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Non-injective representations of a closed surface group into $PSL(2, \mathbb{R})$

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Abstract

Let e denote the Euler class on the space $\mathrm{Hom}(\Gamma_g, PSL(2, \mathbb{R}))$ of representations of the fundamental group Γ_g of the closed surface Σ_g of genus g. Goldman showed that the connected components of $\mathrm{Hom}(\Gamma_g, PSL(2, \mathbb{R}))$ are precisely the inverse images $e^{-1}(k)$, for $2-2g \leqslant k \leqslant 2g-2$, and that the components of Euler class $2-2g$ and $2g-2$ consist of the injective representations whose image is a discrete subgroup of $PSL(2, \mathbb{R})$. We prove that non-faithful representations are dense in all the other components. We show that the image of a discrete representation essentially determines its Euler class. Moreover, we show that for every genus and possible corresponding Euler class, there exist discrete representations.

1. *Introduction*

Let Σ_g be the closed oriented surface of genus $g \geqslant 2$. Let Γ_g denote its fundamental group, and R_g the representation space $\mathrm{Hom}(\Gamma_g, PSL(2, \mathbb{R}))$. Elements of R_g are determined by the images of the $2g$ generators of Γ_g, subject to the single relation defining Γ_g. It follows that R_g has a real algebraic structure (see e.g. [**3**]). Furthermore, being a subset of $(PSL(2, \mathbb{R}))^{2g}$, it is naturally equipped with a Hausdorff topology.

We can define an invariant $e : R_g \to \mathbb{Z}$, called the Euler class, as an obstruction class or as the index of circle bundles associated to representations in R_g (see [**6**, **10**, **14**]).

In [**10**], which may be considered to be the starting point of the subject, Goldman showed that the connected components of R_g are exactly the fibers $e^{-1}(k)$, for $2-2g \leqslant k \leqslant 2g-2$. He also proved that $e^{-1}(2g-2)$ and $e^{-1}(2-2g)$ consist of those injective representations whose image is a discrete subgroup of $PSL(2, \mathbb{R})$. Milnor and Wood had previously proved that the inequality $|e(\rho)| \leqslant 2g-2$ holds for all $\rho \in R_g$ (see [**14**, **16**]). Goldman [**9**] (see also [**11**]) showed that every connected component $e^{-1}(k)$ is a smooth manifold of dimension $6g-3$, except for the component $e^{-1}(0)$, whose singular points are the elementary representations.

These connected components have been studied further. The group $PSL(2, \mathbb{R})$ acts on R_g by conjugation, and the quotient of $e^{-1}(2-2g)$ (respectively $e^{-1}(2g-2)$) under this action is the Teichmüller space of Σ_g (respectively, the space of marked hyperbolic metrics with opposite orientation on Σ_g).

Gallo, Kapovich, Marden and Tan [**5**, **15**] showed that all the representations (in every connected component of R_g) satisfy a weaker metric condition. Indeed, every representation in R_g is the holonomy representation of a branched $\mathbb{C}P^1$-structure on Σ_g, with at most one branched point in Σ_g. Moreover, Tan showed that for every even k such that $|k| \leqslant 2g-4$, there is an explicit non-injective representation in $e^{-1}(k)$, which is not the holonomy representation of any branched hyperbolic structure on Σ_g. However, this explicit representation can be deformed into representations which are holonomy representations of branched hyperbolic structures on Σ_g. It is still unknown whether the holonomy representations of branched hyperbolic structures on Σ_g form a dense subset of R_g.

In this paper we take a more elementary point of view. We often consider representations as products of commutators in $PSL(2,\mathbb{R})$, and most of our results involve explicit representations.

Our first result is the following:

THEOREM 1·1. *For all $g \geqslant 2$ and all k such that $|k| < 2g-2$, non-faithful representations form a dense subset of the connected component $e^{-1}(k)$.*

Recently DeBlois and Kent IV [**4**] proved that the set of faithful representations is also dense, as the intersection of a countable family of open and dense sets. This was independently announced by Breuillard, Gelander, Souto and Storm (see [**1**]). Therefore Theorem 1·1 shows that the set of non-faithful representations should be thought of as $\mathbb{Q}$ in $\mathbb{R}$.

We then show that the Euler class of a discrete representation is essentially determined by its image, as an abstract group. If Γ is a non-cocompact Fuchsian group, set $e(\Gamma)=0$. Otherwise, if $(g; k_1, \ldots, k_r)$ is the signature of the Fuchsian group Γ (with all the k_i's finite), let d be the least common multiple of $k_1, \ldots, k_r$ (put $d=1$ if $r=0$) and set

$$e(\Gamma) = d\left(2g-2+\sum_{i=1}^{r}\left(1-\frac{1}{k_i}\right)\right).$$

We have the following result:

THEOREM 1·2. *Let ρ be a representation (in any $R_{g'}$) whose image is contained in Γ. Then $e(\rho)$ is a multiple of $e(\Gamma)$. Moreover, every multiple of $e(\Gamma)$ is the Euler class of some representation whose image is exactly Γ.*

In particular, there are no representations of non-zero Euler class taking values in $PSL(2,\mathbb{Z})$. We deduce the following proposition:

PROPOSITION 1·1. *Let k be a fixed non-zero integer. There are only finitely many Fuchsian groups Γ such that there exists a representation (of Γ_g, for some $g \geqslant 2$) with image contained in Γ and with Euler class k.*

Moreover, we have:

PROPOSITION 1·2. *Let $g \geqslant 2$ and $k \in \mathbb{Z}$ be such that $|k| \leqslant 2g-3$, $k \neq 0$. The representations whose image is discrete form a nowhere dense closed subset of $e^{-1}(k)$ in R_g.*

In other words, there are few discrete representations of non-zero Euler class. On the other hand, we have the following:

THEOREM 1·3. *For every $g \geqslant 2$ and every k such that $|k| \leqslant 2g-2$, there exists a discrete representation of Euler class k in R_g.*

Those representations are given explicitly in terms of signatures of Fuchsian groups. In particular, using Magnus' results from **[13]**, these representations can be expressed using only matrices in $PSL(2, \mathbb{Z}[\frac{1}{2}])$.

Finally, we deduce a characterization of representations of odd Euler class, which enables us to describe all the subgroups of $PSL(2,\mathbb{R})$ that are the image of some discrete representation of odd Euler class (in some R_g). If Γ is a cocompact Fuchsian group with signature $(g'; k_1, k_2, \ldots, k_r)$, let $m(\Gamma)$ be the maximal power of 2 dividing one of the k_i's. If $m(\Gamma) = 0$, set $n(\Gamma) = 0$. Otherwise, let $n(\Gamma)$ be the number of k_i's which are divisible by $2^{m(\Gamma)}$. We have then the following characterization:

PROPOSITION 1·3. *A Fuchsian group $\Gamma \subset PSL(2,\mathbb{R})$ is the image of a representation (of Γ_g, for some $g \geqslant 2$) of odd Euler class if and only if Γ is a cocompact Fuchsian group such that $n(\Gamma)$ is odd.*

2. *The Lie group $PSL(2,\mathbb{R})$ and Milnor's algorithm*

2·1. *The Lie group $PSL(2,\mathbb{R})$*

We recall first some basic results on $PSL(2,\mathbb{R})$ and we refer to **[12]** for a full treatment of this subject.

The Lie group $SL(2,\mathbb{R})$ is $\{M \in GL(2,\mathbb{R}) | \det M = 1\}$ and $PSL(2,\mathbb{R})$ is the quotient $SL(2,\mathbb{R})/\{\pm 1\}$ by its center. Topologically, $SL(2,\mathbb{R})$ and $PSL(2,\mathbb{R})$ are two three-dimensional solid tori, and the projection map $SL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$ is a 2-sheeted cover. We will denote elements of $PSL(2,\mathbb{R})$ by matrices, as if they were in $SL(2,\mathbb{R})$.

