On the wgsc and qsf tameness conditions for finitely presented groups

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Abstract. A finitely presented group is weakly geometrically simply connected (wgsc) if it is the fundamental group of some compact polyhedron whose universal covering is wgsc, i.e., it has an exhaustion by compact connected and simply connected sub-polyhedra. We show that this condition is almost-equivalent to Brick's qsf property, which amounts to finding an exhaustion approximable by finite simply connected complexes, and also to the tame combability introduced and studied by Mihalik and Tschantz. We further observe that a number of standard constructions in group theory yield qsf groups and analyze specific examples. We show that requiring the exhaustion be made of metric balls in some Cayley complex is a strong constraint, not satisfied by general qsf groups. In the second part of this paper we give sufficient conditions under which groups which are extensions of finitely presented groups by finitely generated (but infinitely presented) groups are qsf. We prove, in particular, that the finitely presented HNN extension of the Grigorchuk group is qsf.

1. Introduction

Casson and Poenaru [40], [25] studied geometric conditions on the Cayley graph of a finitely presented group implying that the universal covering of a compact 3-manifold with given fundamental group is $\mathbb{R}^3$. The proof involves approximating the universal covering by compact, simply connected three-manifolds. This condition was then adapted for arbitrary spaces and finitely presented groups by S. Brick in [8] (see also [42]) under the name *quasi-simply filtered* (abbreviated *qsf* below).

We consider here a related and apparently stronger notion, called *weak geometric simple connectivity* (abbreviated *wgsc*), which came out from the study of the geometric simple connectivity of open manifolds in [22]. Specifically, a polyhedron is wgsc if it admits an exhaustion by compact connected and simply connected polyhedra. The interest of such a strengthening is that it is easier to prove that specific high dimensional polyhedra are not wgsc rather than not qsf. In fact, a major difficulty

encountered when searching for examples of manifolds which are not wgsc is that one has to show that no exhaustion has the required properties, while, in general, non-compact manifolds are precisely described by means of one specific exhaustion. Thus, one needs a method to decide whether a space is not wgsc out of a given (arbitrary) exhaustion. We are not aware about such methods in the qsf setting. However, the criterion given in [22] permits to answer this question for the wgsc condition, at least for non-compact manifolds of high dimensions.

A central issue in geometric group theory is to study classes of groups with various properties of topological nature. The topological properties in question are imported from the realm of infinite complexes by means of the following recipe, which was first used on a large scale by Gromov. Say that a finitely presented (in general infinite) group has a certain property if the universal covering of some finite complex with this fundamental group has the required property. In this setting we can speak about the qsf (or wgsc) of finitely presented groups. In this respect we have three levels of equivalence relations among topological properties. First, the usual one concerning (more or less) arbitrary CW complexes. Second, the *almost-equivalence* which concerns only universal coverings of finite complexes, i.e., finitely presented groups. At last we have the quasi-isometry equivalence relation for finitely presented groups. In this paper we will mostly consider the almost-equivalence of various tameness properties, which will also permit us to draw conclusions about their quasi-isometry invariance. Notice however that qsf and wgsc have different flavors. If one universal covering of a finite complex with given fundamental group is qsf then all such universal coverings are qsf and, in particular, this holds for every Cayley complex. Thus the qsf property is independent on the presentation used in the construction of the Cayley complex. This is not anymore true for the wgsc property. There are examples of presentations of a wgsc group which lead to non wgsc Cayley complexes. However we will see that these two properties define the same class of groups in the sense that a group is qsf if and only if it is wgsc. The qsf is then a group property which is presentation independent and almost-equivalent to the wgsc.

The wgsc property should be compared to a tameness condition which is central in non-compact manifold theory, namely the *simple connectivity at infinity*. Roughly speaking the simple connectivity at infinity expresses the fact that loops which are far away should bound disks which are far away. This topological property have been used for characterizing Euclidean spaces as being the contractible manifolds that are simply connected at infinity by Siebenmann, Stallings and Freedman. Moreover, the simple connectivity at infinity is much stronger than the wgsc in dimensions at least 4, and in particular for finitely presented groups. In fact, M. Davis ([13]) constructed examples of aspherical manifolds whose universal coverings are different from $\mathbb{R}^n$ (for $n \geq 4$). Further one understood that these examples are quite common (see [15]). The groups in these examples are finitely generated Coxeter groups, which act properly co-compactly on some CAT(0) complexes and thus they are wgsc.

In order to give an unified proof that many classes of groups are qsf Mihalik and Tschantz ([38]) introduced the related notion of *tame* 1-*combings* for groups. An usual

combing for a 2-complex is the choice of paths in the 1-skeleton joining a basepoint vertex to every other vertex. Groups whose Cayley graphs admit nice (e.g., bounded) combings have good algorithmic properties, like automatic groups and hyperbolic groups and were the subject of extensive study in the last twenty years. Further a 1-combing corresponds to one dimension higher, namely, to a system of paths joining a basepoint vertex to every point of the 1-skeleton. We refer to the next section for the precise definition (see 2.4) of the enhanced notion of tame 1-combing of 2-complexes (and groups). One of the main results of [38] is that tame 1-combable groups (and in particular asynchronously automatic groups and semi-hyperbolic groups) are actually qsf.

Our aim is to pursue further the study of the qsf condition for groups. The first part of this paper is devoted to finding characterizations of the qsf by means of methods from high dimensional manifold theory. Our first result is the following.

Theorem 1.1. *The wgsc, qsf and tame 1-combability conditions are almost-equivalent topological properties of finitely presented groups.*

In particular, using the results from [6], we obtain:

Corollary 1.2. *A group quasi-isometric to a qsf finitely presented group is qsf.*

In other words, the qsf property of groups is *geometric*. We apply these results to analyze several interesting classes of groups and derive additional examples of qsf groups.

A natural question is whether there is some natural simply connected exhaustion for a wgsc group. A possible candidate is to consider the word metric on the Cayley complex associated to a group presentation and the associated exhaustion by metric balls. We will show in Section 4:

Theorem 1.3. *Finitely presented groups admitting a Cayley complex whose metric balls have fundamental groups generated by loops of uniformly bounded length have linear connectivity radius and solvable word problem.*

In particular such groups are strongly constrained and there are examples of wgsc groups not satisfying these conditions. Therefore the simply connected exhaustions of wgsc Cayley complexes are far from being the ones by metric balls. As application we will give a simple proof for the fact that finitely presented groups admitting complete geodesic rewriting systems are qsf.

The starting point of Section 5 is the result of Brick and Mihalik from [7] which states that extensions of infinite finitely presented groups by finitely presented groups are qsf. This is the group theoretical analog of the fact that products of contractible manifolds are homeomorphic to the Euclidean space. Using the same methods we can prove the following:

Theorem 1.4. *An ascending HNN extension of a finitely presented group is qsf.*

We investigate further extensions of infinite finitely presented groups by suitable infinitely presented groups, for instance torsion groups. The second main result of Section 5 is Theorem 5.1, which gives sufficient (too technical to state here) conditions for such extensions to be qsf.

Then we consider in detail the case of the Grigorchuk group and of its finitely presented HNN extension constructed in ([27], [28]; see also [16]). The main result of the second part (see Section 6) is:

Theorem 1.5. *The finitely presented HNN extension of the Grigorchuk group is qsf.*

These methods could be used in slightly more general situations in order to cover large classes of finitely presented extensions of branch groups having endomorphic presentations, as defined by Bartholdi in [3]. However, the present approach does not permit to prove the qsf of all such extensions, without some additional condition.

At this point we wish to emphasize the difference between the geometric invariants of discrete groups and those of topological nature. Geometric invariants are sensitive to cut and paste operations and thus algebraic constructions can provide a large variety of examples. For instance the set of exponents of polynomial isoperimetric Dehn functions of finitely presented groups is a dense subset of $[2, \infty)$. These correspond to distinct quasi-isometry classes of groups. On the other hand, topological properties are quite stable and thus can be satisfied by very large classes of groups. Two typical cases are the semi-stability at infinity (see e.g. [37]) and the property that $H^2(G, \mathbb{Z}G)$ is free abelian. It is still unknown whether all finitely presented groups satisfy either one of these two properties.

In the same spirit there are still no known examples of finitely presented groups which are not qsf (see [42]). Notice that fundamental groups of compact 3-manifolds are qsf, but the only proof of that is as a consequence of the Thurston geometrization conjecture (settled by Perelman). If non qsf groups do exist they would lay at the opposite extreme to hyperbolic and non-positively curved groups and thus they should be highly non generic. A related question is whether the fundamental group of a closed aspherical manifold could act properly (not necessarily co-compactly) on some non-wgsc contractible manifold, like those described in [22].

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2. Preliminaries on tameness conditions for groups

2.1. The wgsc. The following definition due to C. T. C. Wall came out from the work of S. Smale on the Poincaré conjecture and, more recently, in the work of V. Poenaru ([40]). Moreover, it has been revealed as especially interesting in the non-compact situation, in connection with uniformization problems (see [22]).

Definition 2.1. A non-compact manifold, which might have nonempty boundary, is *geometrically simply connected* (abbreviated *gsc*) if it admits a proper handlebody decomposition without 1-handles, or equivalently, in which every 1-handle is in canceling position with a 2-handle. Alternatively, there exists a proper Morse function $f : W \to \mathbb{R}$, whose critical points are contained within $\text{int}(W)$ such that

(1) f has no index one critical points; and

(2) the restriction $f|_{\partial W} : \partial W \to \mathbb{R}$ is still a proper Morse function without non-fake index one critical points. The non-fake critical points of $f|_{\partial W}$ are those for which the gradient vector field grad f points towards the interior of W, while the fake ones are those for which grad f points outwards.

The gsc condition was shown to be a powerful tameness condition for open three-manifolds and four-manifolds in a series of papers by Poenaru starting with [40].

Remark 2.1. Handle decompositions are known to exist for all manifolds in the topological, PL and smooth settings, except in the case of non-smoothable topological 4-manifolds. Notice that open 4-manifolds are smoothable.

Manifolds and handlebodies considered below are PL.
One has the following combinatorial analog of the gsc for polyhedra:

Definition 2.2. A non-compact polyhedron P is *weakly geometrically simply connected* (abbreviated *wgsc*) if $P = \bigcup_{j=1}^{\infty} K_j$, where $K_1 \subset K_2 \subset \cdots \subset K_j \subset \cdots$ is an exhaustion by compact connected sub-polyhedra with $\pi_1(K_j) = 0$. Alternatively, any compact sub-polyhedron is contained in a simply connected sub-polyhedron.

Notice that a wgsc polyhedron is simply connected. The wgsc notion is the counterpart in the polyhedral category of the gsc of open manifolds and in general it is slightly weaker. The notion which seems to capture the full power of the gsc for non-compact manifolds (with boundary) is the pl-gsc discussed in [23].

Remark 2.2. Similar definitions can be given in the case of topological (respectively smooth) manifolds, where we require the exhaustions to be by topological (respectively smooth) submanifolds.

Remark 2.3. For $n \neq 4$ an open n-manifold is wgsc if and only if it is gsc (see [22] for $n \geq 5$, and for $n = 3$ it follows from the Poincaré conjecture), while in dimension 4 one expects to find open 4-manifolds which are wgsc but not gsc.

Definition 2.3. The finitely presented group Γ is *wgsc* if *there exists some* compact polyhedron X with $\pi_1(X) = \Gamma$ such that its universal covering $\widetilde{X}$ is wgsc.

Remark 2.4. Working with simplicial complexes instead of polyhedra in the definitions above, and thus not allowing subdivisions, yields an equivalent notion of wgsc for finitely presented groups.

Remark 2.5. The fact that a group is not wgsc cannot be read from an arbitrary complex with the given fundamental group. In fact, as F. Lasheras pointed out to us, for any finitely presented group Γ with an element of infinite order, there exists a complex X with $\pi_1 X = \Gamma$ whose universal covering is not wgsc. For example take $\Gamma = \mathbb{Z}$ and the complex X being that associated to the presentation $\mathbb{Z} = \langle a, b \mid baba^{-1}b^{-1} \rangle$. Then the universal covering $\widetilde{X}$ is given in Figure 1.
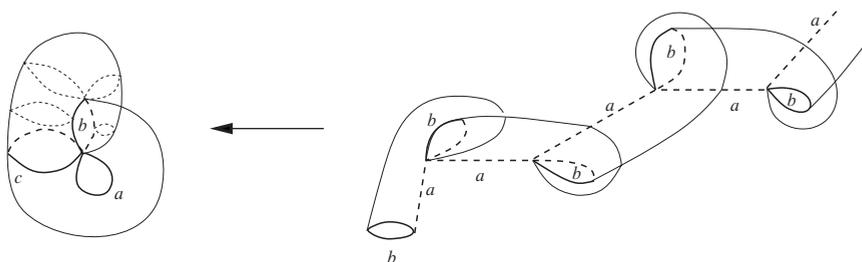
Figure 1. The universal covering $\widetilde{X}$.

One sees that $\widetilde{X}$ is not wgsc because in the process of killing one loop b one creates another one indefinitely.

Further, if Γ is a finitely presented group with an element a of infinite order, we add a new generator b and a new relation as before. The universal covering associated to this presentation is not wgsc, by the same arguments.

Remark 2.6. The wgsc property cannot be extended to arbitrary finitely generated groups, as stated, since any group admits a presentation with infinitely many relations such that the associated 2-complex is wgsc. It suffices to add infinitely many 2-cells, along the boundaries of unions of 2-cells, killing inductively the fundamental group of any compact subset.

Remark 2.7. Recall that there exist uncountably many open contractible manifolds which are not wgsc ([22]). In general, these manifolds are not covering spaces and we do not know whether one could find co-compact universal coverings among the non wgsc manifolds. For instance, if a finitely presented torsion group exists then it is hard to believe that its Cayley complex is wgsc. Swenson has shown that every CAT(0) group has an element of infinite order (see [43]). Notice that manifolds that are simply connected at infinity are automatically wgsc ([41]), but in general not conversely (see examples below).

2.2. The qsf property after Brick and Mihalik. The qsf property is a weaker version of the wgsc, which has the advantage to be independent on the polyhedron we chose. Specifically, Brick ([8]) defined it as follows:

Definition 2.4. The simply connected non-compact PL space X is *qsf* if for any compact sub-polyhedron $C \subset X$ there exists a simply connected compact polyhedron K and a PL map $f : K \to X$ so that $C \subset f(K)$ and $f|_{f^{-1}(C)} : f^{-1}(C) \to C$ is a PL homeomorphism.

Definition 2.5. The finitely presented group Γ is *qsf* if there exists a compact polyhedron P of fundamental group Γ so that its universal covering $\widetilde{P}$ is qsf.

Remark 2.8. It is known (see [8]) that the qsf is a group property and does not depend on the compact polyhedron P we chose in the definition above. In fact, if Q is any compact polyhedron of fundamental group Γ (which is qsf) then $\widetilde{Q}$ is qsf.

Remark 2.9. The qsf is very close to (and a consequence of) the following notion of Dehn exhaustibility (see [40], [22]) which was mainly used in a manifold setting. The polyhedron W is Dehn-exhaustible if for any compact $C \subset W$ there exists a simply connected compact polyhedron K and an *immersion* $f : K \to W$ such that $C \subset f(K)$ and the set of double points of f is disjoint from C. It is known from [40] that a Dehn exhaustible 3-manifold is wgsc.

2.3. Small content and 1-tame groups. Now we consider some other tameness conditions on non-compact spaces, which are closely related to the wgsc. Moreover we will show later that they induce equivalent notions for discrete groups. In many cases it is easier to prove that a specific complex has one of these two properties instead that directly proving the qsf. This will be the case in the second part of this paper for the Grigorchuk group and its extension.

Definition 2.6. The simply connected non-compact polyhedron X has *small content* if for any compact $C \subset X$ there exist two compact connected sub-polyhedra $C \subset D \subset E \subset X$, fulfilling the following properties:

(1) The map $\pi_1(D) \to \pi_1(E)$ induced by the inclusion, is zero.

(2) If two points of D are connected within $E - C$ then they are connected within $D - C$.

(3) Any loop in $E - C$ (based to a point in $D - C$) is homotopic relative to the base point within $X - C$ to a loop which lies entirely inside $D - C$. Alternatively, let us denote by $\iota_Y : \pi_1(Y - C) \to \pi_1(X - C)$ the morphism induced by inclusion (for any compact Y containing D), by fixing a base point (which is considered to be in $D - C$). Then one requires that $\iota_D(\pi_1(D - C)) = \iota_E(\pi_1(E - C))$.

