ON SMOOTH MAPS WITH FINITELY MANY
CRITICAL POINTS

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ABSTRACT

The minimum number of critical points of a small codimension smooth map between two manifolds is computed. Some partial results for the case of higher codimension when the manifolds are spheres are also given.

1. *Introduction*

If M, N are manifolds, possibly with boundary, consider maps $f : M \longrightarrow N$ with $\partial M = f^{-1}(\partial N)$ such that f has no critical points on ∂M. Denote by $\varphi(M, N)$ the minimal number of critical points of such maps. The reader may consult the survey [3] for an account of various features of this invariant (see also [24]). Most of the previously known results consist of sufficient conditions on M and N ensuring that $\varphi(M, N)$ is infinite.

The aim of this paper is to find when non-trivial $\varphi(M^m, N^n)$ can occur if the dimensions m and n of M^m and N^n, respectively, satisfy $m \geqslant n \geqslant 2$. Non-trivial here means finite and non-zero. Our main result is the following.

THEOREM 1.1.   *Assume that M^m, N^n are compact orientable manifolds and $\varphi(M^m, N^n)$ is finite, where $0 \leqslant m - n \leqslant 3$ and $(m, n) \notin \{(2, 2), (4, 3), (4, 2), (5, 3), (5, 2), (6, 3), (8, 5)\}$. If $m - n = 3$, then we also assume that the Poincaré conjecture in dimension 3 holds true.*

Then $\varphi(M^m, N^n) \in \{0, 1\}$ and $\varphi(M^m, N^n) = 1$ precisely when the following two conditions are fulfilled.

(i) *M^m is diffeomorphic to the connected sum $\widehat{N} \sharp \Sigma^m$, where Σ^m is an exotic sphere and $\widehat{N}$ is an m-manifold which fibers over N^n.*

(ii) *M^m does not fiber over N^n.*

Proof.   The statement is a consequence of Propositions 3.1, 4.1 and 5.1.   □

REMARK 1.1.   (i) The second condition is necessary, in general. There exist examples of connected sums $\widehat{N} \sharp \Sigma^m$ which fiber over N, yet they are not diffeomorphic to $\widehat{N}$. In fact, the exotic 7-spheres constructed by Milnor in [20] are pairwise non-diffeomorphic fibrations over S^4 with fiber S^3.

(ii) However, if the codimension $m - n$ is zero, then we believe that the second condition is redundant, that is, if M^m is diffeomorphic to $\widehat{N} \sharp \Sigma^m$ and M^m is not

diffeomorphic to $\widehat{N}$, then M^m cannot be a (smooth) covering of N. This claim holds true when N^n is hyperbolic, for all but finitely many coverings $\widehat{N}$. In fact, Farrell and Jones [13] proved that a finite covering $\widehat{N}$ of sufficiently large degree of a hyperbolic manifold N^n has the property that $\widehat{N}\sharp\Sigma$ admits a Riemannian metric of negative curvature but it does not have a hyperbolic structure. In particular, $\widehat{N}\sharp\Sigma$ is not a covering of N and hence $\varphi(\widehat{N}\sharp\Sigma, N) = 1$. Conversely, if $\varphi(M^n, N^n) = 1$ and N^n is hyperbolic, then M^n cannot be hyperbolic. Otherwise Mostow rigidity would imply that M^n was isometric and hence diffeomorphic to $\widehat{N}$.

(iii) The theorem holds true for non-compact manifolds M and N if we define φ by restricting ourselves to those smooth maps which are proper.

(iv) In most of the cases excluded in the hypothesis of the theorem, one can find examples with non-trivial $\varphi(M^m, N^n) \geqslant 2$ (see below).

(v) One expects that, for all (m, n) with $m - n \geqslant 4$, such examples abound. This is the situation for the local picture. The typical example is a complex projective manifold X admitting non-trivial morphisms into $\mathbb{CP}^1$.

(vi) The case $n = 1$ was analyzed in [25], where the authors proved that $\varphi(M, [0,1]) = 2$, for any non-trivial h-cobordism M.

Most of this paper is devoted to the proof of Theorem 1.1. In the last part we also compute the values of $\varphi(S^m, S^n)$ in a few cases and look for a more subtle invariant which would measure how far a manifold is from being a covering of another one.

We will consider henceforth that all manifolds are closed and connected unless the opposite is stated.

2. Elementary computations for surfaces

Patterson [23] gave necessary and sufficient conditions for the existence of a covering of a surface with prescribed degree and ramification orders. Specifically, his result can be stated as follows.

PROPOSITION 2.1. *Let X be a Riemann surface of genus $g \geqslant 1$. Let $p_1, \ldots, p_k$ be distinct points of X and $m_1, \ldots, m_k$ be strictly positive integers so that*

$$\sum_{i=1}^{k} (m_i - 1) = 0 \pmod 2.$$

Let d be an integer such that $d \geqslant \max_{i=1,\ldots,k} m_i$. Then there exists a Riemann surface Y and a holomorphic covering map $f : Y \longrightarrow X$ of degree d such that there exist k points $q_1, \ldots, q_k$ in Y so that $f(q_j) = p_j$, and f is ramified to order m_j at q_j and is unramified outside the set $\{q_1, \ldots, q_k\}$.

Observe that a smooth map $f : Y \longrightarrow X$ between surfaces has finitely many critical points if and only if it is a ramified covering. Furthermore, $\varphi(Y, X)$ is the minimal number of ramification points of a covering $Y \longrightarrow X$. Estimations can be obtained from the previous result. Denote by Σ_g the oriented surface of genus g. Denote by $[[r]]$ the smallest integer greater than or equal to r. Our principal result in this section is the following.

PROPOSITION 2.2. *Let Σ and Σ' be closed oriented surfaces of Euler characteristics χ and χ', respectively.*

(i) *If $\chi' > \chi$, then $\varphi(\Sigma', \Sigma) = \infty$.*

(ii) *If $\chi' \leqslant 0$, then $\varphi(\Sigma', S^2) = 3$.*

(iii) *If $\chi' \leqslant -2$, then $\varphi(\Sigma', \Sigma_1) = 1$.*

(iv) *If $2 + 2\chi \leqslant \chi' < \chi \leqslant -2$, then $\varphi(\Sigma', \Sigma) = \infty$.*

(v) *If $0 \leqslant |\chi| \leqslant |\chi'|/2$, write $|\chi'| = a|\chi| + b$ with $0 \leqslant b < |\chi|$; then*

$$\varphi(\Sigma', \Sigma) = \left[\left[\frac{b}{a-1}\right]\right].$$

In particular, if $G \geqslant 2(g-1)^2$, then

$$\varphi(\Sigma_G, \Sigma_g) = \begin{cases} 0 & \text{if} & (G-1)/(g-1) \in \mathbb{Z}_+ \\ 1 & \text{otherwise.} \end{cases}$$

Proof. The first claim is obvious.

Further, $\varphi(\Sigma', S^2) \leqslant 3$ because any surface is a covering of the 2-sphere branched at three points (from [2]). A deeper result is that the same inequality holds in the holomorphic framework. In fact, Belyi's theorem states that any Riemann surface defined over a number field admits a meromorphic function on it with only three critical points (see, for example, [30]).