If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R})$, the homography $z \mapsto (az+b)/(cz+d)$ is an isometry of the upper half-plane model of the hyperbolic plane $\mathbb{H}^2$. This isometry of $\mathbb{H}^2$ acts on its boundary $\mathbb{S}^1 = \mathbb{R} \cup \{\infty\}$, still by homography. This defines an injection $PSL(2,\mathbb{R}) \hookrightarrow \mathrm{Homeo}^+(\mathbb{S}^1)$, which we will think of as an inclusion.

On $PSL(2,\mathbb{R})$ only the absolute value of the trace is well-defined. If $M \in SL(2,\mathbb{R})$, we denote $\mathrm{Tr}(M) = |\mathrm{tr}(M)|$. Elements of $PSL(2,\mathbb{R})$ then fall into three types, Ell, Par and Hyp, depending on their traces.

(i) If $\mathrm{Tr}(M) \in [0,2)$ then M fixes a point in $\mathbb{H}^2$ and acts as a rotation around it. The matrix M is conjugate *in* $PSL(2,\mathbb{R})$ to a matrix of the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. It follows that two such elements are conjugate if and only if they have the same trace (in absolute value). Such matrices are called *elliptic*.

(ii) If $\mathrm{Tr}(M) = 2$ and $M \neq I_2$, then M fixes no point in $\mathbb{H}^2$ but fixes a unique point in its boundary. Its orbits in $\mathbb{H}^2$ are the horospheres defined by that fixed point. The matrix is conjugate, in $PSL(2,\mathbb{R})$, to a matrix $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Such elements are called *parabolic*. Elements of Par fall into two conjugacy classes (over $PSL(2,\mathbb{R})$), depending on the sign of t.

(iii) If $\mathrm{Tr}(M) \in (2, +\infty)$ then M is conjugate in $PSL(2,\mathbb{R})$ to $\begin{pmatrix} \operatorname{ch} u & \operatorname{sh} u \\ \operatorname{sh} u & \operatorname{ch} u \end{pmatrix}$ for some $u \in \mathbb{R}$, and two such elements are conjugate if and only if they have the same trace. The action of M on $\mathbb{H}^2$ fixes two points in the boundary of $\mathbb{H}^2$. Those elements are called *hyperbolic*.

Each of these three sets contains (together with I_2) whole one-parameter groups, which are determined by the fixed points described above.

2·2. *Milnor's algorithm*

Now we describe Milnor's algorithm for calculating the Euler class of a representation. We refer to Ghys' article [**6**] for a complete and detailed approach to this important way of computing (or defining) Euler classes.

Topologically, the Lie group $PSL(2,\mathbb{R})$ is a solid torus, and $\pi_1(PSL(2,\mathbb{R})) \cong \mathbb{Z}$. Moreover, for all $A \in PSL(2,\mathbb{R})$, taking a lift $\tilde{A} \in \widetilde{PSL(2,\mathbb{R})}$ is the same as lifting the homeomorphism $A \in \mathrm{Homeo}^+(\mathbb{S}^1)$ to a homeomorphism of the universal cover of the circle. In other words, we have the following short exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{PSL(2,\mathbb{R})} & \longrightarrow & PSL(2,\mathbb{R}) & \longrightarrow & 1. \\ & & & & \cap & & \cap & & \\ & & & & \mathrm{Homeo}^+(\mathbb{R}) & & \mathrm{Homeo}^+(\mathbb{S}^1) & & \end{array}$$

In the diagram above, the sign of the generator in $\mathbb{Z}$ is determined by the choice of an orientation of $\mathbb{S}^1$. As in [**10**], we will denote by $z \in \widetilde{PSL(2,\mathbb{R})}$ the image of this generator $1 \in \mathbb{Z}$.

Note that, for all $A \in PSL(2,\mathbb{R})$, the choice of a lift $\tilde{A} \in \widetilde{PSL(2,\mathbb{R})}$ is well-defined up to a certain number of elementary translations z, which are central in $\widetilde{PSL(2,\mathbb{R})}$.

The Euler class $e(\rho)$ of a representation ρ is computed in the following way. Consider the standard one relator presentation of Γ_g given by

$$\Gamma_g = \langle a_1, a_2, \ldots, a_g, b_1, b_2 \ldots, b_g \mid [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \rangle.$$

For each generator x in the above presentation of Γ_g, choose an arbitrary lift $\widetilde{\rho(x)} \in \widetilde{PSL(2,\mathbb{R})}$. Then $e(\rho)$ is determined by means of the formula:

$$[\widetilde{\rho(a_1)}, \widetilde{\rho(b_1)}] \cdots [\widetilde{\rho(a_g)}, \widetilde{\rho(b_g)}] = z^{e(\rho)}.$$

Obviously, $e(\rho)$ does not depend on the choice of the lifts. Indeed, two lifts differ by powers of z, which disappear in the commutators, because z is central in $\widetilde{PSL(2,\mathbb{R})}$.

3. *Non-injective representations*

In this section we give first an elementary proof of the following proposition, which is a particular case of Theorem 1·1:

PROPOSITION 3·1. *For all k such that $|k| \leqslant 2g-3$, $e^{-1}(k)$ contains an explicit open set E, in which non-injective representations are dense.*

We set $E = \{\rho \in R_g | \rho(a_1) \in Ell, \rho(b_1) \in Ell\} \cap e^{-1}(k)$.

In order to prove the Proposition 3·1, we first need some preliminaries. We then state two lemmas implying the proposition, and finally prove the two lemmas.

3·1. *Preliminaries*

LEMMA 3·1. *Every hyperbolic element in $PSL(2,\mathbb{R})$ is the commutator of two elliptic elements.*

Proof. Let $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $U_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Then $\mathrm{Tr}([A_\theta, U_t A_\theta U_{-t}]) = 2 + (4t^2+t^4)\sin^4\theta$. This trace takes every value from the interval $[2, +\infty)$, and the set $[Ell, Ell]$ is invariant under conjugation. Thus every hyperbolic element is the commutator of two elliptic elements.

Remark 3·1. If $M \in Hyp \cup Par$, or equivalently, if $M \in \text{Homeo}^+(\mathbb{S}^1)$ has a fixed point in $\mathbb{S}^1$, then there is a canonical lift $\widetilde{M}^{\text{can}}$ of M, in $\text{Homeo}^+(\mathbb{R})$, which has fixed points in $\mathbb{R}$.

Remark 3·2. If $A, B \in PSL(2,\mathbb{R})$, $\widetilde{A}$ and $\widetilde{B}$ are defined only up to a power of the elementary translation $z \in \text{Homeo}^+(\mathbb{R})$, but the commutator $[\widetilde{A}, \widetilde{B}] \in \widetilde{PSL(2,\mathbb{R})}$ is well-defined and independent on the choice of lifts, because z is central in $\widetilde{PSL(2,\mathbb{R})}$.

PROPOSITION 3·2. *For all $\varepsilon \in \{-1, 0, 1\}$ and for all $M \in Hyp$, there exist $A, B \in PSL(2,\mathbb{R})$ such that $[\widetilde{A}, \widetilde{B}] = \widetilde{M}^{\text{can}} \cdot z^{\varepsilon}$. Moreover, if $[A, B] = M$ then the only possible values of $[\widetilde{A}, \widetilde{B}]^{-1}\widetilde{M}^{\text{can}}$ are z^{-1}, 1 and z.*

Proofs of this important result can be found in [**10**, **14**].
Another well-known result is the following:

PROPOSITION 3·3. *If $M \in Hyp$, $A, B \in Ell$ and $[A, B] = M$ then $[\widetilde{A}, \widetilde{B}] = \widetilde{M}^{\text{can}}$.*

Proof. This is a corollary of Goldman's main result in [**10**]. Indeed, suppose $[\widetilde{A}, \widetilde{B}] \neq \widetilde{M}^{\text{can}}$. Then $[\widetilde{A}, \widetilde{B}] = \widetilde{M}^{\text{can}} \cdot z^{\delta}$, with $\delta \in \{-1, 1\}$. Now, $M^{-1} \in Hyp$, so, by Proposition 3·2, there exist $C, D \in PSL(2,\mathbb{R})$ such that $[\widetilde{C}, \widetilde{D}] = \widetilde{M^{-1}}^{\text{can}} \cdot z^{\delta}$. Hence $[\widetilde{A}, \widetilde{B}][\widetilde{C}, \widetilde{D}] = z^{2\delta}$, so the formulas

$$\rho(a_1) = A, \rho(b_1) = B, \rho(a_2) = C, \rho(b_2) = D$$

define a representation ρ of the fundamental group $\Gamma_2 = \langle a_1, b_1, a_2, b_2 \,|\, [a_1, b_1][a_2, b_2] \rangle$ of the genus two surface Σ_2, of Euler class 2δ. Therefore, by [**10**, corollary C] this representation is faithful and discrete, which contradicts the assumption $A \in Ell$.