The finitely presented group Γ has *small content* if there exists a compact polyhedron P of fundamental group Γ so that its universal covering $\widetilde{P}$ has small content.

Remark 2.10. An obvious variation would be to ask that the homotopy above might not keep fixed the base point. We do not know whether the new definition is equivalent to the former one.

Definition 2.7. The PL space X is 1-*tame* if any compact sub-polyhedron C is contained in a compact sub-polyhedron $K \subset X$, so that any loop γ in K is (freely) homotopic within K to a loop $\bar{\gamma}$ in $K - C$, while $\bar{\gamma}$ is null-homotopic within $X - C$.

The finitely presented group Γ is 1-*tame* if there exists a compact polyhedron P of fundamental group Γ so that its universal covering $\widetilde{P}$ is 1-tame.

Notice that one does not require that an arbitrary loop in $K - C$ be null-homotopic within $X - C$. This happens only after a suitable homotopy which takes place in K.

2.4. Tame combings and the Tucker property. Group combings were essential ingredients in Thurston's attempt to abstract finiteness properties of fundamental groups of negatively curved manifolds which finally led to automatic groups.

Tame 1-combings of groups were considered by Mihalik and Tschantz in [38] as higher dimensional analogs of usual combings, which are referred of as 0-combings.

Definition 2.8. A 0-*combing* of a 2-complex X is a set of edge-paths $\sigma_p(t), t \in [0, 1]$, joining each vertex p of X to a basepoint vertex x_0. This can be thought of as a homotopy $\sigma : X^0 \times [0, 1] \to X^1$ for which $\sigma(x, 1) = x$ for all $x \in X^0$, and $\sigma(X^0, 0) = x_0$, where X^j denotes the j-dimensional skeleton of X.

A 1-*combing* of the 2-complex X is a continuous family of paths $\sigma_p(t), t \in [0, 1]$, joining each point p of the 1-skeleton of X to a basepoint vertex x_0, whose restriction to vertices is a 0-combing. This is a homotopy $\sigma : X^1 \times [0, 1] \to X$ for which $\sigma(x, 1) = x$ for all $x \in X^1$, $\sigma(X^1, 0) = x_0$, and $\sigma|_{X^0 \times [0,1]}$ is a 0-combing.

Observe that although any connected complex is 0-combable, a 2-complex is 1-combable if and only if it is simply connected.

In order to find interesting consequences in geometric group theory one imposed the boundedness (or fellow traveler condition) on the 0-combing, namely that combing paths of neighbor vertices be at uniformly bounded distance from each other.

In the same spirit Mihalik and Tschantz replaced the boundedness by the following property of topological nature:

Definition 2.9. A 0-combing is called *tame* if for every compact set $C \subseteq X$ there exists a compact set $K \subseteq X$ such that for each $x \in X^0$ the set $\sigma^{-1}(C) \cap (\{x\} \times [0, 1])$ is contained in one path component of $\sigma^{-1}(K) \cap (\{x\} \times [0, 1])$.

A 1-combing is *tame* if its restriction to the set of vertices is a tame 0-combing and for each compact $C \subset X$ there exists a larger compact $K \subset X$ such that for each edge e of X, $\sigma^{-1}(C) \cap (e \times [0, 1])$ is contained in one path component of $\sigma^{-1}(K) \cap (e \times [0, 1])$.

A group is *tame* 1-*combable* if the universal cover of some (equivalently any, see [38]) finite 2-complex with given fundamental group admits a tame 1-combing.

Recall now the following tameness condition of topological spaces:

Definition 2.10. The non-compact PL space X is *Tucker* if the fundamental group of each component of $X - K$ is finitely generated, for any finite subcomplex $K \subset X$.

This definition was motivated by Tucker's work [44] on 3-manifolds. A non-compact manifold is a *missing boundary* manifold if it is obtained from a compact manifold with boundary by removing a closed subset of its boundary. We have the following characterization from [44]: a P^2-irreducible connected 3-manifold is a missing boundary 3-manifold if and only if it is Tucker.

The main results of [38] state the following.

Proposition 2.1 ([38]). *A finitely presented group is tame* 1-*combable if and only if the universal covering of any* (*equivalently, some*) *finite complex with given fundamental group is Tucker. Moreover, a tame* 1-*combable group is qsf.*

All known examples of qsf groups are actually tame 1-combable. We will show in the next section that the two notions are almost-equivalent.

Requiring a tame 0-combing is a very soft condition because of the following result.

Proposition 2.2. *Any connected* 2-*complex X has a tame* 0-*combing.*

Proof. The key point is that any connected 2-complex X is the ascending union of *connected* finite subcomplexes X_n, for instance metric balls.

A 0-combing σ_p is geodesic with respect to $(X_n)_n$ when it satisfies the following properties:

(1) if $p \in X_0$ then σ_p has minimal length among the paths in X_0 joining p to $x_0 \in X_0$;

(2) for $p \in X_n - X_{n-1}$, with $n \geq 1$, there is some $q \in X_{n-1}$ which realizes the distance in X_n from p to X_{n-1}. Let $\eta_p \subset X_n$ be a minimal length curve joining p to q. Then σ_p is the concatenation of η_p and σ_q.

If all X_n are connected then there exist geodesic 0-combings which are defined inductively by means of the two conditions above. Let σ be one of them. It suffices to verify the tameness of σ for large enough finite subcomplexes C and thus to assume that $x_0 \in C$. Set K be the smallest X_n containing C.

We claim that the set $\{t \in [0, 1]; \sigma_p(t) \in K\}$ is connected, which settles the proposition. This is clear when $p \in X_n$. If $p \in X_{n+1} - X_n$ then η_p is contained in $X_{n+1} - X_n$ except for its endpoint $q \in X_n$. Otherwise, we would find a point in X_n closest to p than q contradicting our choice for q. Further $\sigma_q \subset X_n$ is connected, hence $\sigma_p \cap X_n$ is also connected. Using induction on k one shows in the same way that $\sigma_p \cap X_n$ is connected when $p \in X_{n+k} - X_n$. As $X = \bigcup_k X_{n+k}$ the claim follows. $\qquad\square$

Remark 2.11. One says that a 1-combing is weakly tame 1-combable if for each compact $C \subset X$ there exists a larger compact $D \subset X$ such that for every edge e the set $\{(p, t) \in e \times [0, 1]; \sigma_p(t) \in C\}$ is contained in one connected component of $\{(p, t) \in e \times [0, 1]; \sigma_p(t) \in D\}$. Thus one drops from the definition of the tame 1-combing the requirement that the restriction to the vertices be a tame 0-combing. It was mentioned in the last section of [38] that the existence of a weakly tame 1-combing actually implies the existence of a tame 1-combing.

3. Proof of Theorem 1.1

3.1. Comparison of qsf and wgsc conditions.
The subject of this section is the proof of the almost-equivalence of qsf and wgsc conditions from Theorem 1.1. Our result is slightly more general and includes the 1-tameness and small content conditions, which will be used later, in Section 6.

Proposition 3.1. *A wgsc polyhedron has small content and is* 1*-tame. A polyhedron which is either* 1*-tame or else has small content is qsf.*

Proof. Let C be a compact sub-polyhedron of the polyhedron X.

(1) Assume that X is wgsc. Then one can embed C in a compact 1-connected sub-polyhedron $K \subset X$. Taking then $D = E = K$ one finds that X has small content and is 1-tame.

(2) Suppose that X has small content, and D and E are the sub-polyhedra provided by Definition 2.6. Let γ be a loop in E, based at a point in C. We consider the decomposition of γ into maximal arcs $\gamma[j]$ which are (alternatively) contained either in D or in the closure $\overline{E - D}$ of $E - D$, namely $\gamma[1] \subset D, \gamma[2] \subset \overline{E - D}$, and so on. Thus $\gamma[2k] \subset \overline{E - D}$ has its endpoints in D. By hypothesis there exists

another arc $\lambda[k] \subset D - C$ that joins the endpoints of $\gamma[2k]$. The composition $\gamma^k = \gamma[2k]\lambda[k]^{-1}$ is then a loop in $E - C$. Moreover, the composition $\gamma^0 = \gamma[1]\lambda[1]\gamma[3]\lambda[2]\ldots\gamma[2k-1]\lambda[k]\ldots$ is a loop contained in D. Next γ^j (based at one endpoint of $\gamma[2j]$ from $\overline{E-D}$) is homotopic within $X-C$ to a loop $\bar{\gamma}^j \subset D-C$.

Assume now that we chose a system of generators $\gamma_1, \ldots, \gamma_n$ of $\pi_1(E)$. We will do the construction above for each loop γ_j, obtaining the loops γ_j^k in $E - C$ which are homotopic to $\bar{\gamma}_j^k$ in $D - C$. We define first a polyhedron $\widehat{E}$ by adding to E 2-disks along the composition of the loops $\gamma_j^k(\bar{\gamma}_j^k)^{-1}$. Recall that these two loops have the same basepoint (depending on j, k) and so it makes sense to consider their composition.

There is defined a natural map $F: \widehat{E} \to X$, which extends the inclusion $E \hookrightarrow X$, as follows. There exists a homotopy within $X - C$ keeping fixed the base point of γ_j^k between γ_j^k and $\bar{\gamma}_j^k$. Alternatively, there exists a free null-homotopy of the loop $\gamma_j^k(\bar{\gamma}_j^k)^{-1}$ within $X - C$. We then send the 2-disk of $\widehat{E}$ capping off $\gamma_j^k(\bar{\gamma}_j^k)^{-1}$ onto the image of the associated free null-homotopy.

It is clear that F is a homeomorphism over C, since the images of the extra 2-disks are disjoint from C. Moreover, we claim that $\widehat{E}$ is simply connected. In fact, any loop in $\widehat{E}$ is homotopic to a loop within E, and hence to a composition of γ_j. Each γ_j is homotopic relative to the base point, by a homotopy in E, to $\gamma_j[1]\gamma_j^1\lambda_j[1]\gamma_j[3]\gamma_j^2\lambda_j[2]\ldots$, which is homotopic relative to the base point, by a homotopy in $\widehat{E}$, to $\gamma_j[1]\bar{\gamma}_j^1\lambda_j[1]\gamma_j[3]\bar{\gamma}_j^2\lambda_j[2]\ldots$, a loop in D. By hypothesis, this last loop is null-homotopic in E. Therefore $\pi_1(\widehat{E}) = 0$.

(3) Suppose now that X is 1-tame. Let K be the compact associated to an arbitrarily given compact C. Any loop γ in K is freely homotopic to a loop $\bar{\gamma}$ in $K - C$. Consider $\gamma_1, \ldots, \gamma_n$ a system of generators of $\pi_1(K)$. From K we construct the polyhedron $\widehat{K}$ by adding 2-disks along the loops $\bar{\gamma}_j$. There exists a map $F: \widehat{K} \to X$, which extends the inclusion $K \hookrightarrow X$, defined as follows. The 2-disk capping off the loop $\bar{\gamma}_j$ is sent into the null-homotopy of $\bar{\gamma}_j$ within $X - C$. Then F is obviously a homeomorphism over C. Meanwhile, $\widehat{K}$ is simply connected since we killed all homotopy classes of loops from K. $\qquad\square$

Proposition 3.2. *If the open n-manifold M^n is qsf and $n \geq 5$, then M^n is wgsc.*

Proof. It suffices to prove that any compact codimension zero submanifold C is contained in a simply connected compact subspace of M^n. By hypothesis there exists a compact connected and simply connected simplicial complex K and a map $f: K \to M^n$ such that $f: f^{-1}(C) \to C$ is a PL homeomorphism. Assume that f is simplicial, after subdivision. Let L be the 2-skeleton of $\overline{K \setminus f^{-1}(C)}$ and denote by $\partial L = L \cap \partial f^{-1}(C)$. Notice that $f^{-1}(C)$ is a manifold.

The restriction of $f|_L: L \to \overline{M^n \setminus C}$ to the subcomplex $\partial L \subset L$ is an embedding. Since the dimension of L is 2 and $n \geq 5$, general position arguments show that we

can perturb f by a homotopy, which is the identity on ∂L, to a simplicial map $g\colon L \to \overline{M^n \setminus C}$, which is an embedding.

Now $\pi_1(f^{-1}(C) \bigcup_{\partial L} L) \cong \pi_1(K) = 0$, and so $\pi_1(C \bigcup_{f(\partial L)} g(L)) = 0$. Take a small regular neighborhood U of $C \bigcup_{f(\partial L)} g(L)$ inside M^n. Then U is a simply connected compact submanifold of M^n containing C. $\square$

Remark 3.1. A similar result was proved in [22] for Dehn exhaustibility. In particular a n-manifold which is Dehn-exhaustible is wgsc provided that $n \geq 5$.

Definition 3.1. A finitely generated group has the topological property A if *some* Cayley complex has property A. The topological properties A and B are *almost-equivalent* for finitely presented groups if a finitely presented group has A if and only if it has B.

Corollary 3.1. *The wgsc, gsc, qsf, Dehn-exhaustibility, 1-tameness and small content are almost-equivalent for finitely presented groups.*

This also yields the following geometric characterization of the qsf.

Corollary 3.2. *The group Γ is qsf if and only if the universal covering $\widetilde{M^n}$ of any compact manifold M^n with $\pi_1(M^n) = \Gamma$ and dimension $n \geq 5$ is wgsc (or gsc). In particular, a qsf group admits a presentation whose Cayley complex is wgsc.*

Proof. The "if" part is obvious. Assume then that Γ is wgsc, and so there exists a compact polyhedron whose universal covering is wgsc and hence qsf. It is known (see [8]) that the qsf property does not depend on the particular compact polyhedron we chose. Thus, if M^n is a compact manifold with fundamental group Γ then $\widetilde{M^n}$ is also qsf. By the previous proposition, when $n \geq 5$ $\widetilde{M^n}$ is also wgsc, as claimed.

Further, if the group Γ is qsf then consider a compact n-manifold M^n with fundamental group Γ and $n \geq 5$. It is known that $\widetilde{M^n}$ is qsf and thus wgsc.

Consider a triangulation of M^n and T a maximal tree in its 1-skeleton. Since the finite tree T is collapsible it has a small neighborhood $U \subset M^n$ homeomorphic to the n-dimensional disk. The quotient U/T is homeomorphic to the n-disk and thus to U. This implies that the quotient M^n/T is homeomorphic to M^n. Therefore we obtain a finite CW-complex X^n homeomorphic to M^n and having a single vertex. Also X^n is wgsc since it is homeomorphic to a wgsc space.

The wgsc property is inherited by the 2-skeleton, namely a locally finite CW-complex X is wgsc if and only if its 2-skeleton is wgsc. This means that the universal covering of the 2-skeleton of X^n is wgsc. But any finite CW-complex of dimension 2 with one vertex and fundamental group Γ is the Cayley complex associated to a suitable presentation of Γ. Thus the Cayley complex of this presentation is wgsc, as claimed. $\square$

3.2. Qsf and tame 1-combability. The subject of this section is to end the proof of Theorem 1.1 by proving that qsf and tame 1-combability are almost-equivalent for finitely presented groups.

We will consider below open connected manifolds with *finitely many* 1-*handles*, which slightly generalize the gsc condition. In the smooth category this means that there is a proper Morse function with only finitely many index 1 critical points. In the PL category we can ask that the manifold have a proper handlebody decomposition for which 1-handles and 2-handles are in canceling position for all but finitely many pairs.

Proposition 3.3. *Let W^n, $n \geq 5$, be an open connected manifold admitting a proper handlebody decomposition with only finitely many 1-handles. Then W^n is Tucker.*

Proof. We have to prove that for sufficiently large compact subcomplexes K the group $\pi_1(W - K)$ is finitely generated.

Consider a proper handlebody decomposition with a single 0-handle and finitely many 1-handles. We shall assume that K is large enough to include all index 1 handles. Further, by compactness there is a union of handles C containing K. Here C is a manifold with boundary ∂C. We obtain $W - \text{int}(C)$ from $\partial C \times [0, 1]$ by adding inductively handles of index at least 2. In particular $W - \text{int}(C)$ has as many connected components as ∂C. Let F be a connected component of ∂C and Z be the corresponding connected component of $W - \text{int}(C)$.