On the other hand, assume that $f : \Sigma' \longrightarrow S^2$ is a ramified covering with at most two critical points. Then f induces a covering map $\Sigma' - f^{-1}(E) \longrightarrow S^2 - E$, where E is the set of critical values and its cardinality $|E| \leqslant 2$. Therefore one has an injective homomorphism $\pi_1(\Sigma' - f^{-1}(E)) \longrightarrow \pi_1(S^2 - E)$. Now $\pi_1(\Sigma')$ is a quotient of $\pi_1(\Sigma' - f^{-1}(E))$ and $\pi_1(S^2 - E)$ is either trivial or infinite cyclic, which implies that $\Sigma' = S^2$.

Next, the unramified coverings of tori are tori; thus any smooth map $f : \Sigma_G \longrightarrow \Sigma_1$ with finitely many critical points must be ramified, so that $\varphi(\Sigma_G, \Sigma_1) \geqslant 1$, if $G \geqslant 2$. On the other hand, by Patterson's theorem, there exists a covering $\Sigma' \longrightarrow \Sigma_1$ of degree $d = 2G - 1$ of the torus, with a single ramification point of multiplicity $2G - 1$. From the Hurwitz formula, it follows that Σ' has genus G, which shows that $\varphi(\Sigma_G, \Sigma_1) = 1$.

LEMMA 2.1. *$\varphi(\Sigma', \Sigma)$ is the smallest integer k which satisfies*

$$\left[\left[\frac{\chi' - k}{\chi - k}\right]\right] \leqslant \frac{\chi' + k}{\chi}.$$

Proof. Suppose that Σ_G is a covering of degree d of Σ_g, ramified at k points with the multiplicities $m_i = d - \lambda_i$, where $0 \leqslant \lambda_i \leqslant d - 2$. If one sets $\lambda = \sum_i \lambda_i$, then λ satisfies the obvious inequality

$$\lambda \leqslant k(d-2).$$

Further, the Hurwitz formula yields the identity

$$d(k - \chi) = k - \chi' + \lambda.$$

Conversely, if there are solutions (k, λ, d) of the two equations above, with $k, \lambda \geqslant 0$ and $d \geqslant 1$, then one can find integers m_i, λ_i as above and therefore one can construct (using Patterson's theorem) a ramified covering $\Sigma' \longrightarrow \Sigma$ of degree d, with

k ramification points of multiplicities m_i. Therefore, $\varphi(\Sigma', \Sigma)$ is the least integer $k \geqslant 0$ for which there exists a solution $(k, \lambda, d) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}_+$ of the system:

$$0 \leqslant d(k - \chi) + \chi' - k = \lambda \leqslant k(d - 2).$$

That is, for $\chi \leqslant -2$, $\varphi(\Sigma', \Sigma)$ is the least $k \in \mathbb{N}$ for which there exists a positive integer d satisfying

$$\frac{\chi' - k}{\chi - k} \leqslant d \leqslant \frac{\chi' + k}{\chi},$$

and this is clearly equivalent to what is claimed in Lemma 2.1. $\square$

Assume now that $2 + 2\chi \leqslant \chi' < \chi \leqslant -2$. If $f : \Sigma' \longrightarrow \Sigma$ was a ramified covering, then we would have $(\chi' + k)/\chi < 2$, and Lemma 2.1 would imply that $\chi' = \chi$, which is a contradiction. Therefore $\varphi(\Sigma', \Sigma) = \infty$ holds.

Finally, assume that $\chi'/2 \leqslant \chi \leqslant -2$. One has to compute the minimal k satisfying

$$\left[\left[\frac{a\chi - b - k}{\chi - k}\right]\right] \leqslant \frac{a\chi - b + k}{\chi},$$

or, equivalently,

$$\left[\left[\frac{b + (1 - a)k}{\chi - k}\right]\right] \geqslant \frac{b - k}{\chi}.$$

The smallest k for which the quantity in the square brackets is non-positive is $k = [[b/(a - 1)]]$, in which case

$$\left[\left[\frac{b + (1 - a)k}{\chi - k}\right]\right] \geqslant 0 \geqslant \frac{b - k}{\chi}.$$

For k smaller than this value, one has a strictly positive integer on the left-hand side, which is therefore at least 1. However, the right-hand side is strictly smaller than 1; hence the inequality cannot hold. This proves the claim. $\square$

3. Equidimensional case $n \geqslant 3$

The situation changes completely in dimensions $n \geqslant 3$. According to [**8**, II, p. 535], Hopf was the first to notice that a smooth map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ $(n \geqslant 3)$ which has only an isolated critical point p is actually a local homeomorphism at p. Our result below is an easy application of this fact. We outline the proof for the sake of completeness.

PROPOSITION 3.1. *Assume that M^n and N^n are compact manifolds. If $\varphi(M^n, N^n)$ is finite and $n \geqslant 3$, then $\varphi(M^n, N^n) \in \{0, 1\}$. Moreover, $\varphi(M^n, N^n) = 1$ if and only if M^n is the connected sum of a finite covering $\widehat{N^n}$ of N^n with an exotic sphere and M^n is not a covering of N^n.*

Proof. There exists a smooth map $f : M^n \longrightarrow N^n$ which is a local diffeomorphism on the preimage of the complement of a finite subset of points. Notice that f is a proper map.

Let $p \in M^n$ be a critical point and let $q = f(p)$. Let $B \subset N$ be a closed ball intersecting the set of critical values of f only at q. We suppose moreover that q is an interior point of B. Denote by U the connected component of $f^{-1}(B)$ which contains p. As f is proper, its restriction to $f^{-1}(B - \{q\})$ is also proper.

As it is a local diffeomorphism onto $B - \{q\}$, it is a covering, which implies that $f : U - f^{-1}(q) \longrightarrow B - \{q\}$ is also a covering. However, f has only finitely many critical points in U, which shows that $f^{-1}(q)$ is discrete outside this finite set, and so $f^{-1}(q)$ is countable. This shows that $U - f^{-1}(q)$ is connected. As $B - \{q\}$ is simply connected, we see that $f : U - f^{-1}(q) \longrightarrow B - \{q\}$ is a diffeomorphism. This shows that $f^{-1}(q) \cap U = \{p\}$, for otherwise $H_{n-1}(U - f^{-1}(q))$ would not be free cyclic. Thus $f : U - \{p\} \longrightarrow B - \{q\}$ is a diffeomorphism. An alternative way is to observe that $f|_{U - \{p\}}$ is a proper submersion because f is injective in a neigborhood of p (except possibly at p). This implies that $f : U - \{p\} \longrightarrow B - \{q\}$ is a covering and hence a diffeomorphism since $B - \{q\}$ is simply connected.

One can then verify easily that the inverse of $f|_U : U \longrightarrow B$ is continuous at q; hence it is a homeomorphism. In particular, U is homeomorphic to a ball. Since ∂U is a sphere, the results of Smale (for example [**31**]) imply that U is diffeomorphic to the ball for $n \neq 4$.

