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# On the Geometric Simple Connectivity of Open Manifolds

**Louis Funar and Siddhartha Gadgil**



## 1 Introduction

Since the proof of the $h$-cobordism and $s$-cobordism theorems for manifolds whose dimension is at least 5, a central method in topology has been to simplify handle decompositions as much as possible. In the case of compact manifolds, with dimension at least 5, the only obstructions to canceling handles turn out to be well-understood algebraic obstructions, namely, the fundamental group, homology groups, and the Whitehead torsion.

In particular, a compact manifold $M$ whose dimension is at least 5 has a handle decomposition without 1-handles if and only if it is simply connected. Here we study when open manifolds have a handle decomposition without 1-handles.

In the case of 3-manifolds, there is a classical obstruction to a simply connected open manifold having a handle decomposition without 1-handles, namely, the manifold has to be *simply connected at infinity*. However, in higher dimensions, there are several open manifolds having a handle decomposition without 1-handles that are not simply connected at infinity. Thus, there is no classical obstruction, besides the fundamental group (which is far too weak) and the $\pi_1$-stability at infinity (which is stronger), to an open manifold with dimension at least 5 having a handle decomposition without 1-handles. We show here that there are (algebraic) obstructions, which can moreover be seen to be nontrivial in concrete examples.

We show that for open manifolds, there is an algebraic condition, which we call *end-compressibility*, equivalent to the existence of a proper handle decomposition without 1-handles. This is a condition on the behaviour of the manifold at infinity. End-compressibility is defined in terms of fundamental groups related to an exhaustion of the

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open manifold by compact submanifolds. In some sense it is the counterpart of Siebenmann's obstruction to finding a boundary of an open manifold, although our requirements are considerably weaker.

End-compressibility in turn implies a series of conditions, analogous to the lower central series (the first of which is always satisfied). Using these, we show that if $W$ is a Whitehead-type manifold and $M$ a compact (simply connected) manifold, $W \times M$ does not have a handle decomposition without 1-handles.

In the case of manifolds of dimensions 3 and 4, even the compact case is subtle. For 3-manifolds, the corresponding statement in the compact case is equivalent to the Poincaré conjecture, and irreducible open simply connected 3-manifolds have a handle decomposition without 1-handles if and only if they are simply connected at infinity. For a compact, contractible 4-manifold $M$, (an extension of) an argument of Casson shows that if $\pi_1(\partial M)$ has a finite quotient, then any handle decomposition of $M$ has 1-handles.

We study possible handle decompositions of the interior of $M$ and find a standard form for this. As a consequence, if a certain finiteness conjecture, generalizing a classical conjecture about links, is true, then the interior of $M$ also has no handle decomposition without 1-handles.

More explicitly, if all the Massey products of a link $L$ in $S^3$ vanish, then it is conjectured that the link $L$ is a sublink of a homology boundary link. In our case, we have a link $L$ in a manifold $K$, and a degree-one map from $K$ to $\partial M$, so that the image of $L$ in $\pi_1(\partial M)$ consists of homotopically trivial curves. The notions of *vanishing Massey products* and *homology boundary links* generalize to versions relative to $\pi_1(M)$ in this situation. The conjecture mentioned in the above paragraph is that *links with vanishing Massey products* (relative to $\pi_1(M)$) *are sublinks of homology boundary links* (relative to $\pi_1(M)$).

## 2 Statements of the results

The problem we address in this paper is whether 1-handles are necessary in a handle decomposition of a simply connected manifold. Moreover, we investigate when it is possible to kill 1-handles within the proper homotopy type of a given open manifold.

The relation between algebraic connectivity and geometric connectivity (in various forms) was explored first by Zeeman (see [36]) in connection with the Poincaré conjecture. Zeeman's definition of the geometric $k$-connectivity of a manifold amounts to requiring that any $k$-dimensional compact can be engulfed in a ball. His main result was the equivalence of algebraic $k$-connectivity and geometric $k$-connectivity for $n$-manifolds, under the condition $k \leq n-3$. Notice that it makes no difference whether one considers open or compact manifolds.

Later, Wall [33, 34] introduced another concept of geometric connectivity using handle theory which was further developed by Poénaru in his work on the Poincaré conjecture. A similar equivalence between the geometric and algebraic connectivities holds *in the compact case*, but this time one has to replace the previous codimension condition by $k \leq n-4$. In this respect, all results in low codimension are hard results. There is also a noncompact version of this definition which we can state precisely as follows.

Definition 2.1. A (possibly noncompact) manifold, which might have nonempty boundary, is *geometrically* $k$*-connected* (abbreviated *g.*$k$*.c.*) if there exists a proper handlebody decomposition without $j$-handles, for $1 \leq j \leq k$.

One should emphasize that now the compact and noncompact situations are no longer the same. The geometric connectivity is a consequence of the algebraic connectivity only under additional hypotheses concerning the ends. The purpose of this paper is to partially characterize these additional conditions.

Remark 2.2. Handle decompositions are known to exist for all manifolds in the topological PL and smooth settings, except in the case of topological 4-manifolds. In the latter case the existence of a handlebody decomposition is equivalent to that of a PL (or smooth) structure. However, in the open case, such a smooth structure always exists (in dimension 4). Although most results below can be restated and proved for other categories, we will restrict ourselves to considering PL manifolds and handle decompositions in the sequel.

We will be mainly concerned with geometric simple connectivity (abbreviated g.s.c.) in the sequel. A related concept, relevant only in the noncompact case, is introduced below.

Definition 2.3. A (possibly noncompact) polyhedron $P$ is *weakly geometrically simply connected* (abbreviated *w.g.s.c.*) if $P = \cup_{j=1}^{\infty} K_j$, where $K_1 \subset K_2 \subset \cdots \subset K_j \subset \cdots$ is an exhaustion by compact connected subpolyhedra with $\pi_1(K_j) = 0$. Alternatively, any compact subspace is contained in a compact simply connected subpolyhedron.

Notice that a w.g.s.c. polyhedron is simply connected.

Remark 2.4. (i) The w.g.s.c. spaces with which we will be concerned in the sequel are usually manifolds. Similar definitions can be given in the case of topological (resp., smooth) manifolds where we require the exhaustions to be by topological (resp., smooth) submanifolds. All results below hold true for this setting too (provided that handlebodies exist) except those concerning Dehn exhaustibility, since the latter is essentially a PL concept.

(ii) The w.g.s.c. is much more flexible than the g.s.c., the latter making sense only for manifolds, and enables us to work within the realm of polyhedra. However, one can easily show (see below) that w.g.s.c. and g.s.c. are equivalent for noncompact manifolds of dimension different from 4 (under the additional irreducibility assumption for dimension 3). Its invariance under proper homotopy equivalences expresses the persistence of a geometric property (not being g.s.c.) with respect to some higher-dimensional manipulations (as taking the product with a ball) of open manifolds.

The first result of this paper is the following theorem (see Theorem 4.22 and Proposition 4.31).

**Theorem 2.5.** If an open $n$-manifold is w.g.s.c., then it is end-compressible. Conversely, in dimension $n \neq 4$, if an open, simply connected manifold is end-compressible, then it is g.s.c. (assuming irreducibility for $n = 3$). □

Remark 2.6. A similar result holds more generally for noncompact manifolds with boundary, with the appropriate definition of end-compressibility.

End-compressibility (see Definition 4.18) is an algebraic condition which is defined in terms of the fundamental groups of the submanifolds which form an exhaustion. Notice that end-compressibility is weaker than simple connectivity at infinity (s.c.i) for $n > 3$.

Remark 2.7. The result above should be compared with Siebenmann's obstructions to finding a boundary for an open manifold of dimension greater than 5 (see [12, 31] for a thorough discussion of this and related topics). The intermediary result permitting to kill 1-handles in this framework is [31, Theorem 3.10, page 16]. Let $W^n$ be an open smooth $n$-manifold with $n \geq 5$ and $\mathcal{E}$ an isolated end. Assume that the end $\mathcal{E}$ has stable $\pi_1$ and its $\pi_1(\mathcal{E})$ is finitely presented. Then there exist arbitrarily small 1-neighborhoods of $\mathcal{E}$, that is, connected submanifolds $V^n \subset W^n$ whose complements have compact closure, having compact connected boundary $\partial V^n$ such that $\pi_1(\mathcal{E}) \to \pi_1(V)$ and $\pi_1(\partial V) \to \pi_1(V)$ are isomorphisms. It is easy to see that this implies that the 1-neighborhood $V^n$ is g.s.c. This is the principal step towards canceling the handles of $W^n$, hence obtaining a collar. One notices that the hypotheses in Siebenmann's theorem are stronger than the end-compressibility, but the conclusion is stronger too. In particular, an arbitrary w.g.s.c. manifold need not have a well-defined fundamental group at infinity, as is the case for

$\pi_1$-stable ends. However, we think that the relationship between the $\pi_1$-(semi)stability of ends and end-compressibility would deserve further investigation.

The full power of the $\pi_1$-stability is used to cancel more than 1-handles. Actually, Siebenmann considered tame ends, which means that $\mathcal{E}$ is $\pi_1$-stable and it has arbitrarily small neighborhoods which are finitely dominated. The tameness condition is strong enough to insure (see [31, Theorem 4.5]) that all $k$-handles can be canceled for $k \leq n-3$. One more obstruction (the end obstruction) is actually needed in order to be able to cancel the $(n-2)$-handles (which turns out to imply the existence of a collar). There exist tame ends which are not collared (i.e., with nonvanishing end obstruction), as well as $\pi_1$-stable ends with finitely presented $\pi_1(\mathcal{E})$ which are not tame. Thus the obstructions for killing properly the handles of index $1 \leq \lambda \leq k$ should be weaker than the tameness of the end for $k \leq n-3$ and must coincide with Siebenmann's for $k = n-2$.

Remark 2.8. It might be worthy to compare our approach with the results from [12]. First the w.g.s.c. is the analogue of the reverse collaring. According to [12, Proposition 8.5, page 93], a space $W$ is reverse-collared if it has an exhaustion by compacts $K_j$ for which the inclusions $K_j \to W$ are homotopy equivalences (while the w.g.s.c. requires only that these inclusions be 1-connected), and hence a simply connected reverse-collared space is w.g.s.c.

The right extension of the w.g.s.c. to nonsimply connected spaces, which is suitable for applications to 3-manifolds, is the Tucker property (see [4, 21]), which is also a proper homotopy invariant and can be formulated as a group-theoretical property for coverings. This is again weaker than the reverse tameness/collaring (see [12, Propositions 8.9 and 11.13]). Moreover, there is a big difference between the extended theory and the present one: while the w.g.s.c. is the property of having no extra 1-handles, the Tucker property expresses the fact that some handlebody decomposition needs *only finitely many* 1-handles, without any control on their number.

In this respect, our first result is a sharpening of the theory of reverse-collared manifolds specific to the realm of simply connected spaces.

The g.s.c. is mostly interesting in low dimensions, for instance in dimension 3 where it implies the s.c.i. However, its importance relies on its proper homotopy invariance, which has been discovered in a particular form by Poénaru [23] (see also [7, 8]), enabling us to transform the low-dimensional problem "is $W^3$ g.s.c.?" into a high-dimensional one, for example, "is $W^3 \times D^n$ w.g.s.c.?" We provide in this paper a criterion permitting to check the answer to the high-dimensional question in terms of an arbitrary exhaustion by compact submanifolds. This criterion is expressed algebraically as the end-compressibility of the manifold and is closer to the forward tameness from [12] rather than the reverse tameness.

Remark 2.9. If $W^k$ is compact and simply connected, then the product $W^k \times D^n$ with a closed $n$-disk is g.s.c. if $n + k \geq 5$. However, there exist noncompact $n$-manifolds with boundary which are simply connected but not end-compressible (hence not w.g.s.c.) in any dimension $n$, for instance $W^3 \times M^n$ where $\pi_1^\infty W^3 \neq 0$. Notice that $W^k \times \mathrm{int}(D^n)$ is g.s.c. for $n \geq 1$ since $\pi_1^\infty(W^k \times \mathrm{int}(D^n)) = 0$.

We will also prove (see Theorem 5.11) the following theorem.

**Theorem 2.10.** There exist uncountably many open contractible $n$-manifolds for any $n \geq 4$ which are not w.g.s.c. □

The original motivation for this paper was to try to kill 1-handles of open 3-manifolds at least *stably* (i.e., after stabilizing the 3-manifold). The meaning of the word stably in [23], where such results first arose, is to do so at the expense of taking products with some high-dimensional compact ball. This was extended in [7, 8] by allowing the 3-manifold to be replaced by any other polyhedron having the same proper homotopy type. The analogous result is true for $n \geq 5$. For $n = 4$, only a weaker statement holds true (see Theorem 6.3).

**Theorem 2.11.** If a noncompact manifold of dimension $n \neq 4$ is proper homotopically dominated by a w.g.s.c. polyhedron, then it is w.g.s.c. A noncompact 4-manifold proper homotopically dominated by a w.g.s.c. polyhedron is end-compressible. □

Since proper homotopy equivalence implies proper homotopy domination, we obtain the following corollary.

**Corollary 2.12.** If a noncompact manifold of dimension $n \neq 4$ is proper homotopy equivalent to a w.g.s.c. polyhedron, then it is w.g.s.c. □

Remark 2.13. This criterion is an essential ingredient in Poénaru's proof (see [26]) of the covering space conjecture. If $M^3$ is a closed, irreducible, aspherical 3-manifold, then the universal covering space of $M^3$ is $\mathbb{R}^3$. Further developments suggest a similar result in higher dimensions by replacing the s.c.i. conclusion with the weaker w.g.s.c. We will state below a group-theoretical conjecture abstracting this purely 3-dimensional result.

It is very probable that there exist examples of open 4-manifolds which are not w.g.s.c., but their products with a closed ball are w.g.s.c. Thus, in some sense, the previous result is sharp.

The dimension 4 deserves special attention also because one expects that the w.g.s.c. and the g.s.c. are not equivalent. Specifically, Poénaru stated the following conjecture.

**Conjecture 2.14** (Poénaru conjecture)**.** If the interior of a compact contractible 4-manifold with boundary a homology sphere is g.s.c., then the compact 4-manifold is also g.s.c. □

Remark 2.15. (i) A consequence of this conjecture for the particular case of the product of a homotopy 3-disk $\Delta^3$ with an interval is the Poincaré conjecture in dimension 3. This follows from the following two results announced by Poénaru.

(1) If $\Sigma^3$ is a homotopy 3-sphere such that $\Sigma^3 \times [0,1]$ is g.s.c., then $\Sigma^3$ is g.s.c. (hence standard).

(2) If $\Delta^3$ is a homotopy 3-disk, then $\mathrm{int}(\Delta^3 \times [0,1]\sharp_\infty S^2 \times D^2)$ is g.s.c., where $\sharp$ denotes the boundary connected sum.

(ii) The differentiable Poincaré conjecture in dimension 4 is widely believed to be false. One reasonable reformulation of it would be the following. *A smooth homotopy 4-sphere (equivalently homeomorphic to $S^4$) that is g.s.c. should be diffeomorphic to $S^4$.*

(iii) The two conjectures above (Poénaru's and the reformulate Poincaré conjectures) imply also the smooth Schoenflies conjecture in dimension 4, which states that a 3-sphere smoothly embedded in $S^4$ bounds a smoothly embedded 4-ball. In fact, by a celebrated result of Mazur [17], any such Schoenflies ball has its interior diffeomorphic to $\mathbb{R}^4$, hence g.s.c.

An immediate corollary would be that the interior of a Poénaru-Mazur 4-manifold may be w.g.s.c. but not g.s.c., because some (compact) Poénaru-Mazur 4-manifolds are known to be not g.s.c. (the geometrization conjecture implies this statement for all 4-manifolds whose boundary is not a homotopy sphere). The proof is due to A. Casson and it was based on partial positive solutions to the following algebraic conjecture [13, page 117] and [16, page 403].