Lemma 3.1. *The inclusion $F \hookrightarrow Z$ induces a surjective homomorphism $\pi_1(F) \to \pi_1(Z)$.*

Proof. Let Z_k be the result of adding the next k handles of the decomposition to $F \times [0, 1]$ and let F_k be the other boundary of Z_k, namely $\partial Z_k = F \cup F_k$, and $Z_0 = \emptyset$. We claim first that $\pi_1(Z_{k+1} - \text{int}(Z_k), F_k) = 0$, or equivalently, the homomorphism induced by inclusion $\pi_1(F_k) \to \pi_1(Z_{k+1} - \text{int}(Z_k))$ is surjective. In fact one obtains $Z_{k+1} - \text{int}(Z_k)$ from $F_k \times [0, 1]$ by adding one handle of index at least 2. Then van Kampen implies the claim. Further $Z_k = \bigcup_{j=0}^{k-1}(Z_{j+1} - \text{int}(Z_j))$ so that $\pi_1(Z_k)$ is the iterated amalgamated product

$$\pi_1(Z_1) *_{\pi_1(F_1)} \pi_1(Z_2 - \text{int}(Z_1)) *_{\pi_1(F_2)} \pi_1(Z_3 - \text{int}(Z_2)) * \cdots$$
$$\cdots *_{\pi_1(F_{k-1})} \pi_1(Z_k - \text{int}(Z_{k-1})).$$

The previous claim then shows that the inclusion $F \to Z_k$ induces a surjective map $\pi_1(F) \to \pi_1(Z_k)$, for each k. Letting k go to infinity we find that $\pi_1(F)$ surjects onto $\pi_1(Z)$. $\qquad\square$

At last $W - K$ is obtained by gluing $W - C$ and $C - K$. It is clear that $C - K$ has finitely generated fundamental group. As ∂C has finitely many connected components the use of van Kampen and the lemma above imply that $\pi_1(W - K)$ is finitely generated. $\qquad\square$

Corollary 3.3. *If W^n is open gsc manifold of dimension $n \geq 5$, then W^n is Tucker.*

Proposition 3.4. *A finitely presented group is qsf iff it is tame 1-combable.*

Proof. The "if" implication is proved in [38]. Let G be qsf. Choose some closed triangulated 5-manifold M with $\pi_1(M) = G$. According to our previous result $\widetilde{M}$ is an open gsc manifold. In particular, by the corollary above $\widetilde{M}$ is Tucker. Now, a complex X is Tucker if and only if its 2-skeleton X^2 is Tucker. Therefore the 2-skeleton of $\widetilde{M}$ and hence the universal covering $\widetilde{M^{(2)}}$ of the 2-skeleton of the triangulation of M is Tucker. It is clear that $\pi_1(M) = \pi_1(M^{(2)})$. Recall from [38] that G is tame combable if there is some finite 2-complex X with $\pi_1(X) = G$ for which $\widetilde{X}$ has the Tucker property. This proves that G is tame 1-combable. □

In particular we obtain the Corollary 1.2, which we restate here for the sake of completeness:

Corollary 3.4. *The qsf property is a quasi-isometry invariant of finitely presented groups.*

Proof. Brick proved (see [6], and also the refinement from [32], Theorem A) that a group quasi-isometric to a finitely tame 1-combable group is also tame 1-combable. □

3.3. Some examples of qsf groups

3.3.1. General constructions. It follows from [6], [8], [7], [38] that most geometric examples of groups are actually qsf.

Example 3.5. (1) A group G is qsf if and only if a finite index subgroup H of G is qsf.

(2) Let A and B be finitely presented qsf groups and C be a common finitely generated subgroup. Then the amalgamated free product $G = A *_C B$ is qsf. If A is a finitely presented qsf group and $\phi \colon C_1 \to C_2$ is an isomorphism of finitely generated subgroups of A, then the HNN extension $A*_\phi$ is qsf. Conversely, if A, B are finitely presented and C is finitely generated then $A *_C B$ (respectively $A*_\phi$, where $\phi \colon C_1 \to C_2$ is an isomorphism of finitely generated subgroups of A) is qsf implies that A and B are qsf.

(3) All one-relator groups are qsf.

(4) The groups from the class $\mathcal{C}_+$ (combable) in the sense of Alonso–Bridson ([2]) are qsf. In particular automatic groups, small cancelation groups, semi-hyperbolic groups, groups acting properly co-compactly on Tits buildings of Euclidean type, Coxeter groups, fundamental groups of closed non-positively curved 3-manifolds are qsf. Notice that all these groups have solvable word problem.

(5) If a group has a tame 1-combing then it is qsf. In particular, asynchronously automatic groups (see [38]) are qsf.

(6) Groups which are simply connected at infinity are qsf ([8]).

(7) Assume that $1 \to A \to G \to B \to 1$ is a short exact sequence of infinite finitely presented groups. Then G is qsf ([7]). More generally, graph products (i.e., the free product of vertex groups with additional relations added in which elements of adjacent vertex groups commute with each other) of infinite finitely presented groups associated to nontrivial connected graphs are qsf.

Remark 3.2. The last property above is an algebraic analog of the fact that the product of two open simply connected manifolds is gsc. Moreover, if one of them is 1-ended then the product is simply connected at infinity.

Remark 3.3. There exist finitely presented qsf groups with unsolvable word problem. Indeed, in [12] the authors constructed a group with unsolvable word problem that can be obtained from a free group by applying three successive HNN extensions with finitely generated free associated subgroups. Such a group is qsf from (2) of the example above.

3.3.2. Baumslag–Solitar groups: not simply connected at infinity. The Baumslag–Solitar groups are given by the 1-relator presentation

$$B(m, n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle, \quad m, n \in \mathbb{Z}.$$

Since they are 1-relator groups they are qsf. It is known that $B(1, n)$ are amenable, metabelian groups which are neither lattices in 1-connected solvable real Lie groups nor CAT(0) groups (i.e., acting freely co-compactly on a proper CAT(0) space).

Notice that $B(1, n)$ are not almost convex with respect to any generating set and not automatic either, if $n \neq \pm 1$.

Recall that a group G which is simply connected at infinity should satisfies $H^2(G, \mathbb{Z}G) = 0$. Since this condition is not satisfied by $B(1, n)$, for $n > 1$ (see [36]), these groups are not simply connected at infinity.

The higher Baumslag–Solitar groups $B(m, n)$ for $m, n > 1$ are known to be nonlinear, not residually finite (if $m \neq n$), not Hopfian (when m and n are coprime), not virtually solvable. Moreover, they are not automatic if $m \neq \pm n$, but they are asynchronously automatic.

3.3.3. Solvable groups: not CAT(0). Let G be a finitely presented solvable group whose derived series is

$$G \rhd G^{(1)} \rhd G^{(2)} \rhd \cdots \rhd G^{(n)} \rhd G^{(n+1)} = 1.$$

If $G^{(n)}$ is finite then G is qsf if and only if the solvable group $G/G^{(n)}$ (whose derived length is one unit smaller than G) is qsf. Solvable groups with infinite

finitely generated center are qsf by Example 3.5 (7). More generally, if $G^{(n)}$ has an element of infinite order, then Mihalik (see [36]) proved that either G is simply connected at infinity or else there exist $\Lambda \lhd G$, which is a normal subgroup of finite index, and $F \lhd \Lambda$ which is a normal finite subgroup, such that Λ / F is isomorphic to a Baumslag–Solitar group $B(1, m)$. This implies that Λ is qsf and hence G is qsf. This is useful in understanding that qsf groups are far more general than groups acting properly co-compactly and by isometries on CAT(0) spaces. In fact, every solvable subgroup of such a CAT(0) group should be virtually abelian. Thus all solvable groups that are not virtually abelian are not CAT(0) and many of them are qsf (e.g., if their center is not torsion). Notice that there exist solvable groups with infinitely generated centers, as those constructed by Abels (see [1]). In general we do not know whether all solvable groups (in particular those with finite centers) are qsf, but one can prove that Abels' group is qsf since it is an S-arithmetic group.

3.3.4. Higman's group: acyclic examples.
The first finitely presented acyclic group was introduced by G. Higman in [33]:

$$H = \langle x, y, z, w \mid x^w = x^2, y^x = y^2, z^y = z^2, w^z = w^2 \rangle,$$

where $a^b = bab^{-1}$. It is known (see e.g. [18]) that H is an iterated amalgamated product,

$$H = H_{x,y,z} *_{F_{x,z}} H_{z,w,x},$$

with $H_{x,y,z} = H_{x,y} *_{F_y} H_{y,z}$ and $H_{x,y} = \langle x, y, y^x = y^2 \rangle$ is the Baumslag–Solitar group $B(1, 2)$ with generators x, y. Here F_y, $F_{x,z}$ are the free groups in the respective generators. The morphisms $F_y \to H_{x,y}$, $F_{x,z} \to H_{x,y,z}$ and their alike are tautological, i.e., they send each left-hand side generator into the generator denoted by the same letter on the right-hand side. Notice that these homomorphisms are injective. The example above implies that H is qsf. Observe that H is not simply connected at infinity according to [37]. There are more general Higman groups H_n generated by n elements with n relations as above in cyclic order. It is easy to see that H_3 is trivial and the arguments above imply that H_n are qsf for any $n \geq 4$.

3.3.5. The Gromov–Gersten examples.
A slightly related class of groups was considered by Gersten and Gromov (see [29], 4.C$_3$):

$$\Gamma_n = \langle a_0, a_1, \ldots, a_n \mid a_0^{a_1} = a_0^2, a_1^{a_2} = a_1^2, \ldots, a_{n-1}^{a_n} = a_{n-1}^2 \rangle.$$

Note that H_n is obtained from Γ_n by adding one more relation that completes the cyclic order. As above Γ_{n+1} is an amalgamated product $\Gamma_n *_{\mathbb{Z}} B(1, 2)$, and thus Γ_n is qsf for any n. These examples are very instructive since Gromov and Gersten proved that the connectivity radius of Γ_{n+1} is an n-fold iterated exponential (see the next section for a discussion). Moreover, Γ_n is contained in the group

$$\Gamma_* = \langle a_0, b \mid a_0^{a_0^b} = a_0^2 \rangle.$$

Therefore, the connectivity radius of Γ_* is higher than any iterated exponential. Since Γ_* is a 1-relator group it is qsf and has solvable word problem.

3.3.6. Thompson groups. Among the first examples of infinite finitely presented simple groups are those provided by R. Thompson in the sixties. We refer to [11] for a thorough introduction to the groups usually denoted F, T and V. These are by now standard test groups.

According to ([10], [21]) F is a finitely presented group which is an ascending HNN extension of itself. A result of Mihalik ([35], Th.3.1) implies that F is simply connected at infinity and thus qsf.

Remark 3.4. Notice that F is a non-trivial extension of its abelianization $\mathbb{Z}^2$ by its commutator $[F, F]$, which is a simple group. However $[F, F]$ is not finitely presented, although it is still a diagram group, but one associated to an infinite semi-group presentation. Thus one cannot apply directly (7) of the example above.

Moreover, the truncated complex of bases due to Brown and Stein (see [9]) furnishes a contractible complex acted upon freely co-compactly by the Thompson group V. The start-point of the construction is a complex associated to a directed poset which is therefore exhausted by finite simply connected (actually contractible) subcomplexes.

The qsf is preserved through all the subsequent steps of the construction and thus the complex of bases is qsf. In particular the Thompson group V is qsf. We skip the details.

Remark 3.5. It is likely that all diagram groups (associated to a finite presentation of a finite semi-group) considered by Guba and Sapir in [30] (and their generalizations, the picture groups) are qsf. Farley constructed in [19] free proper actions by isometries of diagram groups on infinite dimensional CAT(0) cubical complexes. This action is not co-compact and moreover the respective CAT(0) space is infinite dimensional. However there exists a natural construction of truncating the CAT(0) space X in order to get subspaces X_n of X which are invariant, co-compact and n-connected. Farley's construction works well ([20]) for circular and picture diagrams (in which planar diagrams are replaced by annular diagrams or diagrams whose wires are crossing each other). However, these subcomplexes are not anymore CAT(0), and it is not clear whether they are qsf. Notice that these groups have solvable word problem (see [30]).

3.3.7. Outer automorphism groups. In the case of surface groups these correspond to mapping class groups. Since a finite index subgroup acts freely properly discontinuously on the Teichmüller space it follows that mapping class groups are qsf. The study of Morse type functions on the outer space led to the fact that $Out(F_n)$ is $2n - 5$ connected at infinity and thus qsf as soon as $n \geq 3$ (see [5]).

4. Proof of Theorem 1.3 and applications

4.1. The qsf growth. Let P be a finite presentation of the qsf group G and $C(G, P)$ be the associated Cayley complex. There is a natural word metric on the set of vertices of the Cayley graph $C^1(G, P)$ (the 1-skeleton of the Cayley complex) by setting

$$d(x, y) = \min |w(xy^{-1})|$$

where $|w(a)|$ denotes the length of a word $w(a)$ in the letters s, s^{-1}, for $s \in P$, representing the element a in the group G. By language abuse we call metric complex a simplicial complex whose 0-skeleton is endowed with a metric.

Definition 4.1. The metric ball $B(r, p) \subset C(G, P)$ (respectively metric sphere $S(r, p) \subset C(G, P)$) of radius $r \in \mathbb{Z}_+$ centered at some vertex p is the following subcomplex of $C(G, P)$:

(1) the vertices of $B(r, p)$ (respectively $S(r, p)$) are those vertices of $C(G, P)$ staying at distance at most r (respectively r) from p;

(2) the edges and the 2-cells of $B(r, p)$ (respectively $S(r, p)$) are those edges and 2-cells of $C(G, P)$ whose boundary vertices are at distance at most r (respectively r) from p.

Denote by $B(r)$ (respectively $S(r)$) the metric ball (respectively sphere) of radius r centered at the identity.

Definition 4.2. A π_1-*resolution* of the polyhedron C inside X is a pair (A, f), where A is a CW complex and $f : A \to X$ a PL map such that $f : f^{-1}(C) \to C \subset X$ is a PL-homeomorphism and $\pi_1(A) = 0$.

We want to refine the qsf property for metric complexes. As we are interested in Cayley complexes below, we formulate the definition in this context.

Definition 4.3. The *qsf growth* function $f_{G,P}$ of the Cayley complex $C(G, P)$ is

$$f_G(r) = \inf\{R \text{ such that there exists a } \pi_1\text{-resolution of } B(r) \text{ into } B(R)\}.$$

Recall that the real functions f and g are *rough equivalent* if there exist constants c_i, C_j (with $c_1, c_2 > 0$) such that

$$c_1 f(c_2 R) + c_3 \leq g(R) \leq C_1 f(C_2 R) + C_3.$$

One can show easily that the rough equivalence class of $f_{G,P}(r)$ depends only on the group G and not on the particular presentation, following [8] and [24]. We will write it as $f_G(r)$. We do not know whether the rough equivalence class of f_G is a quasi-isometry invariant. This would be true if we could compare f_G with the tameness function of Hermiller and Meier ([32]).

Recall from ([29], 4.C) that the *connectivity radius* $R_1(r)$ defined by Gromov is the infimal $R_1(r)$ such that $\pi_1(B(r)) \to \pi_1(B(R_1(r)))$ is zero. Notice that the rough equivalence class of R_1 is also well defined and independent on the group presentation we chose for the group.

Remark 4.1. Observe that $\pi_1(B(r)) \to \pi_1(B(f_G(r)))$ is zero. Thus f_G is bounded from below by the connectivity radius R_1.

Recall that the *isodiametric function* of a group G, following Gersten, is the infimal $I_G(k)$ so that loops of length k bound disks of diameter at most $I_G(k)$ in the Cayley complex. The rough equivalence class of I_G is a quasi-isometry invariant of the finitely presented group G.

Proposition 4.1. *A qsf group whose qsf growth f_G is recursive has a solvable word problem.*

Proof. Observe that the growth rate of the qsf is an upper bound for the Gersten isodiametric function I_G, and the word problem is solvable whenever the isodiametric function is recursive. This is standard: if a word w is trivial in the group G presented as $G = \langle S \mid R \rangle$, then it is a product of conjugates of relators uru^{-1}, $r \in R$. By the definition of the isodiametric function one can choose these conjugates in such way that $|u| \leq I_G(|w|) \leq f_G(|w|)$ and this leads to a finite algorithm that checks whether w is trivial or not. $\qquad\qquad\square$

4.2. Metric balls and spheres in Cayley complexes.

We consider now some metric complexes satisfying a closely related property. On one side this condition seems to be slightly weaker than the wgsc since we could have nontrivial (but uniformly small) loops, but on the other side the exhaustions we consider are restricted to metric balls.

Definition 4.4. A metric complex has π_1-*bounded balls* (respectively *spheres*) if there exists a constant C so that $\pi_1(B(r))$ (respectively $\pi_1(S(r))$) is normally generated by loops with length smaller than C.

