tau, \rho_i \tau) \equiv \mathsf{A}_i, \tag{28}$$

$$\mathsf{F}(\tau, \omega_{i,j}\tau) \equiv T_{ij}, \tag{29}$$

$$\mathsf{F}(\tau, \tau^\sigma) \equiv P_\sigma, \quad \forall \sigma \in S_N, \tag{30}$$

where operator P_σ is defined by Eq. (12). Consistency of these equations is ensured by the consistency of Eqs. (13)–(15) with relations (2)–(4).

A particular case of Eq. (26) corresponds to $\tau'' = \tau$:

$$\mathsf{F}(\tau, \tau')\mathsf{F}(\tau', \tau) \in \mathbb{C} \setminus \{0\}. \tag{31}$$

As an example, we can calculate the operator $\mathsf{F}(\tau, \omega_{i,j}^{-1}(\tau))$. Denoting $\tau' \equiv \omega_{i,j}^{-1}(\tau)$ and using Eq. (31), as well as definition (29), we obtain

$$\mathsf{F}(\tau, \omega_{i,j}^{-1}(\tau)) = \mathsf{F}(\omega_{i,j}(\tau'), \tau') \simeq (\mathsf{F}(\tau', \omega_{i,j}(\tau')))^{-1} = T_{ij}^{-1}, \tag{32}$$

where $\simeq$ means equality up to a numerical multiplicative factor.

The operations $\hat{}$ and $\check{}$ at the indices level have the following geometric interpretation. If the distinguished corners of the decorated ideal triangulation are precisely those from Fig. 2 then the quantum functor assigns to the flip on that edge the endomorphism T_{ij}^{-1}. Now, changing the distinguished corner in the triangle labeled i amounts of changing i into $\hat{i}$ or $\check{i}$ (and similarly for j) in the expression of the quantum functor endomorphism. These rules will be intensively used when we compute the expressions of Dehn twists in terms of the generators of the decorated Ptolemy groupoid in the next section.

The quantum functor induces a unitary projective representation of the mapping class group Γ_g^s of Σ as follows:

$$\Gamma_g^s \ni f \mapsto \mathsf{F}\big(\tau, f(\tau)\big) \in \mathrm{End}\big(\mathcal{H}^{\otimes N}\big).$$

Indeed, we have the following relation (up to a non-zero scalar):

$$\mathsf{F}\big(\tau, f(\tau)\big)\mathsf{F}\big(\tau, h(\tau)\big) = \mathsf{F}\big(\tau, f(\tau)\big)\mathsf{F}\big(f(\tau), f\big(h(\tau)\big)\big) \simeq \mathsf{F}\big(\tau, fh(\tau)\big).$$

The main question addressed in this present paper is to identify the central extension of the mapping class group corresponding to this projective representation. Observe that the projective factor lies in the subgroup of $\mathbb{C}^*$ generated by ζ.

In [16,18] one considered only punctured surfaces without boundary. However, the construction extends without essential modifications to the case when Σ is a surface with boundary $\Sigma_{g,r}^s$ when $s \geqslant 1$ and each boundary component contains one boundary puncture. In this case we could define directly the central extension $\widetilde{\Gamma_{g,r}^s}$ by using the decorated Ptolemy groupoid of the punctured surface with boundary, without reference to a larger surface without boundary.

2. Presentation of $\widetilde{\Gamma_{g,r}^s}$

2.1. Generating set for the relations

We start with a number of notations and definitions. Our setup consists of an embedding $\Sigma_{g,r}^s \subset \Sigma_{h,0}^t$ sending punctures into punctures. We assume that each component of $\Sigma_{h,0}^t \setminus \mathrm{int}(\Sigma_{g,r}^s)$ is large, namely it admits ideal triangulations whose vertices are those punctures of $\Sigma_{h,0}^t$ which are not interior punctures of $\Sigma_{g,r}^s$ (hence boundary punctures of $\Sigma_{g,r}^s$ being allowed). In particular, if we discard the boundary punctures of $\Sigma_{g,r}^s$ the complement $\Sigma_{h,0}^t \setminus \mathrm{int}(\Sigma_{g,r}^s)$ contains no disk, punctured disk or cylinder components. According to [23] the surface embedding induces an embedding between the corresponding mapping class groups $\Gamma_{g,r}^s \hookrightarrow \Gamma_{h,0}^t$. The pull-back of the central extension $\widetilde{\Gamma_h^t}$ to $\Gamma_{g,r}^s$ is a central extension $\widetilde{\Gamma_{g,r}^s}$. Our main concern is to study this central extension. The central extension obtained by the present construction is isomorphic to the central extension obtained by the direct quantization of the Teichmüller space associated to $\Sigma_{g,r}^s$ following the procedure of Section 1.4. This follows from the fact that the map between the mapping class groups $\Gamma_{g,r}^s \hookrightarrow \Gamma_{h,0}^t$ is covered by an injective map between the decorated Ptolemy groupoids according to Lemma 1.1.

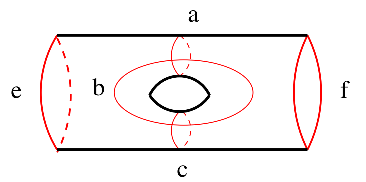
Since the restriction of the Euler class corresponding to the $(s+1)$-th puncture to $\Gamma_{g,r}^s$ vanishes, it is enough to consider $t = s$ below. Our strategy is to compute explicit lifts to $\widetilde{\Gamma_{g,r}^s}$ of a set of relations arising in a group presentation of $\Gamma_{g,r}^s$ by expressing (lifts of) the generators as elements of the decorated Ptolemy groupoid of the larger punctured surface $\Sigma_{h,0}^s$. The independence on the particular embedding of the subsurface $\Sigma_{g,r}^s$, under the assumptions of the main theorem is a consequence of the so-called Grothendieck principle. In the form proved by Gervais in [13] it states that all relations in $\Gamma_{g,r}^s$ are determined by an explicit set of relations among mapping classes supported on small subsurfaces, namely $\Sigma_{0,4}$, $\Sigma_{1,2}$ and $\Sigma_{0,3}$, where $\Sigma_{g,r} = \Sigma_{g,r}^0$. We express then these relations in terms of elements of the decorated Ptolemy groupoids of the surfaces $\Sigma_{0,4}^4$, $\Sigma_{1,2}^2$ and $\Sigma_{0,3}^4$, respectively. According to Lemma 1.1 these relations also hold in $\mathcal{G}_{\Sigma_{h,0}^s}$, provided that $s \geqslant 4$.

If a is a simple closed curve on $\Sigma_{g,r}^s$ we denote by $D_a \in \Gamma_{g,r}^s$ the *right* Dehn twist along a.

Definition 2.1. A *chain relation* C on the surface $\Sigma_{g,r}^s$ is given by an embedding $\Sigma_{1,2} \subset \Sigma_{g,r}^s$ and the standard chain relation on this 2-holed torus, namely

$$(D_a D_b D_c)^4 = D_e D_f$$

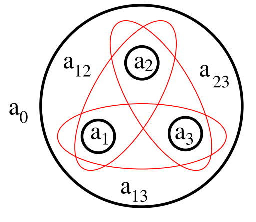
where a, b, c, d, e, f are the following curves of the embedded 2-holed torus:



Definition 2.2. A *lantern relation* L on the surface $\Sigma_{g,r}^s$ is given by an embedding $\Sigma_{0,4} \subset \Sigma_{g,r}^s$ and the standard lantern relation on this 4-holed sphere, namely

$$D_{a_{12}} D_{a_{13}} D_{a_{23}} D_{a_0}^{-1} D_{a_1}^{-1} D_{a_2}^{-1} D_{a_3}^{-1} = 1 \tag{33}$$

where $a_0, a_1, a_2, a_3, a_{12}, a_{13}, a_{23}$ are the following curves of the embedded 4-holed sphere:



Definition 2.3. Consider an embedding $\Sigma_{0,3}^1 \subset \Sigma_{g,r}^s$ such that the boundary components a_1, a_2, a_3 of $\Sigma_{0,3}^1$ are non-separating curves. Let then a_{12}, a_{13}, a_{23} be embedded curves on $\Sigma_{0,3}^1$ so that a_{jk} bounds a pair of pants $\Sigma_{0,3} \subset \Sigma_{0,3}^1$ along with a_j and a_k, for all $1 \leqslant j \neq k \leqslant 3$. Then the *puncture relation* P (supported at the puncture of $\Sigma_{0,3}^1$) on the surface $\Sigma_{g,r}^s$ is:

$$D_{a_{12}} D_{a_{13}} D_{a_{23}} D_{a_1}^{-1} D_{a_2}^{-1} D_{a_3}^{-1} = 1. \tag{34}$$

Remark 2.1. The puncture relation is, in fact, a consequence of the lantern relation and the fact that the Dehn twist along a small loop encircling a puncture is trivial.

The first step in proving Theorem 0.1 is to find an explicit presentation for the central extension $\widetilde{\Gamma_{g,r}^s}$. Specifically, by using Gervais' presentation [13], we have the following description.

Proposition 2.1. *Suppose that $g \geqslant 2$ and $s \geqslant 4$. Then the group $\widetilde{\Gamma_{g,r}^s}$ has the following presentation.*

1. *Generators*:
   (a) *With each non-separating simple closed curve a in $\Sigma_{g,r}^s$ is associated a generator $\widetilde{D}_a$*;
   (b) *One (central) element z.*

2. *Relations*:
   (a) *Centrality*:

$$z\widetilde{D}_a = \widetilde{D}_a z \tag{35}$$

   for any non-separating simple closed curve a on $\Sigma_{g,r}^s$;
   (b) *Braid type* 0*-relations*:

$$\widetilde{D}_a \widetilde{D}_b = \widetilde{D}_b \widetilde{D}_a \tag{36}$$

   for each pair of disjoint non-separating simple closed curves a and b;
   (c) *Braid type* 1*-relations*:

$$\widetilde{D}_a \widetilde{D}_b \widetilde{D}_a = \widetilde{D}_b \widetilde{D}_a \widetilde{D}_b \tag{37}$$

   for each pair of non-separating simple closed curves a and b which intersect transversely at one point;
   (d) *One lantern relation on a* 4*-holed sphere subsurface with non-separating boundary curves*:

$$\widetilde{D}_{a_0} \widetilde{D}_{a_1} \widetilde{D}_{a_2} \widetilde{D}_{a_3} = \widetilde{D}_{a_{12}} \widetilde{D}_{a_{13}} \widetilde{D}_{a_{23}}. \tag{38}$$

   (e) *One chain relation on a* 2*-holed torus subsurface with non-separating boundary curves*:

$$(\widetilde{D}_a \widetilde{D}_b \widetilde{D}_c)^4 = z^{12} \widetilde{D}_e \widetilde{D}_f. \tag{39}$$

   (f) *Puncture relations*:

$$\widetilde{D}_{a_{12(i)}} \widetilde{D}_{a_{13}(i)} \widetilde{D}_{a_{23}(i)} = z\widetilde{D}_{a_1(i)} \widetilde{D}_{a_2(i)} \widetilde{D}_{a_3(i)} \tag{40}$$

   for each puncture p_i *of* $\Sigma_{g,r}^s$, $i \in \{1, 2, \dots, s\}$.
   (g) *Scalar equation*:

$$z^N = 1 \tag{41}$$

   where N is the order of ζ^{-6}, *in the case where* $\zeta \in \mathbb{C}^*$ *is a root of unity.*

2.2. *Proof of Proposition 2.1*

Lemma 2.1. *For any lifts* $\widetilde{D}_a$ *of the Dehn twists* D_a *we have* $\widetilde{D}_a \widetilde{D}_b = \widetilde{D}_b \widetilde{D}_a$, *for any two disjoint simple closed curves a and b, and thus the braid-type* 0*-relations* (b) *are satisfied.*

Proof. The commutativity relations are satisfied for particular lifts coming from a semi-symmetric T-matrix. If we change the lifts by multiplying each lift by some central element the commutativity is still valid. Thus, the commutativity holds for any lifts. □

Lemma 2.2. *There are lifts* $\widetilde{D}_a$ *of the Dehn twists* D_a, *for each non-separating simple closed curve a such that we have* $\widetilde{D}_a \widetilde{D}_b \widetilde{D}_a = \widetilde{D}_b \widetilde{D}_a \widetilde{D}_b$ *for any simple closed curves a, b with one intersection point, and thus the braid type* 1*-relations* (c) *are satisfied. Moreover, the choice of lifts of all* $\widetilde{D}_x$, *with x non-separating, satisfying these requirements is uniquely defined by fixing the lift* $\widetilde{D}_a$ *of one particular Dehn twist.*

Proof. Consider an arbitrary lift of one braid type 1-relation (to be called the fundamental one), which has the form $\widetilde{D}_a\widetilde{D}_b\widetilde{D}_a = z^k\widetilde{D}_b\widetilde{D}_a\widetilde{D}_b$. Change then the lift $\widetilde{D}_b$ into $z^k\widetilde{D}_b$. With the new lift the relation above becomes $\widetilde{D}_a\widetilde{D}_b\widetilde{D}_a = \widetilde{D}_b\widetilde{D}_a\widetilde{D}_b$.

Choose now an arbitrary braid type 1-relation of $\Gamma^s_{g,r}$, say $D_xD_yD_x = D_yD_xD_y$. There exists a 1-holed torus $\Sigma_{1,1} \subset \Sigma^s_{g,r}$ containing x, y, namely a neighborhood of $x \cup y$. Let T be the similar torus containing a, b. Since a, b and x, y are non-separating there exists a homeomorphism $\varphi : \Sigma^s_{g,r} \to \Sigma^s_{g,r}$ such that $\varphi(a) = x$ and $\varphi(b) = y$. We have then

$$D_x = \varphi D_a\varphi^{-1}, \qquad D_y = \varphi D_b\varphi^{-1}.$$

Let us consider now an arbitrary lift $\widetilde{\varphi}$ of φ, which is well-defined only up to a central element, and set

$$\widetilde{D}_x = \widetilde{\varphi}\widetilde{D}_a\widetilde{\varphi}^{-1}, \qquad \widetilde{D}_y = \widetilde{\varphi}\widetilde{D}_b\widetilde{\varphi}^{-1}.$$

These lifts are well-defined since they do not depend on the choice of $\widetilde{\varphi}$ (the central elements coming from $\widetilde{\varphi}$ and $\widetilde{\varphi}^{-1}$ mutually cancel). Moreover, we have then

$$\widetilde{D}_x\widetilde{D}_y\widetilde{D}_x = \widetilde{D}_y\widetilde{D}_x\widetilde{D}_y$$

and so the braid type 1-relations (c) are all satisfied.