3·2. *Proof of Proposition* 3·1

LEMMA 3·2. *For all $\rho \in E$, there exists a path $\rho_t \in R_g$ such that $\rho_0 = \rho$ and $\frac{d}{dt}\text{Tr}(\rho_t(a_1)) \neq 0$.*

Remark 3·3. We are not concerned with the derivability of the function $t \mapsto \text{Tr}(\rho_t(a_1))$, but rather in the fact that it is not locally constant.

LEMMA 3·3. *The set E is not empty.*

An element $A \in Ell$ is conjugate to a unique matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. If $\frac{d}{dt}\text{Tr}(\rho_t(a_1))$ is non-zero, then the angle θ corresponding to $\rho_t(a_1)$ is rational for a dense subset of the possible values of t, hence the representation ρ_t satisfies a relation of the kind $\rho_t(a_1)^n = 1$, so that ρ_t is not injective. Thus these two lemmas imply Proposition 3·1.

The two remaining sections consist of proofs of these two lemmas.

3·3. *Proof of Lemma* 3·2

Let $A = \rho(a_1)$ and $B = \rho(b_1)$. Let $B(t)$ be the one-parameter subgroup of $PSL_2(\mathbb{R})$ containing B. That is, $B(0) = 1$, $B(1) = B$ and for all $t \in \mathbb{R}$, $B(t)$ commutes with B. Set $A_t = AB(t)$. Then $[A_t, B] = AB(t)BB(-t)A^{-1}B^{-1} = [A, B]$. Thus the formulas

$$\rho_t(a_1) = A_t,\ \rho_t(a_i) = \rho(a_i),\ \text{for } i \geqslant 2,\ \rho_t(b_j) = \rho(b_j),\ \text{for } j \geqslant 1$$

define a representation $\rho_t \in \text{Hom}(\Gamma_g, PSL(2,\mathbb{R}))$. Moreover, $\rho_0 = \rho$.

We claim that $\frac{d}{dt}\text{Tr}(\rho_t(a_1)) \neq 0$. Up to conjugation – which does not change the value of the traces – we can write A in an adapted basis under the form $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

In the same basis $B(t)$ reads $\begin{pmatrix} c_1(t) & c_2(t) \\ c_3(t) & c_4(t) \end{pmatrix}$, where $c_j(t)$, $1 \leqslant j \leqslant 4$, are suitable real functions. Set $d_i = c_i'(0)$. We compute

$$\mathrm{tr}(AB(t)) = (c_1(t) + c_4(t))\cos\theta + (c_2(t) - c_3(t))\sin\theta$$

and hence

$$\frac{d}{dt}\mathrm{tr}(AB(t)) = (d_1 + d_4)\cos\theta + (d_2 - d_3)\sin\theta.$$

It suffices to show that this derivative is non-zero. Notice that $d_1 + d_4 = 0$, and $A \in Ell$ (which does not contain 1) so $\sin\theta \neq 0$. Hence, the claim would follow if we proved that $d_2 \neq d_3$. Assume the contrary, namely that $d_2 = d_3$. We develop $B(t)$ up to second order terms:

$$B(t) = \begin{pmatrix} 1 + d_1 t + \alpha t^2 + \mathcal{O}(t^3) & d_2 t + \mathcal{O}(t^2) \\ d_2 t + \mathcal{O}(t^2) & 1 - d_1 t + \beta t^2 + \mathcal{O}(t^3) \end{pmatrix}.$$

Now, $\det B(t) = 1 - (\alpha + \beta - d_1^2 - d_2^2)t^2 + \mathcal{O}(t^3) = 1$, which implies that $\alpha + \beta > 0$. Therefore, for small enough $t > 0$ (and thus for all t), $B(t)$ is hyperbolic, which is a contradiction since we assumed that $B \in Ell$.

3·4. *Proof of Lemma* 3·3

If $|k| \leqslant 2g - 4$, take a representation $\rho' \in R_{g-1}$ of Euler class k. Assume that the standard generators of Γ_{g-1} are denoted by the letters $a_1', b_1', \ldots a_{g-1}', b_{g-1}'$ in order to avoid confusion with the generators $a_1, a_2, \ldots, a_g, b_g$ of Γ_g. Choose an arbitrary $A \in Ell$. Set ρ for the homomorphism defined by the formulas

$$\rho(a_1) = \rho(b_1) = A,\ \rho(a_i) = \rho'(a_{i-1}'),\ \ \rho(b_i) = \rho'(b_{i-1}'),\ \text{ for } i \geqslant 2.$$

Then $\rho \in E$.

If $|k| = 2g - 3$, it suffices to consider the case $k = 2g - 3$. Then take a representation $\rho' \in R_g$ of Euler class $2g - 2$. We can suppose (see e.g. [**10**]) that $\rho'([a_1, b_1])$ is hyperbolic, so by lemma 3.1 there exist $A, B \in Ell$ such that $[A, B] = \rho'([a_1, b_1])$. The formulas

$$\rho(a_1) = A,\ \rho(b_1) = B \text{ and } \rho(a_i) = \rho'(a_i),\ \rho(b_i) = \rho'(b_i),\ \text{ for } 2 \leqslant i \leqslant g$$

define a representation $\rho \in \mathrm{Hom}(\Gamma_g, PSL(2,\mathbb{R}))$. In order to prove that $\rho \in E$, we need to check that $e(\rho) = k$. But by maximality of the Euler class of ρ' and Proposition 3·2, we know that $[\widetilde{\rho'(a_1)}, \widetilde{\rho'(b_1)}] = \widetilde{\rho'([a_1, b_1])}^{\mathrm{can}} \cdot z$, and by Proposition 3·3 we know that $[\widetilde{A}, \widetilde{B}] = \widetilde{\rho'([a_1, b_1])}^{\mathrm{can}}$. Thus $e(\rho) = e(\rho') - 1 = k$, as claimed.

Our proof of Theorem 1·1 uses Theorem 1·3. We postpone the proof of both theorems to the next section.

4. *Representations with discrete image*

4·1. *Presenting lifts of Fuchsian groups into* $\widetilde{PSL(2,\mathbb{R})}$

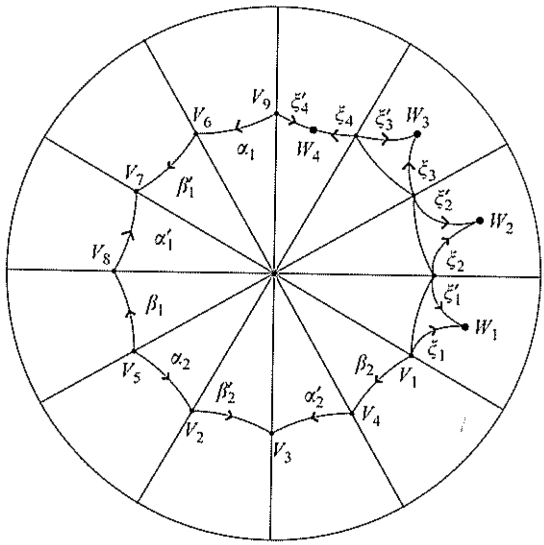
Our main technical result, which will be used throughout this sequel, is the following:

LEMMA 4·1. *Let* Γ *be the Fuchsian group with signature* $(g; k_1, \ldots, k_l, \underbrace{\infty, \ldots, \infty}_{r-l})$.