We obtain that f is a local homeomorphism and hence topologically a covering map. Thus M^n is homeomorphic to a covering of N^n. Let us show now that one can modify M^n by taking the connected sum with an exotic sphere in order to get a smooth covering of N^n.

By gluing a disk to U, using an identification $h : \partial U \longrightarrow \partial B = S^{n-1}$, we obtain a homotopy sphere (possibly exotic) $\Sigma_1 = U \bigcup_h B^n$. Set $M_0 = M - \mathrm{int}(U)$, $N_0 = N - \mathrm{int}(B)$. Given the diffeomorphisms $\alpha : S^{n-1} \longrightarrow \partial U$ and $\beta : S^{n-1} \longrightarrow \partial B$, one can form the manifolds

$$M(\alpha) = M_0 \bigcup_{\alpha \, : \, S^{n-1} \longrightarrow \partial U} B^n, \quad N(\beta) = N_0 \bigcup_{\beta \, : \, S^{n-1} \longrightarrow \partial B} B^n.$$

Set $h = f|_{\partial U} : \partial U \longrightarrow \partial B = S^{n-1}$. A map $F : M(\alpha) \longrightarrow N(h \circ \alpha)$ is then given by

$$F(x) = \begin{cases} x & \text{if } x \in D^n \\ f(x) & \text{if } x \in M_0. \end{cases}$$

The map F has the same critical points as $f|_{M_0}$; hence it has precisely one critical point less than $f : M \longrightarrow N$.

We choose $\alpha = h^{-1}$ and we remark that $M = M(h^{-1}) \sharp \Sigma_1$, where the equality sign '$=$' stands for diffeomorphism equivalence. Denote $M_1 = M(h^{-1})$. We obtained above that $f : M = M_1 \sharp \Sigma_1 \longrightarrow N$ decomposes as follows. The restriction of f to M_0 extends to M_1 without introducing extra critical points, while the restriction to the homotopy ball corresponding to the holed Σ_1 has precisely one critical point.

Thus, iterating this procedure, one finds that there exist possibly exotic spheres Σ_i so that $f : M = M_k \sharp \Sigma_1 \sharp \Sigma_2 \ldots \sharp \Sigma_k \longrightarrow N$ decomposes as follows: the restriction of f to the k-holed M has no critical points, and it extends to M_k without introducing any further critical point. Each critical point of f corresponds to a (holed) exotic Σ_i. In particular, M_k is a smooth covering of N.

Now the connected sum $\Sigma = \Sigma_1 \sharp \Sigma_2 \ldots \sharp \Sigma_k$ is also an exotic sphere. Let $\Delta = \Sigma - \mathrm{int}(B^n)$ be the homotopy ball obtained by removing an open ball from Σ. We claim that there exists a smooth map $\Delta \longrightarrow B^n$ that extends any given diffeomorphism of the boundary and has exactly one critical point. Then one builds up a smooth map $M_k \sharp \Sigma \longrightarrow N$ having precisely one critical point, by putting together the obvious covering on the 1-holed M_k and $\Delta \longrightarrow B^n$. This will show that $\varphi(M, N) \leqslant 1$.

The claim follows easily from the following two remarks. First, the homotopy ball Δ is diffeomorphic to the standard ball by [**31**], when $n \neq 4$. Further, any

diffeomorphism $\varphi \colon S^{n-1} \longrightarrow S^{n-1}$ extends to a smooth homeomorphism with one critical point $\Phi \colon B^n \longrightarrow B^n$, for example

$$\Phi(z) = \exp\left(-\frac{1}{\|z\|^2}\right)\varphi\left(\frac{z}{\|z\|}\right).$$

For $n = 4$, we need an extra argument. Each homotopy ball $\Delta_i^4 = \Sigma_i - \mathrm{int}(B^4)$ is the preimage $f^{-1}(B)$ of a standard ball B. Since f is proper, we can choose B small enough such that Δ_i^4 is contained in a standard 4-ball. Therefore Δ^4 can be engulfed in S^4. Moreover, Δ^4 is the closure of one connected component of the complement of $\partial \Delta^4 = S^3$ in S^4. The result of Huebsch and Morse from [**16**] states that any diffeomorphism $S^3 \longrightarrow S^3$ has a Schoenflies extension to a homeomorphism $\Delta^4 \longrightarrow B^4$ which is a diffeomorphism everywhere except for one (critical) point. This proves the claim.

Remark finally that $\varphi(M^n, N^n) = 0$ if and only if M^n is a covering of N^n. Therefore if M^n is diffeomorphic to the connected sum $\widehat{N^n}\sharp\Sigma^n$ of a covering $\widehat{N^n}$ with an exotic sphere Σ^n, and if it is not diffeomorphic to a covering of N^n, then $\varphi(M^n, N^n) \neq 0$. Now drill a small hole in $\widehat{N^n}$ and glue (differently) an n-disk B^n (respectively a homotopy 4-ball if $n = 4$) in order to get $\widehat{N^n}\sharp\Sigma^n$. The restriction of the covering $\widehat{N^n} \longrightarrow N^n$ to the boundary of the hole extends (by the previous arguments) to a smooth homeomorphism with one critical point over Σ^n. Thus $\varphi(M^n, N^n) = 1$.   □

REMARK 3.1.   (i) We should stress that not all exotic structures on a manifold can be obtained from a given structure by connected sums with an exotic sphere. For example, smooth structures on products of spheres (and sphere bundles of spheres) are well understood (see [**11**, **12**, **17**, **28**, **29**]). All smooth structures on $S^p \times S^q$ are of the form $(S^p \times S^q)\#\Sigma^{p+q}$, where Σ^r denotes a homotopy r-sphere. If $p + 3 \geqslant q \geqslant p$, then it is enough to consider only those manifolds for which $\Sigma^q = S^q$ [**17**], but otherwise there are examples where the number $n(p,q)$ of non-diffeomorphic manifolds among them is larger than the number of homotopy $(p+q)$-spheres. For example, $n(1,7) = 30$, $n(3,10) = 4$, $n(1,16) = n(3,14) = 24$. On the other hand, the connected sum with an exotic sphere does not necessarily change the diffeomorphism type. For example, Kreck [**18**] proved that for any manifold M^m (of dimension $m \neq 4$), there exists an integer r such that either $M\sharp_r S^m$ or $M\sharp\mathrm{ST}(S^{(m-1)/2})\sharp_r S^m$ (if $m = 1 \pmod 4$) has a unique smooth structure, where $\mathrm{ST}(S^k)$ denotes the sphere bundle of the tangent bundle of the sphere S^k. However, results of Farrell and Jones [**13**] show that any hyperbolic manifold has finite coverings for which making a connected sum with an exotic sphere will change the diffeomorphism type.

(ii) Suppose that $M^n = \widehat{N^n}\sharp\Sigma$ is not diffeomorphic to $\widehat{N^n}$. It would be interesting to know under which hypotheses one can insure that M^n is not a smooth covering of N^n.