**Conjecture 2.16** (Kervaire conjecture)**.** Suppose one adds an equal number of generators $\alpha_1, \ldots, \alpha_n$ and relations $r_1, \ldots, r_n$ to a nontrivial group $G$, then the group $G * \langle \alpha_1, \ldots, \alpha_n \rangle / \langle\langle r_1, \ldots, r_n \rangle\rangle$ that one obtains is also nontrivial. □

A. Casson showed that certain 4-manifolds $(W^4, \partial W^4)$ have no handle decompositions without 1-handles by showing that if they did, then $\pi_1(\partial W^4)$ would violate the Kervaire conjecture. Our aim would be to show that most contractible 4-manifolds are not g.s.c., and the method of the proof is to reduce this statement to the compact case.

However, our methods permit us to obtain only a weaker result, in which one shows that the interior of such a manifold cannot have handlebody decompositions without 1-handles, if the decomposition has also some additional properties (see Theorems 8.15 and 8.16 for precise statements).

**Theorem 2.17.** Assume that there exists a proper handlebody decomposition without 1-handles for the interior of a Poénaru-Mazur 4-manifold. If there exists a faraway intermediary-level 3-manifold $M^3$ whose homology is represented by disjoint embedded surfaces and whose fundamental group projects to the trivial group on the boundary, then the compact 4-manifold is also g.s.c. There always exists a collection of immersed surfaces, which might have nontrivial intersections and self-intersections along homologically trivial curves, that fulfills the previous requirements. □

Remark 2.18. Almost all of this paper deals with geometric 1-connectivity. However, the results can be reformulated for higher geometric connectivities within the same range of codimensions.

We wish to emphasize that there is a strong group-theoretical flavour in the w.g.s.c. for universal covering spaces. In this respect the universal covering conjecture in dimension 3 (see [26]) would be a first step in a more general program. We define a finitely presented, infinite group $\Gamma$ to be w.g.s.c. if there exists a compact manifold of dimension $n \geq 5$ with fundamental group $\Gamma$ whose universal covering space is w.g.s.c. It is not hard to show that this definition does not depend on the particular polyhedron one chooses but only on the group. This is part of a more general philosophy, due to M. Gromov, in which infinite groups are considered as geometric objects. This agrees with the idea that killing 1-handles of manifolds is a group-theoretical problem in topological disguise. The authors think that the following might well be true.

**Conjecture 2.19.** Fundamental groups of closed aspherical manifolds are w.g.s.c. □

This will be a far-reaching generalization of the 3-dimensional result announced by Poénaru in [26]. It is worthy to note that all reasonable examples of groups (e.g., word hyperbolic, semihyperbolic, CAT(0), group extensions, one relator groups) are w.g.s.c. It would be interesting to find an example of a finitely presented group which fails to be w.g.s.c. Notice that the well-known examples of M. Davis of Coxeter groups which are fundamental groups of aspherical manifolds whose universal covering spaces are not simply connected at infinity are actually CAT(0), hence w.g.s.c. However, one might expect a direct connection between the semistability of finitely presented groups, the quasi-simply filtrated groups (see [3]), and the w.g.s.c. We will address these questions in a future paper.

Outline of the paper. In Section 3, we compare w.g.s.c., g.s.c., and the s.c.i., showing that w.g.s.c. and g.s.c. are equivalent in high dimensions, and presenting some motivating examples.

Section 4 contains the core of the paper, where we introduce the algebraic conditions and prove their relation to w.g.s.c. We then construct uncountably many Whitehead-type manifolds in Section 5, and show that there are uncountably many manifolds that are not geometrically simply connected.

In Section 6, we show that end-compressibility is a proper homotopy invariant. Finally, in Sections 7 and 8, we turn to the 4-dimensional case.

## 3 On the g.s.c.

### 3.1 Killing 1-handles of 3-manifolds after stabilization

We start with some motivating remarks about the compact 3-dimensional situation for the sake of comparison.

Definition 3.1. The *geometric* 1-*defect* $\epsilon(M^n)$ of the compact manifold $M^n$ is $\epsilon(M^n) = \mu_1(M^n) - \operatorname{rank}\pi_1(M)$, where $\mu_1(M^n)$ is the minimal number of 1-handles in a handlebody decomposition and $\operatorname{rank}\pi_1(M)$ is the minimal number of generators of $\pi_1(M)$.

Remark 3.2. The defect (i.e., the geometric 1-defect) is always nonnegative. There exist examples (see [2]) of 3-manifolds $M^3$ with $\operatorname{rank}\pi_1(M^3) = 2$ and defect $\epsilon(M^3) = 1$. No examples of 3-manifolds with larger defect nor of closed 4-manifolds with positive defect are presently known. However, it's probably true that $\epsilon(M^3 \times [0,1]) = 0$, for all closed 3-manifolds though this might be difficult to settle even for the explicit examples of Boileau and Zieschang [2].

The defect is meaningless in high dimensions because of the following proposition.

**Proposition 3.3.** For a compact manifold, $\epsilon(M^n) = 0$ holds if $n \geq 5$. □

Proof. The proof is standard. Consider a 2-complex $K^2$ associated to a presentation of $\pi_1(M^n)$ with the minimal number $r$ of generators. By general position, there exists an embedding $K^2 \hookrightarrow M^n$ inducing an isomorphism of fundamental groups. Then a regular neighborhood of $K^2$ in $M^n$ has a handlebody decomposition with $r$ 1-handles. Since the complement is 1-connected, then by [28, 33, 34] it is g.s.c. for $n \geq 5$ and this yields the claim. ■

**Corollary 3.4.** For a closed 3-manifold $M^3$, $\epsilon(M^3 \times D^2) = 0$. □

Remark 3.5. As a consequence, if $\Sigma^3$ is a homotopy 3-sphere, then $\Sigma^3 \times D^2$ is g.s.c. Results of Mazur [19] (improved by Milnor in dimension 3) show that $\Sigma^3 \times D^3 = S^3 \times D^3$, but it is still unknown whether $\Sigma^3 \times D^2 = S^3 \times D^2$ holds. An earlier result of Poénaru states that $(\Sigma^3 - nD^3) \times D^2 = (S^3 - nD^3) \times D^2$ for some $n \geq 1$. More recently, Poénaru's program reduced the Poincaré conjecture to the g.s.c. of $\Sigma^3 \times [0, 1]$.

### 3.2 The s.c.i. and g.s.c.

The remarks which follow are intended to (partially) clarify the relation between g.s.c. and the s.c.i., in general.

Recall that a space $X$ is s.c.i. (and one also writes $\pi_1^\infty(X) = 0$) if for any compact $K \subset X$, there exists a larger compact $K \subset L \subset X$ having the property that any loop $l \subset X - L$ is null-homotopic within $X - K$. This is an important tameness condition for the ends of the space. The following result was proved in [27, Theorem 1].

**Proposition 3.6.** Let $W^n$ be an open simply connected $n$-manifold of dimension $n \geq 5$. If $\pi_1^\infty(W^n) = 0$, then $W^n$ is g.s.c. □

Remark 3.7. The converse fails as the following examples show. Namely, for any $n \geq 5$, there exist open $n$-manifolds $W^n$ which are geometrically $(n-4)$-connected but $\pi_1^\infty(W^n) \neq 0$.

There exist compact contractible $n$-manifolds $M^n$ with $\pi_1(\partial M^n) \neq 0$, for any $n \geq 4$ (see [10, 18, 22]). Since $k$-connected compact $n$-manifolds are geometrically $k$-connected if $k \leq n-4$ (see [28, 33, 34]), these manifolds are geometrically $(n-4)$-connected. We consider now $W^n = \mathrm{int}(M^n)$, which is diffeomorphic to $M^n \cup_{\partial M^n \cong \partial M^n \times \{0\}} \partial M^n \times [0, 1)$. Any Morse function on $M^n$ extends over $\mathrm{int}(M^n)$ to a proper one which has no critical points in the open collar $\partial M^n \times [0, 1)$, hence $\mathrm{int}(M^n)$ is also geometrically $(n-4)$-connected. On the other hand, $\pi_1(\partial M^n) \neq 0$ implies $\pi_1^\infty(W^n) \neq 0$.

However, the following partial converse holds.

**Proposition 3.8.** Let $W^n$ be a noncompact simply connected $n$-manifold which has a proper handlebody decomposition (1) without 1- or $(n-2)$-handles, or (2) without $(n-1)$- or $(n-2)$-handles.

Then $\pi_1^\infty(W^n) = 0$. □

Remark 3.9. When $n = 3$, this simply says that 1-handles are necessary unless $\pi_1^\infty(W^3) = 0$.

Proof of Proposition 3.8. Consider the handlebody decomposition $W^n = B^n \cup_{j=1}^{\infty} h_j^{i_j}$, where $h_j^{i_j}$ is an $i_j$-handle ($B^n = h_0^0$). Set $X_m = B^n \cup_{j=1}^{m} h_j^{i_j}$, for $m \geq 0$. Assume that this decomposition has no 1- nor $(n-2)$-handles. Since there are no 1-handles, it follows that $\pi_1(X_j) = 0$ for any $j$ (it is only here that one uses the g.s.c.).

**Lemma 3.10.** If $X^n$ is a compact simply connected $n$-manifold having a handlebody decomposition without $(n-2)$-handles, then $\pi_1(\partial X^n) = 0$. □

Proof. Reversing the handlebody decomposition of $X^n$, one finds a decomposition from $\partial X^n$ without 2-handles. One slides the handles to be attached in increasing order of their indices. Using the van Kampen theorem, it follows that $\pi_1(X^n) = \pi_1(\partial X^n) * \mathbf{F}(r)$, where $r$ is the number of 1-handles, and thus $\pi_1(\partial X^n) = 0$. ■

**Lemma 3.11.** If the compact submanifolds $\cdots \subset X_m \subset X_{m+1} \subset \cdots$ exhausting the simply connected manifold $W^n$ satisfy $\pi_1(\partial X_m) = 0$, for all $m$, then $\pi_1^{\infty}(W^n) = 0$. □

Proof. For $n = 3$, this is clear. Thus we suppose $n \geq 4$. For any compact $K \subset W^n$, choose some $X_m \supset K$ such that $\partial X_m \cap K = \varnothing$. Consider a loop $l \subset W^n - X_m$. Then $l$ bounds an immersed (for $n \geq 5$ embedded) 2-disk $\delta^2$ in $W^n$. We can assume that $\delta^2$ is transversal to $\partial X_m$. Thus it intersects $\partial X_m$ along a collection of circles $l_1, \ldots, l_p \subset \partial X_m$. Since $\pi_1(\partial X_m) = 0$, one is able to cap off the loops $l_j$ by some immersed 2-disks $\delta_j \subset \partial X_m$. Excising the subsurface $\delta^2 \cap X_m$ and replacing it by the disks $\delta_j$, one obtains an immersed 2-disk bounding $l$ in $W^n - K$. ■

This proves the first case of Proposition 3.8. In order to prove the second case, choose some connected compact subset $K \subset W^n$. By compactness, there exists $k$ such that $K \subset X_k$. Let $r$ be large enough (this exists by the properness) such that any handle $h_p^{i_p}$ whose attaching zone touches the lateral surface of one of the handles $h_1^{i_1}, h_2^{i_2}, \ldots, h_k^{i_k}$ satisfies $p \leq r$. The following claim will prove Proposition 3.8. ■

**Proposition 3.12.** Any loop $l$ in $W^n - X_r$ is null-homotopic in $W^n - K$. □

Proof. Actually the following more general engulfing result holds.

**Proposition 3.13.** If $C^2$ is a 2-dimensional polyhedron whose boundary $\partial C^2$ is contained in $W^n - X_r$, then there exists an isotopy of $W^n$ (with compact support), fixing $\partial C^2$ and moving $C^2$ into $W^n - K$. □

Proof. Suppose that $C^2 \subset X_m$. One reverses the handlebody decomposition of $X_m$ and obtains a decomposition from $\partial X_m$ without 1- or 2-handles. Assume that we can move $C^2$ such that it misses the last $j \leq r - 1$ handles. By general position, there exists an isotopy (fixing the last $j$ handles) making $C^2$ disjoint of the cocore ball of the $(j-1)$th handle, since the cocore disk has dimension at most $n-3$. The uniqueness of the regular neighborhood implies that we can move $C^2$ out of the $(j-1)$th handle (see, e.g., [30]), by an isotopy which is identity on the last $j$ handles. This proves Proposition 3.13. ■

This yields the result of Proposition 3.12 by taking for $C^2$ any 2-disk parameterizing a null-homotopy of $l$. ■

### 3.3 The g.s.c. and w.g.s.c.

**Proposition 3.14.** The noncompact manifold $W^n$ $(n \neq 4)$, which one supposes to be irreducible if $n = 3$, is w.g.s.c. if and only if it is g.s.c. □

Proof. For $n = 3$, it is well known that g.s.c. is equivalent to w.g.s.c. which is also equivalent to s.c.i. if the manifold is irreducible. For $n \geq 5$, this is a consequence of Wall's result stating the equivalence of g.s.c. and simple connectivity in the compact case (see [33, 34]). If $W^n$ is w.g.s.c., then it has an exhaustion by compact simply connected submanifolds $M_j$ (by taking suitable regular neighborhoods of the polyhedra). One can also refine the exhaustion such that the boundaries are disjoint. Then the pairs $(\mathrm{cl}(M_{j+1} - M_j), \partial M_j)$ are 1-connected, hence (see [33, 34]), they have a handlebody decomposition without 1-handles. Gluing together these intermediary decompositions, we obtain a proper handlebody decomposition as claimed. ■

## 4 The w.g.s.c. and end-compressibility

In this section, we show that w.g.s.c. is equivalent to an algebraic condition which we call *end-compressibility*. This in turn implies infinitely many conditions, *end* $k$-*compressibility* for ordinals $k$, and is equivalent to all these plus a finiteness condition.

Using the above, we give explicit examples of open manifolds that are not w.g.s.c.

### 4.1 Algebraic preliminaries

In this section, we introduce various algebraic notions of *compressibility* and study the relations between these. This will be applied in a topological context in subsequent sections, where *compressibility* corresponds to being able to attach *enough* 2-handles, and *stable compressibility* refers to the same after possibly attaching some 1-handles.

Definition 4.1. A pair $(\varphi : A \to B, \psi : A \to C)$ of group morphisms is *strongly compressible* if $\varphi(\ker \psi) = \varphi(A)$.

Remark 4.2. Strong compressibility is symmetric in the arguments $(\varphi, \psi)$, that is, $\varphi(\ker\psi) = \varphi(A)$ is equivalent to $\psi(\ker\varphi) = \psi(A)$. The proof is an elementary diagram chase.

Definition 4.3. A pair $(\varphi : A \to B, \psi : A \to C)$ of group morphisms is *stably compressible* if there exist some free group $\mathbf{F}(r)$ on finitely many generators and a morphism $\beta : \mathbf{F}(r) \to C$ such that the pair $(\varphi * 1_{\mathbf{F}(r)} : A * \mathbf{F}(r) \to B * \mathbf{F}(r), \psi * \beta : A * \mathbf{F}(r) \to C)$ is strongly compressible.

We will see that stable incompressibility implies infinitely many conditions, indexed by the ordinals, on the pair of morphisms. We first define a series of groups (analogous to the lower central series).

Definition 4.4. Consider a fixed pair $(\varphi : A \to B, \psi : A \to C)$ of group morphisms. Define inductively a subgroup $G_\alpha \subset C$ for any ordinal $\alpha$. Set $G_0 = C$. If $G_\alpha$ is defined for every $\alpha < \beta$ (i.e., $\beta$ is a limit ordinal), then set $G_\beta = \cap_{\alpha<\beta} G_\alpha$. Further, set $G_{\alpha+1} = \mathcal{N}(\psi(\ker\varphi), G_\alpha) \triangleleft G_\alpha$ for any other ordinal, where $\mathcal{N}(K, G)$ is the smallest normal group containing $K$ in $G$.

The groups $G_\alpha$ form a decreasing sequence of subgroups of $C$. Using Zorn's lemma, there exists an infimum of the lattice of groups $G_\alpha$, ordered by the inclusion, which we denote by $G_\infty = \cap_\alpha G_\alpha$ (over all ordinals $\alpha$).

Definition 4.5. The pair $(\varphi : A \to B, \psi : A \to C)$ is said to be *$\alpha$-compressible* if $\psi(A) \subset G_\alpha$ (where $\alpha$ is an ordinal or $\infty$).