Then the lift $\widetilde{\Gamma}$ *of* Γ *in* $\widetilde{PSL(2,\mathbb{R})}$ *has the following presentation:*

$$\widetilde{\Gamma} = \left\langle q_1, \ldots, q_r, a_1, \ldots, b_g, z \left| \begin{array}{l} zq_1z^{-1}q_1^{-1}, \ldots, zb_gz^{-1}b_g^{-1}, \\ q_1^{k_1}z, \ldots, q_l^{k_l}z, q_1 \cdots q_r[a_1, b_1] \cdots [a_g, b_g]z^{2g-2+r} \end{array} \right. \right\rangle.$$

Proof. Take a Poincaré fundamental domain of Γ in the hyperbolic plane $\mathbb{H}^2$. According to Katok's notations ([**12**]), we have the following picture in the example case where $g = 2$, $r = 4$, $k_i > 2$ for $i = 1, 2, 3$ and $k_4 = 2$:

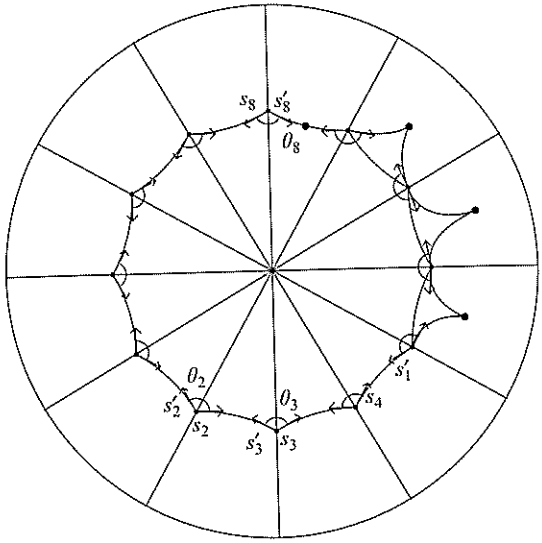


Here q_i is the only element of $PSL(2,\mathbb{R})$ sending ξ_i' to ξ_i, and similarly $a_i(\alpha_i') = \alpha_i$ and $b_i(\beta_i') = \beta_i$.

Now consider the identification of $PSL(2,\mathbb{R})$ with the unit tangent bundle $S\mathbb{H}^2$ of the hyperbolic plane $\mathbb{H}^2$ (see [**12**, theorem 2·1·1]). Then the universal lift $\widetilde{PSL(2,\mathbb{R})}$ of $PSL(2,\mathbb{R})$ can be viewed as the unit tangent bundle $S\mathbb{H}^2$ together with an index number. And lifting elements of $PSL(2,\mathbb{R})$ to $SL(2,\mathbb{R})$ is just taking the index number modulo 2. Given our prefered presentation of Γ, in order to determine a presentation of $\widetilde{\Gamma}$ we just have to know how the relations lift. One may lift q_i to the rotation of positive angle $2\pi/k_i$, so that $q_i^{k_i}$ makes one (positive) turn around W_i. Hence the relation $q_i^{k_i}$ lifts to $q_i^{k_i}z^{-1}$ in $\widetilde{PSL(2,\mathbb{R})}$. We now have to determine the lift of the long relation $q_1 \cdots q_r[a_1, b_1] \cdots [a_g, b_g]$ in $\widetilde{\Gamma}$.

For this, we consider two vectors s_i and s_i' based at each point V_i or W_i.

The element a_i is defined (see [**12**]) as the only element of $PSL(2,\mathbb{R})$ sending α_i' to α_i; we choose its lift in $\widetilde{PSL(2,\mathbb{R})}$ to be the transformation of $S\mathbb{H}^2$ sending $s_{4g-4i+3}$ to $s_{4g-4i+2}'$. The same element also sends $s_{4g-4i+4}'$ to $s_{4g-4i+5}$. Similarly, for the lift of b_i to $\widetilde{PSL(2,\mathbb{R})}$, take the transformation of $S\mathbb{H}^2$ sending $s_{4g-4i+3}'$ to $s_{4g-4i+4}$ and $s_{4g-4i+2}$ to $s_{4g-4i+1}'$. Then $b_g^{-1}(s_1') = s_2, a_g^{-1}(s_2') = s_3, \ldots, q_1(s_{4g+r}') = s_1$. Now define Θ_i: $S\mathbb{H}^2 \to S\mathbb{H}^2$ by $\Theta_i(p, v) = (p, v + \theta_i)$. Then $\Theta_i(s_i) = s_i'$, and Θ_i is central in $\widetilde{PSL(2,\mathbb{R})}$. Moreover, $\Theta_{4g+r} \circ \cdots \circ \Theta_1 = z$, or equivalently, $\sum_{i=1}^{4g+r} \theta_i = 2\pi$ (by the Gauss–Bonnet formula (see also [**12**, theorem 4·3·2])). Now, if the element $q_1 \cdots q_r[a_1, b_1] \cdots [a_g, b_g]$ caused, say, n turns in the index (i.e. $q_1 \cdots q_r[a_1, b_1] \cdots [a_g, b_g] = z^n$), then we make $n + 1$ turns after following the arrows $s_1', s_2, s_2', \ldots, s_{4g+r}', s_1$ and back to s_1'. Now we need just compute this index in order to find n.



The simplest way to do this is to extend our arrows to a vector field whose only singular points in the surface $\mathbb{H}^2/\Gamma$ are the central point O in the center of our fundamental domain, and the point W coming from the V_i's and W_i's. The index of this vector field at the point O is $-(n+1)$ (the arrows point towards O, hence the minus sign) and, since $\sum_{i=1}^{4g+r}\theta_i = 2\pi$, the index at W is 1. Hence, by the classical Poincaré–Hopf index theorem,

$$-(n+1)+1 = \chi\left(\Sigma_g^r\right) = 2g-2+r, \text{ i.e. } n = 2g-2+r.$$

This gives the following presentation for $\widetilde{\Gamma}$:

$$\widetilde{\Gamma} = \left\langle q_1, \ldots, q_r, a_1, \ldots, b_g, z \,\middle|\, \begin{matrix} zq_1z^{-1}q_1^{-1}, \ldots, zb_gz^{-1}b_g^{-1}, \\ q_1^{k_1}z^{-1}, \ldots, q_l^{k_l}z^{-1}, q_1\cdots q_r[a_1,b_1]\cdots[a_g,b_g]z^{2-2g-r} \end{matrix} \right\rangle$$

which is obviously equivalent to our statement (replace z^{-1} with z).

Remark 4·1. We could also count this index geometrically. First suppose for simplicity that $g = 0$. Then the tail of our vector pointed to the center of the fundamental domain exactly once for every rotation, thus r times. And we also turned around this center once negatively, so the index is $r-1$. Thus $n = r-2$, or equivalently, the element $q_1\cdots q_r[a_1,b_1]\cdots[a_g,b_g]$ lifts to $q_1\cdots q_r[a_1,b_1]\cdots[a_g,b_g]z^{-r+2}$ in $\widetilde{PSL}(2,\mathbb{R})$. In the general case ($g \geqslant 1$) one notes that each of the commutators $[a_i,b_i]$ gives two positive turns to our arrow.

4·2. *The Euler class is determined by the image, for discrete representations*

We will present here the proof of Theorem 1·2. Consider a Fuchsian group Γ with signature $(g; k_1, \ldots, k_l, \underbrace{\infty, \ldots, \infty}_{r-l})$. If $r \neq l$, set $e(\Gamma) = 0$. Otherwise, let d be the least common multiple of $k_1, \ldots, k_r$ (put $d = 1$ if $r = 0$) and set

$$e(\Gamma) = d\left(2g-2+\sum_{i=1}^{r}\left(1-\frac{1}{k_i}\right)\right).$$

PROPOSITION 4·1. *Let ρ be a representation of some $\Gamma_{g'}$ whose image is contained in Γ.*

Then $e(\rho)$ is a multiple of $e(\Gamma)$. Moreover, every multiple of $e(\Gamma)$ is the Euler class of some representation whose image is exactly Γ.