COROLLARY 3.1.   *If the dimension $n \in \{3, 5, 6\}$, then $\varphi(M^n, N^n)$ is either 0 or ∞.*

Proof.   In fact, two 3-manifolds which are homeomorphic are diffeomorphic and in dimensions 5 and 6 there are no exotic spheres.   □

REMARK 3.2. A careful analysis of open maps between manifolds of the same dimensions was carried out in [**7**]. In particular, it was proved that an open map of finite degree whose branch locus is a locally tame embedded finite complex (for example if the map is simplicial) has both the branch locus and the critical set of codimension 2 (see [**8**, II]) around each point.

4. *Local obstructions for higher codimension*

Our main result in this section, which is less precise than that for codimension 0, is a simple consequence of the investigation of local obstructions. In fact, the existence of analytic maps $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ with isolated singularities is rather exceptional in the context of smooth real maps (see [**21**]).

PROPOSITION 4.1. *If* $\varphi(M^m, N^n)$ *is finite and* $m = n + 1 \neq 4$, *or* $m = n + 2 \neq 4$ *or* $m = n + 3 \notin \{5, 6, 8\}$ *(when one assumes the Poincaré conjecture to be true), then* M *is homeomorphic to a fibration of base* N. *In particular, if* $m = 3$, $n = 2$, *then* $\varphi(M^3, N^2) \in \{0, \infty\}$, *except possibly for* M^3 *a non-trivial homotopy sphere and* $N^2 = S^2$.

Proof. One first shows the following lemma.

LEMMA 4.1. *Assume that* $\varphi(M^m, N^n) \neq 0$ *is finite for two manifolds* M^m *and* N^n. *Then there exists a polynomial map* $f : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^n, 0)$ *having an isolated singularity at the origin.*

Proof. The hypothesis implies the existence of a smooth map $f : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^n, 0)$ with one isolated singularity at the origin. We can assume that the critical point is not an isolated point of the fiber over 0 (see Remark 4.4). If the restriction $f|_{S^{m-1}}$ is of maximal rank, then the construction goes as follows. One approximates the restriction $f|_{S^{m-1}}$ to the unit sphere, up to the first derivative, by a polynomial map $\tilde{\psi}$ (of some degree d), and one extends the latter to all of $\mathbb{R}^m$ by $\psi(x) = |x|^d \tilde{\psi}(x/|x|)$. If the approximation is sufficiently close, then $\tilde{\psi}$ will be of maximal rank around the unit sphere; hence ψ will have an isolated singularity at the origin. However, some caution is needed when $f|_{S^{m-1}}$ is not of maximal rank. We consider then the restriction $f|_{B^m - B^m_{1-\delta}}$ to the annulus bounded by the spheres of radius 1 and $1 - \delta$, respectively. We claim the following.

LEMMA 4.2. *There exists some* $\delta > 0$ *and a polynomial map* $\tilde{\psi}$ *(of some degree* d) *such that its extension* $\psi(x) = |x|^d \tilde{\psi}(x/|x|)$ *approximates* $f|_{B^m - B^m_{1-\delta}}$ *sufficiently closely.*

Proof. It suffices to see that $f^{-1}(0) \cap (B^m - B^m_{1-\delta})$ has a conical structure. Remark that the function $r(x) = |x|^2$ has finitely many critical values on $f^{-1}(0) \cap (B^m - B^m_{1-\delta})$ since f is smooth and has no critical points in this range. Thus one can choose δ small enough so that r has no critical points. Then [**21**, proof of Theorem 2.10, p. 18] applies in this context. This implies the existence of a good approximation of conical type. □

In particular, our approximating $\psi(x)$ has no critical points in the given annulus. However, if $x_0 \neq 0$ was a critical point of $\psi(x)$ in the ball, then all points of the line $0x_0$ would be critical, since ψ is homogeneous. Therefore ψ has an isolated singularity at the origin.

Notice that one can choose the approximation so that $S^{m-1} \cap \psi^{-1}(0)$ is isotopic to $S^{m-1} \cap f^{-1}(0)$ and therefore non-empty. In particular, the singularities at the origin of f and ψ have the same topological type.

Although ψ is not real analytic, each of its components are algebraic because they can be represented as $\psi_j(x) = P_j(x) + Q_j(x)|x|$, where $P_j(x)$ (respectively $Q_j(x)$) are polynomials of even (respectively odd) degree. The curve selection lemma [**21**, p. 25] can be extended without difficulty to sets defined by equations like the ψ_j above. Then [**21**, proof of Theorem 11.2 (and Lemma 11.3), pp. 97–99] extends to the case of ψ. In particular, there exists a Milnor fibration associated to ψ (the complement of the singular fiber $\psi^{-1}(0)$ in the unit sphere S^{m-1} fibers over S^{n-1}). Alternatively, $(S^{m-1}, S^{m-1} \cap \psi^{-1}(0))$ is a Neuwirth–Stallings pair according to [**19**] and $S^{m-1} \cap \psi^{-1}(0)$ is non-empty. The main theorem from [**19**] then provides a polynomial map with an isolated singularity at the origin, as required. □

Milnor (see [**21**]) called such an isolated singularity *trivial* when its local Milnor fiber is diffeomorphic to a disk. Then it was shown in [**7**, p. 151] that f is trivial if and only if f is locally topologically equivalent to the projection map $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ whenever the dimension of the fiber is $m - n \neq 4, 5$. We recall that the existence of polynomials with isolated singularities was (almost) settled in [**8**, **21**].

PROPOSITION 4.2. *For $0 \leqslant m - n \leqslant 2$, non-trivial polynomial singularities exist precisely for $(2, 2)$, $(4, 3)$ and $(4, 2)$.*

For $m - n \geqslant 4$, non-trivial examples occur for all (m, n).

For $m - n = 3$, non-trivial examples occur for $(5, 2)$ and $(8, 5)$. Moreover, if the 3-dimensional Poincaré conjecture is false, then there are non-trivial examples for all (m, n). Otherwise, all examples are trivial except for $(5, 2)$, $(8, 5)$ and possibly $(6, 3)$.