**Lemma 4.6.** Given a subgroup $L \subset C$, there exists a maximal subgroup $\Gamma = \Gamma(L, C)$ of $C$ so that $L \subset \mathcal{N}(L, \Gamma) = \Gamma$. □

Proof. There exists at least one group $\Gamma$, for instance $\Gamma = L$. Further, if $\Gamma$ and $\Gamma'$ verify the condition $\mathcal{N}(L, \Gamma) = \Gamma$, then their product $\Gamma\Gamma'$ does. Thus, Zorn's lemma says that a maximal element for the lattice of subgroups verifying this property (the order relation is the inclusion) exists. ■

**Lemma 4.7.** The formula $\Gamma(L, C) = G_\infty$ holds, where $L = \psi(\ker\varphi)$. □

Proof. First, $G_\infty$ satisfies the condition $\mathcal{N}(L, \Gamma) = \Gamma$, otherwise the minimality will be contradicted. Pick an arbitrary $\Gamma$ satisfying this condition. If $\Gamma \subset G_\alpha$, it follows that $\Gamma = \mathcal{N}(L, \Gamma) \subset \mathcal{N}(L, G_\alpha) = G_{\alpha+1}$, hence by a transfinite induction we derive our claim. ■

Definition 4.8. It is said that $K$ is *full* in $\Gamma$ if $\mathcal{N}(K, \Gamma) = \Gamma$. If $(\varphi : A \to B, \psi : A \to C)$ is a pair and $\psi(\ker\varphi)$ is full in $\Gamma$, then $\Gamma$ is called *admissible*.

Remark 4.9. If $\Gamma$ is admissible, then $\psi(\ker\varphi) \subset G_\infty$ since $G_\infty$ is the largest group with this property.

**Proposition 4.10.** If the pair $(\varphi : A \to B, \psi : A \to C)$, where $A$, $B$, and $C$ are finitely generated and $\varphi(A)$ is finitely presented, is stably compressible, then it is $\infty$-compressible and there exists a subgroup $\Gamma \subset G_\infty \subset C$ which is normally finitely generated within $C$ and such that $\psi(\ker\varphi)$ is full in $\Gamma$.

Conversely, if the pair $(\varphi : A \to B, \psi : A \to C)$ is $\infty$-compressible and there exists a finitely generated subgroup $\Gamma \subset G_\infty$ such that $\psi(\ker\varphi)$ is full in $\Gamma$, then the pair is stably compressible. □

Proof. We set $K = \ker\varphi$ in the sequel. We first establish the following lemma.

**Lemma 4.11.** If the pair $(\varphi : A \to B, \psi : A \to C)$ is stably compressible, then it is $\infty$-compressible. □

Proof. We will use a transfinite recurrence with the inductive steps provided by the next two lemmas. Set $\beta : \mathbf{F}(r) \to C$ for the morphism making the pair $(\varphi * 1, \psi * \beta)$ strongly compressible.

**Lemma 4.12.** If $\beta(\mathbf{F}(r)) \subset G_i$ and $\psi(A) \subset G_i$, then $\psi(A) \subset G_{i+1}$. □

Proof. By hypothesis, $\varphi * 1(\ker\psi * \beta) \supset \varphi * 1(A * \mathbf{F}(r))$. Alternatively, for any $b \in \varphi(A) \subset B \subset B * \mathbf{F}(r)$, there exists some $x \in A * \mathbf{F}(r)$ such that $\varphi * 1(x) = b$ and $\psi * \beta(x) = 1$. One can write uniquely $x$ in normal form (see [15, Theorem 1.2, page 175]) as $x = a_1 f_1 a_2 f_2 \cdots a_m f_m$, where $a_j \in A$, $f_j \in \mathbf{F}(r)$ are nontrivial (except maybe $f_m$). Then $\varphi * 1(x) = \varphi(a_1) f_1 \varphi(a_2) f_2 \cdots \varphi(a_m) f_m$.

Since the normal form is unique in $B * \mathbf{F}(r)$, one derives that $x$ has the following property. There exists a sequence $p_0 = 1 < p_1 < \cdots < p_l \leq m$ of integers for which

$$\begin{aligned} &\varphi(a_{p_j}) = b_j \neq 1 \in B, \quad \text{where } b_1 b_2 \cdots b_j = b, \\ &\varphi(a_j) = 1, \quad \forall j \notin \{p_0, p_1, \ldots, p_l\}, \\ &f_{p_j} f_{p_j+1} \cdots f_{p_{j+1}-1} = 1, \quad \forall j, \text{ with the convention } p_{l+1} = m. \end{aligned} \tag{4.1}$$

Furthermore, $1 = \psi * \beta(x)$ implies that (recall that $K = \ker\varphi$)

$$\begin{aligned} 1 \in\ &\psi(a_1 K)\beta(f_1)\psi(K)\beta(f_2)\psi(K)\cdots\beta(f_{p_1-1})\psi(a_{p_1}K)\beta(f_{p_1}) \\ &\times \psi(K)\beta(f_{p_1+1})\psi(K)\cdots\beta(f_m). \end{aligned} \tag{4.2}$$

However, each partial product starting at the $p_j$th term and ending at the $(p_{j+1}-1)$th term is a product of conjugates of $\psi(K)$ by elements from the image of $\beta$:

$$\begin{aligned}
&\beta\left(f_{p_j}\right)\psi(K)\beta\left(f_{p_j+1}\right)\psi(K)\cdots\psi(K)\beta\left(f_{p_{j+1}-1}\right)\\
&\quad=\prod_{i=0}^{p_{j+1}-p_j-1}\left(\beta\left(\prod_{k=0}^{i}f_{p_j+k}\right)\psi(K)\beta\left(\prod_{k=0}^{i}f_{p_j+k}\right)^{-1}\right)\subset\mathcal{N}\big(\psi(K),G_i\big)\\
&\quad=G_{i+1}.
\end{aligned}\tag{4.3}$$

We used above the inclusions $\psi(K)\subset\psi(A)\subset G_i$ and $\beta(\mathbf{F}(r))\subset G_i$. Therefore,

$$1\in\psi\big(a_1K\big)G_{i+1}\psi\big(a_{p_1}K\big)G_{i+1}\cdots\psi\big(a_{p_l}K\big)G_{i+1}=\psi(aK)G_{i+1},\tag{4.4}$$

for any $a\in A$ such that $\varphi(a)=b$. This implies that $\psi(aK)\subset G_{i+1}$ and hence $\psi(A)\subset G_{i+1}$. ■

**Lemma 4.13.** If $\beta(\mathbf{F}(r))\subset G_{i-1}$ and $\psi(A)\subset G_i$, then $\beta(\mathbf{F}(r))\subset G_i$. □

Proof. One can use the symmetry of the algebraic compressibility and then the argument from the previous lemma. Alternatively, choose $f\in\mathbf{F}(r)\subset B*\mathbf{F}(r)$ and some $x\in A*\mathbf{F}(r)$ such that $\varphi*1(x)=f$ and $\psi*\beta(x)=1$. Using the normal form as above, we find this time $1\in\beta(f)\mathcal{N}(\psi(K),G_{i-1})$, hence $\beta(\mathbf{F}(r))\subset G_i$. ■

Using in an alternate way the two previous lemmas, one gets Lemma 4.11. ■

**Lemma 4.14.** Let $\beta:\mathbf{F}(r)\to C$ be a homomorphism such that $(\varphi*1,\psi*\beta)$ is strongly compressible. Set $\beta(\mathbf{F}(r))=H$. Then $\psi(K)$ is full in $\psi(K)H$. In particular, if $A$, $B$, and $C$ are finitely generated and $\varphi(A)$ is finitely presented, then the subgroup $\Gamma=\psi(K)H$ is finitely generated. □

Proof. We have already seen that $H\subset G_\infty$. Set $W(L;X)=\{x\mid x=\prod_i g_ix_ig_i^{-1},\ g_i\in X,x_i\in L\}$ for two subgroups $L,X\subset C$. The proof we used to show that $\psi(A)\subset G_\infty$ and $H\subset G_\infty$ actually yields $\psi(A)\subset W(\psi(K),H)$ and $H\subset W(\psi(K),H)$, respectively. We remark now that $W(\psi(K),H)=\mathcal{N}(\psi(K),\psi(K)H)$. The left inclusion is obvious. The other inclusion consists in writing any element $gxg^{-1}$ with $g\in\psi(K)H$ and $x\in\psi(K)$ as a product of conjugates by elements of $H$. This might be done by recurrence on the length of $g$, by using the following trick. If $g=y_1a_1y_2a_2$, $a_i\in\psi(K)$, $y_i\in H$, then $gxg^{-1}=y_1(a_1y_2a_1^{-1})(a_1a_2xa_2^{-1}a_1^{-1})a_1y_2a_1^{-1}y_1^{-1}$.

Consequently, the fact that $H \subset W(\psi(K), H)$ implies that

$$\begin{aligned} W\big(\psi(K), H\big) &\subset W\big(\psi(K), W\big(\psi(K), H\big)\big) = \mathcal{N}\big(\psi(K), \mathcal{N}\big(\psi(K), \psi(K)H\big)\big) \\ &\subset \mathcal{N}\big(\psi(K), \psi(K)H\big) = W\big(\psi(K), H\big), \end{aligned} \tag{4.5}$$

hence all inclusions are equalities. Also, $\psi(K)H \subset \mathcal{N}(\psi(K), \psi(K)H)$ since both components $\psi(K)$ and $H$ are contained in $\mathcal{N}(\psi(K), \psi(K)H)$. This shows that $\mathcal{N}(\psi(K), \psi(K)H) = \psi(K)H$, hence $\psi(K)$ is full in $\psi(K)H$.

We take therefore $\Gamma = \psi(K)H$. It suffices to show now that each of the groups $K$ and $H$ is finitely generated. The group $H$ is finitely generated since it is the image of $\mathbf{F}(r)$. Furthermore, $K$ is finitely generated since $A/K = \varphi(A)$ is finitely presented and $A$ is finitely generated. The theorem of Neumann [1, Theorem 12, page 52] shows that $K$ must be normally finitely generated. This proves the claim. ■

**Lemma 4.15.** Assume that $\psi(K)$ is full in $\Gamma \subset G_\infty$, where $\Gamma$ is finitely generated. If the pair $(\varphi, \psi)$ is $\infty$-compressible, then it is stably compressible. □

Proof. Consider $r$ big enough and a surjective homomorphism $\beta : \mathbf{F}(r) \to \Gamma$. This implies that $\psi * \beta(A * \mathbf{F}(r)) = \psi(K)\Gamma = \Gamma$. We have to show that any $x \in \Gamma$ is in $\psi * \beta(\ker \varphi * 1)$.

Recall that $\mathcal{N}(\psi(K), \Gamma) = \Gamma$. Set ${}^g x = gxg^{-1}$. Then $x = \prod_i {}^{g_i} x_i$ can be written as a product of conjugates of elements $x_i \in \psi(K)$ by elements $g_i \in \Gamma$. Choose $f_i \in \mathbf{F}(r)$ so that $\beta(f_i) = g_i$, and $y_i \in K$ so that $\psi(y_i) = x_i$. Then $\psi * \beta(\prod_i f_i^{-1} y_i f_i) = x$ and $\varphi * 1(\prod_i f_i^{-1} y_i f_i) = \prod_i f_i^{-1} \varphi(y_i) f_i = 1$ since $y_i \in K$. ■

Then Proposition 4.10 follows. ■

## 4.2 End-compressible manifolds

Definition 4.16. The pair of spaces $(T', T)$ is *strongly compressible* (resp., *stably compressible*) if for each component $S_j$ of $\partial T$, a component $V_j$ of $T' - \mathrm{int}(T)$ is chosen such that $S_j \subset V_j$ so that the pair $(*_j \pi_1(S_j) \to \pi_1(T), *_j \pi_1(S_j) \to *_j \pi_1(V_j))$ is strongly compressible (resp., stably compressible). The morphisms are induced by the obvious inclusions. Similarly, the pair of spaces $(T', T)$ is said to be $\alpha$-*compressible* if the pair of morphisms from above is $\alpha$-compressible. Set also $G_\infty(T, T')$ for the $G_\infty$ group associated to this pair of morphisms.

Remark 4.17. These morphisms are not uniquely defined and depend on the various choices of base points in each component. However the compressibility does not depend on the particular choice of the representative.

Definition 4.18. The open manifold $W^n$ is *end-compressible* (resp., *end* $k$*-compressible*) if every exhaustion of $W^n$ by compact submanifolds $T_i^n$, such that $\pi_1(\partial T_i^n) \twoheadrightarrow \pi_1(T_i^n)$ is a surjection, has a refinement

$$W^n = \bigcup_{i=1}^{\infty} T_i^n, \quad T_i^n \subset \operatorname{int}\left(T_{i+1}^n\right), \tag{4.6}$$

such that

(1) all pairs $(T_{i+1}^n, T_i^n)$ are stably compressible (resp., $k$-compressible);

(2) if $S_{i,j}^{n-1}$ denote the components of $\partial T_i^n$, then the homomorphism $*_j : \pi_1(S_{i,j}^{n-1}) \to \pi_1(T_i^n)$ induced by the inclusion is surjective;

(3) any component of $T_{i+1}^n - \operatorname{int}(T_i^n)$ intersects $T_i^n$ along precisely one component.

Remark 4.19. As in the case of compressibility, condition (2) above is independent of the homomorphism we chose, which might depend on the base points in each component.

Remark 4.20. (i) One can require that each connected component of $T_{i+1}^n - \operatorname{int}(T_i^n)$ have exactly one boundary component from $\partial T_i^n$. By adding to an arbitrarily given $T_i^n$ the regular neighborhoods of arcs in $T_{i+1}^n - \operatorname{int}(T_i^n)$ joining different connected components, this condition will be fulfilled.

(ii) Any simply connected manifold $W$ of dimension at least 5 has an exhaustion by $T_i^n$ that have the property that the natural maps $\pi_1(\partial T_i^n) \to \pi_1(T_i^n)$ are surjective for all $i$. A proof will be given in the next section (see Lemmas 4.33 and 4.35). Thus the above condition is never vacuous.

We will henceforth assume that exhaustions have the property that the natural maps $\pi_1(\partial T_i^n) \to \pi_1(T_i^n)$ are surjective for all $i$.

Remark 4.21. The condition that the pair $(T_{i+1}^n, T_i^n)$ is stably compressible is implied by (and later it will be proved that it is equivalent to) the following pair of conditions:

(1) $(T_{i+1}^n, T_i^n)$ is $\infty$-compressible;

(2) there exists an admissible subgroup $\Gamma_i$ of $G_\infty(T_i^n, T_{i+1}^n)$ which is finitely presented.

**Theorem 4.22.** Any w.g.s.c. open $n$-manifold $W^n$ is end-compressible. Conversely, for $n \neq 4$, $W^n$ is end-compressible if and only if it is w.g.s.c. □

Remark 4.23. Notice that the end-compressibility of $W^3$ implies that of $W^3 \times D^2$. As a consequence of this result for $n \geq 5$, we will derive that $W^3 \times D^2$ is w.g.s.c. and the invariance of the w.g.s.c. under proper homotopies (see Theorem 6.3) will imply the result of the theorem for $n = 3$. We will restrict then for the proof to $n \geq 5$.

Remark 4.24. It is an important issue to know whether the stable compressibility of *one particular exhaustion* implies the stable compressibility of some refinement of *any exhaustion*. This is a corollary of Theorem 4.22 and Proposition 4.25. In fact, if $W^n$ is as above, then $W^n \times D^k$ has one stably compressible exhaustion. Take $n + k \geq 5$ to insure that $W^k \times D^n$ is w.g.s.c. and use Proposition 4.25. In particular, any product exhaustion has a stably compressible refinement, and the claim follows.

### 4.3 Proof of Theorem 4.22

We consider an exhaustion $\{T_i^n\}_{i=1,\infty}$ of $W^n$ by compact submanifolds, and fix some index $i$. The following result is the main tool in checking that specific manifolds are not w.g.s.c.