Proof. It follows from Lemma 4·1 that $H^1(\widetilde{\Gamma})$ has the following abelian presentation:

$$H^1(\widetilde{\Gamma}) = \left\langle q_1, \ldots, q_r, a_1, \ldots, b_g, z \;\middle|\; q_1^{k_1} z, \ldots, q_l^{k_l} z, q_1 \cdots q_r z^{2g-2+r} \right\rangle^{\mathrm{ab}}$$

where all generators are supposed to commute with each other, and this is recorded by using the superscript *ab* on the right-hand side.

We just need to show that the subgroup $\langle z\rangle$ generated by z in $H^1(\widetilde{\Gamma})$ is exactly $\mathbb{Z}/e(\Gamma)\mathbb{Z}$, where $e(\Gamma) = d(2g-2+\sum_{i=1}^{r}(1-1/k_i))$. Now $z^N = 1$ in $H^1(\widetilde{\Gamma})$ if and only if z^N is a product of relations $q_1^{k_1} z, \ldots, q_l^{k_l} z$ and $q_1 \cdots q_r z^{2g-2+r}$ in the *abelian* group $H^1(\widetilde{\Gamma})$.

Let us write these relations as column vectors in terms of the generators $q_1, \ldots, q_r$ and z (in the case $l = r$):

$$\begin{pmatrix} k_1 & 0 & \ldots & 0 & 1 \\ 0 & k_2 & \ddots & \vdots & 1 \\ \vdots & \ddots & & 0 & \\ 0 & \ldots & 0 & k_r & 1 \\ 1 & 1 & \ldots & 1 & 2g-2+r \end{pmatrix}.$$

Now, suppose that $z^N = 1$, i.e. there exists a linear relation

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ N \end{pmatrix} = \alpha_1 \begin{pmatrix} k_1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ k_2 \\ 0 \\ \vdots \\ 1 \end{pmatrix} + \cdots + \alpha_r \begin{pmatrix} 0 \\ \vdots \\ 0 \\ k_r \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 2g-2+r \end{pmatrix}$$

with $\alpha_1, \ldots, \alpha_r, \beta$ integers. Then, $k_1\alpha_1 + \beta = 0, \ldots, k_r\alpha_r + \beta = 0$. Hence, β is a multiple of each of the k_i's, i.e. $\beta = nd$ for some $n \in \mathbb{Z}$. Therefore $\alpha_i = -nd/k_i$. Now,

$$\begin{aligned} N &= \alpha_1 + \cdots + \alpha_r + \beta(2g-2+r) \\ &= nd\left(2g-2+r-\sum_{i=1}^{r}\frac{1}{k_i}\right) = nd\left(2g-2+\sum_{i=1}^{r}\left(1-\frac{1}{k_i}\right)\right). \end{aligned}$$

Conversely, we can write $z^{ne(\Gamma)}$ explicitly as a product of commutators, as it will be done in the next remark. This defines a representation of Euler class $ne(\Gamma)$ having its image contained in Γ. Now, to find a representation whose image is exactly Γ, just multiply $z^{ne(\Gamma)}$ by $[q_1, q_1] \cdots [q_r, q_r][a_1, a_1] \cdots [b_g, b_g]$ (as in Remark 4·7). This yields $z^{ne(\Gamma)}$ as a product of commutators, thus defining a representation of the group of some surface $\Sigma_{g'}$, of Euler class $ne(\Gamma)$.

And in the case $l < r$, the rth line simply gives $\beta = 0$, so that $N = 0$, which completes the proof.

Remark 4·2. Given a Fuchsian group Γ, let $G(\Gamma, n)$ be the minimal genus such that there exists a representation $\rho \in R_{G(\Gamma)}$ of image (contained in) Γ and Euler class $ne(\Gamma)$. The

Milnor–Wood inequality together with explicit formulas give the rough inequalities

$$n\frac{e(\Gamma)}{2}+1 \leqslant G(\Gamma,n) \leqslant ndg + r^{nd}.$$

Indeed, write $z^{ne(\Gamma)}$ as a product of $G(\Gamma,n)$ commutators in $\widetilde{\Gamma}$: then $ne(\Gamma) \leqslant 2G(\Gamma,n)-2$, i.e. $G(\Gamma,n) \geqslant ne(\Gamma)/2+1$. Concerning the upper bound, one can check by induction on d that the product $(q_1\cdots q_r)q_1^{-d}\cdots q_r^{-d}$ can be written as a product of r^d commutators. Now in $\widetilde{\Gamma}$, $(q_1\cdots q_r c_g z^{2g-2+r})^{nd}=1$ so $(q_1\cdots q_r c_g)^{nd}q_1^{-nd}\cdots q_r^{-nd}=z^{-ne(\Gamma)}$ and $(q_1\cdots q_r c_g)^{nd}q_1^{-nd}\cdots q_r^{-nd}$ can be written as the product of $ndg+r^{nd}$ commutators.

It is quite easy to refine these inequalities; however, we do not have sharp estimates for $G(\Gamma,n)$.

Remark 4·3. The term $2g-2+\sum_{i=1}^{r}(1-1/k_i)$ is the volume of the Poincaré fundamental domain of the Fuchsian group Γ (see e.g. [**12**]). In particular, it is positive; hence $e(\Gamma)$ is a positive integer if Γ is a cocompact Fuchsian group.

Remark 4·4. Of course, $l \neq r$ if and only if Γ is non-cocompact. In this case, there is another reason for every representation taking values in Γ to have zero Euler class. If Γ is non-cocompact, then $H^2(\Gamma)=0$ (since the surface $\mathbb{H}^2/\Gamma$ is not compact). And we have the following:

4·3. *Restrictions on Fuchsian image groups for fixed Euler class*

PROPOSITION 4·2. *Let Γ be any group injectively mapped in $PSL(2,\mathbb{R})$ such that $H^2(\Gamma)=0$, and let ρ be a representation with image contained in Γ. Then $e(\rho)=0$.*

Proof. The short exact sequence $0\to\mathbb{Z}\to\widetilde{\Gamma}\to\Gamma\to 0$ is a central extension so the corresponding spectral extension is exact at its end (see e.g. [**2**]):

$$\cdots \longrightarrow H^2(\Gamma) \longrightarrow H^1(\mathbb{Z}) \longrightarrow H^1(\widetilde{\Gamma}) \longrightarrow \cdots.$$

Now, $H^1(\mathbb{Z})=\mathbb{Z}$ and the image of any non zero $n\in H^1(\mathbb{Z})$, is not zero since $H^2(\Gamma)=0$. In other words, in $\widetilde{\Gamma}$ no power of z is a product of commutators.

Remark 4·5. However, there exist groups Γ, embedded in $PSL(2,\mathbb{R})$, with $H^2(\Gamma)\neq 0$ such that every representation taking values in Γ has Euler class 0. For instance, every realization of $\mathbb{Z}^2$ in $PSL(2,\mathbb{R})$ is in a one-parameter subgroup of $PSL(2,\mathbb{R})$, and hence every representation taking values in $\mathbb{Z}^2\subset PSL(2,\mathbb{R})$ is elementary and thus has Euler class 0.

On the other hand, many groups are the image of some representation of zero Euler class:

Remark 4·6. For every finitely generated group $G\subset PSL(2,\mathbb{R})$ with generating system $x_1,\ldots,x_g$, the relation $[x_1,x_1]\cdots[x_g,x_g]=1$ defines a representation $\rho\colon\Gamma_g\to PSL(2,\mathbb{R})$ whose image is G, in the following way:

$$\rho(a_i)=x_i,\ \rho(b_i)=x_i,\ \text{for } 1\leqslant i\leqslant g$$

In other words, every finitely generated subgroup of $PSL(2,\mathbb{R})$ is the image of some representation of Euler class 0.