We consider now a smooth map $f : M^m \longrightarrow N^n$ where m, n are as in the hypothesis. For each critical point p, there are open balls $2B^m(p)$ and $2B^n(f(p))$ for which the restriction $f|_{2B^m(p)} : 2B^m(p) \longrightarrow 2B^n(f(p))$ has an isolated singularity at p. One identifies $2B^m(p)$ with the ball of radius 2 in $\mathbb{R}^m$, and let $B^m(p)$ be the preimage of the concentric unit ball. In the proof of Lemma 4.1, we approximated $f|_{\partial B^m(p)}$ by a polynomial map g with isolated singularities, both maps having isotopic links and being close to each other. Assume for simplicity that f (and hence g) is of maximal rank around this (unit) sphere. The general case follows along the same lines. Then there exists an isotopy f_t ($t \in [0, 1]$) between $f|_{\partial B^m(p)}$ and $g|_{\partial B^m(p)}$ which is close to identity. In particular, all f_t are of maximal rank around the unit sphere. Let $\rho : [0, 4] \longrightarrow [0, 1]$ be a smooth decreasing function with $\rho(x) = 0$ if $x \geqslant 1$ and $\rho(x) = 1$ if $x \leqslant \frac{1}{2}$. Let $F : 2B^m(p) \longrightarrow 2B^n(f(p))$ be the map defined by

$$F(x) = \begin{cases} f_{\rho(|x|^2)}(x/|x|) & \text{if } |x| \geqslant \frac{1}{2} \\ g(x) & \text{if } |x| \leqslant \frac{1}{2}. \end{cases}$$

If one replaces $f_{2B^m(p)}$ by F, then one obtains a smooth function with an isolated singularity at p, which must be a topological submersion at p (by Proposition 4.2). An induction on the number of critical points yields a map $F : M^m \longrightarrow N^n$ which is a topological fibration. □

REMARK 4.1.   Notice that there exist real smooth maps f which do not have a Milnor fibration at an isolated singularity. For such f, it is not clear when one should call the singularity trivial. In particular, in this situation we do not know whether f itself must be a topological submersion. Therefore it is necessary to replace f by another map (locally algebraic), in order to be able to apply Proposition 4.2.

REMARK 4.2.   Therefore, within the range $0 \leqslant m - n \leqslant 3$, with the exception of $(2,2)$, $(4,3)$, $(4,2)$, $(5,2)$, $(8,5)$ and $(6,3)$, the non-triviality of φ is related to the exotic structures on fibrations.

One expects that in the case when non-trivial singularities can occur, such examples abound.

EXAMPLE 4.1.   In the remaining cases we have the following.
(1) $(m,n) \in \{(4,3),(8,5)\}$. We will prove below that $\varphi(S^4,S^3) = \varphi(S^8,S^5) = 2$.
(2) $(m,n) = (4,2)$. Non-trivial examples come from Lefschetz fibrations X over a Riemann surface F. For instance X is an elliptic K3 surface and F is $\mathbb{CP}^1$.
(3) $(m,n) = (2k,2)$. More generally, one can consider complex projective k-manifolds admitting morphisms onto an algebraic curve.

Further, one notices that these local obstructions are far from being complete. In fact, the maps $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ arising as restrictions of smooth maps between compact manifolds are quite particular. For instance, if one takes $M = S^m$, then one can obtain by restriction a map $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ which is proper and has only finitely many isolated singularities. However, adding extra conditions can further restrict the range of dimensions.

PROPOSITION 4.3.   *There are no proper smooth functions $f : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^n, 0)$ with one isolated singularity at the origin if $m \leqslant 2n - 3$.*

Proof.   There is a direct proof similar to that of Proposition 6.1. Instead, let us show that the hypothesis implies that $\varphi(S^m, S^n) \leqslant 2$ and so Proposition 6.1 yields the result.

Let $j_k : S^k \longrightarrow \mathbb{R}^k$ denote the stereographic projection from the north pole ∞. There exists an increasing unbounded real function ρ such that $|f(x)| \geqslant \lambda(|x|)$ for all $x \in \mathbb{R}^m$, because f is proper.

We claim that there exists a real function ρ such that $\rho(|x|)f(x)$ extends to a smooth function $F : S^m \longrightarrow S^n$. Specifically, we want the function $F_\rho : S^m \longrightarrow S^n$ defined by

$$F_\rho(x) = \begin{cases} j_n^{-1}(\rho(|j_m(x)|)f(j_m(x))) & \text{if } x \in S^m - \{\infty\} \\ \infty & \text{otherwise} \end{cases}$$

to be smooth at ∞. This is easy to achieve by taking $\rho(x) > \exp(|x|)\lambda^{-1}(|x|)$ for large $|x|$. Now the critical points of F_ρ consist of the two poles, and the claim is proved. $\qquad\square$

REMARK 4.3. Notice that proper maps such as the above for $m = 2n - 2$ exist only for $n \in \{2, 3, 5, 9\}$ (see below).

REMARK 4.4. A special case is when the critical point p is an isolated point in the fiber $f^{-1}(f(p))$. This situation was settled in [**8, 34**], where it was shown that the dimensions (m, k) should be $(2, 2), (4, 3), (8, 5)$ or $(16, 9)$, and the map is locally the cone over the respective Hopf fibration.

5. The global structure for topological submersions

Roughly speaking, the results of the previous section state that maps of low codimension with only finitely many critical points should be topological submersions.

PROPOSITION 5.1. *Assume that there exists a topological submersion* $f : M^m \longrightarrow N^n$ *with finitely many critical points, and* $m > n \geqslant 2$. *Then* $\varphi(M, N) \in \{0, 1\}$ *and* $\varphi(M, N) = 1$ *precisely when* M *is diffeomorphic to the connected sum of a fibration* $\widehat{N}$ *(over* N*) with an exotic sphere, and* M *is not a fibration over* N.

Proof. The first step is to split off one critical point by localizing it within an exotic sphere. Let M_0 be the manifold obtained after excising an embedded ball from M.

LEMMA 5.1. *There exists an exotic sphere* Σ_1 *and a map* $f_1 : M_1 \longrightarrow N$ *such that the following hold.*
(i) *M_0 is a submanifold of both M and M_1. The complements $M_1 - M_0$ and $M - M_0$ are balls and $M = M_1 \sharp \Sigma_1$.*
(ii) *f_1 agrees with f on $M_0 \subset M_1$ and has no other critical points on the ball $M_1 - M_0$.*
(iii) *f has precisely one critical point in $M - M_0$.*

Proof. Let p be a critical point of f, $q = f(p)$, and δ be a small disk around q. We replace f by a map which is locally polynomial around the critical point p, as in the previous section. We show first that the following lemma holds.

LEMMA 5.2. *There exists a neighborhood Z_p of p such that the following conditions are fulfilled.*
(i) *Z_p is diffeomorphic to $D^n \times D^{m-n}$ (for $m \neq 4$).*
(ii) *$\partial Z_p = \partial^h Z \cup \partial^v Z_p$, where the restrictions $f : \partial^v Z_p \longrightarrow D^n$ and $f : \partial^h Z \longrightarrow \partial D^n$ are trivial fibrations, and $\partial^h Z \cap \partial^v Z_p = S^{n-1} \times S^{m-n-1}$.*

Proof. Let $B^m(p)$ be a sufficiently small ball around p, and δ be such that $\delta \subset f(B^m(p))$. We claim that $Z_p = B^m(p) \cap f^{-1}(\delta)$ has the required properties.

One chooses a small ball containing p, $B_0^m(p) \subset Z_p$. Then one uses the argument from [**21**, pp. 97–98], and derives that q is a regular value of the map

$$f : Z_p - \text{int}(B_0^m(p)) \longrightarrow \delta.$$

Therefore the latter is a fibration, and hence a trivial fibration. In particular, the manifold with corners ∂Z_p has a collar whose outer boundary is a smooth sphere. Further, the manifold with boundary Z_p is homeomorphic to $D^n \times D^{m-n}$ and the boundary ∂Z_p is collared as above. The outer sphere bounds a smooth disk (by Smale) and so Z_p is diffeomorphic to $D^n \times D^{m-n}$. $\qquad\square$

Now the proof goes on as in codimension 0. We excise Z_p and glue it back by another diffeomorphism so that the restriction of f extends over the new ball, without introducing any new critical points. The gluing diffeomorphism respects the corner manifold structure. $\qquad\square$

An inductive argument shows that if $\varphi(M, N)$ is finite, then the connected sum $M \sharp \Sigma$ with an exotic sphere is diffeomorphic to a fibration over N.