**Proposition 4.25.** Any exhaustion as above of the w.g.s.c. manifold $W^n$ has a refinement for which consecutive terms fulfill the following conditions:

(1) all pairs $(T_{i+1}^n, T_i^n)$ are stably compressible;

(2) if $S_{i,j}^{n-1}$ denote the components of $\partial T_i^n$, the map $*_j \pi_1(S_{i,j}^{n-1}) \to \pi_1(T_i^n)$ induced by the inclusion is surjective. □

Proof. Since $W^n$ is w.g.s.c., there exists a compact 1-connected submanifold $M^n$ of $W^n$ such that $T_i^n \subset M^n$. We can suppose $M^n \subset T_{i+1}^n$, without loss of generality. From now on, we will focus on the pair $(T_{i+1}^n, T_i^n)$, suppress the index $i$, and denote the pair by $(T', T)$ for the sake of notational simplicity.

**Lemma 4.26.** The pair $(T', T)$ is stably compressible. □

Proof. Let $\varphi : \pi_1(\partial T) \to \pi_1(T)$ and $\psi : \pi_1(\partial T) \to \pi_1(T' - \text{int}(T))$ be the homomorphisms induced by the inclusions $\partial T \hookrightarrow T$ and $\partial T \hookrightarrow T' - \text{int}(T)$, respectively. If $\partial T$ has several components, then we choose base points in each component and set $\pi_1(\partial T) = *_j \pi_1(S_j)$ for notational simplicity.

We consider a handlebody decomposition of $M^n - \text{int}(T)$ (resp., a connected component) from $\partial T$:

$$M^n - \text{int}(T) = \big(\partial T \times [0, 1]\big) \bigcup_{\lambda=1}^{n-1} \big(\cup_j h_j^{\lambda}\big), \tag{4.7}$$

where $h_j^{\lambda}$ is a handle of index $\lambda$. We suppose that the handles are attached in increasing order of their indices. Since the distinct components of $\partial T$ are not connected outside $T$ (by Remark 4.20), the 1-handles which are added have the extremities in the same connected component of $\partial T$. Set $M_2^n \subset M^n$ (resp., $M_1^n$) for the submanifold obtained by attaching to $T$ only the handles $h_j^\lambda$ of index $\lambda \leq 2$ (resp., those of index $\lambda \leq 1$). Then $\pi_1(M_2^n) = 0$ because adding higher-index handles does not affect the fundamental group and we know that $\pi_1(M^n) = 0$.

**Lemma 4.27.** The pair $(T', M_1^n)$ is strongly compressible. □

Proof. Let $\{\gamma_j\}_{j=1,p} \subset \partial M_1^n$ be the set of attaching circles for the 2-handles of $M_2^n$ and let $\{\delta_j^2\}_{j=1,p}$ be the corresponding core of the 2-handles $h_j^2$ $(j = 1, p)$. Since $\delta_j^2$ is a 2-disk embedded in $M^n - \text{int}(T)$, it follows that the homotopy class $[\gamma_j]$ vanishes in $\pi_1(M^n - \text{int}(T))$.

Let $\Gamma \subset \pi_1(\partial M_1^n)$ be the normal subgroup generated by the homotopy classes of the curves $\{\gamma_j\}_{j=1,p}$ which are contained in $\partial M_1^n$. Notice that this amounts to picking base points which are joined to the loops. Therefore, the image of $\Gamma$ under the map $\pi_1(\partial M_1^n) \to \pi_1(M_2^n - \text{int}(M_1^n))$, induced by the inclusion, is zero. In particular, its image in $\pi_1(T' - \text{int}(M_1^n))$ is zero.

On the other hand, the images of the classes $[\gamma_j]$ in $\pi_1(M_1^n)$ normally generate all of the group $\pi_1(M_1^n)$ because $\pi_1(M_2^n) = 1$.

These two properties are equivalent to the strong compressibility of the pair $(M_2^n, M_1^n)$ which in turn implies that of $(T', M_1^n)$. ■

Rest of the proof of Lemma 4.26. Assume now that the number of 1-handles $h_j^1$ is $r$. Notice that $\pi_1(M_1^n - \text{int}(T)) \cong \pi_1(\partial T) * F(r)$ because it can be obtained from $\partial T \times [0, 1]$ by adding 1-handles, and the 1-handles we added do not join distinct boundary components so that each one contributes with a free factor. In particular, the inclusion $\partial T \hookrightarrow M_1^n - \text{int}(T)$ induces a monomorphism $\pi_1(\partial T) \hookrightarrow \pi_1(\partial T) * F(r)$. Observe that $M_1^n - \text{int}(T)$ can also be obtained from $\partial M_1^n \times [0, 1]$ by adding $(n - 1)$-handles, hence the inclusion $\partial M_1^n \hookrightarrow M_1^n - \text{int}(T)$ induces the isomorphism $\pi_1(\partial M_1^n) \cong \pi_1(\partial T) * F(r)$. The same reasoning gives the isomorphism $\pi_1(M_1^n) \cong \pi_1(T) * F(r)$.

In particular, we can view the subgroup $\Gamma$ as a subgroup of $\pi_1(\partial T) * F(r)$. The previous lemma tells us that $\Gamma$ lies in the kernel of $\pi_1(\partial T) * F(r) \to \pi_1(T' - \text{int}(T))$ and also projects epimorphically onto $\pi_1(T) * F(r)$. The identification of the respective maps with the morphisms induced by inclusions yields our claim. ■

This finishes the proof of Proposition 4.25. ■

Conversely, assume that $W^n$ has an exhaustion in which consecutive pairs are stably compressible. Then it is sufficient to show the following proposition.

**Proposition 4.28.** If $(T', T)$ is a stably compressible pair of $n$-manifolds and $n \geq 5$, then $T \subset \text{int}(M^n) \subset T'$, where $M^n$ is a compact submanifold with $\pi_1(M^n) = 0$. □

Proof. One can realize the homomorphism $\beta : \mathbf{F}(r) \to \pi_1(T' - \mathrm{int}(T))$ by a disjoint union of bouquets of circles $\vee^r S^1 \to T' - \mathrm{int}(T)$. There is one bouquet in each connected component of $T' - \mathrm{int}(T)$. One joins each wedge point to the unique connected component of $\partial T$ for which that is possible by an arc, and set $M_1^n$ for the manifold obtained from $T$ by adding a regular neighborhood of the bouquets $\vee^r S^1$ in $T'$ (plus the extra arcs). This is equivalent to adding 1-handles with the induced framing.

**Lemma 4.29.** The kernel $\ker \psi * \beta \subset \pi_1(\partial M_1^n)$ is normally generated by a finite number of elements $\gamma_1, \gamma_2, \dots, \gamma_p$. □

Proof. Consider a finite presentation $\mathbf{F}(k)/H \to \pi_1(\partial T)$. We know that $\pi_1(\partial M_1^n) = \pi_1(\partial T) * \mathbf{F}(r)$. Furthermore, the composition map

$$\lambda : \mathbf{F}(k) * \mathbf{F}(r) \longrightarrow \pi_1(\partial T) * \mathbf{F}(r) \xrightarrow{\psi * \beta} \Gamma \tag{4.8}$$

is surjective (since $\beta$ is). The first map is the free product of the natural projection with the identity. Therefore, $\mathbf{F}(k+r)/\ker \lambda \cong \Gamma$ is a presentation of the group $\Gamma$. The theorem of Neumann (see [1, Theorem 12, page 52]) states that any presentation on finitely many generators of a finitely presented group has a presentation on these generators with only finitely many of the given relations. Applying this to $\Gamma$, one derives that there exist finitely many elements which normally generate $\ker \lambda$ in $\mathbf{F}(k+r)$. Then the images of these elements in $\pi_1(\partial T) * \mathbf{F}(r)$ normally generate $\ker \psi * \beta$ (the projection $\mathbf{F}(k) * \mathbf{F}(r) \to \pi_1(\partial T) * \mathbf{F}(r)$ is surjective). This yields the claim. ■

**Lemma 4.30.** The elements $\gamma_i$ are also in the kernel of $\pi_1(\partial M_1^n) \to \pi_1(T' - \mathrm{int}(M_1^n))$. □

Proof. The map $\pi_1(T' - \mathrm{int}(M_1^n)) \to \pi_1(T' - \mathrm{int}(T))$ induced by the inclusion is injective because $T' - \mathrm{int}(T)$ is obtained from $T' - \mathrm{int}(M_1^n)$ by adding $(n-1)$-handles (dual to the 1-handles from which one gets $M_1^n$ starting from $T$), and $n \geq 5$. Thus the map $\pi_1(\partial M_1^n) \to \pi_1(T' - \mathrm{int}(T))$ factors through

$$\pi_1\left(\partial M_1^n\right) \longrightarrow \pi_1\left(T' - \mathrm{int}\left(M_1^n\right)\right) \longrightarrow \pi_1\left(T' - \mathrm{int}(T)\right), \tag{4.9}$$

and any element in the kernel must be in the kernel of the first map, as stated. ■

The dimension restriction $n \geq 5$ implies that we can assume $\gamma_j$ to be represented by embedded loops having only the base point in common. Then $\gamma_j$ bound singular 2-disks $D_j^2 \subset T' - \mathrm{int}(M_1^n)$. By a general position argument, one can arrange such that the 2-disks $D_j^2$ are embedded in $T' - \mathrm{int}(M_1^n)$ and have disjoint interiors.

As a consequence, the manifold $M^n$ obtained from $M_1^n$ by attaching 2-handles along the $\gamma_j$'s (with the induced framing) can be embedded in $T' - \mathrm{int}(T)$. Moreover, $M^n$ is a compact manifold whose fundamental group is the quotient of $\pi_1(M_1^n)$ by the subgroup normally generated by the elements $\varphi * 1(\gamma_j)$'s. The group $\varphi * 1(\ker\psi * \beta)$ is normally generated by the elements $\varphi * 1(\gamma_j)$. By hypothesis, the pair $(\varphi * 1, \psi * \beta)$ is compressible, hence $\varphi * 1(\ker\psi * \beta)$ contains $\varphi * 1(\pi_1(\partial M_1^n)) = \varphi * 1(\pi_1(\partial T) * F(r)) = \varphi(\pi_1(\partial T)) * F(r)$. Next,

$$\frac{\pi_1\left(M_1^n\right)}{\varphi * 1(\ker\psi * \beta)} = \frac{\pi_1(T) * F(r)}{\varphi\left(\pi_1(\partial T)\right) * F(r)} \cong \frac{\pi_1(T)}{\varphi\left(\pi_1(\partial T)\right)} = 1 \tag{4.10}$$

since $\varphi$ has been supposed surjective. Therefore, the quotient of $\pi_1(M_1^n)$ by the subgroup normally generated by the elements $\varphi * 1(\gamma_j)$ is trivial. ■

## 4.4 End 1-compressibility is trivial for $n \geq 5$

We defined an infinite sequence of obstructions (namely, $k$-compressibility for each $k$) to the w.g.s.c. However, the first obstruction is trivial, that is, equivalent to the simple connectivity, in dimensions $n \neq 4$. In fact, the main result of this section establishes the following proposition.

**Proposition 4.31.** End 1-compressibility and simple connectivity (s.c.) are equivalent for open $n$-manifolds of dimension $n \geq 5$. □

Proof. We first consider a simpler case.

**Proposition 4.32.** The result holds in the case of a manifold $W^n$ of dimension at least 5 with one end. □

Proof. In this case $W^n$ has an exhaustion $T_i$ with $\partial T_i$ connected for all $i$.

**Lemma 4.33.** The manifold $W^n$ has an exhaustion such that the map $\varphi : \pi_1(\partial T_i) \to \pi_1(T_i)$ induced by inclusion is a surjection for each $i$. □

Proof. As $W^n$ is simply connected, by taking a refinement, we can assume that each inclusion map $\pi_1(T_i) \to \pi_1(T_{i+1})$ is the zero map. As usual, we denote $T_i$ and $T_{i+1}$ by $T$ and $T'$, respectively.

Now, take a handle decomposition of $T$ starting with the boundary $\partial T$. Suppose the core of each 1-handle of this decomposition is homotopically trivial in $T$, then it is immediate that $\varphi$ is a surjection. We will enlarge $T$ by adding some 1-handles and 2-handles (that are embedded in $T'$) at $\partial T$ in order to achieve this.

Namely, let $\gamma$ be the core of a 1-handle. By hypothesis, there is a disc $D^2$ in $T'$ bounding the core of each of the 1-handles, which we take to be transversal to $\partial T$. As the dimension of $W^n$ is at least 5, $D^2$ can be taken to be embedded. Notice that the 2-disks corresponding to all 1-handles can also be made disjoint by general position. Thus $D^2$ intersects $T' - \mathrm{int}(T)$ in a collection of embedded disjoint planar surfaces. The neighborhood of each disc component of this intersection can be regarded as a 2-handle (embedded in $T'$) which we add to $T$ at $\partial T$. For components of $D^2 \cap (T' - \mathrm{int}(T)) = D^2 - \mathrm{int}(T)$ with more than one boundary component, we take embedded arcs joining distinct boundary components. We add to $T$ a neighborhood of each arc, which can be regarded as a 1-handle. After doing this for a finite collection of arcs, $D^2 - D^2 \cap T$ becomes a union of discs. Now we add 2-handles as before. The disc $D^2$ that $\gamma$ bounds is now in $T$.

Further, the dual handles to the handles added are of dimension at least 3. In particular, we can extend the previous handle decomposition to a new one for (the new) $T$ starting at (the new) $\partial T$ with no new 1-handles. Thus, after performing the above operation for the core of each 1-handle of the original handle decomposition, the core of each 1-handle of the resulting handle decomposition of $T$ starting at $\partial T$ bounds a disc in $T$. Thus $\phi$ is a surjection. ■

The above exhaustion is in fact 1-compressible by the following algebraic lemma.

**Lemma 4.34.** Suppose that a square of maps verifying the van Kampen theorem is given:

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\varphi"'] & C \arrow[d, "\gamma"] \\ B \arrow[r, "\beta"] & D. \end{tikzcd} \tag{4.11}$$

Let $\xi : A \to D$. Suppose also that $\varphi$ is surjective and $\beta$ is the zero map. Then $\psi(A) \subset \mathcal{N}(\psi(\ker\varphi)) = \mathcal{N}(\psi(\ker\varphi), C)$. □

Proof. Observe that $\xi(\ker\varphi) = 0$, hence $\psi(\ker\varphi) \subset \ker\gamma$. Hence we can define another diagram with $A' = A/\ker\varphi$, $B' = B$, $C' = C/\mathcal{N}(\psi(\ker\varphi))$, and $D' = D$:

$$\begin{tikzcd} A' = A/\ker\varphi \arrow[r, "\psi"] \arrow[d, "\varphi"] & C' = C/\mathcal{N}\big(\psi(\ker\varphi)\big) \arrow[d, "\gamma"] \\ B' = B \arrow[r, "\beta"] & D' = D. \end{tikzcd} \tag{4.12}$$

Notice that the map $A/\ker\varphi \to C/\mathcal{N}(\psi(\ker\varphi))$ is well defined. Again this diagram verifies the van Kampen theorem. For this diagram, the induced $\varphi$ is an isomorphism. It is immediate then that the universal (freest) $D'$ must be $C'$. In fact, $D' = C' * A'/\mathcal{N}(\{\psi(a)a^{-1},\ a \in A'\})$. Consider the map $C' \to C' * A'/\mathcal{N}(\{\psi(a)a^{-1},\ a \in A'\}) \to C'$, where the second arrow consists in replacing any occurrence of $a$ by the element $\psi(a)$ and taking the product in $C'$. This composition is the identity and the first map is a surjection, hence the map $C' \to D'$ is an isomorphism.

The map induced by $\xi$ is zero since $\beta$ is the zero. But the map $\xi : A' \to D'$ is the map $\psi : A' \to C'$ followed by an isomorphism, hence $\psi(A') = 0$. This is equivalent to $\psi(A) \subset \mathcal{N}(\psi(\ker\varphi))$. ■

Note that the above lemma is purely algebraic, and in particular independent of dimension. The two lemmas immediately give us the proposition for one-ended manifolds $W^n$ with $n \geq 5$. ■

*The general case.* We now consider the general case of a simply connected open manifold $W^n$ of dimension at least 5, with possibly more than one end. We will choose the exhaustion $T_i$ with more care in this case.