Remark 4·7. Given two representations $\rho\in R_g$, $\rho'\in R_{g'}$, we have $[\rho(a_1),\rho(b_1)]\cdots[\rho(a_g),\rho(b_g)]=1$ and $[\rho(a_1'),\rho(b_1')]\cdots[\rho(a_{g'}'),\rho(b_{g'}')]=1$ so $[\rho(a_1),\ \rho(b_1)]\cdots[\rho(a_{g'}'),$

$\rho(b'_{g'})] = 1$. This defines naturally a representation in $R_{g+g'}$. This construction, together with the preceding remark, shows that there are also many subgroups of $PSL(2,\mathbb{R})$ arising as the image of representations of non-zero Euler class, even in a fixed R_g.

Now we prove Proposition 1·1.

PROPOSITION 4·3. *Let k be a fixed non-zero integer. There are only finitely many Fuchsian groups Γ such that there exists a representation of Γ_g, for some $g \geqslant 2$, with image contained in Γ and with Euler class k.*

Proof. In other words, we have to show that $e(\Gamma) \leqslant k$ holds true only for finitely many cocompact Fuchsian groups Γ. We will prove that the inequality

$$0 < d\left(2g - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{k_i}\right)\right) \leqslant k$$

implies that g, r and d are bounded in terms of k. Since $d \geqslant k_i$ for all $i \leqslant r$, every integer from the signature of Γ will take only finitely many values.

First, $k_i \geqslant 2$ for all i so $\sum_{i=1}^{r}(1 - 1/k_i) \geqslant r/2$, and $d \geqslant 1$ so $k \geqslant 2g - 2 + r/2$, i.e. $4g + r \leqslant 2k + 4$. Thus r and g are bounded. We claim that $d \leqslant 42k$.

(i) If $g \geqslant 1$ then $k \geqslant d(r/2)$. If $r = 0$ then we have $d = 1$, otherwise $k \geqslant d/2$ so $d \leqslant 42k$.

(ii) Suppose $g = 0$. If $d \geqslant 4k$, then $-2 + \sum_{i=1}^{r}(1 - 1/k_i) \leqslant 1/4$ so $-2 - (r/2) \leqslant 1/4$, hence $r \leqslant 4$. But $-2 + \sum_{i=1}^{r}(1 - 1/k_i) > 0$ (see Remark 4·3) so $r \geqslant 3$. Therefore, it suffices to consider the cases $r = 3$ and $r = 4$.

 (a) If $r = 4$, then $2 - 1/k_1 - 1/k_2 - 1/k_3 - 1/k_4 > 0$ and $k_i \geqslant 2$. Hence, one of the k_i's is greater than 3. Therefore, $2 - 1/k_1 - 1/k_2 - 1/k_3 - 1/k_4 \geqslant 1/6$, and hence $d \leqslant 6k$.

 (b) If $r = 3$, then $1 - 1/k_1 - 1/k_2 - 1/k_3 > 0$. First suppose that $k_1, k_2, k_3 \geqslant 3$. Then one of the k_i's is greater than 4, and hence $1 - 1/k_1 - 1/k_2 - 1/k_3 \geqslant 1/12$, so $d \leqslant 12k$. The only remaining case is when one of the k_i's, say k_3, equals 2. We then have $0 < d\,(1/2 - 1/k_1 - 1/k_2) \leqslant k$. Thus, $k_1, k_2 \geqslant 3$. And if k_1, $k_2 \geqslant 4$, the condition $1/2 - 1/k_1 - 1/k_2 > 0$ forces one of k_1, k_2 to be at least 5, so that $1/2 - 1/k_1 - 1/k_2 \geqslant 1/20$ so $d \leqslant 20k$. Hence, we may suppose that k_1 or k_2 (say, k_2) equals 3. Now, $0 < d\,(1/6 - 1/k_1) \leqslant k$ and hence $d \leqslant 42k$.

COROLLARY 4·1. *No Fuchsian group can be injectively mapped into infinitely many Fuchsian groups. Moreover, if Γ can be a sub-Fuchsian group of Γ' then $e(\Gamma')$ divides $e(\Gamma)$.*

Remark 4·8. However, we can produce chains of inclusions of Fuchsian groups, of arbitrary length. For instance, one may check easily that $(2^k; -) \hookrightarrow (2^{k-1}; 2) \hookrightarrow (2^{k-2}, 4) \hookrightarrow \cdots \hookrightarrow (1; 2^k)$.

4·4. *Paucity of discrete representations*

PROPOSITION 4·4. *Let $g \geqslant 2$ and k be such that $|k| \leqslant 2g - 3$, $k \neq 0$. The representations of discrete image form a nowhere dense closed subset of $e^{-1}(k)$ in R_g.*

Proof. First, we show that the set of discrete representations is closed in $e^{-1}(k)$. For this, we recall Jørgensen's lemma (see e.g. [**12**]). If A, $B \in PSL(2,\mathbb{R})$, we set $J(A, B) = |\mathrm{Tr}^2(A) - 4| + |\mathrm{Tr}([A, B]) - 2|$.

LEMMA 4·2. *If $\langle A, B\rangle$ is a non-elementary discrete group, then $J(A, B) \geqslant 1$.*

Now, let ρ_n be a sequence of discrete representations of Euler class k, which converges pointwise to a representation ρ. Then $e(\rho) = k \neq 0$. Therefore, the representation ρ is not elementary. Thus, there exist $T_1, T_2 \in \rho(\Gamma_g)$ such that $[T_1, T_2] \neq 1$. Suppose that ρ is not discrete. Then there exists a sequence $S_n \in \rho(\Gamma_g)$, $S_n \neq 1$, converging to 1 in $PSL(2, \mathbb{R})$. Hence, there exists $N \geqslant 0$ such that $J(S_N, T_1) < 1$ and $J(S_N, T_2) < 1$. Now, ρ_n converges to ρ pointwise so there exists $n_0 \geqslant 0$ and $s, t_1, t_2 \in \rho_{n_0}(\Gamma_g)$ such that $J(s, t_1) < 1$, $J(s, t_2) < 1$ and $[t_1, t_2] \neq 1$ (since these three conditions are open conditions). But this contradicts the assumption that ρ_{n_0} has discrete image.

Now we show that the closed set of discrete representations of Euler class k is nowhere dense. For this, we show that it is the union of a countable family of closed and nowhere dense sets. Hence, by Baire's theorem, its complement is a dense set.

Every discrete representation splits in the following way: $\Gamma_g \xrightarrow{h} \Gamma \hookrightarrow PSL(2, \mathbb{R})$, where there are only finitely many groups Γ possible. Now, for each Fuchsian group Γ, the set of morphisms $h : \Gamma_g \to \Gamma$ is countable. And the set of injective representations with discrete image $\Gamma \to PSL(2, \mathbb{R})$ has an algebraic structure, similar to that of R_g. When it is pulled back by a morphism $h : \Gamma_g \to \Gamma$, we obtain an algebraic subset of R_g, of codimension at least 1. It follows that for a fixed $h_0 : \Gamma_g \to \Gamma$, the set of representations ρ which factorize through h_0 is a closed and nowhere dense subset of R_g. Now, the union of these sets, over the countable family of possible Γ and h, is the set of discrete representations. As claimed, it is a countable union of nowhere dense subsets of $e^{-1}(k) \subsetneq R_g$.

Remark 4·9. In the set $e^{-1}(0)$, the subset of representations with discrete image is not closed. Indeed, let $h: \Gamma_g \to \mathbb{Z}^2$ be a surjective mapping and let $q \in \mathbb{R}$ be an irrational number. The number q can be approximated by rational numbers φ_n/ψ_n, with $\gcd(\varphi_n, \psi_n) = 1$. Let $R(t)$ be a hyperbolic one-parameter subgroup of $PSL(2, \mathbb{R})$. Define $\phi_n: \mathbb{Z}^2 \to PSL(2, \mathbb{R})$ by $\phi_n(1, 0) = R(1)$ and $\phi_n(0, 1) = R(\varphi_n/\psi_n)$. The representations $\phi_n \circ h: \Gamma_g \to PSL(2, \mathbb{R})$ are elementary hence their Euler class is 0. Moreover, they are all discrete. But they converge to a non-discrete representation.