We want to prove now that one can find another map $M \longrightarrow N$ having precisely one critical point. We first have to put all critical points together inside a standard neighborhood.

LEMMA 5.3.   *If $m > n \geqslant 2$, then the critical points of f are contained in some cylinder $Z^m \subset M$ which is diffeomorphic to $D^n \times D^{m-n}$ (respectively homeomorphic when $m = 4$, by a homeomorphism which is a diffeomorphism on the boundary) such that the fibers of f are either transversal to the boundary (actually to the part $D^n \times \partial D^{m-n}$) or contained in $\partial D^n \times D^{m-n}$.*

Proof.   Pick up a regular point x_0 in M. Let U be the set of regular points which can be joined to x_0 by an arc γ everywhere transversal to the fibers of f (which will be called transversal in the sequel).

We first show that U is open. In a small neighborhood V of $x \in U$, the fibers can be linearized (by means of a diffeomorphism) and identified to parallel $(m-n)$-planes. Let $y \in V$. If x and y are not in the same fiber, then the line joining them is a transversal arc. Otherwise, use a helicoidal arc spinning around the line, which can be constructed since the fibers have codimension at least 2.

At the same time, U is closed in the complement of the critical set. In fact, the previous arguments show that two regular points which are sufficiently closed to each other can be joined by a transversal arc with prescribed initial velocity (provided that this tangent vector is also transversal to the fiber). Thus if y_i converge to a regular point y and $y_i \in U$, then y can be joined to x_0 by first joining x_0 to y_i and then y_i to y (for large enough i) with some prescribed initial velocity, in order to insure the smoothness of the arc. This proves that U is the set of all regular points.

Further, we consider the cylinders Z_{p_i} given by Lemma 5.2. Let $f_i \subset \partial Z_{p_i}$ be some fibers in the boundary. The points $q_i \in f_i$ can be joined by everywhere transversal arcs. Since this is an open condition, one can find disjoint tubes $T_{i,i+1}$ joining neighborhoods of the fibers f_i in ∂Z_{p_i}, and f_{i+1} in $\partial Z_{p_{i+1}}$, and one builds up in this way a cylinder Z containing all critical points. $\qquad\square$

LEMMA 5.4.   *There exists a smooth map* $g : Z \longrightarrow D^n$ *having one critical point such that* $g|_{\partial Z} = f|_{\partial Z}$.

Proof.   We know that each restriction $f : Z_i \longrightarrow D^n$ is topologically a trivial fibration whose restriction to a collar of the boundary sphere ∂D^n is trivial, as a smooth fibration. We claim that $f : Z \longrightarrow D^n$ enjoys the same property. This follows from the fact that the restrictions $f : T_{i,i+1} \longrightarrow D^n$ are also trivial fiber bundles.

The restriction $f|_{\partial^v Z} : \partial^v Z \longrightarrow D^n$ is a trivial fibration over the ball. Thus there exists a diffeomorphism $\partial^v f : \partial^v Z \longrightarrow D^n \times \partial D^{m-n}$, which commutes with the trivial projection $\pi : D^n \times D^{m-n} \longrightarrow D^n$, namely $\pi \circ \partial^v f = f|_{\partial^v Z}$.

Since $f|_{\partial^h Z} : \partial^v Z \longrightarrow \partial D^n$ is a trivial fibration, there exists a diffeomorphism $\partial^h f : \partial^v Z \longrightarrow \partial D^n \times D^{m-n}$ commuting with π. Moreover, these two diffeomorphisms can be chosen to agree on their common intersection, $\partial^h f|_{\partial^h Z \cap \partial^v Z} = \partial^v f|_{\partial^h Z \cap \partial^v Z}$.

We obtain therefore a diffeomorphism ∂f of manifolds with corners $\partial f : \partial Z \longrightarrow \partial D^m$, defined by

$$\partial f(x) = \begin{cases} \partial^h f(x) & \text{if } x \in \partial^h Z \\ \partial^v f(x) & \text{if } x \in \partial^v Z. \end{cases}$$

Assume now that there exists a smooth homeomorphism $\Phi : Z \longrightarrow D^n \times D^{m-n}$ having precisely one critical point, which extends ∂f, that is, such that the following diagram commutes.

$$\begin{array}{ccc} \partial Z & \xrightarrow{\partial f} & \partial(D^n \times D^{m-n}) \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\Phi} & D^n \times D^{m-n} \end{array}$$

We therefore set $g(x) = \pi(\Phi(x))$. It is immediate that g has at most one critical point and g is an extension of $f_{\partial Z}$. Our claim is then a consequence of the following.

LEMMA 5.5.   *Any diffeomorphism of the sphere* S^m, *with the structure of a manifold with corners* $\partial(D^n \times D^{m-n})$, *extends to a smooth homeomorphism of* $D^n \times D^{m-n}$ *with at most one critical point.*

Proof.   Instead of searching for a direct proof, remark that the trivializations leading to ∂f extend over a collar of $\partial(D^n \times D^{m-n})$. This collar is still a manifold with corners, but it contains a smoothly embedded sphere. We then use the standard result (see [**16**]) to extend further the diffeomorphism from the smooth sphere to the ball.   □

Now Lemma 5.4 follows.   □

□

6.   *Maps between spheres*

For spheres, the situation is somewhat simpler than in general, because we can use the global obstructions of topological nature. Our main result settles the case

when the codimension is smaller than the dimension of the base. Specifically, we can state the following proposition.

PROPOSITION 6.1.   (i) *The values of $m > n > 1$ for which $\varphi(S^m, S^n) = 0$ are exactly those arising in the Hopf fibrations, that is, $n \in \{2, 4, 8\}$ and $m = 2n - 1$.*
  (ii) *One has $\varphi(S^4, S^3) = \varphi(S^8, S^5) = \varphi(S^{16}, S^9) = 2$.*
  (iii) *If $m \leqslant 2n - 3$, then $\varphi(S^m, S^n) = \infty$.*
  (iv) *If $\varphi(S^{2n-2}, S^n)$ is finite, then $n \in \{2, 3, 5, 9\}$.*

Proof. (i) Notice first that the existence of the Hopf fibrations $S^3 \longrightarrow S^2$, $S^7 \longrightarrow S^4$, $S^{15} \longrightarrow S^8$ shows that $\varphi(S^3, S^2) = \varphi(S^7, S^4) = \varphi(S^{15}, S^8) = 0$. The converse is already known (see [**7**, Lemma 1; **33**, Lemma 2.7]). We will give a slightly different proof below, on elementary grounds.