Remark 4.2. If the balls in a metric complex are simply connected then the complex is obviously wgsc. However, if the complex is wgsc it is not clear whether we can choose an exhaustion by simply connected metric balls. Thus the main constraint in the definition above is the requirement to work with metric balls.

We actually show that this condition puts strong restrictions on the group:

Proposition 4.2. *If some Cayley complex of a finitely presented group has π_1-bounded balls (or spheres) then the group is qsf with linear qsf growth.*

Proof. Any loop l in the ball $B(r)$ is null-homotopic in the Cayley complex. Thus there exists a disk with holes lying in $B(r)$ such that the outer boundary component is equal to l and the other boundary components $l'_1, \ldots l'_p$ lie in $S(r)$.

If we have π_1-bounded spheres then we can assume that each loop l'_j, $1 \le j \le p$, is made of uniformly small loops on $S(r)$ connected by means of arcs. A loop of length C in the Cayley graph bounds a disk of diameter $I_G(C)$ in the Cayley complex. Thus these disks have uniformly bounded diameters. The loop l'_j bounds therefore a disk $D(l'_j)$ which is disjoint from $B(r - I_G(C))$ and lies within $B(r + I_G(C))$.

We can use this procedure for a system of loops l_j, $1 \le j \le n$, which generate $\pi_1(B(r))$. Thus, to any loop l_j we associate a disk with holes having one boundary component l_j, while the other boundary components are the loops $l'_{j,k}$ that lie on $S(r)$ and then null-homotopy disks $D(l'_{j,k})$ as above. Let D_j denote their union, which is a 2-dimensional subcomplex of $C(G, P)$ providing a null-homotopy of l_j.

Let A denote the simplicial complex made of $B(r)$ union a number of 2-disks $D(j)$ which are attached to $B(r)$ along the loops l_j. As the set of loops l_j generate $\pi_1(B(r))$ the complex A is simply connected.

We define the map $A \to B(r + I_G(C))$ by sending each disk $D(j)$ into the corresponding null-homotopy disk D_j. Since $\pi_1(A) = 0$ this map provides a π_1-resolution of $B(r - I_G(C))$.

The same proof works for π_1-bounded balls. $\qquad\square$

Completion of the proof of Theorem 1.3. We have to show that a group having a Cayley complex with π_1-bounded balls or spheres has linear connectivity radius and solvable word problem. The connectivity radius is at most linear since loops generating $\pi_1(B(r))$ are null-homotopic using uniformly bounded null-homotopies whose size depends only on C. Thus $\pi_1(B(r)) \to \pi_1(B(r + C'))$ is zero for $C' \ge I_G(C)$. This means that a loop in $B(r)$ is null-homotopic in the Cayley complex only if it is null-homotopic within $B(r + C')$. Moreover, the last condition can be checked by a finite algorithm for given r, and in particular one can check whether a given word of length r is trivial or not. $\qquad\square$

Remark 4.3. Some Cayley complexes of hyperbolic groups have π_1-bounded balls and spheres. For instance this is so for any of the Rips complexes, whose metric balls are known to be simply connected. It is likely that any Cayley complex associated to a finite presentation of a hyperbolic group has π_1-bounded balls. Furthermore, if a group G acts properly co-compactly on a CAT(0) space then the metric balls are convex and thus they are simply connected. It seems that this implies that any other space that is acted upon by the group G properly co-compactly (thus quasi-isometric to the CAT(0) space) should have also π_1-bounded balls. This would follow if the π_1-bounded balls property were a quasi-isometry invariant.

Remark 4.4. One can weaken the requirements in the definition of π_1-bounded balls, in the case of a Cayley complex of a group, as follows. We only ask that the group

$\pi_1(B(r))$ be normally generated by loops of length $\rho(r)$, where

$$\lim_{r \to \infty} r - I_G(\rho(r)) = \infty.$$

Note that the limit should be infinite for any choice of the isodiametric function I_G within its rough equivalence class. Then finitely presented groups satisfying this weaker condition are also qsf by means of the same proof.

Notice however that $I_G(r)$ should be non-recursive for groups G with non-solvable word problem, so that $\rho(r)$ grows extremely slow if non-constant. Moreover, if we only ask that the function $r - I_G(\rho(r))$ be recursive then the group under consideration should have again solvable word problem. In fact, we have by the arguments above the inequalities

$$I_G(r - I_G(\rho(r))) \leq f_G(r - I_G(\rho(r))) \leq r + I_G(\rho(r)) < 2r,$$

and so $I_G(r)$ is recursive since it is bounded by the inverse of a recursive function.

Remark 4.5. Recall that the Gersten–Gromov groups Γ_n have n-fold iterated exponential connectivity radius, and thus at least that large qsf growth, while Γ_* has connectivity radius higher than any iterated exponential (see [29], 4.C$_3$). We saw above that all these groups are qsf. However the last corollary shows that the metric balls in their Cayley complexes are not π_1-bounded, and thus their exhaustions by simply connected subcomplexes should be somewhat exotic. On the other hand we can infer from Remark 4.3 that their Cayley complexes have not (group invariant) CAT(0)-metrics, although they are both aspherical and qsf.

4.3. Rewriting systems. Groups admitting a rewriting system form a particular class among groups with solvable word problem (see [31] for an extensive discussion). A *rewriting system* consists of several replacement rules

$$w_j^+ \to w_j^-$$

between words in the generators of the presentation P. We suppose that both s and s^{-1} belong to P. A reduction of the word w consists of a replacement of some subword of w according to one of the replacement rules above. The word is said irreducible if no reduction could be applied anymore. The rewriting system is *complete* if for any word in the generators the reduction process terminates in finitely many steps and is said to be *confluent* if the irreducible words obtained at the end of the reduction are uniquely defined by the class of the initial word, as an element of the group. Thus the irreducible elements are the normal forms for the group elements. If the rules are not length increasing then one calls it a *geodesic rewriting system*. We will suppose that the rewriting system consists of finitely many rules.

Proposition 4.3. *A finitely presented group admitting a complete confluent geodesic rewriting system is qsf.*

Proof. In [31] is proved that such a group is almost convex and thus qsf (by Proposition 4.6.3, see also [39]).

Here is a shorter direct proof. We prove that actually the balls $B(r)$ in the Cayley complex are simply connected. Observe first that in any Cayley complex we have:

Lemma 4.1. *The fundamental group* $\pi_1(B(r))$ *is generated by loops of length at most* $2r + 1$.

Proof. Consider a loop $ep_1p_2 \ldots p_k e$ based at the identity element e and sitting in $B(r)$. Here p_j are the consecutive vertices of the loop. There exists a geodesic γ_j that joins p_j to e, of length at most r. It follows that the initial loop is the product of loops $\gamma_j^{-1} p_j p_{j+1} \gamma_{j+1}$. Since $p_j \in B(r)$ all these loops have length at most $2r + 1$. $\square$

Consider now the Cayley complex of a group presentation that includes all rules from the rewriting system. This means that there is a relation associated to each rule $w_j^+ \to w_j^-$. We claim that the balls $B(r)$ are simply connected. By the previous lemma it suffices to prove that loops of length at most $2r + 1$ within $B(r)$ are null-homotopic in $B(r)$.

Choose such a loop in $B(r)$ which is represented by the word w in the generators. We can assume that the normal form of the identity element is the trivial word. Since the loop is null-homotopic in the Cayley complex, the word w should reduce to identity by the confluent rewriting system. Let us consider some reduction sequence

$$w \to w_1 \to w_2 \to \cdots \to w_N \to e.$$

Each word w_i represents a loop based at the identity in the Cayley graph. Each step $w_j \to w_{j+1}$ is geometrically realized as a homotopy in which the loop associated to the word w_j is slided across a 2-cell associated to a relation from the rewriting system. Further the lengths of these loops satisfy $|w_j| \geq |w_{j+1}|$, since the length of each reduction is non-increasing by assumption. Thus $|w_j| \leq 2r + 1$ and this implies that the loop is contained within $B(r)$. This proves that the reduction sequence above is a null-homotopy of the loop w within $B(r)$. $\square$

Remark 4.6. The Baumslag–Solitar groups $B(1, n)$ and the solvgroups (i.e., lattices in the group SOL) admit rewriting system but not geodesic ones ([31]), since they are not almost convex.

Remark 4.7. More generally one proved in [31] that groups admitting a rewriting system are tame 1-combable and thus qsf by [38].

Remark 4.8. One might wonder whether finitely presented groups that have solvable word problem are actually qsf. Notice that an algorithm solving the word problem does not yield a specific null-homotopy disk for a given loop in the Cayley complex, but rather checks whether a given path closes up.

Remark 4.9. The geometry of null-homotopy disks (size, diameter, area) is controlled by the various filling functions associated to the group. However, in the qsf problem one wants to understand the position of the null-homotopy disks with respect to exhaustion subsets, which is of topological nature. The choice of the exhaustion is implicit but very important and it should depend on the group under consideration.

5. Extensions by finitely generated groups and the Grigorchuk group

5.1. Infinitely presented groups. Although it does not make sense to speak, in general, of the qsf property for an infinitely presented group, one can do it if, additionally, we specify a group presentation.

Recall first that the elementary Tietze transformations of group presentations are the following:

(T1) *Introducing a new generator.* One replaces $\langle x_1, x_2, \ldots \mid r_1, r_2, \ldots \rangle$ with $\langle y, x_1, x_2, \ldots \mid ys^{-1}, r_1, r_2, \ldots \rangle$, where $s = s(x_1, x_2, \ldots)$ is an arbitrary word in the generators x_i.

(T2) *Canceling a generator.* This is the inverse of (T1).

(T3) *Introducing a new relation.* One replaces $\langle x_1, x_2, \ldots \mid r_1, r_2, \ldots \rangle$ with $\langle x_1, x_2, \ldots \mid r, r_1, r_2, \ldots \rangle$, where $r = r(r_1, r_2, \ldots)$ is an arbitrary word in the conjugates of relators r_i and their inverses.

(T4) *Canceling a relation.* This is the inverse of (T3).

Definition 5.1. We say that two infinite presentations are *finitely equivalent* or, they belong to the same finite equivalence class, if there exists a finite sequence of elementary Tietze moves that changes one presentation into the other.

Proposition 5.1. *The qsf property is well defined for groups with a presentation from a fixed finite equivalence class: if the Cayley complex $C(G, P)$ is qsf, then the Cayley complex $C(G, Q)$ is qsf for any presentation Q of G which is finitely equivalent to P.*

Proof. The proof from [8] works in this case word by word. □

Most of the properties shared by the qsf finitely presented groups hold, more generally, for the qsf infinitely presented groups equipped with the convenient finite equivalence class of presentations. For instance, for any infinite groups G, H and group presentations (G, P_G) and (H, P_H), the Cayley complex $C(G \times H, P_G \times P_H)$ is qsf, where $P_G \times P_H$ is the product presentation of $G \times H$.

Remark 5.1. By the method from Remark 2.5, one can obtain an infinite presentation of a group whose Cayley complex is not wgsc. However, it is more difficult to prove that a *specific* infinite presentation of some group is qsf, for instance in the case of Burnside groups.

Remark 5.2. The previous proposition might be extended farther. In fact one could allow infinitely many Tietze moves if they do not accumulate at finite distance but the complete definition is quite involved.

5.2. Extensions by infinitely presented groups. One method for constructing finitely presented groups is to use suitable extensions of finitely presented groups by infinitely presented ones. We did not succeed in proving that *all* such extensions are qsf. However, for finitely presented extensions by *finitely generated* groups things might simplify considerably. We start with the following definition from [3].

Definition 5.2. An *endomorphic* presentation is an expression of the form $P = \langle S|Q|\Phi|R \rangle$, where S is an alphabet (i.e., a set of symbols), Q, R are sets of reduced words in the free group $F(S)$ generated by S and Φ is a set of injective free group homomorphisms $F(S) \to F(S)$. The endomorphic presentation is *finite* if all sets S, Q, Φ, R are finite. This data defines the group

$$G(P) = F(S)/\langle Q \cup \bigcup_{\phi \in \Phi^*} \phi(R) \rangle^{\sharp},$$

where $\langle\,,\,\rangle^{\sharp}$ denotes the normal closure and Φ^* is the monoid generated by Φ, i.e., the closure of $\Phi \cup \{1\}$ under the composition. The endomorphic presentation is said to be *ascending* if $Q = \emptyset$.

Bartholdi observed that groups with finite ascending endomorphic presentations are naturally contained in finitely presented groups constructed as generalized ascending HNN extensions, by adding finitely many stable letters. Each $\phi \in \Phi$ induces a group endomorphism $\varphi \colon G \to G$ and we suppose that the correspondence $\phi \to \varphi$ is one-to-one so Φ is also a set of endomorphisms of G. Then the finitely presented group

$$\bar{G} = \langle S \cup \Phi \mid Q \cup R \cup \{\phi^{-1}s\phi = \phi(s); \ s \in S, \ \phi \in \Phi\}\rangle$$

is a *generalized HNN extension* of G with stable letters corresponding to $\varphi \in \Phi$. If the endomorphic presentation is ascending (i.e., $Q = \emptyset$), and the endomorphisms $\varphi \colon G \to G$ are injective then the natural homomorphism $G \to \bar{G}$ is an embedding and $\bar{G}$ will be what is standardly called an ascending HNN extension with set of stable letters Φ. Further, if the natural map $G \to \bar{G}$ is an embedding, then we can assume that the endomorphic presentation of G is ascending. In fact the relations from Q and the conjugacy relations in $\bar{G}$ imply that the relations $\bigcup_{\phi \in \Phi^*} \phi(Q)$ are satisfied in G. Thus we can replace R by $R \cup Q$ in the endomorphic presentation of G and obtain the same generalized HNN extension group $\bar{G}$.

Set $N(G)$ for the normal subgroup of $\bar{G}$ generated by G. We then have an exact sequence

$$1 \to N(G) \to \bar{G} \to L \to 1,$$

where the quotient $L = \bar{G}/N(G)$ has the presentation, P_L, as follows:

$$L = \langle S \cup \Phi \mid S \cup R \cup \{\phi^{-1}s\phi = \phi(s); s \in S, \phi \in \Phi\}\rangle.$$

Using elementary Tietze moves one sees that L is the free group generated by the set of stable letters Φ.

The images of elements of $\Phi^* \setminus \{1\}$ in L will be called the *positive* elements of L. Let $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_k\}$.

Very interesting examples of groups with finite ascending endomorphic presentations which are also branch groups appeared in the constructions of Bartholdi (see [3]).

Theorem 5.1. *Let G be a finitely generated group admitting a finite ascending endomorphic presentation P_G such that each $\varphi_j \in \Phi$ is injective and $\bar{G}$ be its associated HNN extension. Assume that the group G endowed with the presentation P_G is 1-tame. Then $\bar{G}$ is qsf.*

Remark 5.3. The words from $\phi(R)$, $\phi \in \Phi^*$ are *unreduced* words in the free group $F(S)$, namely one can have adjacent canceling letters. This will be essential for the proof of Lemma 5.6. Allowing unreduced words makes the hypothesis that the presentation P_G of G is 1-tame more difficult to check and potentially more restrictive than in the case when the words from R_∞ are reduced.

5.3. Plan of the proof of Theorem 5.1. Assume that we have a HNN extension as in the hypothesis which induces the exact sequence

$$1 \to N(G) \to \bar{G} \to L \to 1.$$

Consider then an infinite endomorphic presentation P_G: $G = \langle A \mid R_\infty\rangle = \langle a_i \mid R = \{R_j\}, \Phi = \{\phi_1, \ldots, \phi_k\}\rangle$, with an infinite set of relators $R_\infty = \bigcup_{j,\phi \in \Phi^*} \phi(R_j)$ for G and the standard presentation P_L: $L = \langle B \mid \emptyset\rangle$ for the free group L with generators set $B = \{b_1, \ldots, b_k\}$, where every b_j corresponds to the stable letter ϕ_j. One obtains an infinite presentation for $\bar{G}$ by putting together the two presentations above, namely: $P_{\bar{G}}(\infty)$: $\bar{G} = \langle A \cup B \mid R_\infty \cup T\rangle$, where the elements of T express the HNN conditions for stable letters. Thus, each element of T has the form $b^{-1}abw(a, b)$, where $a \in A$, $b \in B$ and $w(a, b) \in F(A)$ is some word in the generators A. We call them *conjugacy relations*.

Henceforth we suppose that T is given by $T = \{b_j^{-1}a_ib_j = \phi_j(a_i) \in A^*, a_i \in A, b_k \in B\}$. Notice that B contains only positive letters. Thus it might not make sense to consider relations of type $b_j^{-1}ab_j$ unless a is a word representing an element of the image of G by the endomorphism associated to b_j.