For the second part of the lemma observe that the choice of $\widetilde{D}_a$ fixes the choice of $\widetilde{D}_b$. If x is a non-separating simple closed curve on $\Sigma^s_{g,r}$, then there exists another non-separating curve y which intersects it in one point. Thus, by the argument which was used above to prove the existence of the lifts the choice of $\widetilde{D}_x$ is unique. □

Lemma 2.3. *One can choose the lifts of Dehn twists in $\widetilde{\Gamma^s_{g,r}}$ so that all braid type relations are satisfied and the lift of the lantern relation* (d) *is trivial, namely*

$$\widetilde{D}_{a_0}\widetilde{D}_{a_1}\widetilde{D}_{a_2}\widetilde{D}_{a_3} = \widetilde{D}_{a_{12}}\widetilde{D}_{a_{13}}\widetilde{D}_{a_{23}}$$

for the non-separating curves on an embedded $\Sigma_{0,4} \subset \Sigma^s_{g,r}$.

Proof. An arbitrary lift of that lantern relation is of the form $\widetilde{D}_{a_0}\widetilde{D}_{a_1}\widetilde{D}_{a_2}\widetilde{D}_{a_3} = z^k\widetilde{D}_{a_{12}}\widetilde{D}_{a_{13}}\widetilde{D}_{a_{23}}$. In this case, we change the lift $\widetilde{D}_{a_0}$ into $z^{-k}\widetilde{D}_{a_0}$ and adjust the lifts of all other Dehn twists along non-separating curves the way that all braid type 1-relations are satisfied. Then, the required form of the lantern relation is satisfied. □

We say that the lifts of the Dehn twists are *normalized* if all braid type relations and one lantern relation are lifted in a trivial way.

Lemma 2.4. *Assume that $s \geqslant 4$. Then a normalized Dehn twist in quantum Teichmüller theory is conjugated to the inverse T-matrix times ζ^{-6} i.e.*

$$\widetilde{D}_\alpha = \mathsf{F}(\tau, D_\alpha\tau) = \zeta^{-6}U_\alpha T^{-1}_{kl}U^{-1}_\alpha.$$

As the computations involved in the proof are rather laborious we postpone it after the proof of Lemma 2.6.

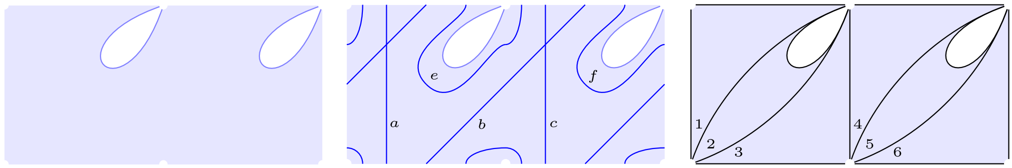


Fig. 5. Torus with two holes and two punctures.

We will suppose henceforth that the lifts of Dehn twists are normalized.

Lemma 2.5. *Let a, b, c, e, f be the five curves appearing in the chain relation $(D_aD_bD_c)^4 = D_eD_f$ on an embedded 2-holed torus sitting inside $\Sigma^s_{g,r}$. If $s \geqslant 2$, then the lifts of Dehn twists in $\widetilde{\Gamma^s_{g,r}}$ satisfy the relation*

$$(\widetilde{D}_a\widetilde{D}_b\widetilde{D}_c)^4 = \zeta^{-72}\widetilde{D}_e\widetilde{D}_f.$$

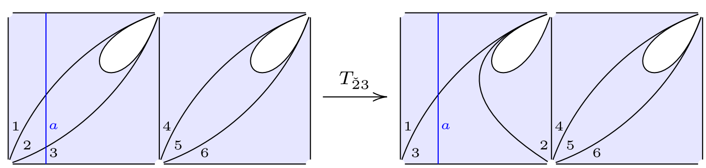
Proof. If $s \geqslant 2$ and $g \geqslant 2$, then there is an embedding $\Sigma^2_{2,1} \subset \Sigma^s_{g,r}$.

We consider a surface S homeomorphic to $\Sigma^2_{1,2}$, i.e. a torus with two holes and two punctures drawn in the left picture of Fig. 5 where the opposite sides of the rectangle are identified. Notice that the two punctures are located on the two boundary components. The central picture of Fig. 5 specifies five simple closed curves a, b, c, e, f in S, the Dehn twists along which enter the chain relation.

We also choose a particular decorated ideal triangulation τ of S given by the right picture of Fig. 5, where the ideal arcs are drawn in black and the positions of the numbers in ideal triangles correspond to the marked corners. Notice that our choice is manifestly symmetric with respect to the exchange of the left and the right halves of the rectangle accompanied with relabeling $(1, 2, 3) \leftrightarrow (4, 5, 6)$. This symmetry will be useful for reducing the amount of calculations in deriving the quantum realizations of the Dehn twists.

The basic procedure in deriving the quantum realization of the Dehn twist D_α along a given simple closed curve α is to use a specific decorated ideal triangulation where the contour α intersects only two ideal arcs, so that the annular neighborhood of α is given by only two ideal triangles. With respect to such (decorated) ideal triangulation the quantum operator realizing D_α is given by a single T-operator. Let us work out this procedure in the case of the curves a, b, c, e, f.

For any simple closed curve α, we denote $\bar{\mathsf{F}}_\alpha = \widetilde{D}_\alpha^{-1} \simeq \mathsf{F}(D_\alpha\tau, \tau)$. To derive the operator representing the Dehn twist D_a, we apply the following change of triangulation:



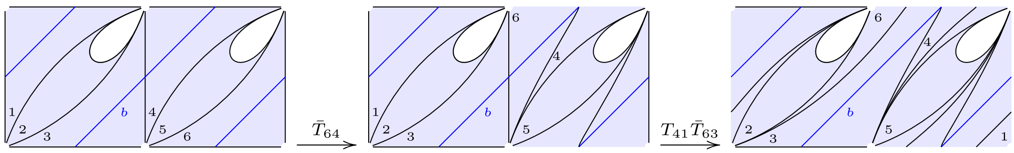
where the operator above the arrow realizes the corresponding element of the groupoid of decorated ideal triangulations within the quantum Teichmüller theory. Thus,

$$\begin{aligned}\zeta^{-6}\bar{\mathsf{F}}_a &= \mathrm{Ad}(T_{\check{2}3})(T_{1\hat{3}})\\ &= T_{\check{2}3}T_{1\hat{3}}\bar{T}_{\check{2}3}\\ &= T_{1\hat{3}}T_{1\hat{2}},\end{aligned}$$

where in the last equality, we have applied once the Pentagon relation, and we use the notation $\bar{T} = T^{-1}$. Here, we use the normalization where the braid-type and the lantern relations are satisfied without projective factors. By the above mentioned left–right symmetry $(1,2,3) \leftrightarrow (4,5,6)$, we immediately get the quantum realization of the Dehn twist D_c:

$$\zeta^{-6}\bar{\mathsf{F}}_c = T_{4\hat{6}}T_{4\hat{5}}.$$

To calculate the quantum realization of D_b we use a two-step chain of transformations of τ:

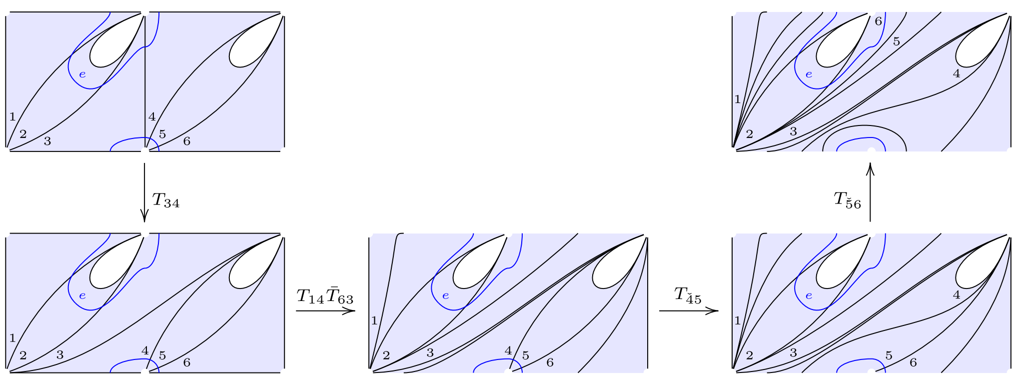


Thus, we have the following sequence of equalities:

$$\begin{aligned}\zeta^{-6}\bar{\mathsf{F}}_b &= \mathrm{Ad}(\bar{T}_{64}T_{41}\bar{T}_{63})(T_{34})\\ &= \bar{T}_{64}T_{41}\underline{\bar{T}_{63}T_{34}T_{63}}\bar{T}_{41}T_{64}\\ &= \underline{\bar{T}_{64}T_{41}T_{64}}T_{34}\bar{T}_{41}T_{64}\\ &= T_{61}\underline{T_{41}T_{34}\bar{T}_{41}}T_{64}\\ &= T_{61}T_{34}T_{31}T_{64},\end{aligned}$$

where in each step the underlined fragment is transformed by using the Pentagon relation.

To calculate the realization of D_e, we consider the following sequence of ideal triangulations:



Thus, we have

$$\zeta^{-6}\bar{\mathsf{F}}_e = \mathrm{Ad}(T_{34}T_{14}\bar{T}_{63}T_{\check{4}5}T_{\check{5}6})(T_{\check{2}\hat{6}})$$

$$
\begin{aligned}
&= T_{34}T_{14}\bar T_{63}T_{\check 45}\underline{T_{\hat 6\hat 5}T_{\check 2\hat 6}\bar T_{\hat 6\hat 5}}\bar T_{\check 45}T_{63}\bar T_{14}\bar T_{34}\\
&= T_{34}T_{14}\bar T_{63}\underline{T_{\hat 5\hat 4}}T_{\check 2\hat 6}\underline{T_{\check 2\hat 5}\bar T_{\hat 5\hat 4}}T_{63}\bar T_{14}\bar T_{34}\\
&= T_{34}\underline{T_{\hat 4\check 1}}\bar T_{63}T_{\check 2\hat 6}T_{\check 2\hat 5}\underline{T_{\check 2\hat 4}}T_{63}\underline{\bar T_{\hat 4\check 1}}\bar T_{34}\\
&= T_{34}\underline{\bar T_{\hat 3\check 6}T_{\check 6\hat 2}}T_{\check 2\hat 5}T_{\check 2\hat 4}T_{\check 2\check 1}\underline{T_{\hat 3\check 6}}\bar T_{34}\\
&= \underline{T_{34}T_{\check 23}}T_{\check 6\hat 2}T_{\check 2\hat 5}T_{\check 2\hat 4}T_{\check 2\check 1}\bar T_{34}\\
&= T_{\check 23}T_{\check 24}\underline{T_{\hat 4\check 3}}T_{\check 6\hat 2}T_{\check 2\hat 5}\underline{T_{\check 2\hat 4}}T_{\check 2\check 1}\underline{\bar T_{\hat 4\check 3}}\\
&= T_{\check 23}T_{\check 24}T_{\check 2\hat 6}T_{\check 2\hat 5}T_{\check 2\hat 4}T_{\check 2\check 3}T_{\check 2\check 1},
\end{aligned}
$$

where, as before, in each step the underlined fragment is transformed by applying the Pentagon relation. We use throughout these computations the fact that T_{ij} and T_{kl} commute if $\{i,j\}\cap\{k,l\}=\emptyset$. Again, using the symmetry $(1,2,3)\leftrightarrow(4,5,6)$, we also have

$$
\zeta^{-6}\bar{\mathsf F}_f = T_{\check 56}T_{\check 51}T_{\check 5\hat 3}T_{\check 5\hat 2}T_{\check 5\hat 1}T_{\check 5\check 6}T_{\check 5\check 4}.
$$

In order to check the Chain relation, we first calculate the following product:

$$
\begin{aligned}
\zeta^{-18}\bar{\mathsf F}_c\bar{\mathsf F}_b\bar{\mathsf F}_a &= T_{4\hat 6}T_{4\hat 5}T_{61}T_{34}\underline{T_{31}}T_{64}\underline{T_{1\hat 3}}T_{1\hat 2}\\
&= T_{4\hat 6}T_{4\hat 5}T_{61}T_{34}T_{64}\zeta P_{(31\hat 3)}T_{1\hat 2}\\
&= \zeta T_{4\hat 6}T_{4\hat 5}T_{61}T_{34}T_{64}T_{\hat 3\hat 2}P_{(31\hat 3)},
\end{aligned}
$$

where we have applied the Inversion relation to the underlined fragment. Next, we calculate

$$
\begin{aligned}
\zeta^{-36}(\bar{\mathsf F}_c\bar{\mathsf F}_b\bar{\mathsf F}_a)^2 &= \zeta^2 T_{4\hat 6}T_{4\hat 5}T_{61}T_{34}T_{64}T_{\hat 3\hat 2}\underline{P_{(31\hat 3)}}T_{4\hat 6}T_{4\hat 5}\underline{T_{61}T_{34}}T_{64}\underline{T_{\hat 3\hat 2}P_{(31\hat 3)}}\\
&= \zeta^2 T_{4\hat 6}T_{4\hat 5}T_{61}T_{34}\underline{T_{64}}T_{\hat 3\hat 2}\underline{T_{4\hat 6}T_{4\hat 5}T_{6\hat 3}T_{14}T_{64}}T_{\hat 1\hat 2}P_{(3\hat 3)(1\hat 1)}\\
&= \zeta^3 T_{4\hat 6}T_{4\hat 5}\underline{T_{61}T_{34}T_{\hat 3\hat 2}T_{\hat 6\hat 5}}T_{4\hat 3}T_{1\hat 6}T_{4\hat 6}T_{\hat 1\hat 2}P_{(64\hat 6)}P_{(3\hat 3)(1\hat 1)}\\
&= \zeta^3 \underline{T_{4\hat 6}T_{4\hat 5}T_{\hat 6\hat 5}}T_{\check 51}\underline{T_{61}}T_{\hat 3\hat 2}T_{\check 24}\underline{T_{34}T_{4\hat 3}T_{1\hat 6}}T_{4\hat 6}T_{\hat 1\hat 2}P_{(64\hat 6)}P_{(3\hat 3)(1\hat 1)}\\
&= \zeta^5 T_{\hat 6\hat 5}T_{4\hat 6}T_{\check 51}T_{\hat 3\hat 2}T_{\check 24}\underline{P_{(61\hat 6)}P_{(34\hat 3)}T_{4\hat 6}T_{\hat 1\hat 2}P_{(64\hat 6)}P_{(3\hat 3)(1\hat 1)}}\\
&= \zeta^5 T_{\hat 6\hat 5}T_{4\hat 6}T_{\check 51}T_{\check 23}T_{\check 24}T_{\hat 3\hat 1}T_{\check 6\hat 2}P_{(1\check 63\hat 4\check 1)},
\end{aligned}
$$

where each equality is obtained by transforming the underlined fragment by applying the Pentagon relation (twice in the forth and once in the fifth equalities), the Inversion relation (once in the third and twice in the fifth equalities), and the extended symmetric group action (in the second, the third, and the sixth equalities). Finally, taking the square of the obtained identity, we have