We will make use of the following construction several times. Start with a compact submanifold $A^n$ of codimension 0, with possibly more than one boundary component. Assume for simplicity (by enlarging $A^n$ if necessary) that no complementary component of $A^n$ is precompact. As $W^n$ is simply connected, we can find a compact submanifold $B^n$ containing $A^n$ in its interior such that the inclusion map on fundamental groups is the zero map. Further, we can do this by thickening and then adding the neighborhood of a 2-complex, that is, a collection of 1-handles and 2-handles. Namely, for each generator $\gamma$ of $\pi_1(A^n)$, we can find a disc $D^2$ that $\gamma$ bounds, and then add 1-handles and 2-handles as in Lemma 4.33. Thus, as $n \geq 5$, the boundary components of $B^n$ correspond to those of $A^n$. We repeat this with $B^n$ in place of $A^n$ to get another submanifold $C^n$.

Observe that as a consequence of this and the simple connectivity of $W^n$, the inclusion map $\pi_1(A^n \cup V^n) \to \pi_1(B^n \cup V^n)$ is the zero map for any component $V^n$ of $W^n - A^n$. More generally, if $Z^n \subset V^n$, then $\ker(\pi_1(A^n \cup Z^n) \to \pi_1(B^n \cup Z^n)) = \ker(\pi_1(A^n \cup Z^n) \to \pi_1(B^n \cup Z^n \cup (W^n - V^n)))$. Similar results hold with $B^n$ and $C^n$ in place of $A^n$ and $B^n$.

Now start with some $A_1^n$ as above and construct $B_1^n$ and $C_1^n$. Thicken $C_1^n$ slightly to get $T = T_1$. We will eventually choose a $T_2 = T'$, but for now we merely note that it can (and so it will) be chosen in such a manner that the inclusion map $\pi_1(T) \to \pi_1(T')$ is the zero map. Let $S_j$, $j = 1, \ldots, n$, be the boundary components of $T$. Let $X_j$ be the union of the components of $T - \mathrm{int}(B^n)$ containing $S_j$ and $B^n$, and define $Y_j$ analogously with $C^n$ in place of $B^n$. Denote the image of $\pi_1(X_j)$ in $\pi_1(Y_j)$ by $\overline{\pi}(X_j)$. We then have a natural map $\varphi : \pi_1(S_j) \to \overline{\pi}(X_j)$. Let $V_j$ be the component of $T' - \mathrm{int}(T)$ containing $S_j$.

**Lemma 4.35.** By adding 1-handles and 2-handles to $S_j$, it can be ensured that $\pi_1(S_j)$ surjects onto $\overline{\pi}(X_j)$. □

Proof. The proof is essentially the same as that of Lemma 4.33. We start with a handle decomposition for $X_j$ starting from $S_j$. We will ensure that the image in $\overline{\pi}(X_j)$ of the core of each 1-handle is trivial. Namely, for each core, we take a disc D that it bounds. By the above remarks, we can, and do, ensure that the disc lies in $Y_j$, and in particular does not intersect any boundary component of T except $S_j$. As in Lemma 4.33, we may now add 1-handles and 2-handles to $S_j$ to achieve the desired result. ■

Notice that the changes made to T in the above lemma do not affect $S_k$, $X_k$, and $Y_k$ for $k \neq j$. Hence, by repeated application of the above lemma, we can ensure that all the maps $\pi_1(S_j) \to \overline{\pi}(X_j)$ are surjections. Also notice that the preceding remarks show that $\ker \varphi = \ker(\pi_1(S_j) \to \pi_1(T))$. Now take $A = \pi_1(S_j)$, $B = \overline{\pi}(X_j)$, and $C = \pi_1(V_j)$, and let D be the image of $\pi_1(V_j \cup X_j)$ in $\pi_1(V_j \cup Y_j)$. Then, by the preceding remarks and Lemma 4.35, the diagram (4.11) satisfies the hypothesis of Lemma 4.34. The 1-compressibility for the pair $(T', T)$ follows.

Now we continue the process inductively. Suppose $T_k$ has been defined and choose $A_{k+1}$ so that it contains $T_k$ and also in such a manner so as to ensure that $A_i$'s exhaust M. Then find $B_{k+1}$, $C_{k+1}$, and $T_{k+1}$ as above. The rest follows as above. ■

Remark 4.36. For the case of simply connected, one-ended (hence contractible) 3-manifolds, a theorem of Luft says that M can be exhausted by a union of homotopy handlebodies. These satisfy the conclusion of Lemma 4.33, hence the proposition still holds. More generally, we can apply the sphere theorem to deduce that we have an exhaustion by connected sums of homotopy handlebodies. It follows that each pair $(T, T')$ of this exhaustion is 1-compressible as we can decompose T and consider each component separately without affecting $\varphi$ or $\psi$.

## 5 Examples of contractible manifolds

### 5.1 Uncountably many Whitehead-type manifolds

Recall the following definition from [35].

Definition 5.1. A *Whitehead link* $T_0^n \subset T_1^n$ is a null-homotopic embedding of the solid torus $T_0^n$ in the (interior of the) unknotted solid torus $T_1^n$ lying in $S^n$ such that the pair $(T_1^n, T_0^n)$ is (boundary) incompressible.

The solid n-torus is $T^n = D^2 \times S^1 \times S^1 \times \cdots \times S^1$. By iterating the ambient homeomorphism which sends $T_0^n$ onto $T_1^n$, one obtains an ascending sequence $T_0^n \subset T_1^n \subset T_2^n \subset \cdots$ whose union is called a Whitehead-type n-manifold. A Whitehead-type manifold is open contractible and not s.c.i. Wright [35] gave a recurrent procedure to construct many Whitehead links in dimensions $n \geq 3$. One shows below that this construction provides uncountably many distinct contractible manifolds.

We introduce an invariant for pairs of solid tori, which generalizes the wrapping number in dimension 3. Moreover, this provides invariants for open manifolds of Whitehead type answering a question raised in [35].

Definition 5.2. A *spine* of the solid torus $T^n$ is an embedded $t^{n-2} = \{*\} \times S^1 \times S^1 \times \cdots \times S^1 \subset T^n$ having a trivial normal bundle in $T^n$. This gives $T^n$ the structure of a trivial 2-disk bundle over $t^{n-2}$.

Remark 5.3. Although the spine is not uniquely defined, its isotopy class within the solid torus is.

Consider a pair of solid tori $T_0^n \subset T^n$. We fix some spine $t^{n-2}$ for $T^n$. To specify the embedding of $T_0^n$ is the same as giving the embedding of a spine $t_0^{n-2}$ of $T_0^n$ in $T^n$. The isotopy class of the embedding $t_0^{n-2} \hookrightarrow T^n$ is therefore uniquely defined by the pair. We pick up a Riemannian metric $g$ on the torus $T^n$ such that $T^n$ is identified with the regular neighborhood of radius $r$ around $t^{n-2}$. We denote this by $T^n = t^{n-2}[r]$, and suppose for simplicity that $r = 1$. Then $t^{n-2}[\lambda]$ for $\lambda \leq 1$ will denote the radius-$\lambda$ tube around $t^{n-2}$ in this metric.

Definition 5.4. The *wrapping number* of the Whitehead link $T_0^n \subset T^n$ is defined as follows:

$$w(T^n, T_0^n) = \lim_{\varepsilon \to 0} \inf_{t_0^n \in \mathcal{I}(t_0^n \subset t^{n-2}[\varepsilon])} \frac{\mathrm{vol}\,(t_0^{n-2})}{\mathrm{vol}\,(t^{n-2})}, \tag{5.1}$$

where $\mathcal{I}(t_0^n \subset T^n)$ is the set of all embeddings of the spine $t_0^{n-2}$ of $T_0^n$ in the given isotopy class, and vol is the $(n-2)$-dimensional volume.

Remark 5.5. Notice that a priori this definition might depend on the particular choice of the spine $t^{n-2}$ and on the metric $g$.

**Proposition 5.6.** The wrapping number is a topological invariant of the pair $(T^n, T_0^n)$. □

Proof. There is a natural projection map on the spine $\pi : T^n \to t^{n-2}$, which is the fiber bundle projection of $T^n$ (with fiber a 2-disk). When both $T^n$ and $t^{n-2}$ are fixed, then such a projection map is also defined only up to isotopy. Set therefore

$$l(T^n, T_0^n) = \inf_{t_0^n \in \mathcal{I}(t_0^n \subset T^n)} \inf_{x \in t^{n-2}} \sharp\{\pi^{-1}(x) \cap t_0^{n-2}\}. \tag{5.2}$$

Since $\inf_{x\in t^{n-2}} \sharp\{\pi^{-1}(x)\cap t_0^{n-2}\}$ does not depend on the particular projection map (in the fixed isotopy class), this number represents a topological invariant of the pair $(T^n, T_0^n)$. Hence the claim follows from the following result.

**Proposition 5.7.** The formula $w(T^n, T_0^n) = l(T^n, T_0^n)$ holds. □

Proof. Consider a position of $t_0^{n-2}$ for which the minimum value $l(T^n, T_0^n)$ is attained. A small isotopy makes $t_0^{n-2}$ transversal to $\pi$. Then, for this precise position of $t_0^{n-2}$, there exists some number $M$ such that

$$\sharp\left\{\pi^{-1}(x)\cap t_0^{n-2}\right\} \leq M, \quad \text{for any } x \in t^{n-2}. \tag{5.3}$$

Denote by $\mu$ the Lebesgue measure on $t^{n-2}$.

**Lemma 5.8.** For any $\varepsilon > 0$, $t_0^{n-2}$ can be moved in $T^n$ by an ambient isotopy such that the following conditions are fulfilled:

$$\begin{aligned} &\sharp\left\{\pi^{-1}(x)\cap t_0^{n-2}\right\} \leq M, \quad \text{for any } x \in t^{n-2}, \\ &\mu\left(\left\{x\in t^{n-2} \mid \sharp\left\{\pi^{-1}(x)\cap t_0^{n-2}\right\} > l\left(T^n, T_0^n\right)\right\}\right) < \varepsilon. \end{aligned} \tag{5.4}$$

□

Proof. The set $U = \{x \in t^{n-2} \mid \sharp\{\pi^{-1}(x)\cap t_0^{n-2}\} = l(T^n, T_0^n)\}$ is an open subset of positive measure. Consider then a flow $\varphi_t$ on the torus $t^{n-2}$ which expands a small ball contained in $U$ into the complement of a measure $\varepsilon$ set (e.g., a small tubular neighborhood of a spine of the 1-holed torus). Extend this flow as $1_{D^2}\times\varphi_t$ all over $T^n$ and consider its action on $t_0^{n-2}$. ■

**Lemma 5.9.** The inequality $w(T^n, T_0^n) \leq l(T^n, T_0^n)$ is satisfied. □

Proof. The map $\pi$ is the projection of the metric tube around $t^{n-2}$ on its spine, hence the Jacobian $\mathrm{Jac}(\pi|_{t_0^{n-2}})$ has bounded norm $|\mathrm{Jac}(\pi|_{t_0^{n-2}})| \leq 1$. It follows that

$$\begin{aligned} \frac{\mathrm{vol}\left(t_0^{n-2}\right)}{\mathrm{vol}\left(t^{n-2}\right)} &= \frac{\int \pi^* d\mu}{\int d\mu} \leq \frac{\int_{\pi^{-1}(U)} \left|\mathrm{Jac}\left(\pi|_{t_0^{n-2}}\right)\right| d\mu}{\int_U d\mu} + M\varepsilon \\ &\leq l\left(T^n, T_0^n\right)(1-\varepsilon) + M\varepsilon, \end{aligned} \tag{5.5}$$

for any $\varepsilon > 0$, hence the claim follows. ■

**Lemma 5.10.** The reverse inequality $w(T^n, T_0^n) \geq l(T^n, T_0^n)$ holds. □

Proof. Set $\lambda_t : t^{n-2}[\delta] \to t^{n-2}[t\delta]$ for the map given in coordinates by $\lambda_t(p,x) = (tp,x)$, $p \in D^2$, $x \in t^{n-2}$. Here the projection $\pi$ provides a global trivialization of $t^{n-2}[\delta] \subset T^n$. Then,

$$\lim_{t\to 0} \left| \operatorname{Jac}\left(\pi|_{t_0^{n-2}} \circ \lambda_t\right)\right| = 1. \tag{5.6}$$

Therefore, for $t$ close enough to $0$, one derives

$$\lim_{t\to 0} \frac{\operatorname{vol}\left(\lambda_t\left(t_0^{n-2}\right)\right)}{\operatorname{vol}\left(t^{n-2}\right)} = \lim_{t\to 0} \frac{\int_{\lambda_t(t_0^{n-2})} \left|\operatorname{Jac}\left(\pi|_{t_0^{n-2}}\right)\right| d\mu}{\int_{t^{n-2}} d\mu} \geq l\left(T^n, T_0^n\right). \tag{5.7}$$

Since the position of $t_0^{n-2}$ was chosen arbitrary, this inequality survives after passing to the infimum and the claim follows. ∎

This ends the proof of Proposition 5.7. ∎

In particular, the wrapping number is a topological invariant, as claimed in Proposition 5.6. ∎

**Theorem 5.11.** There exist uncountably many Whitehead-type manifolds for $n \geq 5$. □

Proof. The proof here follows the same pattern as that given by McMillan [20] for the 3-dimensional case. We first establish the following useful property of the wrapping number.

**Proposition 5.12.** If $T_0^n \subset T_1^n \subset T_2^n$, then $w(T_2^n, T_0^n) = w(T_2^n, T_1^n)w(T_1^n, T_0^n)$. □

Proof. This is a consequence of the two lemmas below.

**Lemma 5.13.** The inequality $l(T_2^n, T_0^n) \leq l(T_2^n, T_1^n)l(T_1^n, T_0^n)$ holds. □

Proof. Consider $t_0^{n-2} \subset T_1^n \subset T_2^n$, where $T_1^n$ is a very thin tube around $t_1^{n-2}$, and the two projections to $\pi_2 : t_1^{n-2} \to t_2^{n-2}$ and $\pi_1 : t_0^{n-2} \to t_1^{n-2}$, respectively. Using Lemma 5.8, one can assume that the conditions

$$\begin{aligned} &\mu\left(\left\{x \in t_1^{n-2} \mid \sharp\{\pi_1^{-1}(x)\} = l\left(T_1^n, T_0^n\right)\right\}\right) > 1 - \varepsilon, \\ &\mu\left(\left\{x \in t_2^{n-2} \mid \sharp\{\pi_2^{-1}(x)\} = l\left(T_2^n, T_1^n\right)\right\}\right) > 1 - \varepsilon \end{aligned} \tag{5.8}$$

hold. For a small enough $\epsilon$, one derives that

$$\mu\left(\left\{x \in t_2^{n-2} \mid \sharp\left\{\left(\pi_2 \circ \pi_1\right)^{-1}(x)\right\} = l\left(T_2^n, T_1^n\right)l\left(T_1^n, T_0^n\right)\right\}\right) > 0. \tag{5.9}$$

This proves that the minimal cardinal of $(\pi_2 \circ \pi_1)^{-1}(x)$ is not greater than $l(T_2^n, T_1^n)l(T_1^n, T_0^n)$, hence the claim follows. ∎

**Lemma 5.14.** The wrapping number satisfies the inequality $w(T_2^n, T_0^n) \geq w(T_2^n, T_1^n)w(T_1^n, T_0^n)$. □

Proof. We can assume that $w(T_2^n, T_1^n) \neq 0$. Consider an embedding of the $(n-2)$-torus $s_1^{n-2} \subset T_2 = t_2^{n-2}[\varepsilon]$ for which the value of $\mathrm{vol}(t_1^{n-2})/\mathrm{vol}(t_2^{n-2})$ (as a function of $t_1^{n-2}$) is close to the infimum in the isotopy class. We will assume that in all the formulas below the tori lay in their respective isotopy classes. Then

$$\begin{aligned}\left(\inf_{t_0^{n-2}\subset s_1^{n-2}[2\varepsilon]} \frac{\mathrm{vol}\left(t_0^{n-2}\right)}{\mathrm{vol}\left(s_1^{n-2}\right)}\right)&\left(\inf_{t_1^{n-2}\subset t_2^{n-2}[\varepsilon]} \frac{\mathrm{vol}\left(t_1^{n-2}\right)}{\mathrm{vol}\left(t_2^{n-2}\right)}\right)\\ &\leq \left(\inf_{t_0^{n-2}\subset s_1^{n-2}[2\varepsilon]} \frac{\mathrm{vol}\left(t_0^{n-2}\right)}{\mathrm{vol}\left(t_1^{n-2}\right)}\right)\frac{\mathrm{vol}\left(s_1^{n-2}\right)}{\mathrm{vol}\left(t_2^{n-2}\right)}\\ &= \inf_{t_0^{n-2}\subset s_1^{n-2}[2\varepsilon]} \frac{\mathrm{vol}\left(t_0^{n-2}\right)}{\mathrm{vol}\left(t_2^{n-2}\right)}\\ &\leq \inf_{t_0^{n-2}\subset t_2^{n-2}[\varepsilon]} \frac{\mathrm{vol}\left(t_0^{n-2}\right)}{\mathrm{vol}\left(t_2^{n-2}\right)}.\end{aligned} \tag{5.10}$$

The last inequality follows from the fact that $s_1^{n-2}[2\varepsilon] \subset t_2^{n-2}[\varepsilon]$. In fact, $w(T_2^n, T_1^n) \neq 0$ implies that $s_1^{n-2}$ intersects any 2-disk $D^2 \times \{*\}$ (i.e., any fiber of the projection $\pi_2 : T_2^n \to t_2^{n-2}$) of $T_2^n = t_2^{n-2}[\varepsilon]$ in at least one point. Then the transversal disk $D^2 \times \{*\}$ of radius $\varepsilon$ is therefore contained in the tube $s_1^{n-2}[2\varepsilon]$ of radius $2\varepsilon$ around $s_1^{n-2}$, establishing the claimed inclusion.