Using the Serre exact sequence for an $(n-1)$-connected basis, one finds that the homology of the fiber F (of the fibration $f: S^m \longrightarrow S^n$) agrees with that of S^{n-1} up to dimension $n-1$, and a subsequent application of the same sequence shows that F is a homology $(n-1)$-sphere. Therefore $m = 2n - 1$. In particular, we infer that the transgression map $\tau^*: H^{n-1}(F) \longrightarrow H^n(S^n)$ is an isomorphism.

Let $i_F: F \longrightarrow S^{2n-1}$ and $j_F: S^{2n-1} \longrightarrow (S^{2n-1}, F)$ denote the inclusion maps. Denote by $C^*(X)$ the cochain complex of the space X.

LEMMA 6.1.   *The composition of maps*

$$C^{n-1}(S^{2n-1}) \xrightarrow{i_F^*} C^{n-1}(F) \xrightarrow{\tau^*} C^n(S^{2n}) \xrightarrow{f^*} C^n(S^{2n-1})$$

is the boundary operator $d: C^{n-1}(S^{2n-1}) \longrightarrow C^n(S^{2n-1})$.

Proof. The transgression map τ^* can be identified (see for example [**36**, pp. 648–651]) with the composition

$$H^{n-1}(F) \xrightarrow{\partial^*} H^n(S^{2n-1}, F) \xrightarrow{f^{*-1}} H^n(S^n),$$

where ∂^* is the boundary homomorphism in the long exact sequence of the pair (S^{2n-1}, F). One then sees that

$$C^n(S^{2n-1}, F) \xrightarrow{f^{*-1}} C^n(S^n) \xrightarrow{f^*} C^n(S^{2n-1})$$

agrees with j_F^*. Further, the composition of maps from the statement of the lemma is equivalent to

$$C^{n-1}(S^{2n-1}) \xrightarrow{i_F^*} C^{n-1}(F) \xrightarrow{\partial^*} C^n(S^{2n-1}, F) \xrightarrow{j_F^*} C^n(S^{2n-1}),$$

which acts as the boundary operator d, as claimed. $\square$

Let u be an $(n-1)$-form on S^{2n-1} such that $i_F^* u$ is a generator for $H^{n-1}(F, \mathbb{Z}) \subset H^{n-1}(F, \mathbb{R})$. Then

$$\langle i_F^* u, [F] \rangle = \int_F u = 1.$$

Since τ^* is an isomorphism, it follows that $\tau^* i_F^* u = v$, where v is the generator of $H^n(S^n, \mathbb{Z})$. Thus v is the volume form on S^n, normalized so that $\int_{S^n} v = 1$.

Let us recall the definition of the Hopf invariant $H(f)$. Consider any $(n-1)$-form w on S^{2n-1} satisfying $f^*v = dw$. Then

$$H(f) = \int_{S^{2n-1}} w \wedge dw = \int_{x \in S^n} \left(\int_{f^{-1}(x)} w \right) v.$$

According to the lemma, one has $f^*v = du$. However, it is clear that the function

$$x \longrightarrow \int_{f^{-1}(x)} w$$

is constant (more generally it is locally constant on the set of regular values for an arbitrary f), and this constant in our case is $\int_F u = 1$. Therefore $H(f) = 1$ and the Adams theorem (see [1]) implies the claim.

REMARK 6.1. The above result holds true if one relaxes the assumptions by asking f to be only a Serre fibration. One replaces in the proof the integral in the definition of the Hopf invariant by the intersection of chains (see for example [36, pp. 509–510]).

(ii) We will show now that by suspending the Hopf fibrations, we obtain examples of pairs with non-trivial φ. In fact, choose a Hopf map $f : S^{2n-1} \longrightarrow S^n$, and extend it to $B^{2n} \longrightarrow B^{n+1}$ by taking the cone and smoothing it at the origin. Then glue together two copies of B^{2n} along the boundary. One gets a smooth map having two critical points. The previous result implies that

$$1 \leqslant \varphi(S^4, S^3), \varphi(S^8, S^5), \varphi(S^{16}, S^9) \leqslant 2.$$

Let us introduce some more notations: set $p_1, \ldots, p_r$ for the critical points of the map $f : S^m \longrightarrow S^n$ under consideration if there are only finitely many. Let $F_{e_i} = f^{-1}(f(p_i))$ denote the singular fibers, let $F_e = \bigcup_{i=1}^r F_{e_i}$ stand for their union, and let F stand for the generic fiber which is a closed oriented $(m-n)$-manifold.

LEMMA 6.2. Each component of F_e is either a smooth $(m-n)$-manifold around each point which is not in the critical set $\{p_1, \ldots, p_r\}$, or else an isolated p_i.

Proof. In fact, f is a submersion at all points but p_i. □

LEMMA 6.3. If $m < 2n - 1$, then $\varphi(S^m, S^n) \geqslant 2$.

Proof. Assume that there is a map $f : S^m \longrightarrow S^n$ with precisely one critical point p. Then $f : S^m - F_e \longrightarrow B^n$ is a fibration, so that $S^m - F_e = B^n \times F$.

One rules out the case when the exceptional fiber is one point by observing that $H_{m-n}(F)$ is not trivial. Using an $(n-1)$-cycle linking once a component of F_{e_i}, one can show that $H_{n-1}(S^m - F_e)$ is non-trivial. Since $n - 1 > m - n$, the equality above is impossible, and the claim is proved. □

Now the equalities from the statement follow.

REMARK 6.2. This might be used to construct other examples with finite φ in the respective dimensions. For instance, one finds that $\varphi(\Sigma^8, S^5) = \varphi(\Sigma^{16}, S^9) = 2$, where Σ^n denotes an exotic n-sphere.

(iii) Assume that there is a smooth map $f : S^m \longrightarrow S^n$ with r critical points. We suppose, for simplicity, that the critical values q_i are distinct. One uses the Serre exact sequence for the fibration $S^m - F_{\mathrm{e}} \longrightarrow S^n - \{q_1, \ldots, q_r\}$, and one derives

$$H_i(F) = H_i(S^m - F_{\mathrm{e}}) \quad \text{if } i \leqslant n - 3.$$

Further, $H^{m-i}(F_{\mathrm{e}}) = 0$ for $i \leqslant n - 1$, because F_{e} has dimension at most $(m - n)$. Then Alexander's duality, $\widetilde{H}_{i-1}(S^m - F_{\mathrm{e}}) = \widetilde{H}^{m-i}(F_{\mathrm{e}})$ and the previous equality imply that $H_i(F) = 0$, for all $i \leqslant n - 3$. This is impossible, because the fiber F is a compact $(m - n)$-manifold and $m - n \leqslant n - 3$.