$$
\begin{aligned}
\zeta^{-72}(\bar{\mathsf F}_c\bar{\mathsf F}_b\bar{\mathsf F}_a)^4 &= \zeta^{10}T_{\hat 6\hat 5}T_{4\hat 6}T_{\check 51}T_{\check 23}T_{\check 24}T_{\hat 3\hat 1}T_{\check 6\hat 2}\underline{P_{(1\check 63\hat 4\check 1)}T_{\hat 6\hat 5}T_{4\hat 6}T_{\check 51}T_{\check 23}T_{\check 24}T_{\hat 3\hat 1}T_{\check 6\hat 2}P_{(1\check 63\hat 4\check 1)}}\\
&= \zeta^{10}T_{\check 56}T_{4\hat 6}T_{\check 51}T_{\check 23}T_{\check 24}\underline{T_{\hat 3\hat 1}}T_{\check 6\hat 2}\underline{T_{\check 5\hat 3}}T_{\hat 1\check 3}T_{\check 5\check 6}T_{\check 2\hat 4}T_{\check 2\hat 1}T_{\check 46}T_{3\hat 2}P_{(13\check 1)}P_{(46\check 4)}\\
&= \zeta^{10}T_{\check 56}T_{4\hat 6}T_{\check 51}T_{\check 23}T_{\check 24}T_{\check 6\hat 2}T_{\check 5\hat 3}T_{\check 5\hat 1}\underline{T_{\hat 3\hat 1}T_{\hat 1\check 3}}T_{\check 5\check 6}T_{\check 2\hat 4}\underline{T_{\check 2\hat 1}}T_{\check 46}\underline{T_{3\hat 2}P_{(13\check 1)}}P_{(46\check 4)}\\
&= \zeta^{11}T_{\check 56}\underline{T_{4\hat 6}}T_{\check 51}T_{\check 23}\underline{T_{\check 24}}T_{\check 6\hat 2}T_{\check 5\hat 3}T_{\check 5\hat 1}T_{\check 5\check 6}T_{\check 2\hat 4}T_{\check 2\check 3}T_{\check 46}T_{1\hat 2}P_{(46\check 4)}\\
&= \zeta^{11}T_{\check 56}T_{\check 51}T_{\check 23}T_{\check 24}T_{\check 2\hat 6}\underline{T_{4\hat 6}T_{\check 6\hat 2}}T_{\check 5\hat 3}T_{\check 5\hat 1}T_{\check 5\check 6}\underline{T_{\check 2\hat 4}}T_{\check 46}T_{\check 2\check 3}T_{1\hat 2}P_{(46\check 4)}
\end{aligned}
$$

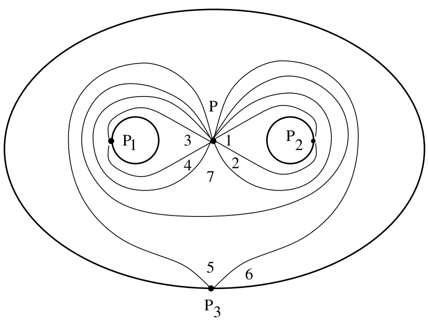
$$\begin{aligned}
&= \zeta^{11} T_{\check{5}6} T_{\check{5}1} T_{\check{2}3} T_{\check{2}4} T_{\check{2}\hat{6}} T_{\check{5}\hat{3}} T_{\check{5}\hat{1}} T_{\check{2}\hat{4}} \underline{T_{4\hat{6}} T_{\check{5}\check{6}}} T_{\check{4}6} T_{\check{2}\check{3}} T_{1\hat{2}} P_{(46\check{4})} \\
&= \zeta^{11} T_{\check{5}6} T_{\check{5}1} T_{\check{2}3} T_{\check{2}4} T_{\check{2}\hat{6}} T_{\check{5}\hat{3}} T_{\check{5}\hat{1}} T_{\check{2}\hat{4}} T_{\check{5}\check{6}} T_{\check{5}\check{4}} \underline{T_{4\hat{6}} T_{\check{4}6}} T_{\check{2}\check{3}} T_{1\hat{2}} \underline{P_{(46\check{4})}} \\
&= \zeta^{12} T_{\check{5}6} T_{\check{5}1} \underline{T_{\check{2}3}} T_{\check{2}4} \underline{T_{\check{2}\hat{6}} T_{\check{5}\hat{3}}} T_{\check{5}\hat{1}} \underline{T_{\check{2}\hat{4}} T_{\check{5}\check{6}} T_{\check{5}\check{4}}} T_{\check{2}\check{3}} T_{1\hat{2}} \\
&= \zeta^{12} T_{\check{5}6} T_{\check{5}1} T_{\check{5}\hat{3}} T_{\check{5}\hat{2}} T_{\check{2}3} \underline{T_{\check{2}4}} T_{\check{5}\hat{1}} T_{\check{5}\check{6}} \underline{T_{\check{5}\hat{2}}} T_{\check{2}\hat{6}} \underline{T_{\check{5}\check{4}}} T_{\check{5}\hat{2}} T_{\check{2}\hat{4}} T_{\check{2}\check{3}} T_{1\hat{2}} \\
&= \zeta^{12} \underline{T_{\check{5}6} T_{\check{5}1} T_{\check{5}\hat{3}} T_{\check{5}\hat{2}}} T_{\check{2}3} \underline{T_{\check{5}\hat{1}} T_{\check{5}\check{6}} T_{\check{5}\check{4}}} T_{\check{2}4} T_{\check{2}\hat{6}} T_{\check{5}\hat{2}} T_{\check{2}\hat{4}} T_{\check{2}\check{3}} T_{1\hat{2}} \\
&= \bar{\mathsf{F}}_f \bar{\mathsf{F}}_e,
\end{aligned}$$

where each equality, except for the last one, is obtained by transforming the underlined fragment by applying the Pentagon relation (one time in the third, the fifth, the sixth, the seventh, the tenth, and three times in the ninth equalities), the Inversion relation (in the forth and the eighth equalities), and the extended symmetric group action (in the second, the forth, and the eighth equalities), while in the last equality the underlined (respectively the non-underlined) fragment corresponds to the operator $\bar{\mathsf{F}}_f$ (respectively $\bar{\mathsf{F}}_e$). □

Lemma 2.6. *Suppose that $s \geqslant 4$. Then the lift of each puncture relation is ζ^6.*

Proof. Observe first that the central element P_i which is the lift of the puncture relation at the puncture p_i is independent of the particular subsurface $S_{0,3}^1$. If we consider another subsurface, there exists a homeomorphism $\varphi : S_{g,r}^s \to S_{g,r}^s$ fixing the puncture p_i and sending it to the initial subsurface, because the boundary components are non-separating. The new puncture relation is then conjugate of P_i by $\widetilde{\varphi}$ and hence they coincide, as they are elements of the center.

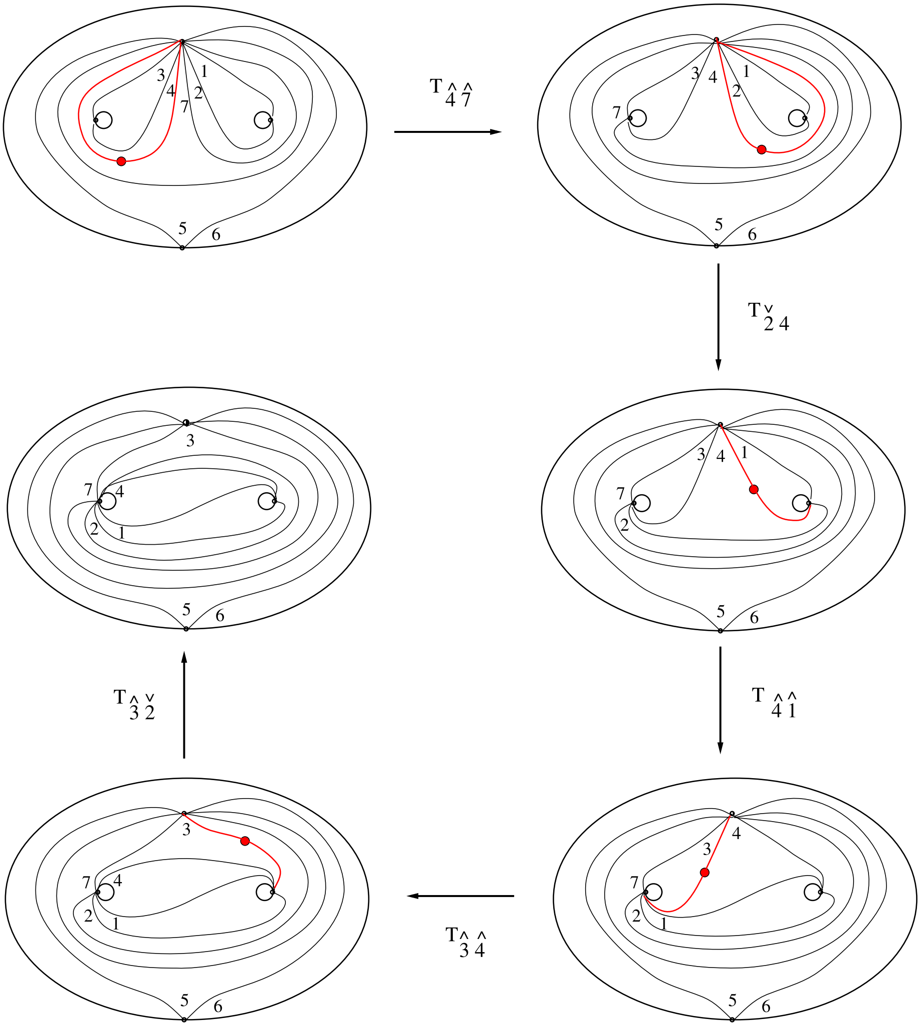
If $s \geqslant 4$ then there is an embedding $S_{0,3}^4 \subset S_{g,r}^s$, such that each boundary component of $S_{0,3}^4$ has a puncture on it. Consider first the following decomposition τ of the punctured pair of pants into triangles. The position of the label of each triangle indicates also the marked corner.



Then we can express easily the action of each Dehn twist D_{a_j} on the triangulation τ as a composition of flips. If we set $\mathsf{F}_{a_j} = \mathsf{F}(\tau, D_{a_j}(\tau))$ then we have:

$$\mathsf{F}_{a_1} = T^{-1}_{\check{3}\check{4}}, \qquad \mathsf{F}_{a_2} = T^{-1}_{\check{1}2}, \qquad \mathsf{F}_{a_3} = T^{-1}_{\check{5}6}.$$

Further we use the sequence of transformations below, in order to change the triangulation τ into a triangulation which intersects the curve a_{12} in only two points.



Here and in the pictures below we marked by a dot the edges where a flip occurs, in order to help the reader visualize the sequence of transformations. Then the method outlined above permits to compute the Dehn twist $\mathsf{F}_{a_{12}} = \mathsf{F}(\tau, D_{a_{12}}(\tau))$ as follows:

$$\mathsf{F}_{a_{12}} = \mathrm{Ad}(T_{\hat{4}\hat{7}} T_{\check{2}4} T_{\hat{4}\hat{1}} T_{\hat{3}\hat{4}} T_{\check{3}\check{2}})\big(T_{\check{3}\hat{7}}^{-1}\big).$$

Let us first simplify the formula for $\mathsf{F}_{a_{12}}$. We have

$$\begin{aligned}
\bar{\mathsf{F}}_{a_{12}} &= T_{\check{7}4} T_{\check{2}4} T_{\hat{4}\hat{1}} T_{\hat{3}\hat{4}} \underline{T_{\check{3}\check{2}} T_{\check{7}\hat{3}} \bar{T}_{\check{3}\check{2}}} \bar{T}_{\hat{3}\hat{4}} \bar{T}_{\hat{4}\hat{1}} \bar{T}_{\check{2}4} \bar{T}_{\check{7}4} = T_{\check{7}4} T_{\check{2}4} T_{\hat{4}\hat{1}} \underline{T_{\hat{3}\hat{4}} T_{\check{7}\hat{3}}} T_{\check{7}\check{2}} \underline{\bar{T}_{\hat{3}\hat{4}}} \bar{T}_{\hat{4}\hat{1}} \bar{T}_{\check{2}4} \bar{T}_{\check{7}4} \\
&= T_{\check{7}4} T_{\check{2}4} \underline{T_{\hat{4}\hat{1}}} T_{\check{7}\hat{3}} \underline{T_{\check{7}\hat{4}}} T_{\check{7}\check{2}} \underline{\bar{T}_{\hat{4}\hat{1}}} \bar{T}_{\check{2}4} \bar{T}_{\check{7}4} = T_{\check{7}4} T_{\check{2}4} T_{\check{7}\hat{3}} T_{\check{7}\hat{4}} T_{\check{7}\hat{1}} \underline{T_{\check{7}\check{2}} \bar{T}_{\check{2}4} \bar{T}_{\check{7}4}} \\
&= T_{\check{7}4} \underline{T_{\check{2}4}} T_{\check{7}\hat{3}} \underline{T_{\check{7}\hat{4}}} T_{\check{7}\hat{1}} \underline{\bar{T}_{\check{2}4}} T_{\check{7}\check{2}} = T_{\check{7}4} T_{\check{7}\hat{3}} T_{\check{7}\hat{4}} T_{\check{7}\hat{2}} T_{\check{7}\hat{1}} T_{\check{7}\check{2}}
\end{aligned}$$

where in each step the underlined fragment is transformed by using the Pentagon equation, and in the last equality it is also combined with the symmetry relation $T_{\check{2}4} = T_{\hat{4}\hat{2}}$.