However, the group $\bar{G}$ could be defined by the same set of generators $A \cup B$ and a finite subset of relations from above. We may assume that this finite presentation is $P_{\bar{G}}$: $\bar{G} = \langle A \cup B \mid R \cup T\rangle$, where $R \subset R_\infty$, is a finite set of relations.

The plan of the proof is as follows. The exact sequence induces a kind of foliation of the Cayley complex $C(\bar{G}, P_{\bar{G}})$ by horizontal leaves associated to $N(G)$. These

leaves are connected by means of vertical tubes associated to conjugacy relations and going upward. Given a compact C, we can use these vertical tubes to push up loops in C and find a larger compact K whose fundamental group is generated by loops lying in a top horizontal leaf K_u far from C. If G were finitely presented then the horizontal leaves would be simply connected so that loops could be homotopically killed inside the top leaf. When G is not finitely presented, the fundamental group of a (connected component of a) horizontal leaf is generated by the relations in $R_\infty - R$. Thus loops in the top horizontal leaf are now freely homotopic to relation loops expressing words from $R_\infty - R$. The 1-tameness of $C(G, P_G)$ enables us to consider only relation loops which are either contained in a larger compact E of the horizontal leaf and which are disjoint from K_u or else contained in K_u. Each such loop has a particularly nice null-homotopy in $C(\bar{G}, P_{\bar{G}})$ by expressing the relation as an element of $\Phi^*(R)$. Namely, there exists a canonical vertical tube going downward from that loop to a loop which is null-homotopic in the bottom horizontal leaf. We add then more material to K so that all canonical homotopies of loops from the top leaf be either contained in K or else disjoint from C. Here one makes use of the fact that the monoid of positive elements in L defines an order on the set of horizontal leaves. Then loops in K are freely homotopic to loops which are null-homotopic in $C(\bar{G}, P_{\bar{G}}) - C$, and so the Cayley complex is 1-tame.

5.4. Preliminaries from Brick and Mihalik and the proof of Theorem 1.4.

Our aim is to prove that the Cayley complex $C(\bar{G}, P_{\bar{G}})$ is qsf. We follow closely the proof given by Brick and Mihalik in [7] for the fact that the extension of an infinite finitely presented group by an infinite finitely presented group is qsf. First, we state below the necessary adjustments for the main lemmas from [7] work now for HNN extensions. Then we will point out the arguments which have to be modified in the present setting.

Let $C(H, P_H)$ denote the Cayley 2-complex associated to the presentation P_H of the group H. Consider now the subcomplex $X(G) \subset C(\bar{G}, P_{\bar{G}})$ spanned by the vertices of $N(G) \subset \bar{G}$. Observe that the subcomplex $X_0(G)$ spanned by $G \subset N(G)$ can be obtained from $C(G, P_G)$ by removing the 2-cells corresponding to the relations from $R_\infty - R$. Moreover, $X(G)$ is the disjoint sum of copies of $X_0(G)$. In fact for each coset $w \in N(G)/G$ we have a copy $wX_0(G) \subset X(G)$ obtained by left translating by w. These copies are disjoint because any edge of $X(G)$ corresponds to a generator of G and so a path in $X(G)$ corresponds to an element of G. Thus $wX_0(G)$ intersects $w'X_0(G)$ only if $wG = w'G$, for $w, w' \in N(G)$, and in this case they coincide.

To each $x \in L$ we associate the *horizontal slice* $xX(G) \subset C(\bar{G}, P_{\bar{G}})$ obtained by a left translation of $X(G)$ so that it projects down onto $x \in L$ under the map $\bar{G} \to L$. The 0-skeleton of the Cayley complex $C(\bar{G}, P_{\bar{G}})$ is then decomposed as the disjoint union of 0-skeleta of slices $xX(G)$, over $x \in L$. The paths (edges) which are contained in such a horizontal slice $xX(G)$ will be called *A-paths* (respectively *A*-edges). The *B-edges* are those edges of $C(\bar{G}, P_{\bar{G}})$ which project onto the generators B of L. The 2-cells corresponding to relators in T will be called *conjugation*

cells. Notice that the attaching map of a conjugation cell is of the form $b_1 a b_2^{-1} w$, where b_1 and b_2 are B-edges corresponding to the same $b \in B$, and w is the A-path corresponding to the word $w = w(a, b)$ appearing in the respective conjugacy relation. The loops in $C(\bar{G}, P_{\bar{G}})$ are called of *type* 1 if they are conjugate to A-loops and of *type* 2 otherwise.

Those subcomplexes of $C(\bar{G}, P_{\bar{G}})$ which are finite, connected and which intersect each $w X_0(G)$, for $w \in \bar{G}$, in a connected – possibly empty – subset are called *admissible*.

Let $C \subset C(\bar{G}, P_{\bar{G}})$ be a finite connected subcomplex. By adding finitely many edges we may assume that C is admissible. We want to show that there is a larger subcomplex K, obtained by adjoining finitely many edges and conjugation cells such that $\pi_1(K)$ is normally generated by finitely many loops in K which are null-homotopic in $C(\bar{G}, P_{\bar{G}}) - C$.

Lemma 5.1. *Let Z be an admissible subcomplex of $C(\bar{G}, P_{\bar{G}})$. Let $\{u_1, u_2, \ldots, u_n\}$ be a system of generators of $\pi_1(Z)$.*

(1) *If e is an A-edge that meets Z then $Z \cup e$ is admissible. Further, $\pi_1(Z \cup e)$ is generated by $\{u_1, u_2, \ldots, u_n, \tau \lambda \tau^{-1}\}$, where λ is an A-loop in the copy of $X(G)$ containing e.*

(2) *If Δ is a conjugation cell with $\partial \Delta = b_1^{-1} a b_2 w^{-1}$, and $b_1 \subset Z$, then $Z \cup \Delta$ is admissible. Further $\pi_1(Z \cup \Delta)$ is generated by $\{u_1, u_2, \ldots, u_n, v_1, \ldots, v_m\}$, where each v_i is a type 1 generator. Moreover, if $a \subset Z$ then each v_i is of the form $\tau \lambda \tau^{-1}$, where λ is an A-loop in the copy of $X(G)$ containing the endpoint of b_1.*

Proof. Lemma 2.1 from [7] extends trivially to the present situation. $\square$

Proposition 5.2. *There exists a complex C_1 obtained from C by adjoining finitely many A-edges and conjugation cells as in Lemma 5.1 such that $\pi_1(C_1)$ is generated by classes of loops of type 1.*

Proof. We want to transform each loop of type 2 into a loop of type 1 by using homotopies which can be realized after adjoining finitely many A-edges and conjugation cells to C (satisfying the requirements of Lemma 5.1). Since one does not create any additional type 2 loop, we end up with a complex C_1 whose fundamental group is generated by classes of loops of type 1.

Consider first the case when there is only one stable letter, $B = \{b\}$. We use the conjugacy relation $b^{-1} a = \phi(a) b$, for $a \in A$, to move the B-edge labeled b^{-1} to the right of the adjacent A-edge labeled $\phi(a)$. In meantime use the conjugacy relation $ab = b\phi(a)$ to move the B-edge labeled b to the left of the adjacent A-edge labeled $\phi(a)$. Keep moving B-edges this way until two B-edges labeled b^{-1} and b become adjacent, in which case the two edges will be removed as their labels cancel and resume the process. This procedure eventually stops when the initial loop is

transformed into the composition of an A-path with a B-path (or vice-versa). Now the extension $\bar{G} \to L$ splits (because L is free) and hence the B-path above should be a loop. This B-loop is then homotopically trivial in its own image, since L is free and so we eventually obtain an A-loop.

Suppose now that the number of stable letters is $k \geq 2$. Set $\widetilde{B} = B \setminus \{b_k\}$, $\widetilde{\Phi} = \Phi \setminus \{\phi_k\}$, and $\widetilde{G}$ for the HNN extension associated to the set of stable letters $\widetilde{\Phi}$ (which is not necessarily finitely presented). Then G has a natural injective homomorphism $i \colon G \to \widetilde{G}$, and thus φ_k induces an isomorphism $\tilde{\varphi}_k \colon i(G) \to i(\varphi_k(G))$ between two subgroups of $\widetilde{G}$.

Therefore the group $\bar{G}$ is also the HNN extension $\widetilde{G} *_{\tilde{\varphi}_k}$ with base group $\widetilde{G}$, stable letter b_k and associated subgroups $i(G)$ and $i(\varphi_k(G))$.

Any loop in $C(\bar{G}, P_{\bar{G}})$ corresponds to a word W in $\bar{G}$ representing the trivial element in the group. Britton's lemma tells us that either the letter b_k does not occur in W or else W contains an unreduced word with respect to the stable letter b_k, namely either

(1) a subword of the form $b_k w b_k^{-1}$, with w a word representing an element of $\tilde{\varphi}_k(G)$; or

(2) a subword of the form $b_k^{-1} w b_k$, with w a word representing an element of $i(G)$.

Thus w is an A-word, i.e., a word using only the letters A (constrained to belong to the image of ϕ_k in case (1)). In the first situation we choose an A-word z so that $\phi_k(z)$ represents the same element in G as w. If α is an edge loop representing a generator of $\pi_1(C)$ and $b_k w b_k^{-1}$ is an unreduced subword of α (with respect to the HNN structure with stable letter b_k) then write α as $\tau w \delta$. Add the type 1 generator $\tau \phi_k(z) w^{-1} \tau^{-1}$ to the list of generators of $\pi_1(C)$ and replace the generator α, with the product of the two generators $\tau \phi_k(z) w^{-1} \tau^{-1}$ and α (which is $\tau b_k \phi_k(z) b_k^{-1}$). This change replaces the subword $b_k w b_k^{-1}$ of α by $b_k \phi_k(z) b_k^{-1}$. Now, in either case the occurrence of the unreduced subword can be eliminated by adjoining conjugacy 2-cells along the paths labeled z and respectively w. The new loop has fewer B-edges than the former one and we keep eliminating unreduced subwords until all occurrences of the letter b_k are removed. The same method permits to get rid of all stable letters and hence to transform the loop into a composition of A-loops and hence a loop of type 1. $\square$

Proposition 5.3. *Suppose that C_1 satisfies the requirements of Proposition 5.2. Then there exists a finite complex K obtained from C_1 by adjoining finitely many conjugation cells and finitely many B-edges e_j, each e_j having one endpoint $u_j \notin C_1$, such that $\pi_1(K)$ is generated by the classes of loops $\{v_1, \ldots, v_m\}$ so that each v_s is freely homotopic in K to an A-loop v_s based at some u_j and lying entirely in the layer $K \cap u_j X_0(G)$. Moreover, each v_j is disjoint from C_1.*

Proof. Given a compact K we define the *layer* $K_x = K \cap x X(G)$. We want to adjoin conjugacy cells in order to homotop all (type 1) generators of $\pi_1(C_1)$ into a disjoint union of layers.

Define an *order* on L by setting $y < x$ if $y^{-1}x \in L$ is positive. This order extends to the set of layers, by saying that the (non-empty) layer K_y is *below* the layer K_x if $y < x$. We extend this terminology to (oriented) B-edges, by declaring them positive if their label is positive.

The proof of this proposition follows along the lines of ([7], Section 4). We define first an oriented graph $\Gamma = \Gamma_{C_1}$ whose vertex set Γ^0 is the set of non-empty layers of C_1, i.e., slices intersecting non-trivially C_1.

The vertices w_1, w_2 of Γ are joined by an oriented edge of Γ if there exists some positive B-edge joining the slice w_1 to the slice w_2. Notice that we do not ask that the respective layers be connected by a positive edge. We will consider subcomplexes C_2 obtained from C_1 by adding conjugacy 2-cells. Given such a complex C_2 and a subset $\mathcal{A}$ of Γ^0 we say that $\mathcal{A}$ *carries* the loops of C_2 if $\pi_1(C_2)$ has a set of generators $\{v_1, v_2, \ldots, v_m\}$, where all v_i are $\mathcal{A}$-loops freely homotopic in C_2 to $\mathcal{A}$-loops that are in the union of slices in $\mathcal{A}$.

A vertex v of Γ is *extremal* if there is no outgoing edge of Γ issued from it. The key step is the following:

Lemma 5.2. *The set of extremal vertices of Γ carries the loops for a suitable chosen C_2 which is obtained from C_1 by adjoining conjugacy cells.*

Proof. Let us start with $C_2 = C_1$. Then the loops of C_2 are carried by the set of all vertices of Γ. Let $\mathcal{T}$ be a maximal subtree of Γ. A vertex of $\mathcal{T}$ is $\mathcal{T}$-*extremal* if all its adjacent edges in $\mathcal{T}$ are incoming. Let x be a non $\mathcal{T}$-extremal vertex of $\mathcal{T}$ and xy be an oriented (outgoing) edge of $\mathcal{T}$ labeled by $b \in B$. Let u be an $\mathcal{A}$-loop contained in the slice x. We add conjugacy 2-cells to C_2 along the $\mathcal{A}$-loop u in the direction given by b. In the new complex, still called C_2, we can freely homotop u to an $\mathcal{A}$-loop in the slice y. Notice that the slice x might intersect C_2 in a non-connected subcomplex. By adjoining conjugacy 2-cells one might create additional type 1 generators, but according to Lemma 5.1 the new loops are $\mathcal{A}$-loops in the slice y. Proceed in the same way for each non $\mathcal{T}$-extremal vertex of $\mathcal{T}$. We obtain that $\mathcal{A}$-loops in C_2 are carried by the subset of $\mathcal{T}$-extremal vertices of any maximal subtree $\mathcal{T}$.

Assume first that there is only one stable letter b. Then any maximal subtree $\mathcal{T}$ (and actually the graph Γ) is an oriented chain since otherwise we would have a vertex of it with two incoming (or outgoing) B-edges, which is impossible. Moreover the terminal vertex of the chain is both $\mathcal{T}$-extremal and the unique extremal vertex. The claim follows.

In general we can have several $\mathcal{T}$-extremal vertices which might not be extremal. We will show that we can get rid of those vertices which are $\mathcal{T}$-extremal but not extremal by changing the subtree $\mathcal{T}$ and adjoining conjugacy 2-cells. Let x be such a vertex of $\mathcal{T}$. By hypothesis there exists some positive B-edge e joining x to a vertex y of Γ. Since $\mathcal{T}$ was a maximal subtree the graph $\mathcal{T} \cup e$ admits a (non-oriented) minimal length circuit which passes through e. Let $f \subset \mathcal{T}$ be the (incoming) edge of this circuit adjacent to the vertex x and distinct from e. Consider then the new

maximal subtree $\mathcal{T}' = \mathcal{T} \cup \{e\} \setminus \{f\}$. The vertex x is not anymore $\mathcal{T}'$-extremal and the set $\mathcal{A}$ is replaced by $\mathcal{A}' = \mathcal{A} - \{x\} \cup \{y\}$. If y is $\mathcal{T}'$-extremal but not extremal then we continue this process and get the sequences x_n of vertices of Γ, and $\mathcal{T}^n$ of maximal subtrees. At some point we will find a vertex which is either

(1) non $\mathcal{T}^n$-extremal, and then we can reduce the number of $\mathcal{T}^n$-extremal vertices, i.e., the size of the set carrying the loops of C_2; or

(2) an extremal vertex, and we are done; or else

(3) we turn back to a vertex which has been considered before during this process. In this case this means the sequence of vertices that we meet contains an oriented circuit made of B-edges, which is impossible since $(L, <)$ is ordered.

This proves that extremal vertices of Γ carry the loops of C_2, as claimed. $\qquad\square$

A layer is said to be *extremal* if it lies in a slice corresponding to an extremal vertex of Γ. Since C_2 is connected it can be arranged so that all extremal layers are connected. In fact, two points of an extremal layer can be joined by a path contained in C_2. By adding conjugacy cells we transform this path into an A-path followed by a B-path (or vice-versa). The B-path should be a loop and hence homotopically trivial in its own image since L is a free group. Thus the two points are joined by a path contained in the layer. Although adding conjugacy cells can create new loops, these are contained in the same connected component of the extremal layer.

Remark 5.4. If there is only one stable letter then the extremal layer provided by the lemma above is unique and so Proposition 5.3 follows.