On the other hand, the equation

$$\lim_{\varepsilon\to 0} \inf_{t_0^{n-2}\subset s_1^{n-2}[2\varepsilon]} \frac{\mathrm{vol}\left(t_0^{n-2}\right)}{\mathrm{vol}\left(s_1^{n-2}\right)} = w\left(T_1^n, T_0^n\right) \tag{5.11}$$

holds due to the topological invariance of the wrapping number. Letting $\varepsilon$ go to 0 in (5.10) yields the claim. ■

The above two lemmas prove Proposition 5.12. ■

**Proposition 5.15.** There exist Whitehead links whose wrapping number has the form $2^{n-2}p$ for any natural number $p$. □

Proof. The claim is well known for $n = 3$. One uses Wright's construction [35] of Whitehead links by induction on the dimension. If $T_0^n \subset T_1^n$ is a Whitehead link, then set $T^{n+1} = T_1^n \times S^1$. Consider the projection $q$ of the solid torus $T_0^n \times S^1 \cong D^2 \times S^1 \times \cdots \times S^1$ onto $D^2 \times S^1$ (the first and the last factors). Choose some Whitehead link $L^3 \subset D^2 \times S^1$ and set then $Q^{n+1} = q^{-1}(L^3)$. The pair $Q^{n+1} \subset T^{n+1}$ is a Whitehead link of dimension $n+1$. The proposition then is an immediate consequence of the following lemma.

**Lemma 5.16.** The equation $w(T^{n+1}, Q^{n+1}) = w(T_1^n, T_0^n)w(D^2 \times S^1, L^3)$ holds. □

Proof. From the multiplicativity of $w$ and the triviality of the projection $q$, it is sufficient to prove that $w(T_1^n \times S^1, T_0^n \times S^1) = w(T_1^n, T_0^n)$. This formula can be checked directly using $l$ instead of $w$. ■

By using Lemma 5.16, one constructs examples of Whitehead links, as claimed in Proposition 5.15, by recurrence on the dimension. ■

**Proposition 5.17.** For any sequence $\mathbf{p} = p_0, p_1, \ldots$ of positive integers, consider a Whitehead-type manifold $W^n(\mathbf{p}) = \cup_{k=1}^{\infty} T_k^n$, where $w(T_{k+1}^n, T_k^n) = 2^{n-2}p_k$. If the sequences $\mathbf{p}$ and $\mathbf{q}$ have infinitely many nonoverlapping prime factors, then the manifolds $W^n(\mathbf{p})$ and $W^n(\mathbf{q})$ are not PL homeomorphic. □

Proof. The proof is similar to that of [20, page 375]. Set $W^n(\mathbf{p}) = \cup_{k=1}^{\infty} T_k^n$, $W^n(\mathbf{q}) = \cup_{k=1}^{\infty} s_k^n$, where $T_k^n, \widetilde{T}_k^n$ are tori, as above. If $h : W^n(\mathbf{q}) \to W^n(\mathbf{p})$ is a PL homeomorphism, there exist integers $j, k$ such that $T_0^n \subset \mathrm{int}(h(\widetilde{T}_j^n))$, $q_k$ has a prime factor which occurs in $\mathbf{q}$ but not in $\mathbf{p}$, $k > j + 1$, and $h(\widetilde{T}_k^n) \subset \mathrm{int}(T_m^n)$. We have therefore

$$w\big(T_m^n, T_0^n\big) = w\big(T_m^n, h\big(\widetilde{T}_k^n\big)\big)w\big(h\big(\widetilde{T}_k^n\big), h\big(\widetilde{T}_j^n\big)\big)w\big(h\big(\widetilde{T}_j^n, T_0^n\big)\big). \tag{5.12}$$

We have obtained a contradiction because $q_k$ divides $w(h(\widetilde{T}_k^n), h(\widetilde{T}_j^n))$ but not the left-hand side (which is nonzero also). ■

This proves Theorem 5.11. ■

## 5.2 Open manifolds which are not w.g.s.c.

In general, the tower of obstructions we defined in the previous sections is not trivial as is shown below.

**Theorem 5.18.** For uncountably many Whitehead-type manifolds $W^n$ of dimension $n \geq 3$, the manifolds $W^n \times N^k$ are not $\infty$-compressible for any closed simply connected $k$-manifold $N^k$. □

Proof. It is sufficient to consider the case of the Whitehead-type manifolds since the pairs of groups appearing in the product exhaustions are the same as in this case.

We start with the 3-dimensional case and take for $W^3$ the classical Whitehead manifold. Recall that $W^3$ is an increasing union of solid tori $T_i$, with $T_i$ embedded in $T_{i+1}$ as a neighborhood of a Whitehead link.

We will first show that the pair $(T_{i+1}, T_i)$ is not 2-weakly compressible, and hence not stably compressible. We then extend this argument to show that any pair of the form $(T_{i+n}, T_i)$ is not $(n+1)$-weakly compressible, and hence not stably compressible. By Proposition 4.25, it follows that $W^3 \times N^k$ is not $\infty$-compressible, and hence not w.g.s.c.

Let $T$ and $T'$ be as usual, let $M^3 = T' - \mathrm{int}(T)$, and fix a base point $p \in \partial T$. Then $C = G_0 = \pi_1(M^3)$ in our usual notation. Note that $\ker(\varphi)$ is normally generated by the meridian of $T$ and hence $\pi_1(M^3)/\mathcal{N}(\ker(\varphi), C) = \pi_1(T') = \mathbb{Z}$. Thus $G_1 = \mathcal{N}(\ker(\varphi), C)$ consists of the homologically trivial elements in $\pi_1(M^3)$.

Consider now the cover $\widetilde{M^3}$ of $M^3$ with fundamental group $G_1$. This is $\mathbb{R}^3$ with the neighborhood of an infinite component link, say indexed by the integers, deleted. Further, each pair of adjacent components has linking number 1. Pick a lift $p'$ of the base point $p$, which we use for all the fundamental groups we consider.

In this cover, $\psi(A)$ is the image of the bounding torus $T$ of the component of this link containing $p'$, and $\psi(\ker(\varphi))$ is generated by the meridian of this component. Thus, $G_1/\mathcal{N}(\psi(\ker(\varphi)), G_1)$ is the fundamental group of $\widetilde{M^3} \cup_T D^2 \times S^1$, that is, of $\widetilde{M^3}$ with a solid torus glued along $T$ to kill the meridian. But, because of the linking, the longitude $\lambda \subset T$ is not trivial in this group, that is, $\lambda \notin G_2 = \mathcal{N}(\psi(\ker(\varphi)), G_1)$. Since $\lambda \in \psi(A)$, we see that the Whitehead link is not 2-compressible.

We will now consider a pair $(T', T)$ in some refinement of the given exhaustion. This is homeomorphic to a pair of the form $(T_n, T_1)$ for some $n$. As before, pick a base point $p \in \partial T$.

Let $M_i = T_i - T_1$ and let $M = M_n$. Note that $M_i \subset M_{i+1}$. In terms of earlier notation, $M_n = T' - T$ and $\pi_1(M) = C = G_0$. Further, $\ker(\varphi)$ is normally generated by the meridian of $T_1$.

We have a sequence of subgroups $G_k \subset G_0 = \pi_1(M)$ and hence covers $M^j$ of $M$ corresponding to these subgroups. Pick lifts $p^k$ of the base point $p$ to these covers. Then $\mathcal{N}(\psi(\ker(\varphi)), G_k)$ is generated by the meridian of the component of the inverse image of $\partial T_1$ containing $p^k$. As the meridian is in $\ker(\varphi)$, and each $G_i$ is the normal subgroup generated by $\ker(\varphi)$ in $G_{i-1}$, we see inductively that the lift of the meridian is a closed curve in $M^k$ so that the previous sentence makes sense.

Let $N^i$ be the result of gluing a solid torus or cylinder to $M^i$ along the component containing $p^k$ so that the meridian is killed. Then by the above, $G_k/\mathcal{N}(\psi(\ker(\varphi)), G_k) = \pi_1(N^i)$.

We will prove by induction the following lemma.

**Lemma 5.19.** The solid torus $T_{n-k}$ lifts to $N^k$, or equivalently, $M_{n-k}$ lifts to $M^k$. Furthermore, the longitude of the lift of $\partial T_{n-k}$ is a nontrivial element in $\pi_1(N^k)$. □

Proof. The case when $k = 1$ is the above special case. Suppose now that the statement is true for $k$.

As the longitude of the lift of $\partial T_{n-k}$ is a nontrivial element in $\pi_1(N^k)$, in $M^{k+1}$ the inverse image of $M_{n-k}$ is a cylinder with a sequence of linked lifts of $M_{n-(k+1)}$ deleted. Thus, $M_{n-(k+1)}$ lifts to $M^{k+1}$, and its longitude is linked with other lifts. It follows that the longitude of the lift of $\partial T_{n-(k+1)}$ is nontrivial in $\pi_1(N^{k+1})$. ■

As a subgroup of $G_{n-1}$, $\psi(A)$ is the image of the lift of $\partial T_1$ containing the base point. As in the special case, as the longitude of this torus is a nontrivial element of $\pi_1(N^{n-1})$, it follows that $\psi(A) \not\subset G^n$. Thus $(T_n, T_1)$ is not $n$-compressible.

This ends the proof of the claim for the Whitehead manifold. Observe however that the same proof works for uncountably many similar manifolds—namely, we may embed $T_i$ in $T_{i+1}$ as a link similar to the Whitehead link that winds around the solid torus several times.

We will now use a recurrence on the dimension and the results of the previous section in order to settle the higher-dimensional situation. Consider for simplicity $n = 4$ and a Whitehead-type manifold $W^4$ which is the ascending union of solid tori as in Wright's construction. We use the notations from Lemma 5.16. Then the pair of tori $(T^4, Q^4)$ is constructed out of the two 3-dimensional Whitehead links $(T_1^3, T_0^3)$ and $(D^2 \times S^1, L^3)$.

As above, $\ker(\varphi)$ is normally generated by the meridian and $G_1$ consists of homologically trivial elements of $\pi_1(T^4 - \mathrm{int}(Q^4))$, by using the van Kampen theorem and the fact that $\pi_1(T^4)$ is abelian. The cover $\widetilde{M^4}$ of $M^4 = T^4 - \mathrm{int}(Q^4)$ with fundamental group $G_1$ is $\mathbb{R}^4$ with a thick infinite link deleted. There is an obvious $\mathbb{Z}^2$ action on the components of this link, and so we can label the boundary tori as $T_{i,j}$ for integer numbers $i, j$.

Let $\lambda$ be the longitude curve having the parameters $(n, k)$ on the torus $T_{0,0}$. Then one can compute the linking numbers $\mathrm{lk}(\lambda, T_{0,1}) = k$ and $\mathrm{lk}(\lambda, T_{1,0}) = n$. This follows because both links used in the construction were the standard Whitehead link. Variations which yield nonzero linking numbers are also convenient for our purposes. Consequently, for nonzero $n, k$, we obtained an element which is nontrivial in $G_2$, hence the pair of solid tori is not 2-compressible. A similar argument goes through the higher compressibility as well. Using suitable variations in choosing the links and mixing the pairs of solid tori as in the previous section yields uncountably many examples as in the theorem. ■

Remark 5.20. It follows that $W^3 \times D^k$ is not w.g.s.c. using the criterion from [7, 8]. However, the previous theorem is more precise regarding the failure of g.s.c. for these product manifolds.

## 6 The proper homotopy invariance of the w.g.s.c.

### 6.1 Dehn exhaustibility

We study in this section to what extent the w.g.s.c. is a proper homotopy invariant.

Definition 6.1. A polyhedron $M$ is *(proper) homotopically dominated* by the polyhedron $X$ if there exists a map $f : M \to X$ such that the mapping cylinder $Z_f$ (properly) retracts on $M$.

Remark 6.2. A proper homotopy equivalence is the simplest example of a proper homotopically domination.

The main result of this section is the following theorem.

**Theorem 6.3.** For $n \neq 4$, a noncompact $n$-manifold is w.g.s.c. if and only if it is proper homotopically dominated by a w.g.s.c. polyhedron. □

Remark 6.4. It seems that the result does not hold, as stated, for $n = 4$ (see also Section 6.2).

Proof of Theorem 6.3. The main ingredient of the proof is the following notion, weaker than the w.g.s.c., introduced by Poénaru.

Definition 6.5. The simply connected $n$-manifold $W^n$ is *Dehn exhaustible* if, for any compact $K \subset W^n$, there exist some simply connected compact polyhedron $L$ and a commutative diagram

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[dr, "i"'] & L \arrow[d, "g"] \\ & W^n, \end{tikzcd} \tag{6.1}$$

where $i$ is the inclusion, $f$ is an embedding, $g$ is an immersion, and $f(K) \cap M_2(g) = \varnothing$. Here $M_2(g)$ is the set of double points, namely, $M_2(g) = \{x \in L;\ \sharp g^{-1}(g(x)) \geq 2\} \subset L$. If $n = 3$, then it is required that the map $g$ be a *generic immersion*, which means here that it has no triple points.

The first step is to establish the following proposition.

**Proposition 6.6.** An open simply connected manifold which is proper homotopically dominated by a w.g.s.c. polyhedron is Dehn exhaustible. □

Proof. The proof given in [7] for the 3-dimensional statement extends without any essential modification, and we skip the details. ■

Remark 6.7. Poénaru proved a Dehn-type lemma (see [23, pages 333–339]) which states that a Dehn exhaustible 3-manifold is w.g.s.c. This settles the dimension 3 case.

**Lemma 6.8.** If the open simply connected $n$-manifold $W^n$ is Dehn exhaustible and $n \geq 5$, then it is w.g.s.c. □

Proof. Consider a connected compact submanifold $K \subset W^n$. Assume that there exists a compact polyhedron $M^n$ with $\pi_1(M^n) = 0$ and an immersion $F$:

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[dr, "i"'] & L \arrow[d, "F"] \\ & W^n \end{tikzcd} \tag{6.2}$$

such that $M_2(F) \cap K = \varnothing$.