Our triangulation is invariant under the following simultaneous cyclic permutations

$$\pi : P_1 \mapsto P_2 \mapsto P_3 \mapsto P_1, \quad 1 \mapsto \check{6} \mapsto 3 \mapsto 1, \quad 2 \mapsto \hat{5} \mapsto \check{4} \mapsto 2, \quad 7 \mapsto \check{7},$$

so that the contours a_j and a_{kl} are transformed as follows:

$$\pi : a_1 \mapsto a_2 \mapsto a_3 \mapsto a_1, \quad a_{12} \mapsto a_{23} \mapsto a_{31} \mapsto a_{12}.$$

Thus, it suffices to know the explicit formula for $\mathsf{F}_{a_{12}}$ in order to write out the other two without any further calculation:

$$\begin{aligned}\bar{\mathsf{F}}_{a_{23}} &= \pi(\bar{\mathsf{F}}_{a_{12}}) \\ &= \pi(T_{\check{7}4} T_{\check{7}\hat{3}} T_{\check{7}\hat{4}} T_{\check{7}\hat{2}} T_{\check{7}\hat{1}} T_{\check{7}\check{2}}) \\ &= T_{\hat{7}\hat{2}} T_{\hat{7}\hat{1}} T_{\hat{7}\check{2}} T_{\hat{7}\check{5}} T_{\hat{7}6} T_{\hat{7}5},\end{aligned}$$

and

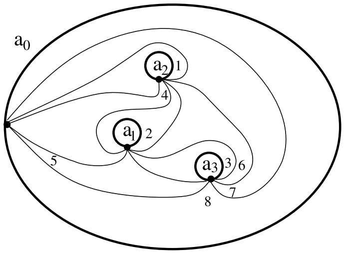
$$\begin{aligned}\bar{\mathsf{F}}_{a_{31}} &= \pi(\bar{\mathsf{F}}_{a_{23}}) \\ &= \pi(T_{\hat{7}\hat{2}} T_{\hat{7}\hat{1}} T_{\hat{7}\check{2}} T_{\hat{7}\check{5}} T_{\hat{7}6} T_{\hat{7}5}) \\ &= T_{7\check{5}} T_{76} T_{75} T_{74} T_{7\hat{3}} T_{7\hat{4}}.\end{aligned}$$

Now, we have

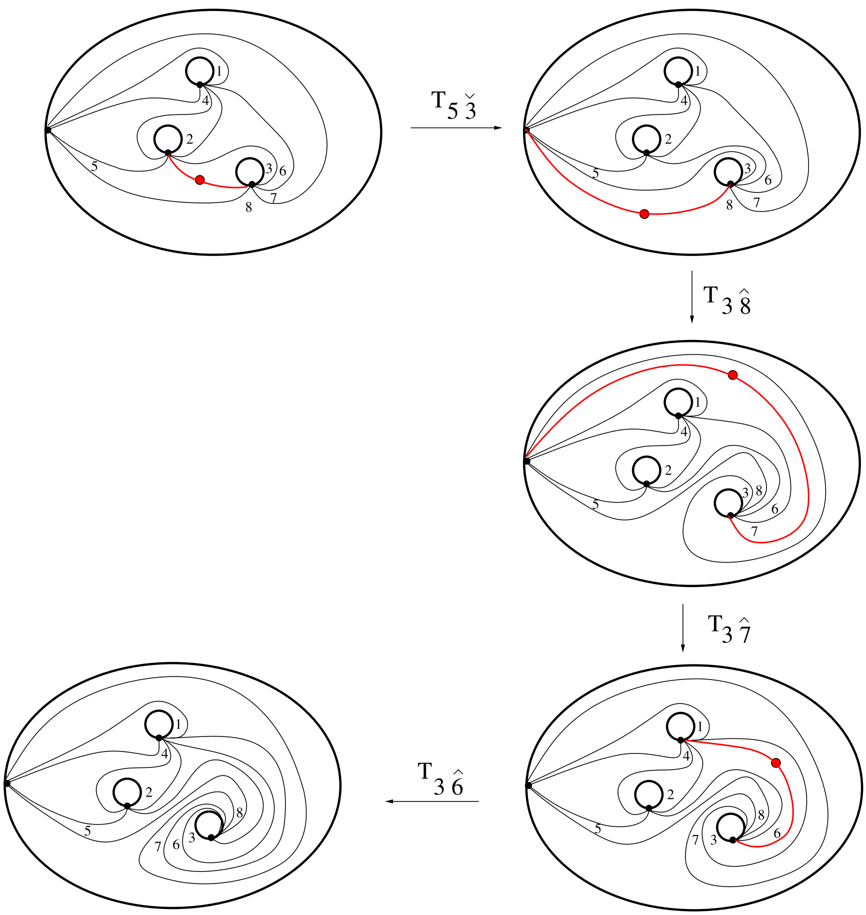
$$\begin{aligned}\bar{\mathsf{F}}_{a_{12}} \bar{\mathsf{F}}_{a_{23}} \bar{\mathsf{F}}_{a_{31}} &= T_{\check{7}4} T_{\check{7}\hat{3}} T_{\check{7}\hat{4}} T_{\check{7}\hat{2}} T_{\check{7}\hat{1}} \underline{T_{\check{7}\check{2}} T_{\hat{7}\hat{2}}} T_{\hat{7}\hat{1}} T_{\hat{7}\check{2}} T_{\hat{7}\check{5}} T_{\hat{7}6} \underline{T_{\hat{7}5} T_{7\check{5}}} T_{76} T_{75} T_{74} T_{7\hat{3}} T_{7\hat{4}} \\ &= T_{\check{7}4} T_{\check{7}\hat{3}} T_{\check{7}\hat{4}} T_{\check{7}\hat{2}} T_{\check{7}\hat{1}} \zeta P_{(2\hat{7}\hat{2})} T_{\hat{7}\hat{1}} T_{\hat{7}\check{2}} T_{\hat{7}\check{5}} T_{\hat{7}6} \zeta P_{(\hat{5}7\check{5})} T_{76} T_{75} T_{74} T_{7\hat{3}} T_{7\hat{4}} \\ &= \zeta^2 T_{\check{7}4} T_{\check{7}\hat{3}} T_{\check{7}\hat{4}} \underline{T_{\check{7}\hat{2}} T_{\check{7}\hat{1}} T_{\hat{2}\hat{1}}} T_{\hat{2}7} \underline{T_{\hat{2}\check{5}} T_{\hat{2}6} T_{\check{5}6}} T_{\check{5}\check{2}} T_{\check{5}4} T_{\check{5}\hat{3}} T_{\check{5}\hat{4}} P_{(2\hat{7}5\check{2})} \\ &= \zeta^2 T_{\check{7}4} T_{\check{7}\hat{3}} T_{\check{7}\hat{4}} T_{\hat{2}\hat{1}} \underline{T_{\check{7}\hat{2}} T_{\hat{2}7}} T_{\check{5}6} \underline{T_{\hat{2}\check{5}} T_{\check{5}\check{2}}} T_{\check{5}4} T_{\check{5}\hat{3}} T_{\check{5}\hat{4}} P_{(2\hat{7}5\check{2})} \\ &= \zeta^2 T_{\check{7}4} T_{\check{7}\hat{3}} T_{\check{7}\hat{4}} T_{\hat{2}\hat{1}} \zeta P_{(\check{7}\hat{2}7)} T_{\check{5}6} \zeta P_{(\hat{2}\check{5}\check{2})} T_{\check{5}4} T_{\check{5}\hat{3}} T_{\check{5}\hat{4}} P_{(2\hat{7}5\check{2})} \\ &= \zeta^4 T_{\check{7}4} T_{\check{7}\hat{3}} \underline{T_{\check{7}\hat{4}}} T_{\hat{2}\hat{1}} T_{\check{5}6} \underline{T_{\hat{7}4}} T_{\hat{7}\hat{3}} T_{\hat{7}\hat{4}} P_{(7\hat{7})} \\ &= \zeta^4 T_{\check{7}4} T_{\check{7}\hat{3}} T_{\hat{2}\hat{1}} T_{\check{5}6} \zeta P_{(\check{4}\hat{7}4)} T_{\hat{7}\hat{3}} T_{\hat{7}\hat{4}} P_{(7\hat{7})} \\ &= \zeta^5 \underline{T_{\check{7}4} T_{\check{7}\hat{3}}} T_{\hat{2}\hat{1}} T_{\check{5}6} \underline{T_{4\hat{3}}} T_{47} P_{(74\check{7})} \\ &= \zeta^5 T_{\hat{2}\hat{1}} T_{\check{5}6} T_{4\hat{3}} \underline{T_{\check{7}4} T_{47}} P_{(74\check{7})} \\ &= \zeta^5 T_{\hat{2}\hat{1}} T_{\check{5}6} T_{4\hat{3}} \zeta P_{(\check{7}47)} P_{(74\check{7})} \\ &= \zeta^6 T_{\hat{2}\hat{1}} T_{\check{5}6} T_{4\hat{3}} \\ &= \zeta^6 \bar{\mathsf{F}}_{a_2} \bar{\mathsf{F}}_{a_3} \bar{\mathsf{F}}_{a_1}\end{aligned}$$

where in the underlined fragments the Pentagon equation is used twice in the forth and once in the ninth equalities, the Inversion relation is used twice in the second and the fifth, and once in the seventh and the tenth equalities, while in the third, sixth, eighth, and eleventh equalities the permutation operators are moved to the right and the powers of ζ, to the left. □

Proof of Lemma 2.4. The idea of the proof is to calculate the lift of the lantern relation. Consider the following decorated triangulation τ of the 4-holed disk with 4 punctures:



The trick used in [16,18] for computing D_a is to use a sequence of flips to change the triangulation into one which intersects some curve isotopic to a into two points. Then the Dehn twist along a can be expressed as the flip of one of the two edges of the latter triangulation intersecting a. This recipe generalizes to the case where the curve a intersects several edges of the triangulation, if a is a boundary component with one puncture on it. Specifically, let $e_1, \dots, e_s$ be the edges issued from the puncture, in counterclockwise order. Then the Dehn twist D_a can be expressed as the result of composing the flips of $e_1, e_2, \dots, e_{s-1}$. We illustrate this procedure with the case of the left Dehn twist $D_{a_3}^{-1}$ on the triangulation τ above:



In particular, we find the following expression for the right Dehn twist along a_3:

$$\begin{aligned}\bar{\mathsf{F}}_{a_3} &= \bar{\mathsf{F}}(\tau, D_{a_3}\tau) \\ &= T_{3\check{5}} T_{3\hat{8}} T_{3\hat{7}} T_{3\hat{6}}. \end{aligned} \tag{42}$$

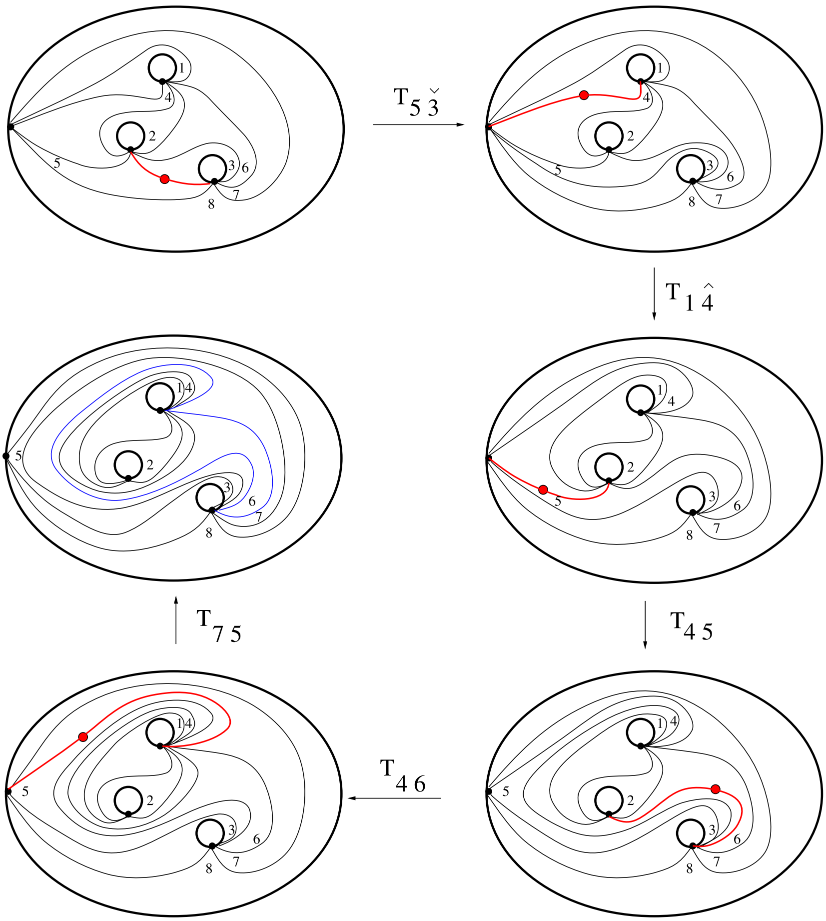
We used above the symmetry property of the T-matrix $T_{3\check{5}} = T_{5\check{3}}$ (see Remark 1.3, Eq. (25)). The same recipe for the remaining Dehn twists along boundary components gives us:

$$\bar{\mathsf{F}}_{a_2} = \bar{\mathsf{F}}(\tau, D_{a_2}\tau) = T_{24} T_{25} T_{2\check{3}} T_{26}, \tag{43}$$

$$\bar{\mathsf{F}}_{a_1} = \bar{\mathsf{F}}(\tau, D_{a_1}\tau) = T_{1\hat{4}} T_{1\check{2}} T_{1\check{6}} T_{17}, \tag{44}$$

$$\bar{\mathsf{F}}_{a_0} = \bar{\mathsf{F}}(\tau, D_{a_0}\tau) = T_{\check{8}\hat{5}} T_{\check{8}\check{4}} T_{\check{8}\check{1}} T_{\check{8}\check{7}}. \tag{45}$$

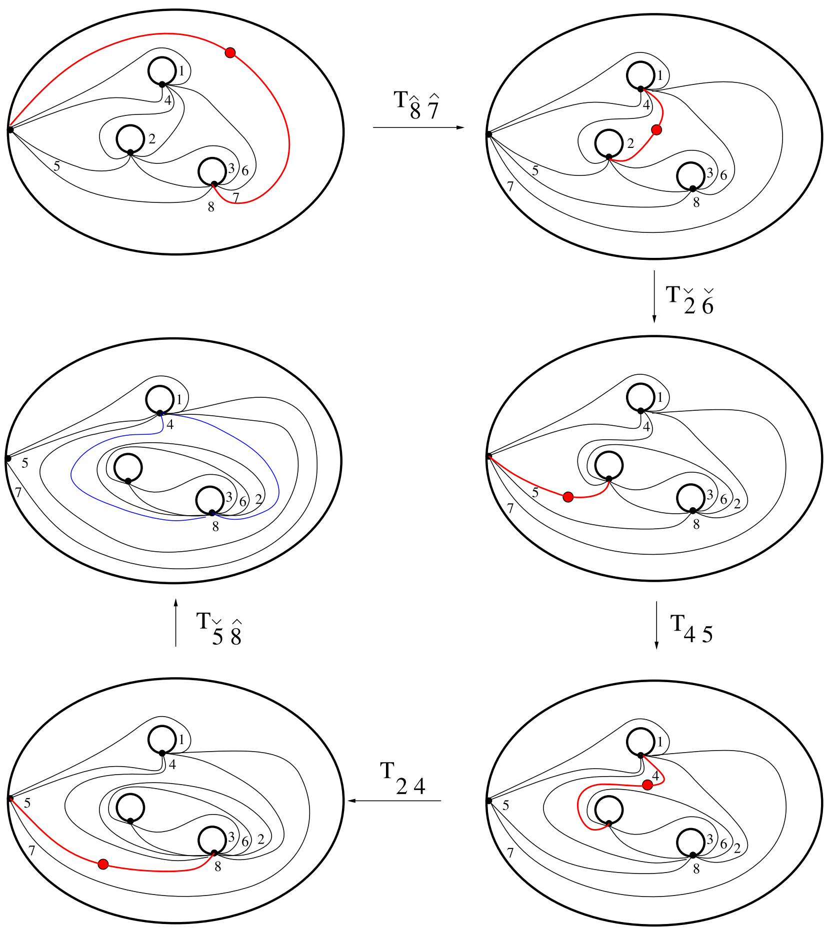
In order to compute $F_{a_{12}}$ we need to transform the triangulation τ into one which intersects a curve isotopic to a_{12} into precisely two points. This can be done as follows:



Therefore we have:

$$\begin{aligned}\bar{\mathsf{F}}_{a_{12}} &= \bar{\mathsf{F}}(\tau, D_{a_{12}}\tau)\\ &= \mathrm{Ad}(T_{3\check{5}}T_{1\check{4}}T_{45}T_{46}T_{75})(T_{\check{6}7}).\end{aligned} \tag{46}$$

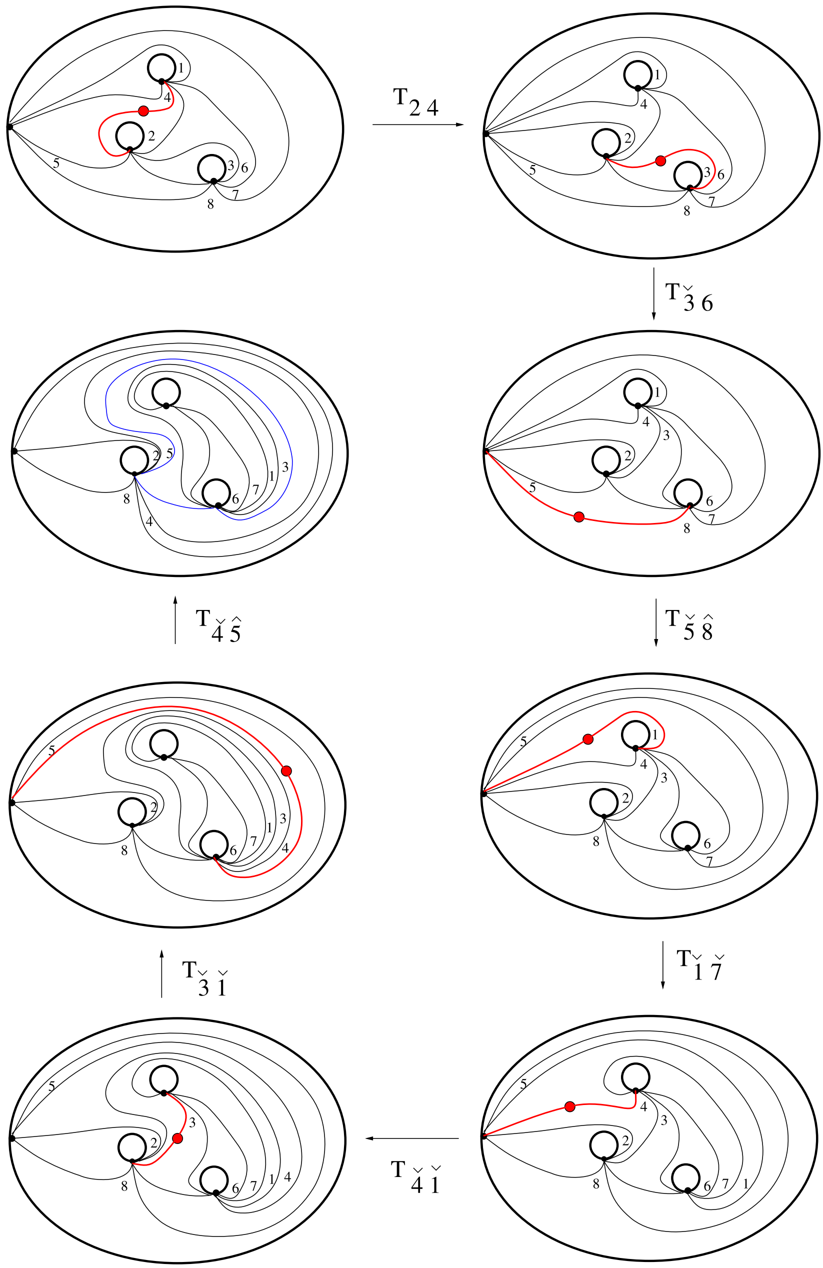
The following sequence of transformations



can be used to compute:

$$\begin{aligned}\bar{\mathsf{F}}_{a_{13}} &= \bar{\mathsf{F}}(\tau, D_{a_{13}}\tau)\\ &= \mathrm{Ad}(T_{\hat{8}\hat{7}}T_{\check{2}\check{6}}T_{45}T_{24}T_{\check{5}\hat{8}})(T_{\check{4}\check{5}}).\end{aligned} \tag{47}$$

Eventually use the transformations



in order to obtain:

$$\bar{\mathsf{F}}_{a_{23}} = \bar{\mathsf{F}}(\tau, D_{a_{23}}\tau) = \mathrm{Ad}(T_{24}T_{\check{3}6}T_{\check{5}\hat{8}}T_{\check{1}\check{7}}T_{\check{4}\check{1}}T_{\check{3}\check{1}}T_{\check{4}\hat{5}})(T_{3\check{5}}). \tag{48}$$

The next step is to simplify the expression of the last three Dehn twist, as follows:

$$\bar{\mathsf{F}}_{a_{12}} = T_{3\check{5}}T_{1\check{4}}T_{45}\underline{T_{46}T_{75}}T_{6\check{7}}\underline{\bar{T}_{75}\bar{T}_{46}}\bar{T}_{45}\bar{T}_{1\check{4}}\bar{T}_{3\check{5}} = T_{3\check{5}}T_{1\check{4}}T_{45}T_{75}\underline{T_{46}T_{6\check{7}}\bar{T}_{46}}\bar{T}_{75}\bar{T}_{45}\bar{T}_{1\check{4}}\bar{T}_{3\check{5}}$$

$$
\begin{aligned}
&= T_{3\check{5}}T_{1\check{4}}T_{45}T_{75}T_{\check{6}7}\underline{\bar{T}_{47}\bar{T}_{75}\bar{T}_{45}}\bar{T}_{1\check{4}}\bar{T}_{3\check{5}} = T_{3\check{5}}T_{1\check{4}}T_{45}\underline{T_{75}T_{\check{6}7}\bar{T}_{75}}T_{47}\bar{T}_{1\check{4}}\bar{T}_{3\check{5}}\\
&= T_{3\check{5}}T_{1\check{4}}T_{45}T_{\check{6}7}T_{\check{6}5}T_{47}\bar{T}_{1\check{4}}\bar{T}_{3\check{5}} = T_{1\check{4}}T_{\check{6}7}T_{3\check{5}}T_{45}T_{\check{6}5}\bar{T}_{3\check{5}}T_{47}\bar{T}_{1\check{4}}\\
&= T_{1\check{4}}T_{\check{6}7}\underline{T_{3\check{5}}T_{45}\bar{T}_{3\check{5}}}\,\underline{T_{3\check{5}}T_{\check{6}5}\bar{T}_{3\check{5}}}T_{47}\bar{T}_{1\check{4}} = T_{1\check{4}}T_{\check{6}7}T_{45}T_{3\check{4}}T_{\check{6}5}T_{3\hat{6}}T_{47}\bar{T}_{1\check{4}}.
\end{aligned}
$$

The first equality above corresponds to the commutativity of T_{ij} and T_{kl} in the case when the two sets of indices are disjoint, for each one of the underlined fragments. We further also made use of the symmetry property from (Remark 1.3, relation (25)) in order to be able to use the Pentagon relation, as in the last equality above. Specifically, the rightmost reduction consists of the following steps:

$$
T_{3\check{5}}T_{\check{6}5}\bar{T}_{3\check{5}} = \underline{T_{5\check{3}}T_{\check{6}5}}\bar{T}_{3\check{5}} = \underline{T_{\check{6}5}T_{3\hat{6}}T_{5\check{3}}}\bar{T}_{3\check{5}} = T_{\check{6}5}T_{3\hat{6}}T_{3\check{5}}\bar{T}_{3\check{5}} = T_{\check{6}5}T_{3\hat{6}}. \tag{49}
$$

Similar simplifications lead to:

$$
\begin{aligned}
\bar{\mathsf{F}}_{a_{13}} &= T_{\hat{8}\hat{7}}T_{\check{2}\check{6}}T_{45}\underline{T_{24}T_{\check{5}\hat{8}}}T_{\check{4}\check{5}}\underline{\bar{T}_{\check{5}\hat{8}}\bar{T}_{24}}\bar{T}_{45}\bar{T}_{\check{2}\check{6}}\bar{T}_{\hat{8}\hat{7}} = T_{\hat{8}\hat{7}}T_{\check{2}\check{6}}T_{45}T_{\check{5}\hat{8}}\underline{T_{24}T_{\check{4}\check{5}}\bar{T}_{24}}\bar{T}_{\check{5}\hat{8}}\bar{T}_{45}\bar{T}_{\check{2}\check{6}}\bar{T}_{\hat{8}\hat{7}}\\
&= T_{\hat{8}\hat{7}}T_{\check{2}\check{6}}\underline{T_{45}T_{\check{5}\hat{8}}T_{\check{4}\check{5}}}T_{5\check{2}}\bar{T}_{\check{5}\hat{8}}\bar{T}_{45}\bar{T}_{\check{2}\check{6}}\bar{T}_{\hat{8}\hat{7}} = \zeta\, T_{\hat{8}\hat{7}}T_{\check{2}\check{6}}T_{\check{5}\hat{8}}T_{4\hat{8}}P_{(45\hat{4})}\bar{T}_{5\check{2}}\bar{T}_{\check{5}\hat{8}}\bar{T}_{45}\bar{T}_{\check{2}\check{6}}\bar{T}_{\hat{8}\hat{7}}\\
&= \zeta\, T_{\hat{8}\hat{7}}T_{\check{2}\check{6}}T_{\check{5}\hat{8}}\underline{T_{4\hat{8}}\bar{T}_{\hat{4}\check{2}}\bar{T}_{4\hat{8}}}\bar{T}_{5\hat{4}}\bar{T}_{\check{2}\check{6}}\bar{T}_{\hat{8}\hat{7}}P_{(45\hat{4})} = \zeta\, T_{\hat{8}\hat{7}}\underline{T_{\check{2}\check{6}}T_{\check{5}\hat{8}}}T_{24}T_{2\hat{8}}\underline{\bar{T}_{5\hat{4}}\bar{T}_{\check{2}\check{6}}}\bar{T}_{\hat{8}\hat{7}}P_{(45\hat{4})}\\
&= \zeta\, T_{\hat{8}\hat{7}}T_{\check{5}\hat{8}}T_{\check{2}\check{6}}T_{24}T_{2\hat{8}}\bar{T}_{\check{2}\check{6}}\bar{T}_{5\hat{4}}\bar{T}_{\hat{8}\hat{7}}P_{(45\hat{4})} = \zeta\, T_{\hat{8}\hat{7}}T_{\check{5}\hat{8}}\underline{T_{\check{2}\check{6}}T_{24}\bar{T}_{\check{2}\check{6}}}\,\underline{T_{\check{2}\check{6}}T_{2\hat{8}}\bar{T}_{\check{2}\check{6}}}\bar{T}_{5\hat{4}}\bar{T}_{\hat{8}\hat{7}}P_{(45\hat{4})}\\
&= \zeta\, T_{\hat{8}\hat{7}}T_{\check{5}\hat{8}}T_{24}T_{\hat{4}\check{6}}T_{2\hat{8}}T_{\check{8}\check{6}}\bar{T}_{5\hat{4}}\bar{T}_{\hat{8}\hat{7}}P_{(45\hat{4})},\\
\bar{\mathsf{F}}_{a_{23}} &= T_{24}T_{\check{3}6}T_{\check{5}\hat{8}}T_{\check{1}\check{7}}\underline{T_{\check{4}\check{1}}T_{\check{3}\check{1}}}T_{\check{4}\hat{5}}T_{3\check{5}}\underline{\bar{T}_{\check{4}\hat{5}}\bar{T}_{\check{3}\check{1}}}\bar{T}_{\check{4}\check{1}}\bar{T}_{\check{1}\check{7}}\bar{T}_{\check{5}\hat{8}}\bar{T}_{\check{3}6}\bar{T}_{24}\\
&= T_{24}T_{\check{3}6}T_{\check{5}\hat{8}}T_{\check{1}\check{7}}T_{\check{4}\check{1}}T_{\check{4}\hat{5}}\underline{T_{\check{3}\check{1}}T_{3\check{5}}\bar{T}_{\check{3}\check{1}}}\bar{T}_{\check{4}\hat{5}}\bar{T}_{\check{4}\check{1}}\bar{T}_{\check{1}\check{7}}\bar{T}_{\check{5}\hat{8}}\bar{T}_{\check{3}6}\bar{T}_{24}\\
&= T_{24}T_{\check{3}6}T_{\check{5}\hat{8}}T_{\check{1}\check{7}}T_{\check{4}\check{1}}T_{\check{4}\hat{5}}T_{3\check{5}}\underline{T_{5\check{1}}\bar{T}_{\check{4}\hat{5}}\bar{T}_{\check{4}\check{1}}}\bar{T}_{\check{1}\check{7}}\bar{T}_{\check{5}\hat{8}}\bar{T}_{\check{3}6}\bar{T}_{24}\\
&= T_{24}T_{\check{3}6}T_{\check{5}\hat{8}}T_{\check{1}\check{7}}T_{\check{4}\check{1}}\underline{T_{\check{4}\hat{5}}T_{3\check{5}}\bar{T}_{\check{4}\hat{5}}}T_{5\check{1}}\bar{T}_{\check{1}\check{7}}\bar{T}_{\check{5}\hat{8}}\bar{T}_{\check{3}6}\bar{T}_{24}\\
&= T_{24}T_{\check{3}6}T_{\check{5}\hat{8}}T_{\check{1}\check{7}}T_{\check{4}\check{1}}T_{\check{4}\hat{5}}T_{3\check{5}}\underline{T_{5\check{1}}\bar{T}_{\check{4}\hat{5}}\bar{T}_{\check{4}\check{1}}}\bar{T}_{\check{1}\check{7}}\bar{T}_{\check{5}\hat{8}}\bar{T}_{\check{3}6}\bar{T}_{24}\\
&= T_{24}T_{\check{3}6}T_{\check{5}\hat{8}}T_{\check{1}\check{7}}T_{\check{4}\check{1}}T_{3\check{5}}T_{3\hat{4}}T_{5\check{1}}\bar{T}_{\check{1}\check{7}}\bar{T}_{\check{5}\hat{8}}\bar{T}_{\check{3}6}\bar{T}_{24}.
\end{aligned}
$$