We can suppose, without loss of generality, that C_2 contains the identity element of the group. Recall that $p \colon \bar{G} \to L$ denotes the natural group epimorphism. We will need further the following technical lemma:

Lemma 5.3. *If C_{2x} is an extremal layer then we can write $p(x) = by$, where $b \in B$ is a positive generator of L and y is an element of L represented by a reduced word in B not starting with b^{-1}. Moreover, the layer associated to y is not empty.*

Proof. We have a cellular map $C(\bar{G}, P_{\bar{G}}) \to C(L, P_L)$ between the respective Cayley complexes induced by p. Here P_L is the presentation of L induced from $\bar{G}$. Further, we have a cellular map $C(L, P_L) \to C(L)$, where $C(L)$ is the tree associated to the presentation $L = \langle \Phi \rangle$ with empty set of relations. We denote by $p \colon C(\bar{G}, P_{\bar{G}}) \to C(L)$ the composition of the two cellular projections. Observe then that the layer C_{2x} is below the layer C_{2y} if and only if there is a positive path from $p(x)$ to $p(y)$ in $C(L)$.

Since C_2 is connected its image $p(C_2)$ is also connected in the tree $C(L)$. Thus, for any $x \in C_2$ the geodesic in $C(L)$ joining the origin 1 to $p(x)$ is contained in $p(C_2)$. We can suppose that the distance from $p(x)$ to the origin is at least 1. Let y be the vertex of this geodesic at distance 1 from $p(x)$. If C_{2x} is an extremal layer of

C_2, then the oriented edge $yp(x)$ of $C(L)$ is positive. In fact, $y \in p(C_2)$ and thus the layer C_{2y} is non-empty. If the edge $yp(x)$ were negative then the layer C_{2x} would be below the layer C_{2y}, contradicting the extremality of C_{2x}. This proves that we can write $p(x) = by$, where $b \in B$ is a positive generator and y is a reduced word not starting with b^{-1}. $\qquad \square$

For each extremal layer of C_2 choose a vertex w_j of it and a positive B-edge $e_j = w_j u_j$ issued from w_j. We adjoin the edges e_j to C_2 and call K the new complex.

Lemma 5.4. *The layers K_{u_j} are pairwise disjoint, disjoint from C_2 and carry the loops of $\pi_1(K)$.*

Proof. First, the slice $u_j X(G)$ through u_j is disjoint from C_2. If this were not true, then C_{2w_j} and $C_2 \cap u_j X(G)$ would represent vertices of Γ connected by a positive (outgoing) edge, contradicting the extremality of C_{2w_j}.

Next, the slices $u_{j_1} X(G)$ and $u_{j_2} X(G)$ are disjoint for distinct j_1, j_2. Otherwise the two slices must coincide so that $p(u_{j_1}) = p(u_{j_2})$. Observe that $p(u_j) = b_{m(j)} p(w_j)$, where $b_{m(j)}$ is the positive generator from B associated to the B-edge e_j. Lemma 5.3 shows that $p(w_j) = b_{n(j)} y_j$, where $b_{n(j)} \in B$ is a positive generator and y_j is a reduced word in the B-letters not starting with $b_{n(j)}^{-1}$. Then we have the identity $b_{m(j_1)} b_{n(j_1)} y_{j_1} = b_{m(j_2)} b_{n(j_2)} y_{j_2}$ in the free group L. This implies that $j_1 = j_2$.

Eventually, u_j are extremal vertices for the graph Γ_K associated to K. It suffices to show that there is no positive B-edge connecting the slices $u_j X(G)$ and $u_k X(G)$. If such an edge, labeled b_i, exists then $p(u_j) = b_i^{-1} p(u_k)$. Recall that $p(u_j) = b_{m(j)} p(w_j) = b_{m(j)} b_{n(j)} y_j$, where y_j is a reduced word in the B-letters not starting with $b_{n(j)}^{-1}$ and $p(u_k) = b_{m(k)} p(w_j) = b_{m(k)} b_{n(k)} y_k$, where y_k is a reduced word in the B-letters not starting with $b_{n(k)}^{-1}$. Then we have the equality $b_{m(j)} b_{n(j)} y_j = b_i^{-1} b_{m(k)} b_{n(k)} y_k$ holding in L. This forces $i = m(k)$, and hence $b_{m(j)} p(w_j) = p(w_k)$ holds in L, which implies that there is a positive B-edge labeled $b_{m(j)}$ joining the slices through w_j and through w_k. In particular, C_{2x_j} is not an extremal layer of K. This contradiction shows that K_{u_j} are extremal layers of K.

Further $\pi_1(K)$ is isomorphic to $\pi_1(C_2)$ (since we simply added a number of disjoint edges) and the loops lying in the extremal slices of C_2 can be homotopically pushed into the slices through the u_j. Thus the set of layers K_{u_j} is the set of all extremal layers of K. This proves the lemma. $\qquad \square$

Moreover, since K is connected, it can be arranged so that $K \cap u_j X(G) = K \cap u_j X_0(G)$ is connected for every j. Then the previous lemmas prove the proposition. $\qquad \square$

End of the proof of Theorem 1.4. We have to prove that an ascending HNN extension $\bar{G}$ of a finitely presented group G is qsf. This is a consequence of Proposition 5.3.

If G is finitely presented, then each connected component $X_0(G)$ of $X(G)$ is simply connected as being the Cayley complex associated to G. Therefore the loops ν_j are null-homotopic in $uX(G)$ and thus in $C(\bar{G}, P_{\bar{G}}) - C$. This implies immediately that the complex $C(\bar{G}, P_{\bar{G}})$ is qsf.   □

However, when P_G is infinite this argument does not work anymore and we need additional ingredients.

5.5. Constructing homotopies using extra 2-cells from $R_\infty - R$. Consider now a loop l in $K_u \subset uX(G)$, for an extremal layer K_u. Now $X(G)$ is the disjoint union of copies of $X_0(G)$ and each $X_0(G)$ embeds into the simply connected Cayley complex $C(G, P_G)$ of G. Therefore $uX(G)$ can be embedded in the disjoint union of copies of the simply connected complex $C(G, P_G)$. The later complex can be viewed as having the same 0- and 1-skeleton as $uX(G)$, the 2-cells from $uX(G)$ and the additional 2-cells coming from the relations in $R_\infty - R$. Moreover the loop l should be contained into one connected component of the disjoint union. Thus there exists a null-homotopy of l inside the respective $C(G, P_G)$. It is then standard that this implies the existence of a simplicial map $f : D^2 \to C(G, P_G)$ from the 2-disk D^2 suitably triangulated, whose restriction to the boundary is the loop l. The image $f(D^2)$ intersects only finitely many cells of $C(G, P_G)$ by compactness, thus there are only finitely many open 2-cells e of $C(G, P_G) - uX(G)$ for which the inverse image $f^{-1}(e)$ is non-empty. Consider the set $\{e_1(l), \ldots, e_m(l)\}$ of the 2-cells with this property. Each such 2-cell corresponds to a relation from $R_\infty - R$. Since f was supposed to be simplicial, $f^{-1}(e_{i,u}(l))$ is a finite union of 2-cells $e_{i_j,u}$ of the triangulated D^2. Moreover, the boundary paths $\partial e_{i,u}(l)$ are contained in $uX(G)$.

We say that a set of m loops $\{l_j\}$ is *null-bordant* in X if there exists a continuous map σ, called a *null-bordism*, from the m-holed 2-sphere to the space X such that its restrictions to the boundary circles is $\{l_j\}$. In particular, the union of loops $l \bigcup_i \partial e_i(l)$ obtained above is null-bordant in $uX(G)$. Thus there exists a map $\sigma(l)$ from the m-holed sphere to $uX(G)$ whose restriction to the boundary is $l \bigcup_i \partial e_i(l)$, and we write $l \bigcup_i \partial e_i(l) = \partial \sigma(l)$. We will make use of this argument further on.

Recall now that $C(G, P_G)$ was supposed to be 1-tame. Thus $K_u \subset \widetilde{E}_u$, where the compact $\widetilde{E}_u$ has the property that any loop $l \subset \widetilde{E}_u$ is homotopic within $\widetilde{E}_u$ to a loop l' lying in $\widetilde{E}_u - K_u$ and which is further null-homotopic in $C(G, P_G) - K_u$. Let $E_u = \widetilde{E}_u \cap uX(G)$, so that E_u can be written as $E_u = \widetilde{E}_u - \bigcup_{j=1}^k e_{j,u}$, where $\{e_{j,u}\}_{j=1,\ldots,k}$ is a suitable finite set of 2-cells of (the disjoint union of copies of) $C(G, P_G)$ which are not 2-cells of $uX(G)$. Notice that $\partial e_{j,u} \subset E_u$ for any j. A homotopy between the loop l and the loop l' within $\widetilde{E}_u$ induces by the argument above a null-bordism $H(l, l')$ between l' and $l \cup \partial e_{j,u}(l)$ within E_u, where the set $\{e_{j,u}(l)\}$ is a suitable subset of $\{e_{j,u}\}_{j=1,\ldots,k}$. Furthermore a null-homotopy of l' in $C(G, P_G) - K_u$ induces a null-bordism $N(l')$ of $l' \cup \partial \delta_{j,u}(l)$ within $uX(G) - K_u$, where $\delta_{j,u}(l)$ are (finitely many) 2-cells from $C(G, P_G) - K$ (which are not in $uX(G)$).

We consider now a finite set $J_u = \{l_{j,u}\}$ of loops which are normal generators of $\pi_1(E_u)$ and let $\{\delta_{1,u}, \ldots, \delta_{N,u}\}$ be the set of all 2-cells $\delta_{i,u}(l)$ obtained by considering all $l \in J_u$.

The key point is that $\partial e_{j,u}$ are either contained in E_u or else disjoint from K_u (and hence from K), while $\partial \delta_{j,u}$ are disjoint from K. Notice that it is the 1-tameness of $C(G, P_G)$ which permitted us to discard the 2-cells of $C(G, P_G)$ whose boundaries are not contained in E_u but intersect K_u.

5.6. Standard null-homotopies. The boundary paths $\partial e_{i,u}, \partial \delta_{i,u} \subset uX(G) \subset C(\bar{G}, P_{\bar{G}})$ are null-homotopic within $C(\bar{G}, P_{\bar{G}})$ and thus bound 2-disks $D(e_{i,u})$, $D(\delta_{i,u}) \subset C(\bar{G}, P_{\bar{G}})$. However, there exist some special null-homotopies for them, which are canonical up to the choice of a basepoint. At this place we will make use of the fact that the presentation P_G is an endomorphic presentation.

Consider $\{\lambda_i\}$ be the set of loops of the form $\partial e_{i,u}$ or $\partial \delta_{j,u}$, for unifying the notations in the construction below. The loops λ_j represent words which are relations from P_G and thus can be written in the form

$$\lambda_j = \varphi_{j_1} \varphi_{j_2} \ldots \varphi_{j_{k_j}} (r_{\alpha_j})$$

where $r_{\alpha_j} \in R$ and the j_i's depend on j. We have implicitly chose the convenient orientation of the loops λ_j in order to be recovered from r_{α_j} and not from $r_{\alpha_j}^{-1}$. It is important to notice that all φ_j appear only with positive exponents in the expression above. Recall that R is the set of relations that survive in $P_{\bar{G}}$. We can identify a loop with the word that represents that loop in the Cayley complex. Thus, by abuse of notation, we can speak of $\varphi_k(l)$ where l is a loop. Observe that the loop $\varphi_m(l)$ is freely homotopic to the loop l, since it is associated to a specific conjugate in terms of words. This homotopy is the cylinder $C_m(l)$ which is the union of all conjugacy cells based on elements of l and using the vertical element b_m. The loop corresponding to $\varphi_{j_1} \varphi_{j_2} \ldots \varphi_{j_{k_j}} (r_{\alpha_j})$ is one boundary of the cylinder $C_{j_1}(\lambda_j^{(1)})$. The other boundary of this cylinder is the loop $\lambda_j^{(1)} = \varphi_{j_2} \ldots \varphi_{j_{k_j}} (r_{\alpha_j})$. The second loop has, in some sense, smaller complexity than the former one and we can continue to simplify it. The cylinder $C_{j_s}(\lambda_j^s)$ interpolates between $\lambda_j^{(s)} = \varphi_{j_{s+1}} \ldots \varphi_{j_{k_j}} (r_{\alpha_j})$ and $\lambda_j^{(s-1)}$. Set $C(\lambda_j) = \bigcup_{1 \le s \le k_j} C_{j_s}(\lambda_j^{(s)})$. Finally recall that $r_{\alpha_j} \in R$, and thus the corresponding loop bounds a 2-cell ε_{α_j} of $X(G)$. Thus $D(\lambda_j) = C(\lambda_j) \cup \varepsilon_{\alpha_j}$ is a specific 2-disk giving an explicit null-homotopy of λ_j within $C(\bar{G}, P_{\bar{G}})$.

5.7. Saturation of layers. Given a compact K we considered the layers $K_x = K \cap xX(G)$. Observe, following [8], that we can suppose that all intersections $K \cap xX_0(G)$ are connected for all x where non-empty, and $K_x = K \cap xX_0(G)$ if K_x is an extremal layer.

The finite complex K is said to be *saturated* if it has the following property. For each vertex c of C and positive B-path at c that ends at c' in an extremal layer of K the endpoint c' is in K.

Lemma 5.5. *We can assume that the complex K obtained in Proposition* 5.3 *has saturated layers.*

Proof. It suffices to add finitely many conjugacy cells in order to achieve the saturation. Moreover, when adjoining conjugacy cells we do not create extra loops of type 2 and hence the requirements in Proposition 5.3 are still satisfied. $\qquad\square$

Recall now that $\partial \delta_{j,u}$ are disjoint from K_u, for any extremal layer K_u. We then have:

Lemma 5.6. *If K is saturated then $D(\partial \delta_{j,u}) \cap C = \emptyset$.*

Proof. If we had a point c belonging to $D(\partial \delta_{j,u}) \cap C$, then there would exist a path from c to a point c' in $\partial \delta_{j,u}$, which contains only vertical segments from the cylinder $C(\partial \delta_{j,u})$. This is then a positive B-path and thus its endpoint c' belongs to K_u, by the saturation hypothesis, but this contradicts the fact that $\partial \delta_{j,u} \subset E_u - K_u$. $\qquad\square$

Remark 5.5. The analogous statement fails in the case when we take for R_∞ the set of *reduced* words in the free group $F(A)$ coming from iterating the ϕ_i on the set R, in general.

Although $\partial e_{j,u}$ might intersect K they are contained in E_u. Consider then $W = \{(\alpha, u); D(\partial e_{\alpha,u}) \cap C \neq \emptyset\}$. We construct therefore the set

$$ Z = K \cup \bigcup_u E_u \cup \bigcup_{(\alpha,u) \in W} D(\partial e_{\alpha,u}). $$

Lemma 5.7. *If K is saturated then the inclusion $K \cup \bigcup_u E_u \hookrightarrow Z$ induces a surjection $\pi_1(K \cup \bigcup_u E_u) \to \pi_1(Z)$.*

Proof. The only new loops appearing when we added the cylinders $C(\partial e_{\alpha,u})$ come either from their intersections with K or else from their pairwise intersections.

In the first case consider $q \in D(\partial e_{\alpha,u}) \cap K \neq \emptyset$. The new loop λ created this way is the composition of an A-path from a vertex $*$ of K_u to a vertex of $\partial e_{\alpha,u}$ followed by a B-path in $C(\partial e_{\alpha,u})$ and then by a path in K to the point $*$. Now $(\alpha, u) \in W$, so that $C(\partial e_{\alpha,u}) \cap C \neq \emptyset$. Any vertex of $D(\partial e_{\alpha,u}) \cap K$ belongs therefore to a positive B-path starting at a point of C and ending at the extremal layer K_u. Thus we can homotopically push such a loop λ using the conjugacy cells – that are contained both in the cylinders $C(\partial e_{\alpha,u})$ and in K, since K is saturated – until they reach the layer $Z \cap uX(G) = E_u$. Thus the subgroup generated by images of A-loops in K and loops in E contains the loops of the form λ from $\pi_1(Z)$.

In the second case assume that $C(\partial e_{\alpha,u}) \cap C(\partial e_{\beta,v}) \neq \emptyset$. If $K_u = K_v$ the proof from above applies without essential modifications. This proves the lemma for the case when we have only one stable letter.

Assume now that $K_u \neq K_v$. Let q be an intersection point of these cylinders. A loop λ created by this double point is then the composition of an A-path joining a point $*$ of K_u to some point of $E_u - K_u$, followed by a B-path in $C(\partial e_{\alpha,u})$ reaching q then a B-path in $C(\partial e_{\beta,v})$ to a point in $E_v - K_v$, followed by an A-path to a point of K_v and eventually by a path in K joining it to $*$. The only problem, with respect to the previous case, is that we cannot push directly the loop λ along conjugacy cells since we have two extremal layers. The idea is to decompose it as the composition of two loops, each one of them which can be homotopically pushed into one extremal layer.