4·5. *Existence of discrete representations with prescribed Euler class*

Now, we prove Theorem 1·3.

PROPOSITION 4·5. *For all $g \geqslant 2$ and $2-2g \leqslant k \leqslant 2g-2$, there exists a representation $\rho \in R_g$ with discrete image and Euler class k.*

Proof. First, if ρ is a representation of Γ_g of Euler class k, then $\rho'(a_i) = \rho(b_{g-i})$, $\rho'(b_i) = \rho(a_{g-i})$ defines a representation of Γ_g of Euler class $-k$, by Milnor's algorithm. Thus we just need to consider the case $k \geqslant 0$.

If k is even, set $k = 2l$. Notice that $l + 1 \leqslant g$, by the Milnor–Wood inequality. Take a representation in the Teichmüller component of the surface of genus $l + 1$. Its Euler class is $2(l+1)-2 = k$. Now extend it by identity elements (as in Remark 4·7) to get a representation of Γ_g. It still has discrete image, and by Milnor's algorithm its Euler class is k.

Similarly, for an arbitrary g, it suffices to find a representation of Euler class $2g - 3$ in order to cover the case of odd k. We will extend it to representations of higher genus and the same Euler class, by using identity elements.

(i) If g is even, say, $g = 2g'$, consider the Fuchsian group Γ of signature $(g'; 2)$. We then have the following relations that hold in the lift $\widetilde{\Gamma}$ of Γ to $\widetilde{PSL(2, \mathbb{R})}$: $q^2 = z$,

$qc = z^{2g'-1}$, where c is the product of the g' commutators defining the Fuchsian group Γ. Now, z is central in $\widetilde{PSL}(2,\mathbb{R})$ so $cq = z^{2g'-1}$, and hence $cq^2c = z^{4g'-2}$, i.e. $c^2 = z^{4g'-3} = z^{2g-3}$. This relation enables us to write the element z^{2g-3} using $2g' = g$ commutators in the Fuchsian group Γ of signature $(g'; 2)$, and this defines a representation of Γ_g of Euler class $2g-3$. Its image is a subgroup of Γ, hence discrete. Actually, its image is precisely Γ.

(ii) If g is odd, say $g = 2g'+1$, consider the Fuchsian group Γ of signature $(g'; 2, 2, 2)$. In its lift $\widetilde{\Gamma}$ in $\widetilde{PSL}(2,\mathbb{R})$ we have the following relations: $q_1^2 = z$, $q_2^2 = z$, $q_3^2 = z$, $q_1q_2q_3c = z^{2g'+1}$, where c is once again a product of g' commutators. Now, z is central so $q_2q_3cq_1 = z^{2g'+1}$. Thus $q_2q_3cq_2q_3c = z^{4g'+1}$ and hence

$$z^{4g'-1} = q_2^{-1}q_3^{-1}cq_2q_3c = q_2^{-1}q_3^{-1}q_2q_3q_3^{-1}q_2^{-1}cq_2q_3c = \left[q_2^{-1}, q_3^{-1}\right]\cdot((q_2q_3)^{-1}c(q_2q_3))\cdot c.$$

This implies that $z^{4g'-1}$ is the product of $2g'+1$ commutators, because the conjugation by q_1q_2 enters the product of commutators c. Thus we obtain a representation of Γ_g of Euler class $4g'-1 = 2g-3$. Moreover, we can check that the image of this representation is the group Γ.

4·6. *Non-faithful representations are dense in every non-Teichmüller component*

We will now prove Theorem 1·1, which, for the sake of completeness, we restate here:

PROPOSITION 4·6. *For all $g \geqslant 2$ and all k such that $|k| < 2g-2$, non-injective representations form a dense subset of the connected component $e^{-1}(k)$.*

Proof. Suppose the contrary. Then there exists an open set $V \subset R_g$ consisting only of injective representations of Euler class k ($|k| < 2g-2$). Let $\rho_0 \in V$. The representation ρ_0 is not discrete, so there exists $x \in \Gamma_g$ such that $\rho_0(x) \in Ell$ (see e.g. [**12**]). And ρ_0 is faithful so $\mathrm{Tr}(\rho_0(x)) \in [0, 2) = 2\cos\theta_0$, with θ_0 irrational (otherwise, $\rho_0(x)$ would be of finite order, which is impossible since Γ_g is torsion-free). Now $\rho \mapsto \mathrm{Tr}(\rho(x))$ is continuous, so the angle θ has to be constant on V in order to keep being irrational. Hence the algebraic function $\mathrm{Tr}(\rho(x))$ is constant on the open set V of the algebraic set R_g. Hence it is constant on the whole connected component $e^{-1}(k)$, since this connected component is contained within one irreducible component of the algebraic set R_g (see [**4**]).

It follows that for every $\rho \in e^{-1}(k)$, ρ sends x to an elliptic element corresponding to an irrational angle. In particular, ρ cannot have a discrete image. But this contradicts Theorem 1·3.

Remark 4·10. As it was pointed out in [**1, 4**], injective representations form a dense subset of R_g. Therefore one may understand the set of non-faithful representations in $e^{-1}(k)$, for $|k| \leqslant 2g-1$ as $\mathbb{Q}$ in $\mathbb{R}$. A very comparable result has been recently proved by Glutsyuk in [**7**], for the set of representations of free groups into general Lie groups, thereby answering a question of Ghys.

5. *Representations of odd Euler class*

The case of odd Euler class is somewhat simpler than the general case. Even though the following results can be deduced from the latest part, the approach is often more elementary.

5·1. *A cohomological criterion*

The commutator map

$$SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \longrightarrow SL(2,\mathbb{R})$$
$$A, B \longmapsto [A,B]$$

defines an application

$$PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R}) \longrightarrow PSL(2,\mathbb{R})$$
$$A, B \longmapsto [A,B].$$

Hence, there exists a continuous function $\varepsilon\colon R_g \to \{-1, 1\}$ such that for all $\rho \in R_g$,

$$[\rho(a_1), \rho(b_1)] \cdots [\rho(a_g), \rho(b_g)] = \varepsilon(\rho).$$

PROPOSITION 5·1. *For all $\rho \in \mathrm{Hom}(\Gamma_g, PSL(2,\mathbb{R}))$, $\varepsilon(\rho) = (-1)^{e(\rho)}$, where $e(\rho)$ is the Euler class of ρ.*

Proof. This follows directly from Milnor's algorithm: the element z maps to -1 in $SL(2,\mathbb{R})$.

In particular, this gives a decomposition of R_g into two sub-varieties of $(SL(2,\mathbb{R}))^{2g}$. One of them is defined by the equation $[x_1, y_1] \cdots [x_g, y_g] = 1$ and the other is defined by $[x_1, y_1] \cdots [x_g, y_g] = -1$. These two algebraic varieties are irreducible. In fact the invariant $\varepsilon(\rho)$ is precisely the second Stiefel–Whitney class $w_2(\rho)$ (see [**4**, **9**, **10**]).

COROLLARY 5·1. *Let Γ be a subgroup of $PSL(2,\mathbb{R})$ of finite type and let $\widetilde{\Gamma}$ be its lift to $SL(2,\mathbb{R})$ containing -1. (Note that this is well defined.) Then the two following assertions are equivalent:*

(i) *there exists $\rho \in R_g$ of odd Euler class and whose image is Γ;*
(ii) *-1 is a product of commutators in $\widetilde{\Gamma}$, i.e. -1 maps to the neutral element in $H^1(\widetilde{\Gamma})$.*

Proof. If there exists a representation $\rho \in R_g$ of odd Euler class whose image is Γ then by Proposition 4·1, $[\rho(a_1), \rho(b_1)] \cdots [\rho(a_g), \rho(b_g)] = -1$ so -1 is indeed a product of commutators in the image of ρ.