Putting all these together we obtain:

$$
\begin{aligned}
&\bar{\mathsf{F}}_{a_{12}}\bar{\mathsf{F}}_{a_{23}}\bar{\mathsf{F}}_{a_{13}}\\
&\quad= \zeta\, T_{1\check{4}}T_{\check{6}7}T_{45}T_{3\check{4}}T_{\check{6}5}T_{3\hat{6}}T_{47}\bar{T}_{1\check{4}}T_{24}T_{\check{3}6}T_{\check{5}\hat{8}}T_{\check{1}\check{7}}T_{\check{4}\check{1}}T_{3\check{5}}T_{3\hat{4}}T_{5\check{1}}\bar{T}_{\check{1}\check{7}}\underline{\bar{T}_{\check{5}\hat{8}}}\bar{T}_{\check{3}6}\underline{\bar{T}_{24}}\,\underline{T_{\hat{8}\hat{7}}T_{\check{5}\hat{8}}}\underline{T_{24}}\\
&\qquad\times T_{\hat{4}\check{6}}T_{2\hat{8}}T_{\check{8}\check{6}}\bar{T}_{5\hat{4}}\bar{T}_{\hat{8}\hat{7}}P_{(45\hat{4})}\\
&\quad= \zeta\, T_{1\check{4}}T_{\check{6}7}T_{45}T_{3\check{4}}T_{\check{6}5}T_{3\hat{6}}T_{47}\bar{T}_{1\check{4}}T_{24}T_{\check{3}6}T_{\check{5}\hat{8}}T_{\check{1}\check{7}}T_{\check{4}\check{1}}T_{3\check{5}}T_{3\hat{4}}\underline{T_{5\check{1}}\bar{T}_{\check{1}\check{7}}}\bar{T}_{\check{3}6}\underline{T_{\check{5}\hat{7}}}T_{\hat{8}\hat{7}}T_{\hat{4}\check{6}}T_{2\hat{8}}T_{\check{8}\check{6}}\bar{T}_{5\hat{4}}\\
&\qquad\times \bar{T}_{\hat{8}\hat{7}}P_{(45\hat{4})}\\
&\quad= \zeta^2 T_{1\check{4}}T_{\check{6}7}T_{45}T_{3\check{4}}T_{\check{6}5}T_{3\hat{6}}T_{47}\bar{T}_{1\check{4}}T_{24}T_{\check{3}6}T_{\check{5}\hat{8}}\underline{T_{\check{1}\check{7}}T_{\check{4}\check{1}}}T_{3\check{5}}T_{3\hat{4}}\underline{\bar{T}_{\check{1}\check{7}}}T_{5\check{1}}\underline{P_{(5\check{7}\hat{5})}}\bar{T}_{\check{3}6}T_{\hat{8}\hat{7}}T_{\hat{4}\check{6}}T_{2\hat{8}}T_{\check{8}\check{6}}\\
&\qquad\times \bar{T}_{5\hat{4}}\bar{T}_{\hat{8}\hat{7}}P_{(45\hat{4})}\\
&\quad= \zeta^2 T_{1\check{4}}T_{\check{6}7}T_{45}T_{3\check{4}}T_{\check{6}5}T_{3\hat{6}}T_{47}\underline{\bar{T}_{1\check{4}}T_{24}}T_{\check{3}6}T_{\check{5}\hat{8}}\underline{T_{\check{4}\check{1}}}T_{\check{4}\check{7}}T_{3\check{5}}T_{3\hat{4}}T_{5\check{1}}\bar{T}_{\check{3}6}T_{\hat{4}\check{6}}T_{\hat{8}5}T_{2\hat{8}}T_{\check{8}\check{6}}\bar{T}_{\hat{8}5}\bar{T}_{\check{7}\hat{4}}\\
&\qquad\times P_{(5\check{7}\hat{5})}P_{(45\hat{4})}\\
&\quad= \zeta^2 T_{1\check{4}}T_{\check{6}7}T_{45}T_{3\check{4}}T_{\check{6}5}T_{3\hat{6}}\underline{T_{47}}T_{1\check{2}}\underline{T_{24}}T_{\check{3}6}T_{\check{5}\hat{8}}\underline{T_{\check{4}\check{7}}}T_{3\check{5}}T_{3\hat{4}}T_{5\check{1}}\bar{T}_{\check{3}6}T_{\hat{4}\check{6}}T_{\hat{8}5}T_{2\hat{8}}T_{\check{8}\check{6}}\bar{T}_{\hat{8}5}\bar{T}_{\check{7}\hat{4}}
\end{aligned}
$$

$$
\begin{aligned}
&\times P_{(5\check7\hat5)}P_{(45\hat4)}\\
&= \zeta^3 T_{1\check4}\underline{T_{\check67}T_{45}T_{3\check4}T_{\check65}T_{3\hat6}}\,\underline{T_{1\check2}}T_{24}T_{27}P_{(47\hat4)}T_{\check36}T_{\check5\hat8}T_{3\check5}T_{3\hat4}T_{5\check1}\bar T_{\check36}T_{\hat4\check6}T_{\hat85}T_{2\hat8}T_{\check8\check6}\bar T_{\hat85}\bar T_{\check7\hat4}\\
&\quad\times P_{(5\check7\hat5)}P_{(45\hat4)}\\
&= \zeta^3 T_{1\check4}T_{1\check2}\underline{T_{1\check6}T_{17}\bar T_{17}\bar T_{1\check6}}T_{\check67}\underline{T_{24}\bar T_{24}}T_{45}T_{3\check4}\underline{T_{\check65}T_{3\hat6}}\,\underline{T_{24}}T_{27}P_{(47\hat4)}T_{\check36}T_{\check5\hat8}T_{3\check5}T_{3\hat4}T_{5\check1}\bar T_{\check36}T_{\hat4\check6}\\
&\quad\times T_{\hat85}T_{2\hat8}T_{\check8\check6}\bar T_{\hat85}\bar T_{\check7\hat4}P_{(5\check7\hat5)}P_{(45\hat4)}\\
&= \zeta^3\bar{\mathsf F}_{a_2}\underline{\bar T_{17}\bar T_{\check67}\bar T_{1\check6}}T_{24}\underline{\bar T_{24}T_{45}T_{24}}\,\underline{\bar T_{24}T_{3\check4}T_{24}}T_{\check65}\underline{T_{3\hat6}T_{27}}\,\underline{P_{(47\hat4)}T_{\check36}}T_{\check5\hat8}T_{3\check5}T_{3\hat4}T_{5\check1}\bar T_{\check36}T_{\hat4\check6}\\
&\quad\times T_{\hat85}T_{2\hat8}T_{\check8\check6}\bar T_{\hat85}\bar T_{\check7\hat4}P_{(5\check7\hat5)}P_{(45\hat4)}\\
&= \zeta^3\bar{\mathsf F}_{a_2}T_{\check67}\bar T_{1\check6}T_{24}T_{25}\underline{T_{45}T_{2\check3}}T_{4\check3}\underline{T_{\check65}T_{27}T_{3\hat6}T_{\check36}}P_{(47\hat4)}T_{\check5\hat8}T_{3\check5}T_{3\hat4}T_{5\check1}\bar T_{\check36}T_{\hat4\check6}\\
&\quad\times T_{\hat85}T_{2\hat8}T_{\check8\check6}\bar T_{\hat85}\bar T_{\check7\hat4}P_{(5\check7\hat5)}P_{(45\hat4)}\\
&= \zeta^4\bar{\mathsf F}_{a_2}\underline{T_{24}T_{25}T_{2\check3}T_{26}}\,\underline{\bar T_{26}T_{\check67}}\bar T_{1\check6}T_{45}T_{4\check3}\underline{T_{27}}T_{\check65}P_{(\check6\check36)}P_{(47\hat4)}T_{\check5\hat8}T_{3\check5}T_{3\hat4}T_{5\check1}\bar T_{\check36}T_{\hat4\check6}\\
&\quad\times T_{\hat85}T_{2\hat8}T_{\check8\check6}\bar T_{\hat85}\bar T_{\check7\hat4}P_{(5\check7\hat5)}P_{(45\hat4)}\\
&= \zeta^4\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}T_{\check67}\bar T_{26}\bar T_{1\check6}T_{45}T_{4\check3}\underline{T_{\check65}T_{\check5\hat8}T_{\hat6\check5}}\,\underline{T_{6\hat7}}T_{5\check1}\underline{\bar T_{63}T_{\hat7\check3}}T_{\hat85}T_{2\hat8}T_{\check8\check3}\bar T_{\hat85}\bar T_{4\hat7}\\
&\quad\times P_{(\check6\check36)}P_{(47\hat4)}P_{(5\check7\hat5)}P_{(45\hat4)}\\
&= \zeta^6\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}T_{\check67}\bar T_{26}\underline{\bar T_{1\check6}}T_{45}T_{4\check3}T_{\check5\hat8}\underline{T_{\check6\hat8}T_{6\check1}}\bar T_{\hat53}T_{\check5\hat7}T_{\hat86}T_{2\hat8}T_{\check8\check7}\bar T_{\hat86}\bar T_{4\bar3}\\
&\quad\times P_{(\check656)}P_{(\hat3\hat7\check3)}P_{(\check6\check36)}P_{(47\hat4)}P_{(5\check7\hat5)}P_{(45\hat4)}\\
&= \zeta^6\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}T_{\check67}\bar T_{26}T_{45}T_{4\check3}T_{\check5\hat8}T_{1\hat8}\underline{T_{\check6\hat8}}\bar T_{\hat53}T_{\check5\hat7}\underline{T_{\hat86}}T_{2\hat8}T_{\check8\check7}\bar T_{\hat86}\bar T_{4\bar3}\\
&\quad\times P_{(\check656)}P_{(\hat3\hat7\check3)}P_{(\check6\check36)}P_{(47\hat4)}P_{(5\check7\hat5)}P_{(45\hat4)}\\
&= \zeta^7\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}T_{\check67}\bar T_{26}T_{45}T_{4\check3}T_{\check5\hat8}T_{1\hat8}\bar T_{\hat53}T_{\check5\hat7}P_{(\check6\hat86)}T_{\hat8}T_{\check8\check7}\bar T_{\hat86}\bar T_{4\bar3}\\
&\quad\times P_{(\check656)}P_{(\hat3\hat7\check3)}P_{(\check6\check36)}P_{(47\hat4)}P_{(5\check7\hat5)}P_{(45\hat4)}\\
&= \zeta^7\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}\underline{T_{\check67}}\,\underline{\bar T_{26}}T_{45}T_{4\check3}T_{\check5\hat8}T_{1\hat8}\bar T_{\hat53}\underline{T_{\check5\hat7}}\,\underline{T_{26}}\,\underline{T_{\hat6\check7}}\bar T_{6\check8}\bar T_{4\bar3}\\
&\quad\times P_{(\check6\hat86)}P_{(\check656)}P_{(\hat3\hat7\check3)}P_{(\check6\check36)}P_{(47\hat4)}P_{(5\check7\hat5)}P_{(45\hat4)}\\
&= \zeta^8\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}T_{45}T_{4\check3}T_{\check5\hat8}T_{1\hat8}\bar T_{\hat53}T_{\check5\hat7}T_{\check6\hat5}P_{(\check676)}\bar T_{6\check8}\bar T_{4\bar3}\\
&\quad\times P_{(\check6\hat86)}P_{(\check656)}P_{(\hat3\hat7\check3)}P_{(\check6\check36)}P_{(47\hat4)}P_{(5\check7\hat5)}P_{(45\hat4)}\\
&= \zeta^8\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}\bar{\mathsf F}_{a_0}\bar T_{\check8\check7}\bar T_{\check8\check1}\bar T_{\check8\check4}\underline{\bar T_{\check8\hat5}T_{45}}T_{4\check3}\underline{T_{\check5\hat8}}T_{1\hat8}\bar T_{\hat53}T_{\check5\hat7}T_{\check6\hat5}\bar T_{\hat7\check8}\bar T_{4\bar3}\\
&\quad\times P_{(\check676)}P_{(\check6\hat86)}P_{(\check656)}P_{(\hat3\hat7\check3)}P_{(\check6\check36)}P_{(47\hat4)}P_{(5\check7\hat5)}P_{(45\hat4)}\\
&= \zeta^8\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}\bar{\mathsf F}_{a_0}\bar T_{\check8\check7}\underline{\bar T_{\check8\check1}}\,\underline{\bar T_{\check8\check4}}\,\underline{T_{4\hat8}}T_{45}T_{4\check3}\underline{T_{1\hat8}}\bar T_{\hat53}T_{\check5\hat7}T_{\check6\hat5}\bar T_{\hat7\check8}\bar T_{4\bar3}\\
&\quad\times P_{(\check676)}P_{(\check6\hat86)}P_{(\check656)}P_{(\hat3\hat7\check3)}P_{(\check6\check36)}P_{(47\hat4)}P_{(5\check7\hat5)}P_{(45\hat4)}\\
&= \zeta^8\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}\bar{\mathsf F}_{a_0}\underline{\bar T_{\check8\check7}}T_{45}T_{4\check3}\bar T_{\hat53}\underline{T_{\check5\hat7}}T_{\check6\hat5}\underline{\bar T_{\hat7\check8}}\bar T_{4\bar3}\\
&\quad\times P_{(\check676)}P_{(\check6\hat86)}P_{(\check656)}P_{(\hat3\hat7\check3)}P_{(\check6\check36)}P_{(47\hat4)}P_{(5\check7\hat5)}P_{(45\hat4)}.
\end{aligned}
$$

In the previous lines we used both the Pentagon relation coupled with the symmetry property several times and the commutativity relations corresponding to the underlined fragments. Sometimes several simplifications are recorded in the same line, as in the first equality above where the underlined factors $\bar T_{24}$ and T_{24} commute with $T_{\hat8\hat7}T_{\check5\hat8}$ and therefore cancel each other, so that along with the first underlined factor we obtain a subproduct $\bar{T}_{\check{5}\hat{8}}T_{\hat{8}\hat{7}}T_{\check{5}\hat{8}}$ and the Pentagon relation can be applied.