The subset $p(K) \subset C(L)$ is connected and thus the geodesic γ in $C(L)$ joining $p(u)$ and $p(v)$ is contained in $p(K)$. The cylinder $C(\partial e_{\alpha,u})$ (respectively $C(\partial e_{\beta,v})$) is made of conjugacy cells starting from some relation in R in the direction given by a positive B-path γ_1 (respectively γ_2), as explained in Section 5.6. Then γ_j are positive paths and hence geodesics in the tree $C(L)$ having a common vertex, namely $p(q)$. Therefore γ, γ_1 and γ_2 have a common vertex, say y. Further the positive B-subpaths $\gamma_1[p(q)y]$ and $\gamma_2[p(q)y]$ coincide. We can push q along this positive B-subpath and find that $C(\partial e_{\alpha,u}) \cap C(\partial e_{\beta,v})$ contains also a vertex t in the slice associated to y, namely with $p(t) = y$. It then suffices to consider the case where $p(q) = y$.

Furthermore, we know that $y \in p(K)$, which implies that the layer K_t is not empty. We claim:

Lemma 5.8. *There exists a vertex of $z \in K_t$ which is in the same connected component of $tX(G)$ as t.*

Proof. Let $t_u \in E_u$ (respectively $t_v \in E_v$) be the endpoint of the B-path given by the word $\gamma_1[yp(u)]$ (respectively $\gamma_2[yp(v)]$) starting at t. Recall now that E_u and E_v are each connected and thus we can find vertices $w_u \in K_u$, $w_v \in K_v$ which are joined to t_u and t_v respectively by A-paths corresponding to words a_u and a_v in the A-letters.

Observe that the B-subpaths $\gamma[p(u)y]$ and $\gamma_1[p(u)y]$ joining $p(u)$ and y in $C(L)$ coincide, as well as $\gamma[p(v)y]$ and $\gamma_2[p(v)y]$.

Consider then a path joining w_u to w_v in the connected subcomplex $p^{-1}(\gamma) \cap K$. This path is given by a word of the form

$$U = a_{2k+1}b_{i_{2k}}a_{2k}b_{i_{2k-1}}\ldots a_{k+2}b_{i_{k+1}}a_{k+1}b_{i_k}^{-1}a_k b_{i_{k-1}}^{-1}\ldots a_2 b_{i_1}^{-1}a_1,$$

where a_j are words in the A-letters and $b_i \in B$ are the positive generators. Furthermore the B-path $\gamma[yp(v)]$ is given by the word $B_+ = b_{i_{2k}}b_{i_{2k-1}}\ldots b_{i_{k+1}}$, while the B-path $\gamma[yp(u)]$ is given by the word $B_- = b_{i_k}^{-1}b_{i_{k-1}}^{-1}\ldots b_{i_1}^{-1}$. Notice that $b_{i_{k+1}} \neq b_{i_k}$ since γ is a geodesic. We then have a loop λ_0 in the Cayley graph of $\bar{G}$ realizing the word $B_+^{-1}a_v^{-1}Ua_u^{-1}B_-^{-1}$. This word must therefore represent the identity in $\bar{G}$. We use induction on k and Britton's lemma to obtain that the only way that this word can be simplified to the empty word is by means of reductions of the type

$bab^{-1} = c$, where $\phi_j(c) = a$, for $b \in B$ and $a, c \in A$. This means that there exist families of conjugacy cells in $C(\bar{G}, P_{\bar{G}})$, where the first family touches the extremal slice along the path $a_1 a_u^{-1}$ (respectively $a_{2k+1} a_v^{-1}$) in the direction $b_{i_1}^{-1}$ (respectively $b_{i_{2k}}^{-1}$) and the next ones use inductively the directions given by $b_{i_2}^{-1}, \ldots, b_{i_k}^{-1}$ (respectively $b_{i_{2k-1}}^{-1}, \ldots, b_{i_{k+1}}^{-1}$). Each family connects one slice to the slice below it. We can therefore push homotopically in $C(\bar{G}, P_{\bar{G}})$ the loop λ_0 along these conjugacy cells to the lowest slice $tX(G)$. But, at each step, the pushed loop has at least one vertex from K. Thus there exists a vertex $z \in K_v$ which is connected by an A-path to the vertex t, as claimed. $\qquad\square$

We turn back now to the loop λ. Since $p(\lambda) \subset C(L)$ contains both $p(u)$ and $p(v)$, it should contain the geodesic γ and then, by the previous arguments, there exists a point z' of λ in the layer K_v. Using Lemma 5.8 for the points z and z' instead of z and t it follows that z' and z (and hence t) are in the same connected component $tX_0(G)$ of $tX(G)$ so that z' can be joined by an A-path ζ to t. Therefore we can split λ as the composition of two loops $\lambda_1 \lambda_2$ by inserting ζ between z' and t. But now each one of the two loops λ_i can be homotopically pushed within K in the directions given by the B-subpaths of $\gamma[yp(u)]$ and respectively $\gamma[yp(v)]$ to one of the extremal slices E_u or E_v.

This proves that A-paths in K and loops in $\bigcup_u E_u$ generate all of $\pi_1(Z)$. This settles Lemma 5.7. $\qquad\square$

5.8. End of the proof of Theorem 5.1. Take a loop l in $\pi_1(Z)$. It can be supposed that l is either from the set $\sqcup_u J_u$ that normally generates $\pi_1(\coprod_u E_u)$ (recall that E_u are disjoint) or else from the set of A-loops ν_j furnished by Proposition 5.3. We observed that $l \subset E_u$ and $l' \bigcup_j \partial e_{j,u}(l) \subset E_u - K_u$ are null-bordant in $E_u \subset uX(G)$ using the null-bordism $H(l, l')$. Moreover l and $l'' = l' \bigcup_{j;(j,u)\notin W} e_{j,u}(l)$ are null-bordant in Z by means of the modified null-bordism $H(l, l') \bigcup_{(j,u)\in W} D(e_{j,u}(l))$, since the boundaries $\partial e_{j,u}$, with $(j, u) \in W$, are null-homotopic in Z. Moreover, $l' \bigcup_{j;(j,u)\notin W} e_{j,u}(l) \bigcup_k \partial \delta_{k,u}$ is furthermore null-homotopic in $uX(G) - K_u \subset C(\bar{G}, P_{\bar{G}}) - C$. We then adjoin the 2-disks $D(e_{j,u}(l))$ and $D(\partial \delta_{k,u})$ and obtain a null-homotopy of l'' within $C(\bar{G}, P_{\bar{G}}) - C$. This means that $C(\bar{G}, P_{\bar{G}})$ is 1-tame and thus qsf. This proves Theorem 5.1.

Remark 5.6. (1) We can always add new relations to the group presentation P_G in order to make it 1-tame. However the new presentation is not necessarily a finite endomorphic presentation. Thus the second assumption in Theorem 5.1 seems nontrivial.

(2) We do not know whether the 1-tameness of a presentation P with infinitely many relations which are unreduced words is equivalent to the 1-tameness of the presentation P_r consisting of the reduced words arising in the relations of P. It does so, for instance, when the length of the canceling subwords (i.e., subwords of the form $a_1 a_2 \ldots a_k a_k^{-1} \ldots a_2^{-1} a_1^{-1}$) is uniformly bounded.

6. Proof of Theorem 1.5

6.1. The Grigorchuk group. Grigorchuk constructed in the 1980s a finitely generated infinite torsion group of intermediate growth having solvable word problem (see [26]). This group is not finitely presented, but Lysënok obtained ([34]) a nice recursive presentation of G as follows:

$$G = \langle a, c, d \mid \sigma^n(a^2), \sigma^n((ad)^4), \sigma^n((adacac)^4) \rangle, \quad n \geq 0,$$

where $\sigma : \{a, c, d\}^* \to \{a, c, d\}^*$ is the substitution that transforms words according to the rules

$$\sigma(a) = aca, \quad \sigma(c) = cd, \quad \sigma(d) = c.$$

We denote by A^* the set of positive nontrivial words in the letters of the alphabet A, i.e., the free monoid generated by A without the trivial element.

The finitely presented HNN extension $\bar{G}$ of the Grigorchuk group G was constructed for the first time in [27], [28] as a group with 5 generators and 12 short relations. The group $\bar{G}$ is a finitely presented example of a group which is amenable but not elementary amenable. Bartholdi transformed this presentation in the form of a presentation with 2 generators and 5 relations, as described in [17]. Later Bartholdi presented (see [3]) some general method of getting endomorphic presentations for branch groups.

The endomorphism of G defining the HNN extension is induced by the substitution σ and thus the new group $\bar{G}$ has the finite presentation

$$\bar{G} = \langle a, c, d, t \mid a^2 = c^2 = d^2 = (ad)^4 = (adacac)^4 = 1,$$
$$a^t = aca, c^t = dc, d^t = c \rangle,$$

where $x^y = yxy^{-1}$. Theorem 5.1 is the main ingredient needed for proving Theorem 1.5, which we restate here for the sake of completeness:

Theorem 6.1. *The HNN extension of the Grigorchuk group is qsf.*

Remark 6.1. Relations in the Lysënok endomorphic presentation of Grigorchuk's group are given by iterating the substitution σ and thus involve only words with positive exponents on the generators which are reduced words.

6.2. The Lysenök presentation is 1-tame. We want to use Theorem 5.1. Using the notations from Section 5 the group L is the infinite cyclic group generated by the endomorphism σ. Since the endomorphism σ is expansive there are only finitely many positive paths between two elements of L. Further, the map $M \to L$ is obviously injective.

In the next section we will show:

Proposition 6.1. *The group G with the Lysënok presentation P_G is 1-tame.*

This proposition and Theorem 5.1 will settle the proof of Theorem 1.5.

The main idea is that the group G is commensurable with $G \times G$ (see e.g. [16], Theorem 28, p. 229). Further, the qsf property is invariant under commensurability. Moreover, the proof in [7] which shows that extensions of infinite finitely presented groups are qsf works also in the infinitely presented case. Even more, [7] shows that extensions of infinite finitely presented groups are actually 1-tame. Thus $G \times G$ with any direct product presentation is 1-tame. In particular, this happens if we consider the presentation $P_{G \times G} = P_G \times P_G$, defined as follows:

- take two copies of the generators a_j, b_j, c_j, d_j, $j \in \{1, 2\}$, corresponding to $G \times \{1\}$ and $\{1\} \times G$ respectively;

- take two copies of the Lysënok relations corresponding to each group of generators;

- add the commutativity relations between generators from distinct groups, namely,

$$[a_1, a_2] = [a_1, b_2] = [a_1, c_2] = [a_1, d_2] = 1$$

and the similar ones involving b_1, c_1 and d_1.

Since G is commensurable to $G \times G$ we will show that the presentation $P_{G \times G}$ induces a presentation P_G^* of G. The induction procedure consists of transferring presentations towards – or from – a finite index normal subgroup and transport it by some isomorphism. In particular, G with the induced presentation P_G^* is 1-tame.

We will show below that the P_G^* (up to finitely many relations) consists of P_G and finitely many families of relations, each family being conjugated to the family of standard relations in P_G. The later relations can be simply discarded from P_G^* without affecting the qsf property of the associated Cayley complex. In particular, the presentation obtained after that is in the same finite equivalence class as P_G. This will imply that the group (G, P_G) is qsf and thus its HNN extension $\bar{G}$ is also qsf, according to the Theorem 5.1.

Remark 6.2. Other examples of groups with endomorphic presentations including branch groups are given in [3]. Our present methods do not permit handling all of them. It is very probable that a general method working for this family will actually yield the fact that any finitely presented group admitting a normal (infinite) finitely generated subgroup of infinite index should be qsf.

6.3. Preliminaries concerning G following [16], VIII.B. It is customary to use the following presentation of G with 4 generators:

$$P_G(a, b, c, d): G = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = 1,$$
$$\sigma^n((ad)^4), \sigma^n((adacac)^4) \rangle,$$

where $n \geq 0$ and $\sigma : \{a, b, c, d\}^* \to \{a, b, c, d\}^*$ is the substitution that transforms words according to the rules

$$\sigma(a) = aca, \quad \sigma(b) = d, \quad \sigma(c) = b, \quad \sigma(d) = c,$$

from which we can drop either the generator b or else c and get the equivalent presentations with three generators

$$P_G(a, c, d) : G = \langle a, c, d \mid a^2 = c^2 = d^2 = 1, \sigma^n((ad)^4), \sigma^n((adacac)^4) \rangle,$$

where $n \geq 0$ and $\sigma : \{a, c, d\}^* \to \{a, c, d\}^*$ is the substitution that transforms words according to the rules

$$\sigma(a) = aca, \quad \sigma(c) = cd, \quad \sigma(d) = c,$$

or else

$$P_G(a, b, d) : G = \langle a, b, d \mid a^2 = b^2 = d^2 = 1, \sigma^n((ad)^4), \sigma^n((adabdabd)^4) \rangle,$$

where $n \geq 0$ and $\sigma : \{a, b, d\}^* \to \{a, b, d\}^*$ is the substitution that transforms words according to the rules

$$\sigma(a) = abda, \quad \sigma(b) = d, \quad \sigma(d) = bd.$$

Define G^0 be the subgroup consisting of words in a, b, c, d having an even number of occurrences of the letter a. This is the same as the subgroup denoted $St_G(1)$ in [16], p. 221. It is clear that $G^0 \lhd G$ is a normal subgroup and we have an exact sequence

$$1 \to G^0 \to G \to G/G^0 = \mathbb{Z}/2 = \langle a \rangle \to 1,$$

where G/G^0 is generated by aG^0. It follows that G^0 is the subgroup of G generated by the 6 elements

$$G^0 = \langle b, c, d, aba, aca, ada \rangle \subset G.$$

There exists an injective homomorphism $\psi : G^0 \to G \times G$ given by the formulas

$$\psi(b) = (a, c), \quad \psi(c) = (a, d), \quad \psi(d) = (1, b),$$
$$\psi(aba) = (c, a), \quad \psi(aca) = (d, a), \quad \psi(ada) = (b, 1).$$

Let $B \lhd G$ be the normal subgroup generated by b. It is known that $B = \langle b, aba, (bada)^2, (abad)^2 \rangle$. We then have an exact sequence

$$1 \to B \to G \to G/B = D_8 = \langle a, d \rangle \to 1,$$

where G/B is the dihedral group of order 8, denoted D_8. Moreover D_8 is generated by the images of the generators a and d. Since the subgroup $D = \langle a, d \rangle \subset G$ is the

dihedral group D_8, it follows that the extension above is split. Consider further the group $D^{\text{diag}} = \langle (a,d), (d,a) \rangle \subset G \times G$, which is isomorphic to the group D_8. Then we can describe the image of ψ as $\psi(G^0) = (B \times B) \ltimes D^{\text{diag}} \subset G$. Notice that the latter is a subgroup (although not a normal subgroup) of $G \times G$ having index 8.

It is easier to work with normal subgroups below since we want to track explicit presentations in the commensurability process. Therefore we will be interested in the subgroup $B \times B \subset \psi(G^0) \subset G \times G$ which is a normal subgroup. Denote by A the inverse image $\psi^{-1}(B \times B)$, which is a normal subgroup of G^0. It follows that $G^0/A \to (B \times B) \ltimes D^{\text{diag}}/B \times B = D$ is an isomorphism, and G^0/A is generated by the images of c and aca. Moreover the subgroup $\langle c, aca \rangle \subset G^0$ is dihedral of order 8 and thus there is a split exact sequence

$$1 \to A \to G^0 \to G^0/A = D_8 = \langle c, aca \rangle \to 1.$$

Collecting these facts it follows that A is the normal subgroup of G generated by d, and we have a split exact sequence

$$1 \to A \to G \to G/A = D_{16} = \langle a, c \rangle \to 1,$$

where G/A is generated by the images of a and c and it is isomorphic to the group $E = \langle a, c \rangle \subset G$, which is the dihedral group of order 16.

6.4. Inducing group presentations.
The presentation $P_{G \times G}$ of $G \times G$ induces a presentation $P_{B \times B}$ of its normal finite index subgroup $B \times B$. The isomorphism $\psi \colon A \to B \times B$ transports $P_{B \times B}$ into the presentation P_A of A. Eventually we can recover the presentation P_G^* of G from that of its normal subgroup A. In order to proceed we need to know how to induce presentations from and to normal finite index subgroups.