**Lemma 6.9.** One can suppose that $M^n$ is a manifold. □

Proof. The polyhedron $M^n$ is endowed with an immersion $F$ into the manifold $W^n$. Among all abstract regular neighborhoods (i.e., thickenings) of $M^n$, there is an $n$-dimensional one $U(M^n, F)$, which is called the regular neighborhood determined by the immersion, such that the following conditions are fulfilled:

(1) $F : M^n \to W^n$ extends to an immersion $\widetilde{F} : U(M^n, F) \to W^n$;

(2) the image of $\widetilde{F}(U(M^n, F)) \subset W^n$ is the regular neighborhood of the polyhedron $F(M^n)$ in $W^n$.

The construction of the PL regular neighborhood determined by an immersion of polyhedra is given in [14]. The authors were building on the case of an immersion of manifolds, considered previously in [11]. Moreover, if one replaces $M^n$ by the manifold $U(M^n, F)$ and $F$ by $\widetilde{F}$, we are in the conditions required by the Dehn-type lemma. ■

Consider now a handlebody decomposition of $M^n - f(K)$ and let $N_2^n$ be the union of $f(K)$ with the handles of index 1 and 2. Then $\pi_1(N_2^n) = 0$. Let $\delta_j^2, \delta_j^1$ be the cores of these extra 1- and 2-handles. By using a small homotopy of $F$, one can replace $F(\delta_j^2) \subset W^n$ by some embedded 2-disks $d_j^2 \subset W^n$ with the same boundary. Also by general position, these 2-disks can be chosen to have disjoint interiors. Both assertions follow from the assumptions $n \geq 5$ and $M_2(F) \cap f(K) = \varnothing$. This implies that the restriction of the new map $F'$, obtained by perturbing $F$, to $\delta_j^2$ (and $\delta_j^1$) is an embedding into $W^n - K$. Using the uniqueness of the regular neighborhood, it follows that $F'$ can be chosen to be an embedding on $N_2^n$. In particular, $K$ is engulfed in the 1-connected compact $F'(N_2^n)$. ■

### 6.2 Dehn exhaustibility and end-compressibility in dimension 4

**Proposition 6.10.** An open 4-manifold is end-compressible if and only if it is Dehn exhaustible. □

Proof. We have to reconsider the proof of Proposition 4.28. Everything works as above except that the disks $\delta_j^2$ cannot be anymore embedded, but only (generically) immersed. They may have finitely many double points in their interior. Then the manifold $M^4$ obtained by adding 2-handles along $\gamma_j$ has a generic immersion $F : M \to T'$, whose double points $M_2(F)$ are outside of $T$. This implies that $W^4$ is Dehn exhaustible.

Conversely, assume that $W^4$ is Dehn exhaustible. Let $K^4$ be a compact submanifold of $W^4$ and let $M^4$ be the immersible simply connected polyhedron provided by the Dehn exhaustibility property. Lemma 6.9 allow us to assume that $M^4$ and $F(M^4)$ are 4-manifolds. Consider now the proof of the first claim from Theorem 6.3. It is sufficient to consider the case when $M^4 = M_2^4$, that is, $M^4$ is obtained from $K^4$ by adding 1- and 2-handles. If $\Gamma \subset \pi_1(\partial M_1^4)$ is the normal subgroup generated by the attaching curves of the 2-handles of $M^4$, then the same argument yields

$$\Gamma \subset \ker\left(\pi_1\left(\partial M_1^4\right) \longrightarrow \pi_1\left(M^4 - \operatorname{int}\left(K^4\right)\right)\right). \tag{6.3}$$

Since $F$ is a generic immersion, we can suppose that $F$ is an embedding of the cores of the 1-handles, and so $F|_{M_1^4}$ is an embedding.

We have $F(M^4 - \operatorname{int}(K^4)) \subset F(M^4) - \operatorname{int}(K^4)$ because the double points of $F$ are outside of $K^4$. Now the homomorphism induced by $F$ on the left-hand side of the diagram

$$\begin{tikzcd}
\pi_1\left(\partial M_1^4\right) \arrow[r] \arrow[d, "F"] & \pi_1\left(M^4 - \operatorname{int}\left(K^4\right)\right) \arrow[d, "F"] \\
\pi_1\left(\partial F\left(M_1^4\right)\right) \arrow[r] & \pi_1\left(F\left(M^4\right) - \operatorname{int}\left(K^4\right)\right)
\end{tikzcd} \tag{6.4}$$

is an isomorphism, and we derive that

$$F(\Gamma) \subset \ker\left(\pi_1\left(\partial F\left(M_1^4\right)\right) \longrightarrow \pi_1\left(F\left(M^4\right) - \operatorname{int}\left(K^4\right)\right)\right). \tag{6.5}$$

Meanwhile, $F(\Gamma)$ surjects onto $\pi_1(F(M_1^4))$ under the map $\pi_1(\partial F(M_1^4)) \to \pi_1(F(M_1^4))$. But $F(M_1^4)$ is homeomorphic to $M_1^4$, hence it is obtained from $K^4$ by adding 1-handles. This shows that the pair $(F(M^4), K^4)$ is stably compressible, from which one obtains the end-compressibility of $W^4$ as in the proof of Theorem 4.22. ■

Remark 6.11. If the open 4-manifold $W^4$ is Dehn exhaustible, then $W^4 \times [0, 1]$ is also Dehn exhaustible, hence w.g.s.c. Therefore, an example of an open 4-manifold $W^4$ which is end-compressible but which is not w.g.s.c. will show that the result of Theorem 6.3 cannot be extended to dimension 4, as stated. Such examples are very likely to exist as the Dehn lemma is known to fail in dimension 4 (by S. Akbulut's examples).

### 6.3 Proper homotopy invariance of the end-compressibility

It would be interesting to have a soft version of Theorem 6.3 for the end-compressibility situation. Notice that the definition of the end-compressibility extends word by word to noncompact polyhedra. One uses instead of the boundary of manifolds the frontier of a polyhedron.

Remark 6.12. One expects that the following will be true. If a polyhedron $M$ is proper homotopically dominated by an end-compressible polyhedron $X$, then $M$ is also end-compressible. The only ingredient lacking for the complete proof is the analogue of Remark 4.24 for polyhedra: if one exhaustion is stably compressible, then all exhaustions have stably compressible exhaustions.

Along the same lines we have the following proposition.

**Proposition 6.13.** If there is a degree-one map $f : X^n \to M^n$ between one-ended manifolds of the same dimension, then if $X^n$ is end-compressible, so is $M^n$. □

Proof. We use the fact that degree-one maps are surjective on fundamental group. Given an exhaustion $\{L_j\}$ of $M^n$, pull it back to $\{f^{-1}(L_j)\}$ of $X^n$. Notice that $\partial f^{-1}(L_j) = f^{-1}(\partial L_j)$, where $\partial$ stands for the frontier.

One needs then the following approximation-by-manifolds result. Given two $n$-complexes $K_1 \subset \text{int}(K_2) \subset \mathbb{R}^n$, there exist regular neighborhoods $K_j^\varepsilon \subset \mathbb{R}^n$ such that $K_j \subset \text{int}(K_j^\varepsilon)$, $K_j$ is homotopy equivalent to $K_j^\varepsilon$, $K_2 - \text{int}(K_1)$ is homotopy equivalent to $K_2^\varepsilon - K_1^\varepsilon$, and, moreover, $\partial K_1^\varepsilon$ is homotopy equivalent to $\partial K_1$. This uses essentially the fact that $K_j$ are of codimension zero in $\mathbb{R}^n$.

Now, the hypothesis applied to the approximating exhaustion consisting of submanifolds implies the existence of a stably compressible refinement of $\{f^{-1}(L_j)\}$. Since degree-one maps are surjective on fundamental groups, the lemma below permits to descend to $M^n$.

**Lemma 6.14.** Suppose that the triple $(A, B, C)$ of groups surjects onto $(A', B', C')$, that is, there exist three surjections inducing pairwise commuting diagrams. Then if $(A, B, C)$ is strongly (or stably) compressible, then so is $(A', B', C')$. □

The proof is straightforward.

The only subtlety above is to make sure that the inverse image of boundary components is connected (or else we can connect them up in the one-ended case). ■

Remark 6.15. In the many-ended case, we need to say that we have a degree-one map between each pair of ends (not 2 ends mapping to one, with one of them having degree 2 and the other $-1$). This holds in particular for a proper map that has degree one and is injective on ends.

## 7 The g.s.c. for 4-manifolds

### 7.1 The w.g.s.c. versus g.s.c.

Definition 7.1. A *geometric Poénaru-Mazur-type manifold* $M^4$ is a compact simply connected 4-manifold satisfying the following conditions:

(1) $H_2(M^4) = 0$;

(2) the boundary $\partial M^4$ is connected and $\pi_1$-dominates a virtually geometric 3-manifold group, that is, there exists a surjective homomorphism

$$\pi_1\left(\partial M^4\right) \longrightarrow \pi_1\left(N^3\right), \tag{7.1}$$

onto the (nontrivial) fundamental group of a virtually geometric 3-manifold $N^3$.

**Proposition 7.2.** The interior $\mathrm{int}(M^4)$ of a geometric Poénaru-Mazur-type manifold $M^4$ does not have a proper handlebody decomposition without 1-handles with the boundary of a cofinal subset of the intermediate manifolds obtained on a finite number of handle additions being homology spheres. □

### 7.2 Casson's proof of Proposition 7.2

The main ingredient is the following proposition extending an unpublished result of A. Casson.

**Proposition 7.3.** Consider the 4-dimensional (compact) cobordism $(W^4, M^3, N^3)$ such that $(W^4, M^3)$ is 1-connected. Assume moreover that the following conditions are satisfied:

(1) $H_2(W^4, M^3; \mathbb{Q}) = 0$, both $M^3$ and $N^3$ are connected;

(2) $\pi_1(N^3)$ is a group which $\pi_1$-dominates a virtually geometric nontrivial 3-manifold group; let $K$ be the kernel of this epimorphism;

(3) $b_1(W^4) \leq b_1(N^3)$, where $b_1$ denotes the first Betti number;

(4) the map $\pi_1(N^3) \to \pi_1(W^4)$ induced by the inclusion $N^3 \hookrightarrow W^4$ has kernel strictly bigger than the subgroup $K$. In particular, this is true if this map is trivial.

Then any handlebody decomposition of $W^4$ from $M^3$ has 1-handles, that is, the pair $(W^4, M^3)$ is not g.s.c. □

Remark 7.4. A necessary condition for the g.s.c. of $(W^4, M^3)$ is that the map $\pi_1(M^3) \to \pi_1(W^4)$, induced by the inclusion $M^3 \hookrightarrow W^4$, be onto. In fact, adding 2-handles amounts to introducing new relations to the fundamental group of the boundary, whereas the latter is not affected by higher-dimensional handle additions.

Casson's result was based on partial positive solutions to the Kervaire conjecture (Conjecture 2.16). One proves that certain 4-manifolds $(N, \partial N)$ have no handle decompositions without 1-handles by showing that if they did, then $\pi_1(\partial N)$ would violate the Kervaire conjecture. Casson's argument works to the extent that the Kervaire conjecture is known to be true. Casson originally applied it using a theorem of Gerstenhaber and Rothaus [9], which said that the Kervaire conjecture holds for subgroups of a compact Lie group. Subsequently, Rothaus [29] showed that the conjecture in fact holds for residually finite groups. Since residual finiteness for all 3-manifold groups is implied by the geometrization conjecture, Casson's argument works in particular for all manifolds satisfying the geometrization conjecture. A simple argument (Proposition 7.5) extends further the class of groups for which the Kervaire conjecture is known.

**Proposition 7.5.** If some nontrivial quotient $Q$ of a group $G$ satisfies the Kervaire conjecture, then so does $G$. In particular, if a finitely generated group $G$ has a proper finite-index subgroup, then $G$ satisfies the Kervaire conjecture (since finite groups satisfy the Kervaire conjecture by [9]). □

Proof. Let $\phi : G \twoheadrightarrow Q$ be the quotient map. Assume that $Q$ satisfies the Kervaire conjecture. Suppose that $G$ violates the Kervaire conjecture. Then we have generators $\alpha_1, \ldots, \alpha_n$ and relations such that $G * \langle \alpha_1, \ldots, \alpha_n \rangle / \langle\langle r_1, \ldots, r_n \rangle\rangle$ is the trivial group. Let $\phi^* : G * \langle \alpha_1, \ldots, \alpha_n \rangle \to Q * \langle \overline{\alpha_1}, \ldots, \overline{\alpha_n} \rangle$ be the map extending $\phi$ by mapping $\alpha_i$ to $\overline{\alpha_i}$. This is clearly a surjection, and induces a surjective map $\overline{\phi} : G * \langle \alpha_1, \dots, \alpha_n \rangle / \langle\langle r_1, \dots, r_n \rangle\rangle \twoheadrightarrow Q * \langle \overline{\alpha_1}, \dots, \overline{\alpha_n} \rangle / \langle\langle \phi^*(r_1), \dots, \phi^*(r_n) \rangle\rangle$. But since the domain of the surjection $\overline{\phi}$ is trivial, so is the codomain. But this means that $Q * \langle \overline{\alpha_1}, \dots, \overline{\alpha_n} \rangle / \langle\langle \phi^*(r_1), \dots, \phi^*(r_n) \rangle\rangle$ is trivial, and so $Q$ violates the Kervaire conjecture, a contradiction. ■

Proof of Proposition 7.3. Suppose that

$$W^4 = M^3 \times [0,1] \bigcup_k 2\text{-handles} \bigcup_r 3\text{-handles} \tag{7.2}$$

(with some 0-handle or 4-handle added if one boundary component is empty). It is well known (see [30]) that the homology groups $H_*(W^4, M^3)$ are the same as those of a differential complex $C_*$, whose component $C_j$ is the free module generated by the j-handles. Therefore, this complex has the form

$$0 \longrightarrow \mathbb{Z}^r \longrightarrow \mathbb{Z}^k \longrightarrow 0. \tag{7.3}$$

Thus $H_2(W^4, M^3; \mathbb{Q}) = 0$ implies that $k \leq r$ holds.

Now, consider that the handlebody decomposition is turned upside down:

$$W^4 = N^3 \times [0,1] \bigcup_r 1\text{-handles} \bigcup_k 2\text{-handles} \tag{7.4}$$

(plus possibly one 0-handle or 4-handle if the respective boundary component is empty).

By the van Kampen theorem, it follows that $\pi_1(W^4)$ is obtained from $\pi_1(N^3) = \pi_1(N^3 \times [0,1])$ by adding one generator for each 1-handle and one relation for each 2-handle. Therefore,

$$\pi_1\left(W^4\right) = \pi_1\left(N^3\right) * \mathbf{F}(r)/W(k), \tag{7.5}$$

where $\mathbf{F}(r)$ is the free group on $r$ generators $x_1, \dots, x_r$ and $W(k)$ is a normal subgroup of the free product generated also by $k$ words $Y_1, \dots, Y_k$.

Consider a virtually geometric 3-manifold $L^3$ such that $\pi_1(M^3) \to \pi_1(L^3)$ is surjective. If $L^3$ is a geometric 3-manifold, then its fundamental group is residually finite (see, e.g., [32, Theorem 3.3, page 364]). Let $d_{ij}$ be the degree of the letter $x_j$ in the word representing $Y_i$. The result of Rothaus (see [29, Theorem 18, page 611]) states that for any locally residually finite group G and choice of words $Y_i$ such that $\mathbf{d} = (d_{ij})_{i,j}$ is of (maximal) rank k, the natural morphism $G \to G * \mathbf{F}(r)/W(k)$ is an injection. We have therefore a commutative diagram

$$\begin{tikzcd} \pi_1(M^3) \arrow[r] \arrow[d] & \pi_1(W^4) \arrow[d] \\ \pi_1(L^3) \arrow[r, hook] & \pi_1(L^3) * \mathbf{F}(r)/W(k) \end{tikzcd} \tag{7.6}$$

whose vertical arrows are surjections. The kernel of the map induced by inclusion, $\pi_1(M^3) \to \pi_1(W^4)$, is contained in K. This contradicts our hypothesis.