Use now the identity:

$$\bar{T}_{\check{8}\check{7}}T_{\check{5}\hat{7}}\bar{T}_{\hat{7}\check{8}} = T_{\check{5}\hat{8}}T_{\check{5}\hat{7}}\bar{T}_{\check{8}\check{7}}\bar{T}_{\hat{7}\check{8}} = \zeta^{-1}T_{\check{5}\hat{8}}T_{\check{5}\hat{7}}P_{(\check{7}\check{8}\hat{7})}$$

and introduce above to find that:

$$\begin{aligned}
\bar{\mathsf{F}}_{a_{12}}\bar{\mathsf{F}}_{a_{23}}\bar{\mathsf{F}}_{a_{13}} &= \zeta^{7}\bar{\mathsf{F}}_{a_2}\bar{\mathsf{F}}_{a_1}\bar{\mathsf{F}}_{a_0}T_{45}T_{4\check{3}}\bar{T}_{\hat{5}3}T_{\check{5}\hat{8}}T_{\check{5}\hat{7}}T_{\check{6}\hat{5}}\bar{T}_{4\bar{3}} \\
&\quad \times P_{(\check{7}\check{8}\hat{7})}P_{(\check{6}76)}P_{(\check{6}\hat{8}6)}P_{(\check{6}56)}P_{(\hat{3}\hat{7}\check{3})}P_{(\check{6}\check{3}6)}P_{(47\hat{4})}P_{(5\check{7}\hat{5})}P_{(45\hat{4})} \\
&= \zeta^{7}\bar{\mathsf{F}}_{a_2}\bar{\mathsf{F}}_{a_1}\bar{\mathsf{F}}_{a_0}\bar{\mathsf{F}}_{a_3}\bar{T}_{3\hat{6}}\bar{T}_{3\hat{7}}\bar{T}_{3\hat{8}}\underline{\bar{T}_{3\check{5}}T_{45}T_{4\check{3}}}\bar{T}_{\hat{5}3}T_{\check{5}\hat{8}}T_{\check{5}\hat{7}}T_{\check{6}\hat{5}}\bar{T}_{4\bar{3}} \\
&\quad \times P_{(\check{7}\check{8}\hat{7})}P_{(\check{6}76)}P_{(\check{6}\hat{8}6)}P_{(\check{6}56)}P_{(\hat{3}\hat{7}\check{3})}P_{(\check{6}\check{3}6)}P_{(47\hat{4})}P_{(5\check{7}\hat{5})}P_{(45\hat{4})} \\
&= \zeta^{7}\bar{\mathsf{F}}_{a_2}\bar{\mathsf{F}}_{a_1}\bar{\mathsf{F}}_{a_0}\bar{\mathsf{F}}_{a_3}\bar{T}_{3\hat{6}}\bar{T}_{3\hat{7}}\bar{T}_{3\hat{8}}T_{45}\underline{\bar{T}_{3\check{5}}\bar{T}_{\hat{5}3}}T_{\check{5}\hat{8}}T_{\check{5}\hat{7}}T_{\check{6}\hat{5}}\bar{T}_{4\bar{3}} \\
&\quad \times P_{(\check{7}\check{8}\hat{7})}P_{(\check{6}76)}P_{(\check{6}\hat{8}6)}P_{(\check{6}56)}P_{(\hat{3}\hat{7}\check{3})}P_{(\check{6}\check{3}6)}P_{(47\hat{4})}P_{(5\check{7}\hat{5})}P_{(45\hat{4})} \\
&= \zeta^{6}\bar{\mathsf{F}}_{a_2}\bar{\mathsf{F}}_{a_1}\bar{\mathsf{F}}_{a_0}\bar{\mathsf{F}}_{a_3}\bar{T}_{3\hat{6}}\bar{T}_{3\hat{7}}\bar{T}_{3\hat{8}}T_{45}P_{(\check{5}3\hat{5})}T_{\check{5}\hat{8}}T_{\check{5}\hat{7}}T_{\check{6}\hat{5}}\bar{T}_{4\bar{3}} \\
&\quad \times P_{(\check{7}\check{8}\hat{7})}P_{(\check{6}76)}P_{(\check{6}\hat{8}6)}P_{(\check{6}56)}P_{(\hat{3}\hat{7}\check{3})}P_{(\check{6}\check{3}6)}P_{(47\hat{4})}P_{(5\check{7}\hat{5})}P_{(45\hat{4})} \\
&= \zeta^{6}\bar{\mathsf{F}}_{a_2}\bar{\mathsf{F}}_{a_1}\bar{\mathsf{F}}_{a_0}\bar{\mathsf{F}}_{a_3}\underline{\bar{T}_{3\hat{6}}}\,\underline{\bar{T}_{3\hat{7}}}\,\underline{\bar{T}_{3\hat{8}}}\,\underline{T_{45}}\,\underline{T_{3\hat{8}}}\,\underline{T_{3\hat{7}}T_{\check{6}\check{3}}}\,\underline{\bar{T}_{45}} \\
&\quad \times P_{(\check{5}3\hat{5})}P_{(\check{7}\check{8}\hat{7})}P_{(\check{6}76)}P_{(\check{6}\hat{8}6)}P_{(\check{6}56)}P_{(\hat{3}\hat{7}\check{3})}P_{(\check{6}\check{3}6)}P_{(47\hat{4})}P_{(5\check{7}\hat{5})}P_{(45\hat{4})} \\
&= \zeta^{6}\bar{\mathsf{F}}_{a_2}\bar{\mathsf{F}}_{a_1}\bar{\mathsf{F}}_{a_0}\bar{\mathsf{F}}_{a_3}.
\end{aligned}$$

Thus the lift of the lantern relation is ζ^6. Therefore we have to renormalize each right Dehn twist by taking $\widetilde{D_\alpha} = \zeta^{-6}\mathsf{F}_\alpha$, as claimed. □

The following lemma is a simple consequence of a deep result of Gervais from [13]:

Lemma 2.7. *Let $g \geqslant 2$ and $s \geqslant 0$. Then the group $\Gamma_{g,r}^s$ is presented as follows*:

1. *Generators are all Dehn twists D_a along the non-separating simple closed curves a on $\Sigma_{g,r}^s$.*
2. *Relations*:
   (a) *Braid type* 0*-relations*:

   $$D_a D_b = D_b D_a$$

   for each pair of disjoint non-separating simple closed curves a and b;

   (b) *Braid type* 1*-relations*:

   $$D_a D_b D_a = D_b D_a D_b$$

   for each pair of non-separating simple closed curves a and b which intersect transversely in one point;

   (c) *One lantern relation for a* 4*-hold sphere embedded in $\Sigma_{g,r}^s$ so that all boundary curves are non-separating*;
   (d) *One chain relation for a* 2*-holed torus embedded in $\Sigma_{g,r}^s$ so that all boundary curves are non-separating*;
   (e) *A puncture relation for each puncture.*

Proof. According to [13, Theorem B] we have a presentation of $\Gamma_{g,s+r}$ with the generators above and all but the puncture relations. Now, the kernel of $\Gamma_{g,s+r} \to \Gamma_{g,r}^{s}$ is the free Abelian group generated by the Dehn twists along the boundary curves to be pinched to punctures. Such a Dehn twist is expressed (using the lantern relation) by the left hand side of the puncture relation. This proves the claim. □

Proof of Proposition 2.1. According to the normalization coming from the braid relations and the lantern relations the images of the standard Dehn twist generators of the mapping class group are products of ζ^6 and elements T_{ij}, where i, j are the labels of the triangles (possibly with ˆ or ˇ). Thus the projective factors that appear belong to the subgroup A generated by ζ^6. The only non-trivial lift of a relation from Lemma 2.7 is the chain relation which lifts to ζ^{-72}. Set z for the element ζ^{-6} of $\widetilde{\Gamma_{g,r}^{s}}$. Then the presentation of the central extension $\widetilde{\Gamma_{g,r}^{s}}$ is given by the claimed relations. □

2.3. Cohomological consequences

Recall from [20, Corollary 4.4] that the 2-cohomology classes χ and e_i are defined for any $g \geqslant 3$, $s, r \geqslant 0$ and they span a free Abelian subgroup $\mathbb{Z}^{s+1} \subset H^2(\Gamma_{g,r}^{s})$. This inclusion is actually an isomorphism when $g \geqslant 4$.

We will denote by $\widehat{\Gamma_{g,r}^{s}}$ the group defined by the presentation given in Proposition 2.1, for all values of s, g, r. Thus, according to Proposition 2.1 the extension $\widehat{\Gamma_{g,r}^{s}}$ is isomorphic to $\widetilde{\Gamma_{g,r}^{s}}$ if $s \geqslant 4$ and $g \geqslant 2$.

Lemma 2.8. *If $g \geqslant 2$, then we have $c_{\widehat{\Gamma_{g,r}}} = 12\chi \in H^2(\Gamma_{g,r}; A)$.*

Proof. Consider first the case where ζ is not a root of unity, so that the group A is isomorphic to $\mathbb{Z}$. Gervais proved in [13, Theorem 3.6] that $\widehat{\Gamma_{g,r}}$ (namely, where $s = 0$) is isomorphic to the so-called p_1-central extension of $\Gamma_{g,r}$. Further in [13,21] the authors identified the class of the p_1-central extension of $\Gamma_{g,r}$ to the class 12χ and thus $c_{\widehat{\Gamma_{g,r}}} = 12\chi$.

Here is a more direct argument. Set $\Gamma_{g,r}(1)$ for the subgroup of $\widehat{\Gamma_{g,r}}$ generated by the lifts $\widetilde{D}_a$ of the Dehn twists and the central element $u = z^{12}$. Then $\Gamma_{g,r}(1)$ is the universal central extension considered by Harer (see [13,14]) and thus $c_{\Gamma_{g,r}(1)}$ is the generator χ of $H^2(\Gamma_{g,r}) \cong \mathbb{Z}$.

The cohomology class $c_{\Gamma_{g,r}(1)}$ is represented by some explicit 2-cocycle $C_{\Gamma_{g,r}(1)} : \Gamma_{g,r} \times \Gamma_{g,r} \to \mathbb{Z}$ which arises as follows. Let $S : \Gamma_{g,r} \to \Gamma_{g,r}(1)$ be a set-wise section. Let also $i : \ker(\Gamma_{g,r}(1) \to \Gamma_{g,r}) \to \mathbb{Z}$ be the group isomorphism defined by $i(u) = 1$. It is well known that the 2-cocycle

$$C_{\Gamma_{g,r}(1)}(x, y) = i\big(S(xy)S(x)^{-1}S(y)^{-1}\big) \in \mathbb{Z}$$

represents the cohomology class $c_{\Gamma_{g,r}(1)}$.

Let us construct now a 2-cocycle representing the extension $\widehat{\Gamma_{g,r}}$. Consider the set-wise section $\iota \circ S : \Gamma_{g,r} \to \widehat{\Gamma_{g,r}}$, where $\iota : \Gamma_{g,r}(1) \to \widehat{\Gamma_{g,r}}$ is the obvious inclusion. Let also $j : \ker(\widehat{\Gamma_{g,r}} \to \Gamma_{g,r}) \to \mathbb{Z}$ be the isomorphism given by $j(z) = 1$. Then

$$C_{\widehat{\Gamma_{g,r}}}(x, y) = j\big((\iota \circ S)(xy)(\iota \circ S)(x)^{-1}(\iota \circ S)(y)^{-1}\big) = j\big(\iota\big(S(xy)S(x)^{-1}S(y)^{-1}\big)\big) \in \mathbb{Z}$$

is a 2-cocycle representing $c_{\widehat{\Gamma_{g,r}}}$. Since $j(\iota(u)) = j(z^{12}) = 12i(u)$ and $S(xy)S(x)^{-1}S(y)^{-1}$ belongs to the cyclic subgroup of $\Gamma_{g,r}(1)$ generated by u, it follows that

$$C_{\widehat{\Gamma_{g,r}}}(x,y) = 12C_{\Gamma_{g,r}(1)}$$

and thus $c_{\widehat{\Gamma_{g,r}}} = 12\chi$, where χ is one fourth of the Meyer signature class, which is a generator of $H^2(\Gamma_{g,1}) \subset H^2(\Gamma_g^1)$.

When ζ is a root of unity of order N then the class of the extension $\widehat{\Gamma_{g,r}}$ is the image of 12χ in $H^2(\Gamma_{g,r}; \mathbb{Z}/N\mathbb{Z})$ by the reduction mod N. □

The next step is to prove a similar statement when the number s of punctures is non-zero.

Definition 2.4. For $(m_1, m_2, \ldots, m_s) \in \mathbb{Z}^s$ let $\Gamma_{g,r}^s(m_1, m_2, \ldots, m_s)$ be the central extension of $\Gamma_{g,r}^s$ by A having the following presentation:

1. Generators are the $\widetilde{D_\alpha}$, where D_α are Dehn twist generators of $\Gamma_{g,r}^s$ and the central element z of the same order as ζ^{-6};
2. Relations are as follows. For each puncture p_i the lift of the corresponding puncture relation reads:
$$\widetilde{D_{a_1(i)}}^{-1}\widetilde{D_{a_2(i)}}^{-1}\widetilde{D_{a_3(i)}}^{-1}\widetilde{D_{a_{12}(i)}}\widetilde{D_{a_{13}(i)}}\widetilde{D_{a_{23}(i)}} = z^{m_i}$$
where $\widetilde{D_a}$ are lifts of Dehn twists. Furthermore the chain and lantern relations have trivial lifts.

Proposition 2.2. *Suppose that $g \geqslant 0$. Then $c_{\Gamma_{g,r}^s(m_1,\ldots,m_s)} \in A^{n+1} \subset H^2(\Gamma_{g,r}^s; A)$ is the vector $m_1e_1 + m_2e_2 + \cdots + m_se_s$, where e_i is the Euler class of the i-th puncture.*

Proof. This is folklore. Consider first that ζ is not a root of unity. Let $\Sigma_{g,r+1;i}^{s-1}$ denote the subsurface of $\Sigma_{g,r}^s$ obtained by removing a one-punctured disk centered at the puncture p_i and thus creating a new boundary component b_i. We have then a central extension

$$\mathbb{Z} \to \Gamma_{g,r+1;i}^{s-1} \to \Gamma_{g,r}^s \to 1$$

induced by the inclusion map $\Sigma_{g,r+1;i}^{s-1} \hookrightarrow \Sigma_{g,r}^s$. It is well known that its cohomology class is $c_{\Gamma_{g,r+1;i}^{s-1}} = e_i$.

Lemma 2.9. *The extension $\Gamma_{g,r+1;i}^{s-1}$ is isomorphic to $\Gamma_{g,r}^s(0,\ldots,1,0\ldots,0)$, where 1 is on the i-th position.*

Proof. There is a natural set-wise section $S_i : \Gamma_{g,r}^s \to \Gamma_{g,r+1;i}^{s-1}$, given by $S_i(D_\alpha) = D_\alpha$, for any Dehn twist D_α. In order to make sense, we might suppose that a simple closed curve α disjoint from the puncture p_i is actually disjoint from b_i so that it lies within $\Sigma_{g,r+1;i}^{s-1}$.