(iv) As above, the Serre exact sequence shows that F is an $(n - 2)$-homology sphere. Further, the generalized Gysin sequence yields

$$\widetilde{H}^{2n-3-j}(F_{\mathrm{e}}) = \widetilde{H}_j(S^{2n-2} - F_{\mathrm{e}}) = \begin{cases} \mathbb{Z}^r & \text{if } j = 2n - 3 \\ \mathbb{Z}^{r-2} & \text{if } j = n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $H_{n-2}(F_{\mathrm{e}})$ (or equivalently $H_{n-1}(S^{2n-2} - F_{\mathrm{e}})$) cannot be of rank $r - 2$ unless some (more precisely two such) exceptional fiber in F_{e} consists of one point. In fact, if we have q connected components of F_{e} of dimension $(n - 2)$, then the rank of $H_{n-2}(F_{\mathrm{e}})$ is at least q.

Furthermore, such a critical point p is isolated in $f^{-1}(f(p))$. Then [**33**, Proposition 3.1] yields the claim. ☐

REMARK 6.3.   Notice that F^{m-n} is $(n-3)$-connected. In particular, if $m \leqslant 3n - 6$ ($n \geqslant 5$), then F is homeomorphic to S^{m-n}. In fact, one can obtain S^m from the complement $S^m - \mathrm{int}(N(F_e))$ of a neighborhood of the exceptional fibers by adding cells of dimension at least n, with one $(n + i)$-cell for each i-cell of F_{e}. Therefore $\pi_j(S^m - \mathrm{int}(N(F_{\mathrm{e}}))) = 0 = \pi_j(S^m) = 0$, for $j \leqslant n - 2$. The base space of the fibration $f|_{S^m - \mathrm{int}(N(F_e))}$ is S^n with small open neighborhoods of the critical values deleted; thus it is homotopy equivalent to a bouquet of S^{n-1} (at least one critical value). The long exact sequence in homotopy shows then that the fiber is $(n-3)$-connected.

REMARK 6.4.   (i) If there exists a non-trivial proper smooth $F : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^n, 0)$ having only one isolated singularity at the origin, then $\varphi(S^m, S^n) \leqslant 2$ (see the proof of Proposition 4.3), and this condition seems to be quite restrictive, in view of Proposition 6.1.

(ii) The explicit computation of $\varphi(S^m, S^n)$ for general m, n seems to be difficult. Further steps towards the answer would be to prove that $\varphi(S^{2n-1}, S^n)$ is finite only if $n \in \{1, 2, 4, 8\}$, and that $\varphi(S^m, S^n) = \infty$ if $n \geqslant 5$ and $2n - 1 < m \leqslant 3n - 6$.

7. *Remarks concerning a substitute for φ in dimension 3*

One saw that $\varphi(M^n, N^n)$ is less interesting if $n \geqslant 3$. One would like to have an invariant of the pair (M^n, N^n) measuring how far M^n is from being an unramified covering of N^n. First one has to know whether there is a branched covering $M^n \longrightarrow N^n$, and next if the branch locus could be empty.

REMARK 7.1.   A classical theorem of Alexander [**2**] states that any n-manifold is a branched covering of the sphere S^n. Moreover, one can assume that the ramification locus is the $(n - 2)$-skeleton of the standard n-simplex.

REMARK 7.2. There exists an obvious obstruction to the existence of a ramified covering $M^n \longrightarrow N^n$, namely the existence of a map of non-zero degree $M^n \longrightarrow N^n$. In particular, a necessary condition is $\|M\| \geqslant \|N\|$, where $\|M\|$ denotes the simplicial volume of M (see [**15**, **22**]). However, this condition is far from being sufficient. Take M with finite fundamental group and N with infinite amenable fundamental group (for instance of polynomial growth); then $\|M\| = \|N\| = 0$, while it is elementary that there does not exist a non-zero degree map $M \longrightarrow N$.

A possible candidate for replacing φ in dimension 3 is the ratio of simplicial volumes mod $\mathbb{Z}$, namely

$$v(M, N) = \frac{\|M\|}{\|N\|} \ (\mathrm{mod} \ \mathbb{Z}) \in [0, 1),$$

which is defined when N has non-zero simplicial volume. Notice that (for closed manifolds M) the simplicial volume $\|M\|$ depends only on the fundamental group $\pi_1(M)$ of M. In particular, it vanishes for simply connected manifolds, making it less useful in dimensions at least 4.

REMARK 7.3. If M^n covers N^n, then $v(M, N) = 0$ (see [**15**]). The converse holds true for surfaces of genus at least 2, from the Hurwitz formula.

The norm ratio has been extensively studied for hyperbolic manifolds in dimension 3, where it coincides with the volume ratio, in connection with commensurability problems (see, for example, [**33**]). In particular, the values $v(M^3, N^3)$ accumulate on 1 since the set of volumes of closed hyperbolic 3-manifolds has an accumulation point. The simplicial volume is zero for a Haken 3-manifold if and only if the manifold is a graph manifold (from [**32**]), and conjecturally the simplicial volume is the sum of (the hyperbolic) volumes of the hyperbolic components of the manifold.

However, it seems that this invariant is not appropriate in dimensions higher than 3 (even if one restricts to aspherical manifolds). Here are two arguments in favour of this claim.

PROPOSITION 7.1. Let us suppose that M^n is a ramified covering of N^n over the complex K^{n-2}. Assume that both the branch locus K^{n-2} and its preimage in M^n can be engulfed in a simply connected codimension 1 submanifold. Then $v(M^n, N^n) = 0$.

Assume that there is a map $f : M^n \longrightarrow N^n$ such that the kernel $\ker(f_* : \pi_1(M) \longrightarrow \pi_1(N))$ is an amenable group. Then $v(M^n, N^n) = 0$.

Proof. One uses the fact that for any simply connected codimension 1 submanifold $A^{n-1} \subset M^n$, one has $\|M\| = \|M - A\|$ (see [**15**, p. 10, 3.5]). The second part follows from [**27**, Remark 3.5], which states that $\|M\| = \deg(f)\|N\|$, where $\deg(f)$ stands for the degree of f. □

REMARK 7.4. (i) For all $n \geqslant 4$, Sambusetti [**27**] constructed examples of manifolds M^n and N^n that satisfy the second condition (and hence v vanishes), but which are not fibrations.

(ii) It seems that there are no such examples in dimension 3. At least for Haken hyperbolic N^3, any M^3 dominating N^3 with amenable kernel must be a covering, according to [**27**, Remark 3.5] and the rigidity result of Soma and Thurston (see [**32**]).

REMARK 7.5. One could replace the simplicial volume by any other volume, as defined by Reznikov [**26**]. For instance, the $SL(2, R)$-volume is defined for Seifert fibered 3-manifolds, and it behaves multiplicatively under finite coverings (compare with [**35**]). In particular, one can define an appropiate $v(M, N)$ for graph manifolds using this volume. Other topological invariants which behave multiplicatively under finite coverings are the l^2-Betti numbers.

REMARK 7.6. If there is a branched covering $f : M^n \longrightarrow N^n$, then the branch locus is of codimension 2. This yields a heuristical explanation for the almost triviality of $\varphi(M^n, N^n)$ in high dimensions. A possible extension of φ would have to take into account the minimal complexity of the branch locus, (for example its Betti number) over all branched coverings. For a given N, this complexity must be bounded from above, as does happen in the case when N is a sphere by Alexander's theorem. However, it seems that such invariants are not easily computable.

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