First we have the following well-known lemma of Hall:

Lemma 6.1. *Assume that we have an exact sequence*

$$1 \to K \to G \to F \to 1$$

and $K = \langle k_i \mid R_j \rangle$, $F = \langle m_j \mid S_n \rangle$ are group presentations in the generators k_i (respectively m_j) and relations R_j (respectively S_n). Then G has a presentation of the form

$$G = \langle k_i, m_j \mid R_j, S_n(m_j) = A_n(k_i), m_j k_i m_j^{-1} = B_{ji}(k_i) \rangle,$$

where A_n, B_{ji} are suitable words in the generators k_i. Specifically the relations using A_n express the relations between the lifts of the generators m_j to G, while the last relations express the normality of K within G.

Inducing presentations to a normal subgroup seems slightly more complicated. For the sake of simplicity we formulate the answer in the case where the relations

are positive (i.e., there are no negative exponents) and the extension is split, as it is needed for our purposes. Observe however that the result can be extended to the general situation.

Lemma 6.2. *Assume that we have a split exact sequence*

$$1 \to K \to G \to F \to 1,$$

where $G = \langle \{x_i\}_{i=1,\dots,N} \mid R_j \rangle$ *and the group* F *is finite. Let* $F = \{1 = f_0, f_1, f_2, \dots, f_n\}$ *be an enumeration of its elements. Assume further that the projection map* $p \colon G \to F$ *takes the form* $p(x_i) = f_{p(i)}$, *where* p *is a map* $p \colon \{1, 2, \dots, N\} \to \{0, 1, \dots, n\}$. *Assume that the relations* R_j *read as*

$$R_j = x_{i_1} \cdot x_{i_2} \dots x_{i_{k+1}}.$$

We choose lifts $\widehat{f}_j \in G$ *for the elements* f_j, *using the splitting homomorphism. Set then* $y_j = x_j \widehat{f_{p(j)}}^{-1}$ *and denote by* $^{f_k}y_j = \widehat{f}_k y_j \widehat{f}_k^{-1}$ *the conjugation. We consider below* $^{f_k}y_j$ *as being distinct symbols, called* y*-letters, for all* $k \neq 0$ *and* j.

  Then the group K *admits the following presentation:*

- *The generating set is the set of* $N(n + 1)$ *elements* y_j, $^{f_k}y_j$, $k \in \{1, 2, \dots n\}$, $j \in \{1, 2, \dots, N\}$.
- *Relations are obtained using the following procedure.*

  - *Each relation* $R_j = x_{i_1} \cdot x_{i_2} \dots x_{i_k}$ *gives rise to a basic relation in the* y*-letters alphabet,*

    $$R'_j = y_{i1} \cdot (^{f_{i_1}}y_{i_2}) \cdot (^{f_{i_1} \cdot f_{i_2}}y_{i_2}) \dots (^{f_{i_1} \cdot f_{i_2} \dots f_{i_k}}y_{i_k}),$$

    where each superscript product $f_{i_1} \cdot f_{i_2} \dots f_{i_s}$ *is replaced by its value, as an element* $f_{\lambda(i_1, i_2, \dots, i_s)} \in F$.

  - *Next one considers all images of the basic relations* R'_j *under the action of* F *(by conjugacy). Specifically, for any basic relation in* y*-letters*

    $$R = {}^{f_{j_1}}y_{j_1} \cdot {}^{f_{j_2}}y_{j_2} \dots {}^{f_{j_p}}y_{j_p}$$

    and any element $f \in F$ *one associates the relation*

    $$^{f}R = {}^{f f_{j_1}}y_{j_1} \cdot {}^{f f_{j_2}}y_{j_2} \dots {}^{f f_{j_p}}y_{j_p},$$

    in which each superscript is considered as an element of F.

  Here we set the notation $^a y$ in order to emphasize that these are abstract symbols, which will be viewed as elements of K. They will be equal to the usual conjugacies y^a only when seen as elements of G.

Proof. Any element of G is a product of y-elements and some $\widehat{f}_j$. Thus an element of K should involve no $\widehat{f}_j$.

Notice now that expressing R_j using the elements $^{f_k}y_j$ we obtain

$$R_j = (y_{i1} \cdot {}^{f_{i_1}}y_{i_2} \cdot {}^{f_{i_1} \cdot f_{i_2}} y_{i_2} \ldots {}^{f_{i_1} \cdot f_{i_2} \cdots f_{i_k}}y_{i_k}) \widehat{f}_{i_1} \cdot \widehat{f}_{i_2} \ldots \widehat{f}_{i_k}.$$

Moreover, the product of the first k terms in the right-hand side is an element of K. Since the extension is split we should have $\widehat{f}_{i_1} \cdot \widehat{f}_{i_2} \ldots \widehat{f}_{i_k} = 1$ coming as a relation in F. Thus $R'_j = 1$, as claimed. It is then clear that $^f R'_j = 1$ holds true also because K is a normal subgroup.

In order to see that these relations define K, consider the 2-complex Y_G associated to the given presentation of G. Thus Y_G has one vertex v. Then K is the fundamental group of the finite covering $\widehat{Y}_G$ (with deck group F) of Y_G, that is associated to the projection map $G \to F$. This is a non-ramified covering of degree $|F|$, the order of F. Thus each open 2-cell of Y_G is covered by precisely $|F|$ 2-cells of $\widehat{Y}_G$. It would suffices now to read the presentation of $\pi_1(\widehat{Y}_G)$ on the cell structure of $\widehat{Y}_G$. The only problem is that loops in Y_G lift to paths in $\widehat{Y}_G$ which are not closed. Now $\widehat{Y}_G$ has $|F|$ vertices that are permuted among themselves by F, let us call them v^f for $f \in F$, such that the deck transformations act as $g \cdot v^f = v^{gf}$. The vertex v^1 will be the base point of $\widehat{Y}_G$. The loop l_j based at v that corresponds to the generator x_j lifts to a path c_j joining v^1 to $v^{p(j)}$. Moreover the inverse image of the loop l_j under the covering is the union of all translated copies fc_j (joining v^f to $v^{fp(j)}$) of this path, which should be distinct as the covering is non-ramified. In this setting we have a natural presentation of $\pi_1(\widehat{Y}_G)$ as a fundamental groupoid with basepoints v^f for all $f \in F$. Simply take all (oriented) edges fc_j as generators and all 2-cells as relations. The 2-cells are all disjoint and permuted among themselves by F and in each F-orbit the 2-cell based at v^1 corresponds to one 2-cell of Y_G. One could choose now a maximal tree (corresponding to the choice of the elements $\widehat{f}_j$) in the 1-skeleton of $\widehat{Y}_G$ and collapse it in order to find a complex which comes from a group presentation. Alternatively we can transform the groupoid presentation into a group presentation by choosing a fixed set of paths $l(f)$ joining v^1 to v^f. The choice of this system amounts to choose lifts $\widehat{f}_j$ in G. Then the paths $l(f) \cdot fc_jl(fp(j))^{-1}$ are now based at v^1 and represent a generator system for the loops in $\widehat{Y}_G$. This loop represents the generator $^f y_j$ of K under the identification with $\pi_1(\widehat{Y}_G)$. Furthermore, the 2-cell based at v^1 corresponds to the basic relation associated to a relation in G and its images under the deck transformations are those described in the statement. Thus the fundamental group $\pi_1(\widehat{Y}_G)$ based at v^1 has the claimed presentation. $\qquad\square$

6.5. Carrying on the induction for the Grigorchuk group.
We will consider first the group G with its presentation $P_G(a, b, d)$ and the normal subgroup B normally generated by B. According to the induction lemma above we have a natural system

of generators given by $^{G/B}b = {}^{\langle a,d \rangle}d$ which is simply a notation for

$$\{^x b; x \in G/B = \langle a, d \rangle\} = \{b, {}^a b, {}^d b, {}^{ad} b, {}^{da} b, {}^{ada} b, {}^{dad} b, {}^{adad} b\}$$

The infinite set of words $w_n = \sigma^n((ad)^4)$, $z_n = \sigma^n((adabdabd)^4)$ are relations in G that induce relations $T(w_n)$ and $T(z_n)$ in B by the procedure above. This amounts to the following. Write first w_n (and z_n) as a word in a, b, d as

$$w_n = w_{n,0}(a, d) \, b \, w_{n,1}(a, d) \, b \, \ldots w_{n,k}(a, d) \, b \, w_{n,k+1}(a, d),$$

where $w_{n,j}(a, d)$ are words in a and d and thus can be reduced as elements of $D = G/B$. Then the basic relation corresponding to w_n is now

$$T(w_n) = (^{w_{n,0}}b) \; (^{w_{n,0}w_{n,1}}b) \ldots (^{w_{n,0}w_{n,1}\ldots w_{n,k}}b),$$

where the right-hand side is interpreted as a word in the alphabet $^{G/B}b$ and all products in $\langle a, d \rangle$ are reduced to the canonical form as elements of the generators set above. The D-action on relations yields the additional set of relations, for each $x \in D = \langle a, d \rangle$,

$$^x T(w_n) = (^{x w_{n,0}}b)(^{x w_{n,0}w_{n,1}}b) \ldots (^{x w_{n,0}w_{n,1}\ldots w_{n,k}}b).$$

The same procedure computes $^x T(z_n)$.

A presentation for the group $B \times B$ is now obtained by using the generating set $^{G/B}b \times {}^{G/B}b$ and the following families of relations (coming either from relations in B or from the commutativity of the two factors):

$$(^x T w_n, 1) = 1, \quad (1, {}^x T w_n) = 1, \quad (^x T z_n, 1) = 1,$$
$$(1, {}^x T z_n) = 1, \quad (^x b, 1)(1, {}^y b) = (1, y^b)(^x b, 1).$$

The next step is to obtain a presentation P_A for A and then using Hall's lemma to recover the presentation of G. Several remarks are in order. Since we seek for the finite equivalence class we can discard or adjoin finitely many relations at the end. When shifting from A to G we have to add the extra generators from $G/A = \langle a, c \rangle$, thus the generators a and c. We have also to add finitely many conjugation relations corresponding to the normality of A and lifts of relations in G/A. However the previous remark enables us to ignore all these and keep track only of the following (four) infinite families of relations in $B \times B$ expressed by $(T w_n, 1) = 1, (1, T w_n) = 1, (T z_n, 1) = 1, (1, T z_n) = 1$.

In order to understand the isomorphism ψ we have to shift to the presentation $P_G(a, c, d)$ of G. A natural system of generators for A is given in the spirit of the induction lemma by the set $^{G/A}d = {}^{\langle a,c \rangle}d$ which is simply a notation for

$$\{^x d; x \in G/A = \langle a, c \rangle\} = {}^{G^0/A}d \cup {}^{G^0 a/A}d.$$

This system of generators is convenient because ψ has now a simple expression:

Lemma 6.3. *The isomorphism* $\psi : A \to B \times B$ *takes the form*

$$\psi_0 : \{^{G^0/A}d\} \to \{1\} \times \{^{G/B}b\}, \quad \psi_1 : \{^{G^0a/A}d\} \to \{^{G/B}b\} \times \{1\},$$

where

- ψ_0 *is given by*
  $$\psi_0^{-1}(b, 1) = d, \quad \psi_0^{-1}(^xb, 1) = {}^{\varphi_0(x)}d,$$

  where $\varphi_0 : G/B = \langle a, d \rangle \to G^0/A = \langle c, aca \rangle$ *is the isomorphism*

  $$\varphi_0(d) = c, \quad \varphi_0(a) = aca;$$

- ψ_1 *is given by*
  $$\psi_1^{-1}(1, b) = {}^a d, \quad \psi_0^{-1}(1, {}^x b) = {}^{\varphi_1(x)a}d,$$

  where $\varphi_1 : G/B = \langle a, d \rangle \to G^0/A = \langle c, aca \rangle$ *is the conjugated isomorphism*

  $$\varphi_0(d) = aca, \quad \varphi_0(a) = c.$$

Proof. This is a direct calculation. For instance, $\psi(^c d) = (a, d)(1, b)(a, d) = (1, {}^d b)$. $\square$

Let us transport now the relation $(1, T w_n) = 1$ from $B \times B$ to A.
This relation reads

$$(1, {}^{w_{n,0}} b)(1, {}^{w_{n,0}w_{n,1}} b) \dots (1, {}^{w_{n,0}w_{n,1}\dots w_{n,k}} b) = 1.$$

According to the previous lemma this relation now reads in A as

$$({}^{\varphi_0(w_{n,0})}d)({}^{\varphi_0(w_{n,0}w_{n,1})}d) \dots ({}^{\varphi_0(w_{n,0}w_{n,1}\dots w_{n,k})}d) = 1.$$

Further we interpret these relations in G (as part of the presentation P_G^*), where we restored also the generators a and c. Writing down the terms by developing each conjugation we obtain

$$\varphi_0(w_{n,0})d \cdot \varphi_0(w_{n,1})d \dots \varphi_0(w_{n,k})d(\varphi_0(w_{n,0}w_{n,1} \dots w_{n,k})^{-1}) = 1.$$

The key point is the fact that the map φ_0 acts like σ on the letters a, d; actually, if one extends φ_0 to a monoid homomorphism sending b into d we obtain σ. Thus the relation above is the same as

$$\sigma(w_n) = 1.$$

But $\sigma(w_n) = w_{n+1}$ and thus we have no additional relation induced in P_G^* other than those already existing in P_G.

Let us look now at the transformations of the relation $(1, {}^x T w_n) = 1$ for $x \in D$. This relation reads in A as

$$({}^{\varphi_0(x w_{n,0})} d)({}^{\varphi_0(x w_{n,0} w_{n,1})} d) \ldots ({}^{\varphi_0(x w_{n,0} w_{n,1} \ldots w_{n,k})} d) = 1$$

and, by developing it again in G,

$$\varphi_0(x w_{n,0}) d \cdot \varphi_0(w_{n,1}) d \ldots \varphi_0(w_{n,k}) d (\varphi_0(w_{n,0} w_{n,1} \ldots w_{n,k})^{-1}) \varphi_0(x)^{-1} = 1.$$

This is precisely the relation

$$\varphi_0(x) \sigma(w_n) \varphi_0(x)^{-1} = 1,$$

which is a conjugation of the already existing relation $w_{n+1} = 0$.

The same reasoning shows that starting from $(1, {}^x T z_n)$ we obtain in P_G^* the relation $z_{n+1} = 1$ (or conjugacies of it).

Eventually we consider the relations $(T w_n, 1) = 1$ in $B \times B$, namely,

$$({}^{w_{n,0}} b), 1)({}^{w_{n,0} w_{n,1}} b, 1) \ldots ({}^{w_{n,0} w_{n,1} \ldots w_{n,k}} b, 1) = 1.$$

The image of ψ^{-1} of this relation in A is therefore

$$({}^{\varphi_1(w_{n,0} A)} d)({}^{\varphi_1(w_{n,0} w_{n,1} a)} d) \ldots ({}^{\varphi_1(w_{n,0} w_{n,1} \ldots w_{n,k} a)} d) = 1.$$

But $\varphi_1(x) = a \varphi_0(x) a$ and thus this relation is the same as

$$({}^{a \varphi_0(w_{n,0})} d)({}^{a \varphi_0(w_{n,0} w_{n,1})} d) \ldots ({}^{a \varphi_1(w_{n,0} w_{n,1} \ldots w_{n,k})} d) = 1,$$

which, by developing all terms, yields in G

$$a \varphi_0(w_{n,0})) d \cdot \varphi_0(w_{n,1}) d \ldots \varphi_0(w_{n,0} w_{n,k}) d (\varphi_0(w_{n,0} w_{n,1} \ldots w_{n,k})^{-1}) a = 1.$$

However, this is the same as $a w_{n+1} a = 1$, which is a consequence of $w_{n+1} = 1$. The same holds true for the relations induced by $(T z_n, 1) = 1$. Starting from $({}^x T w_n, 1) = 1$ or $({}^x T z_n, 1) = 1$ we obtain again conjugated relations.

Lemma 6.4. *Consider two presentations of some group G of the form $P_1 = \langle S \mid R \rangle$ and $P_2 = \langle S \mid R \cup a R a^{-1} \rangle$. If P_2 is qsf then P_1 is qsf.*

Proof. We can assume that $a \in S$. Every homotopy involving $a R a^{-1}$ can be realized using only relations from R. This proves the claim. $\qquad \square$

Now P_G^* is finitely equivalent to the presentation consisting of P_G with finitely many additional families, each additional family being conjugated to the family of relations $\{w_n = z_n = 1, \ n \geq 1\}$. If we remove the additional relations we obtain P_G. The previous lemma and Proposition 5.1 show that P_G is qsf. This settles Proposition 6.1.

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