Braid, chain and lantern relations are then lifted trivially. A puncture relation at p_j is lifted trivially if $j \neq i$. Consider next a puncture relation at p_i in $\Sigma_{g,r}^s$, which is supported on some subsurface $\Sigma_{0,3}^1$. The three boundary curves of $\Sigma_{0,3}^1$ lie within $\Sigma_{g,r+1;i}^{s-1}$ and together with b_i bound a 4-holed sphere in $\Sigma_{g,r+1;i}^{s-1}$. The lantern relation associated to this 4-holed sphere on $\Sigma_{g,r+1;i}^{s-1}$ is then the lift of the puncture relation at p_i. The Dehn twist along b_i is the generator z of the central factor $\ker(\Gamma_{g,r+1;i}^{s-1} \to \Gamma_{g,r}^s)$. Thus the lift of a puncture relation at p_i is the factor z. □

Lemma 2.10. *Let $L_{\mathbf{m}} : \mathbb{Z}^s \to \mathbb{Z}$ denote the linear map $L_{\mathbf{m}}(n_1, \dots, n_s) = \sum_{i=1}^{s} m_i n_i$, where $\mathbf{m} = (m_1, \dots, m_s)$. Consider the central extension*

$$1 \to \mathbb{Z}^s \to \Gamma_{g,r+s} \to \Gamma_{g,r}^s \to 1.$$

Then the map $L_{\mathbf{m}}$ induces a quotient of $\Gamma_{g,r+s}$, which is a central extension $\Gamma_{g,r}^s(\mathbf{m})$ of $\Gamma_{g,r}^s$ by $\mathbb{Z}$ which is isomorphic to $\Gamma_{g,r}^s(m_1, m_2, \dots, m_s)$ and gives rise to the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}^s & \longrightarrow & \Gamma_{g,r+s} & \longrightarrow & \Gamma_{g,r}^s & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle L_{\mathbf{a}}} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathbf{1}} & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Gamma_{g,r}^s(\mathbf{m}) & \longrightarrow & \Gamma_{g,r}^s & \longrightarrow & 1
\end{array}$$

Proof. The class of the central extension $c_{\Gamma_{g,r+s}}$ belongs to $H^2(\Gamma_{g,r}^s; \mathbb{Z}^s) = \bigoplus_s H^2(\Gamma_{g,r}^s, \mathbb{Z})$. By functoriality we derive that $c_{\Gamma_{g,r+s}} = (e_1, e_2, \dots, e_s) \in H^2(\Gamma_{g,r}^s; \mathbb{Z}^s)$. Then the class $c_{\Gamma_{g,r}^s(\mathbf{a})}$ is the image of $c_{\Gamma_{g,r+s}}$ into $H^2(\Gamma_{g,r}^s)$ by the homomorphism of coefficients rings $L_{\mathbf{m}} : \mathbb{Z}^s \to \mathbb{Z}$. There is an obvious set-wise section S defined in the same way as the S_i from above. Then $c_{\Gamma_{g,r+s}}$ is the class of the 2-cocycle $L_{\mathbf{m}}C$, where C is the 2-cocycle associated to S and so

$$\begin{aligned}
L_{\mathbf{m}}C(x, y) &= \pi\bigl(S(x)^{-1}S(y)^{-1}S(xy)\bigr) = L_{\mathbf{m}}\bigl(\bigl(S_i(x)^{-1}S_i(y)^{-1}S_i(xy)\bigr)_{i=1,s}\bigr) \\
&= \sum_{i=1}^{s} m_i C_i(x, y)
\end{aligned}$$

where C_i is the 2-cocycle associated to S_i. Since the class of C_i is e_i it follows that the class of $L_{\mathbf{m}}C$ is $\sum_{i=1}^{s} m_i e_i$.

On the other hand the lifts of relations in $\Gamma_{g,r}^s(\mathbf{m})$ are the same as in $\Gamma_{g,r}^s(m_1, \dots, m_s)$ and thus they are isomorphic. In fact the lifts of braid, chain and lantern relations to $\Gamma_{g,r+s}$ are trivial. The lift of a puncture relation at p_i is the i-th generator of the central factor $\mathbb{Z}^s$, according to Lemma 2.9. Therefore its image into $\Gamma_{g,r}^s(\mathbf{m})$ is z^{m_i}, namely the lift of the puncture relation in $\Gamma_{g,r}^s(m_1, \dots, m_s)$. □

When ζ is a root of unity the extensions by $\mathbb{Z}$ above are replaced by extensions by $\mathbb{Z}/N\mathbb{Z}$ and all arguments go through without essential modifications.

This proves the proposition. □

Proof of the theorem. Assume first that A is cyclic infinite. Consider the operation $\otimes$ (which is a push-out, or a fibered product) on central extensions defined as follows. If $f_i : G_i \to G$ are the projections homomorphisms of the central extensions G_i of G by $\mathbb{Z}$ then $G_1 \otimes G_2$ is the extension $f_1^* G_2$ (or equivalently $f_2^* G_1$) of G by $\mathbb{Z}^2$. The class $c_{G_1 \otimes G_2} \in H^2(G, \mathbb{Z}^2)$ is the direct sum of the classes $c_{G_i} \in H^2(G, \mathbb{Z})$ under the identification of $H^2(G, \mathbb{Z}^2)$ with the sum of two copies of $H^2(G, \mathbb{Z})$.

Let f denote the surjective homomorphism $f : \Gamma_{g,r}^s \to \Gamma_{g,r}$. Consider then the central extension

$$1 \to \mathbb{Z}^2 \to f^*(\widehat{\Gamma_{g,r}}) \otimes \Gamma_{g,r}^s(1, 1, \dots, 1) \to \Gamma_{g,r}^s \to 1.$$

Using the map $L : \mathbb{Z}^2 \to \mathbb{Z}$ given by $L(x, y) = x + y$ we find a quotient of $f^*(\widehat{\Gamma_{g,r}}) \otimes \Gamma^s_{g,r}(1, 1, \dots, 1)$, which is a central extension by $\mathbb{Z}$ isomorphic to $\widehat{\Gamma^s_{g,r}}$. In fact, there is a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & f^*(\widehat{\Gamma_{g,r}}) \otimes \Gamma^s_{g,r}(1,1,\dots,1) & \longrightarrow & \Gamma^s_{g,r} & \longrightarrow & 1 \\ & & \downarrow L & & \downarrow \pi & & \downarrow \mathbf{1} & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widehat{\Gamma^s_{g,r}} & \longrightarrow & \Gamma^s_{g,r} & \longrightarrow & 1 \end{array}$$

The central extension from the lower row is isomorphic to $\widehat{\Gamma^s_{g,r}}$ because the lifts of relations are the same. Braid and lantern relations lift trivially. Chain relations lift to z^{12} in $f^*(\widehat{\Gamma_{g,r}})$ and trivially to $\Gamma^s_{g,r}(1, 1, \dots, 1)$ and thus the image of the lift by L (or π) is z^{12}. Puncture relations at p_i lift trivially to $f^*(\widehat{\Gamma_{g,r}})$ and to z in the factor $\Gamma^s_{g,r}(1, 1, \dots, 1)$, so that its image by L (or π) is z. As a consequence of this description the class $c_{\widehat{\Gamma^s_{g,r}}}$ is the image by L of the class of $f^*(\widehat{\Gamma_{g,r}}) \otimes \Gamma^s_{g,r}(1, 1, \dots, 1)$, namely $c_{f^*(\widehat{\Gamma_{g,r}})} + c_{\Gamma^s_{g,r}(1,1,\dots,1)}$.

On the other hand, by functoriality, the class $c_{f^*(\widehat{\Gamma_{g,r}})}$ is $f^*(12\chi) = 12\chi \in H^2(\Gamma^s_{g,r})$, because the map f^* is the standard embedding of $H^2(\Gamma_{g,r}) = \mathbb{Z}\chi$ into $H^2(\Gamma^s_{g,r})$. Proposition 2.2 proves the theorem for $g \geqslant 3$.

When $g = 2$ one does not know the group $H^2(\Gamma^s_{2,r})$, but for $s = 0$ and $r \leqslant 1$. Nevertheless, the classes χ and e_j are still defined. It suffices to prove that:

Lemma 2.11. *The subgroup of $H^2(\Gamma^s_{2,r})$ generated by χ and $e_1, \dots, e_s$ is isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}^s$.*

Proof. By the universal coefficients theorem we have

$$1 \to H_1(\Gamma^s_{2,r}) \to H^2(\Gamma^s_{2,r}) \to \mathrm{Hom}(H_2(\Gamma^s_{2,r}), \mathbb{Z}) \to 1.$$

From [20, Proposition 1.6] we have $H_1(\Gamma^s_{2,r}) = \mathbb{Z}/10\mathbb{Z}$. The Meyer class χ in genus 2 is one half of the class of Meyer's cocycle from [22] and it generates the image of $H_1(\Gamma^s_{2,r})$ into $H^2(\Gamma^s_{2,r})$.

Consider next the extensions $\Gamma^s_{2,r}(\mathbf{m})$ for integral vectors $\mathbf{m}$. According to the previous description lifts of puncture relations are of the form z^{m_i}. Suppose that there exists an isomorphism between the extensions $\Gamma^s_{2,r}(\mathbf{m})$ and $\Gamma^s_{2,r}(\mathbf{u})$. Such an isomorphism of extensions should send $\widetilde{D_\alpha}$ into $z^{n(\alpha)}\widetilde{D_\alpha}$, because it has to induce the identity on $\Gamma^s_{2,r}$. Since lifts of braid relations are trivial in both extension groups it follows that $n(\alpha) = n$ does not depend on the non-separating curve α. But puncture relations are homogeneous, and so they do not depend on n. This shows that $\mathbf{m} = \mathbf{u}$. In particular the classes e_i span a free $\mathbb{Z}$-submodule of $H^2(\Gamma^s_{2,r})$.

Since the class χ is of order 10 and both subgroups $\mathbb{Z}/10\mathbb{Z}$ (generated by χ) and $\mathbb{Z}^s$ (generated by $e_1, \dots, e_s$) inject into $H^2(\Gamma^s_{2,r})$, the claim follows. □

Then the arguments used above for $g \geqslant 3$ work as well for $g = 2$ and the theorem follows. When ζ is a root of unity the associated cohomology class is the reduction mod N of the corresponding integral cohomology class. □

Proof of Corollary 0.2. Consider the extension $\widehat{\Gamma_{g,r+s}}$ of class 12χ. The corollary claims that there is an exact sequence:

$$1 \to A^{s-1} \to \widehat{\Gamma_{g,r+s}} \to \widehat{\Gamma_{g,r}^s} \to 1.$$

This can be verified by using the explicit presentations of the two groups involved. The kernel is generated by the products of two opposite Dehn twists on the s blown up boundary components. □

Proof of Corollary 0.3. It suffices to understand the map $H^2(\Gamma_{g,r}^s; A) \to H^2(\Gamma_{g,r}^s, \mathbb{C}^*)$ induced by $z \to \zeta^{-6}$. This map is injective, when $g \geqslant 3$.

The Universal Coefficients Theorem states that, for any Abelian group W, the following exact sequence is exact:

$$1 \to \mathrm{Ext}\big(H_0\big(\Gamma_{g,r}^s\big), W\big) \to H^1\big(\Gamma_{g,r}^s; W\big) \to \mathrm{Hom}\big(H_1\big(\Gamma_{g,r}^s\big), W\big) \to 1.$$

Now $\mathrm{Ext}(\mathbb{Z}, W) = 0$, for any Abelian group W. This implies that $H^1(\Gamma_{g,r}^s; \mathbb{C}^*) = H^1(\Gamma_{g,r}^s; \mathbb{C}^*/A) = 0$, if $g \geqslant 3$. From the Bockstein exact sequence

$$H^1\big(\Gamma_{g,r}^s; \mathbb{C}^*\big) \to H^1\big(\Gamma_{g,r}^s; \mathbb{C}^*/A\big) \xrightarrow{\beta} H^2\big(\Gamma_{g,r}^s; A\big) \xrightarrow{\nu} H^2\big(\Gamma_{g,r}^s; \mathbb{C}^*\big)$$

we derive the claim.

When $g = 2$ the Universal Coefficients Theorem shows, as above, that $H^1(\Gamma_{2,r}^s; \mathbb{C}^*) = \mathrm{Hom}(H_1(\Gamma_{2,r}^s), \mathbb{C}^*)$ and $H^1(\Gamma_{2,r}^s; \mathbb{C}^*/A) = \mathrm{Hom}(H_1(\Gamma_{2,r}^s), \mathbb{C}^*/A)$. Thus $H^1(\Gamma_{2,r}^s; \mathbb{C}^*) = \mathrm{Hom}(\mathbb{Z}/10\mathbb{Z}, \mathbb{C}^*) = U_{10}$, where U_{10} is the subgroup of roots of unity of order 10. The last isomorphism sends a homomorphism into its value on the generator 1. Next $H^1(\Gamma_{2,r}^s; \mathbb{C}^*/A) = \mathrm{Hom}(\mathbb{Z}/10\mathbb{Z}, \mathbb{C}^*/A) = U_{10} \times A/10A$. To explain the last isomorphism, each element $f \in H^1(\Gamma_{2,r}^s; \mathbb{C}^*/A)$ is determined by its value $f(1) = As$, for some $s \in \mathbb{C}^*$. Here $s^{10} = a^n \in A$, where a is the generator of A. Fix some 10-th root $a^{1/10} \in \mathbb{C}^*$ of the generator of A. Then the isomorphism above associates to f the element $(sa^{-n/10}, s^{10}) \in U_{10} \times A/10A$, which is well-defined and independent of the choice of the representative s in its A-coset. In particular the map $H^1(\Gamma_{2,r}^s, \mathbb{C}^*) \to H^1(\Gamma_{2,r}^s, \mathbb{C}^*/A)$ sends U_{10} onto the factor U_{10} of the second group.

Let $\hat{f}$ be a lift of f to $\hat{f} : \mathbb{Z}/10\mathbb{Z} = H_1(\Gamma_{g,r}^s) \to \mathbb{C}^*$, for instance $\hat{f}(k) = s^k$, where $k \in \mathbb{Z}/10\mathbb{Z}$. Then $F(k_1, k_2) = \hat{f}(k_1)\hat{f}(k_2)\hat{f}(k_1k_2)^{-1} \in A$ is a 2-cocycle on $H_1(\Gamma_{2,r}^s)$ with values in A. The pull-back in $H^2(\Gamma_{2,r}^s, A)$ of the class of F by the map $\Gamma_{2,r}^s \to H_1(\Gamma_{2,r}^s)$ is the element $\beta(f)$. It is well known that $H^2(\mathbb{Z}/10\mathbb{Z}, A) = A/10A$ is generated by the Euler class. Specifically, the cohomology class of the 2-cocycle F in $H^2(\mathbb{Z}/10\mathbb{Z}, A)$ is the element $s^{10} \in A/10A$, under the previous isomorphism.

The Universal Coefficients Theorem shows that

$$1 \to \mathrm{Ext}\big(H_1\big(\Gamma_{2,r}^s\big), A\big) \to H^2\big(\Gamma_{2,r}^s; A\big) \to \mathrm{Hom}\big(H_2\big(\Gamma_{2,r}^s\big), A\big) \to 1.$$

Further $\mathrm{Ext}(H_1(\Gamma_{g,r}^s), A) = A/10A$ is generated by the class χ (as an A-valued cohomology class). Using the definition of Ext one identifies the class χ with the generator of $H^2(\mathbb{Z}/10\mathbb{Z}; A)$. This implies that the image of β is the subgroup generated by χ within $H^2(\Gamma_{2,r}^s; A)$. Then Corollary 0.3 follows. □

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