Conversely, suppose -1 is a product of commutators in the group Γ, namely that $[x_1, y_1] \cdots [x_g, y_g] = -1$. Let $z_1, \ldots, z_n$ be a system of generators of Γ. Then using the relation $[x_1, y_1] \cdots [x_g, y_g][z_1, z_1] \cdots [z_n, z_n] = 1$, one defines a representation ρ of $\pi_1(\Sigma_{g+n})$, whose image is exactly Γ.

COROLLARY 5·2. *There are no representations of odd Euler class whose image is in $PSL(2,\mathbb{Z})$.*

Proof. The element $-1 \in SL(2,\mathbb{Z})$ maps to a non-trivial element in the group $H^1(SL(2,\mathbb{Z})) \cong \mathbb{Z}/12\mathbb{Z}$.

5·2. *An explicit example*

We can use this characterization to give an explicit example of a discrete representation of odd Euler class as follows. Let Γ be the Fuchsian triangular group of signature $(0; 2, 3, 7)$. Then Γ has the following presentation:

$$\Gamma = \left\langle q_1, q_2, q_3 \,\middle|\, q_1^2 = 1, q_2^3 = 1, q_3^7 = 1, q_1 q_2 q_3 = 1 \right\rangle.$$

Its lift $\widetilde{\Gamma}$ then has the following presentation:

$$\widetilde{\Gamma} = \left\langle q_1, q_2, q_3, h \,\middle|\, \begin{array}{l} h^2 = 1, hq_1 = q_1 h, hq_2 = q_2 h, hq_3 = q_3 h, \\ q_1^2 = h^{\beta_1}, q_2^3 = h^{\beta_2}, q_3^7 = h^{\beta_3}, q_1 q_2 q_3 = h^{\beta} \end{array} \right\rangle.$$

One can replace q_2 and q_3, respectively, by $q_2 h$ and $q_3 h$, to get $\beta_2 = \beta_3 = 1$, and then replace q_1 by $q_1 h$ to get $\beta = 1$ in this presentation. Now, if we had $\beta_1 = 0$, this would mean that q_1 has order 2 in $SL(2, \mathbb{R})$ and thus is equal to 1 or -1, and hence is 1 in $PSL(2, \mathbb{R})$, which is not the case. Thus $\beta_1 = 1 \pmod 2$. Finally, we get the following presentation of $\widetilde{\Gamma}$:

$$\widetilde{\Gamma} = \left\langle q_1, q_2, q_3, h \,\middle|\, \begin{array}{l} h^2 = 1, hq_1 = q_1 h, hq_2 = q_2 h, hq_3 = q_3 h, \\ q_1^2 = h, q_2^3 = 1, q_3^7 = 1, q_1 q_2 q_3 = 1 \end{array} \right\rangle.$$

Consider now $H^1(\widetilde{\Gamma})$. In this group, $q_1 q_2 q_3 = 1$ so $(q_1 q_2 q_3)^{42} = 1 = q_1^{42} q_2^{42} q_3^{42} = h^{21} q_2^{3\times 14} q_3^{7\times 6} = h.h^{2\times 10} = h$ so $h = 1$ in $H^1(\widetilde{\Gamma})$. Equivalently, -1 is a product of commutators in $\widetilde{\Gamma}$. In the same way as in Remark 4·3, we can explicitly write -1 as a product of commutators, thus defining a representation of some Γ_g in $PSL(2, \mathbb{R})$.

5·3. *Discrete representations of odd Euler class*

If Γ is a cocompact Fuchsian group with signature $(g; k_1, k_2, \ldots, k_r)$, let $m(\Gamma)$ be the maximal power of 2 dividing one of the k_i's. If $m(\Gamma) = 0$, set $n(\Gamma) = 0$. Otherwise, let $n(\Gamma)$ be the number of k_i's which are divisible by $2^{m(\Gamma)}$. We then have the following characterization:

PROPOSITION 5·2. *A Fuchsian group $\Gamma \subset PSL(2, \mathbb{R})$ is the image of a representation of Γ_g, for some g, with odd Euler class, if and only if Γ is a cocompact Fuchsian group such that $n(\Gamma)$ is odd.*

Proof. This may be deduced directly from Proposition 4·1, but we give a slightly different proof here. First, by Remark 4·4 (or Proposition 4·1), Γ has to be cocompact (otherwise $H^2(\Gamma) = 0$ and $e(\rho) = 0$).

Now, let Γ be a cocompact Fuchsian group, with signature $(g; k_1, k_2, \ldots, k_r)$. This means that Γ has the following presentation:

$$\Gamma = \langle q_1, q_2, \ldots, q_r, a_1, \ldots, b_g \,|\, q_1^{k_1}, \ldots, q_r^{k_r}, q_1 \cdots q_r [a_1, b_1] \cdots [a_g, b_g] \rangle.$$

Now using Lemma 4·1 and the fact that $h^2 = 1$ here (we are in $SL(2, \mathbb{R})$ and not in $PSL(2, \mathbb{R})$), we get the following presentation for the lift $\widetilde{\Gamma}$ in $SL(2, \mathbb{R})$:

$$\widetilde{\Gamma} = \left\langle q_1, \ldots, q_r, a_1, \ldots, b_g, h \,\middle|\, \begin{array}{l} hq_1 h^{-1} q_1^{-1}, \ldots, hb_g h^{-1} b_g^{-1}, h^2, q_1^{k_1} h, \ldots, q_r^{k_r} h, \\ q_1 \cdots q_r [a_1, b_1] \cdots [a_g, b_g] h^r \end{array} \right\rangle.$$

It follows that the abelianisation of $\widetilde{\Gamma}$ has the following abelian presentation:

$$H^1(\widetilde{\Gamma}) = \langle q_1, \ldots, q_r, h \,|\, h^2, q_1^{k_1} h, \ldots, q_r^{k_r} h, q_1 \cdots q_r h^r \rangle^{\mathrm{ab}}.$$

Now reorder the q_i's so that the powers of 2 dividing k_i are decreasing. For $1 \leqslant i \leqslant r$, let $k_i = 2^{u_i} v_i$ with v_i odd. In particular, $u_i = m(\Gamma)$ if $1 \leqslant i \leqslant n(\Gamma)$.

(i) First suppose that $n(\Gamma)$ is odd. Then $(q_1 \cdots q_r h^r)^{2^{m(\Gamma)} v_1 \cdots v_r} = h^{n(\Gamma)} = 1$ so $h = 1$ in $H^1(\widetilde{\Gamma})$.

(ii) Assume now that $n(\Gamma)$ is even. Then we will define a morphism $\phi \colon H^1(\widetilde{\Gamma}) \to \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. First, let $\phi(h) = -1$. Next, if $i \geqslant 2$, let $\phi(q_i) = \exp(i\pi/2^{u_i})$. Then $\phi(q_i^{k_i} h) = -\exp(i\pi k_i/2^{u_i}) = \exp(i\pi(1 + v_i)) = 1$. If $n(\Gamma) = 0$ (or equivalently, if all the k_i's are odd), set $\phi(q_1) = -1$. Then one may easily check that $\phi(q_1^{k_1}h) = 1$ and $\phi(q_1 \cdots q_r h^r) = 1$, so our morphism ϕ is well-defined. Otherwise, let $\phi(q_1) = \exp(-i\pi/2^m((n-1) + 2^m r + \sum_{i=n+1}^{r} 2^{m-n_i}))$. Then once again we can check that $\phi(q_1^{k_1}h) = 1$ and that $\phi(q_1 \cdots q_r h^r) = 1$, so our morphism is well defined. This proves that $h \neq 1$ in $H^1(\widetilde{\Gamma})$.

Remark 5·1. As in Remark 4·3, in the case $h = 1$ in $H^1(\widetilde{\Gamma})$, we can define an explicit representation of odd Euler class, whose image is Γ. This is done by writing $(q_1 \cdots q_r)^d q_1^{-d} \cdots q_r^{-d}$, with $d = 2^{m(\Gamma)} v_1 \cdots v_r$, as a product of commutators.

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