On the other hand, if the rank of $\mathbf{d}$ is not maximal, then by considering the abelianizations, one derives $H_1(G * \mathbf{F}(r)/W(k)) \subset H_1(G) \oplus \mathbb{Z}$, hence $b_1(W^4) \geq b_1(N^3) + 1$, which is also false. ■

**Corollary 7.6.** Consider a 4-manifold $W^4$ which is compact connected simply connected with nonsimply connected boundary $\partial M$. If the boundary is (virtually) geometric and $H_2(W^4) = 0$, then $W^4$ is not g.s.c. □

Proof of Proposition 7.2. Assume now that $\mathrm{int}(M^4)$ admitted a proper handlebody decomposition without 1-handles. One identifies $\mathrm{int}(M^4)$ with $M^4 \cup_{\partial M \cong \partial M \times \{0\}} \partial M \times [0, 1)$. We can truncate the handle decomposition at a finite stage in order to obtain a manifold $Q^4$ such that $\partial Q^4 \subset \partial M^4 \times (0, 1)$ because the decomposition is proper. We can suppose that $\partial Q^4$ is connected since $\mathrm{int}(M^4)$ has one end. Then $Q^4$ is g.s.c., hence $\pi_1(Q^4) = 0$.

By hypothesis, we can choose $\partial Q^4$ to be a homology sphere. Then $\partial Q^4$ separates the cylinder $\partial M^4 \times [0, 1]$ into two manifolds with boundary which, by Mayer-Vietoris, have the homology of $S^3$. This implies that $H_2(Q^4) = 0$ (again by Mayer-Vietoris).

We now consider the map $f : \partial Q^4 \hookrightarrow \partial M^4 \times [0, 1] \to \partial M^4$, the composition of the inclusion with the obvious projection.

**Lemma 7.7.** The map f has degree one, hence induces a surjection on the fundamental groups. □

Proof. The 3-manifold $\partial Q^4$ separates the two components of the boundary. In particular, the generic arc joining $\partial M^4 \times \{0\}$ to $\partial M^4 \times \{1\}$ intersects transversally $\partial Q^4$ in a number of points, which counted with the sign sum up to 1 (or $-1$). If properly interpreted, this is the same as claiming the degree of f is one.

It is well known that a degree-one map between orientable 3-manifolds induces a surjective map on the fundamental group (more generally, the image of the homomorphism induced by a degree $d$ is a subgroup whose index is bounded by $d$). ■

This shows that $\pi_1(\partial Q^4) \to \pi_1(\partial M^4)$ is surjective. On the other hand, $\pi_1(\partial M^4)$ surjects onto a nontrivial residually finite group. Since $\pi_1(\partial Q^4) \to \pi_1(Q^4) = 1$ is the trivial map, the argument we used previously (from Rothaus' theorem) gives us a contradiction. This settles our claim. ■

## 8 Handle decompositions without 1-handles in dimension 4

### 8.1 Open tame 4-manifolds

Definition 8.1. An exhaustion of a 4-manifold is *g.s.c.* if it corresponds to a proper sequence of handle additions with no 1-handles. Alternatively, there exists a proper Morse function, which will be referred to as *time*, with words like *past* and *future* having obvious meanings, with no critical points of index 1. The inverse images of regular points are 3-manifolds, which are referred to as the *manifolds at that time*.

We assume henceforth that we have a g.s.c. handle decomposition of the interior $\mathrm{int}(W^4)$ of $(W^4, \partial W^4)$, a compact 4-manifold with boundary, a homology 3-sphere, and $H_2(W^4) = 0$.

Now let $(K^4_i, \partial K^4_i)$, $i \in \mathbb{N}$, denote the 4-manifolds obtained by successively attaching handles to the 0-handle $(B^4, S^3)$, that is, if $t : (W^4, \partial W^4) \to \mathbb{R}$ is the Morse function time, then $(K^4_i, \partial K^4_i) = t^{-1}((\infty, a_i])$, with $a_i$ being points lying between pairs of critical values of the Morse function.

**Lemma 8.2.** The 3-manifold $\partial K^4_{i+1}$ is obtained from $\partial K^4_i$ by one of the following:

(i) a 0-frame surgery about a homologically trivial knot in $\partial K^4_i$;

(ii) cutting along a nonseparating 2-sphere in $\partial K^4_i$ and capping off the result by attaching a 3-ball.

These correspond respectively to attaching 2-handles and 3-handles to $(K^4_i, \partial K^4_i)$. □

Proof. Since attaching 2-handles and 3-handles corresponds to surgery and cutting along 2-spheres, respectively, we merely have to show that the surgery is 0-frame about a homologically trivial curve and the spheres along which one cuts are nonseparating.

First, note that the absence of 1-handles implies that $H_1(K^4_i) = 0 = \pi_1(K^4_i)$, for all $i$. Further, each $\partial K^4_i$ is connected because $\mathrm{int}(W^4)$ has one end. Thus the 2-spheres along which any $\partial K^4_i$ is split have to be nonseparating.

Using Mayer-Vietoris, the fact that $H_2(W^4) = 0$, and the long exact sequence in homology, we derive that $H_2(K_i^4) = H_2(\partial K_i^4)$. Also adding a 3-handle decreases the rank of $H_2(\partial K_i^4)$ by one, hence every surgery increases the rank of $H_2(\partial K_i^4)$ by one unit. But this means that the surgery must be a 0-frame surgery about a homologically trivial curve. ■

For $i$ large enough, $\partial K_i^4$ lies in a collar $\partial W^4 \times [0, \infty)$, hence we have a map $f_i : \partial K_i^4 \to \partial W^4$ which is the composition of the inclusion with the projection. By Lemma 7.7, the maps $f_i$ are of degree one and induce surjections $\phi_i : \pi_1(\partial K_i^4) \to \pi_1(\partial W^4)$. Here and henceforth we always assume that the index $i$ is large enough so that $f_i$ is defined.

**Lemma 8.3.** The homotopy class of a curve along which surgery is performed is in the kernel of $\phi_i : \pi_1(\partial K_i^4) \to \pi_1(\partial W^4)$. □

Proof. If a surgery is performed along a curve $\gamma$, this means that a 2-handle is attached along the curve in the 4-manifold $W^4$. Hence, $\gamma$ bounds a disk in $\partial W^4 \times [0, \infty)$, which projects to a disk bounded by $f_i(\gamma)$ in $\partial W^4$. ■

Remark 8.4. The maps $\phi_i$ and $\phi_{i+1}$ are related in a natural way. To define the map $\phi_{i+1}$, take a generic curve $\gamma$ representing any given element of $\pi_1(\partial K_{i+1}^4)$. If $\partial K_{i+1}^4$ is obtained from $\partial K_i^4$ by splitting along a sphere, then $\gamma$ is a curve in $\partial K_i^4$, and so we can simply take its image. On the other hand, if a surgery was performed, then we may assume that $\gamma$ lies off the solid torus that has been attached, and hence lies in $\partial K_i^4$, so we can take its image as before. This map is well defined by Lemma 8.3.

Definition 8.5. A curve $\gamma' \subset \partial K_i^4$ is a *descendant* of the surgery curve $\gamma \subset \partial K_i^4$ if it is homotopic to it in $\partial K_i^4$ (though not in general homotopic to $\gamma$ after the surgery). A curve $\gamma \subset \partial K_i^4$ is said to *persist till* $\partial K_{i+n}^4$ if some descendant of $\gamma$ persists, that is, we can homotope $\gamma$ in $\partial K_i^4$ so that it is disjoint from all the future 2-spheres on which 3-handles are attached while passing from $\partial K_i^4 = M_i^3$ to $\partial K_{i+n}^4 = M_{i+n}^3$.

Definition 8.6. A curve $\gamma \subset \partial K_i^4$ is said to *die by* $\partial K_{i+n}^4$ if it is homotopically trivial in the 4-manifold obtained by attaching 2-handles to $K_i^4$ along the curves in $\partial K_i^4$ where surgeries are performed in the process of passing to $\partial K_{i+n}^4$, or equivalently, $\gamma$ is trivial in the group obtained by adding relations to $\pi_1(M_i^3)$ corresponding to curves along which the surgery is performed.

We prove now a key property of the sequence $\partial K_i^4$.

**Lemma 8.7.** For each $i$, there is a uniform $n = n(i)$ such that any curve $\gamma \subset \partial K_i^4$, $\gamma \in \ker \phi_i$, that persists till $\partial K_{i+n}^4$ dies by $\partial K_{i+n}^4$. □

Proof. We can find $x \in [0,\infty)$ so that $\partial W^4 \times \{x\}$ is entirely after $\partial K_i^4$, and $n_1$ so that $\partial W^4 \times \{x\} \subset K_{i+n_1^4}$, because the handlebody decomposition is proper. We then define $n$ by repeating this process once, that is, $\partial W^4 \times \{x_1+\varepsilon\} \subset K^4_{i+n}$, for some $x_1+\varepsilon > x_1 > x$ for which $\partial W^4 \times \{x_1\}$ is entirely after $\partial K^4_{i+n_1}$. Consider $\gamma \in \ker \phi_i$ which persists till $\partial K^4_{i+n_1}$. This means that there is an annulus properly embedded in $K^4_{i+n} - \mathrm{int}(K_i^4)$, whose boundary curves are $\gamma$ and $\tilde{\gamma} \in \partial K^4_{i+n} \subset \partial W^4 \times [x_1, x_1+\varepsilon)$. Since $\gamma \in \ker \phi_i$, it bounds a disc in $\partial W^4 \times [x_1,\infty)$. This disc together with the above annulus ensure that $\gamma$ dies by $\partial K^4_{i+n}$, as they bound together a disc entirely in $K^4_{i+n} - \mathrm{int}(K_i^4)$, and 3-handles do not affect the fundamental group. ■

## 8.2 The structure theorem

Suppose henceforth that we have a sequence of connected 3-manifolds $M_i^3 \subset \partial W^4 \times [0,\infty)$ and associated maps onto $f_i : M_i^3 \to \partial W^4$ that satisfies the properties of $\partial K_i$ stated above. Specifically one requires that

(i) the maps $f_i : M_i^3 \to \partial W^4$ be of degree one, hence inducing surjection $\phi_i : \pi_1(M_i^3) \to \pi_1(\partial W^4)$;

(ii) $M_{i+1}$ be obtained from $M_i^3$ either by a 0-frame surgery along a homologically trivial knot in $M_i^3$, or else by cutting along a nonseparating 2-sphere in $M_i^3$;

(iii) the surgery curves in $M_i^3$ belong to $\ker \phi_i$;

(iv) the maps $\phi_i$ and $\phi_{i+1}$ be related as in Remark 8.4;

(v) for any $i$, there exists some $n = n(i)$ such that any curve in $M_i^3$ which persists till $M^3_{i+n}$ dies by $M^3_{i+n}$.

We show in this section that, after possibly changing the order of attaching handles, any handle decomposition without 1-handles is of a particular form.

We first describe a procedure for attempting to construct a handle decomposition for $\mathrm{int}(W^4)$ starting with a partial handle decomposition, with boundary $M_i^3$. In general, $M_i^3$ has nontrivial homology. It follows readily from the proof of Lemma 8.2 that $H_1(M_i^3)$ is a torsion-free abelian group. The only way by which we can remove homology is by splitting along spheres. To this end, we take a collection of surfaces representing the homology, perform surgeries along curves in these surfaces so that they compress down to spheres, and then split along these spheres. By doing the surgeries, we have created new homology, and hence have to take new surfaces representing this homology and continue this procedure. In addition to this, we may need to perform other surgeries to get rid of the *homologically trivial portion* of the kernel of $\phi_i : \pi_1(M_i^3) \to \pi_1(\partial W^4)$.

The above construction may meet obstructions since the surgeries have to be performed about curves that are homologically trivial as well as lie in the kernel of $\phi_i$, hence it may not be always possible to perform enough of them to compress the surfaces to spheres. The construction terminates at some finite stage if at that stage all the homology is represented by spheres and no surgery off these surfaces is necessary.

**Theorem 8.8.** After possibly changing the order of attaching handles, any handle decomposition without 1-handles may be described as follows. There exists a collection of surfaces $F_j^2(i)$, with disjoint simple closed curves $l_{j,k} \subset F_j^2(i)$ and a generic immersion $\psi_i : \cup_j F_j^2(i) \to M_i^3$ such that the following conditions hold.

(i) The surfaces represent the homology of $M_i^3$, that is, $\psi_i$ induces a surjection $\psi_i : H_2(\cup_j F_j^2(i)) \to H_2(M_i^3)$.

(ii) The immersion $\psi_i$ has only ordinary double points and the restriction to each individual surface $F_j(i)$ is an embedding. The double curves of $\psi_i$ are among the curves $l_{j,k}$. Their images $\psi_i(l_{j,k})$ are called seams.

(iii) When compressed along the seams (i.e., by adding 2-handles along them), the surfaces $\psi_i(F_j^2(i))$ become unions of spheres.

(iv) The seams are homologically trivial curves in $M_i^3$ and lie in the kernel of $\phi_i$.

(v) The pullbacks (see the definition below) of the surfaces $\psi_m(F_k^2(m)) \subset M_m^3$ for $m > i$, which are surfaces with boundary in $M_i^3$, can only intersect the $F_j(i)$'s either transversely at the seams or by having some boundary components along the seams.

First, 2-handles are attached along all the seams of $M_i^3$, and possibly also along some curves that are completely off the surfaces $F_j^2(i)$ in $M_i^3$ and have no intersection with any future surface $F_k(m)$, $m > i$. Then 3-handles are attached along the 2-spheres obtained by compressing the surfaces $\psi_i(F_j^2(i))$. Iterating this procedure gives us the handle decomposition. □

We will see that once we construct the surfaces, all of the properties will follow automatically.

Let $F^2 \subset M_{i+n}^3$ be an embedded surface. We let $M_i^3 = \partial K_i^4$, where $K_i^4$ is the bounded component in $\mathrm{int}(W^4)$. Then $K_{i+n}^4 - \mathrm{int}(K_i^4) = M_i^3 \times [0, \varepsilon] \cup h_j^2 \cup h_k^3$, where $h_j^m$ are the attached $m$-handles.

**Lemma 8.9.** There exists an isotopy of $K_{i+n}^4 - \mathrm{int}(K_i^4)$ such that $F^2 \subset M_i^3 \times \{\varepsilon\} \cup h_j^2$ and $F^2 \cap h_j^2 = \cup_k \delta_{j,k}^2$, where $\delta_{j,k}^2$ are disjoint 2-disks properly embedded in the pair $(h_j^2, \partial_a h_j^2)$ (here $\partial_a h_j^2$ denotes the attachment zone of the handle, which is a solid torus), which are parallel to the core of the handle. Moreover, $\partial\delta_{j,k}^2 \subset \partial(\partial_a h_j^2)$ are concentric circles on the torus, parallel to the 0-framing of the attaching circle. □

Proof. It follows from a transversality argument that the image of $F$ intersects only the 2-handles, along 2-disks. Further, it is sufficient to see that the circles $\partial\delta^2_{j,k}$ are homotopic to the 0-framing since in $K^4_{i+n} - \mathrm{int}(K^4_i)$, homotopy implies isotopy for circles. If one circle is nullhomotopic, then it can be removed by means of an ambient isotopy. If a circle turns $p$ times around the longitude, then it cannot bound a disk in $h^2_j$ unless $p = 1$. ■

Definition 8.10. Consider a parallel copy in $M^3_i = M^3_i \times \{0\}$ of the surface with boundary $F'^2 = F^2 - \cup_{j,k}\delta^2_{j,k} \subset M^3_i \times \{\varepsilon\} - \cup_j \partial_a h^2_j$, and use standardly embedded annuli in the torus $\partial_a h^2_j$, which join the parallel circles to the central knot in order to get a surface with boundary on the surgery loci. This is called a *pullback* of the surface $F^2 \subset M^3_{i+n}$.

**Lemma 8.11.** Let $\alpha : \pi_1(M^3_i) \to H_1(M^3_i)$ be the Hurewicz map. Then $\phi_i(\ker(\alpha)) = \pi_1(\partial W^4)$, that is, the pair $(\alpha, \phi_i)$ is strongly compressible. □

Proof. Consider the diagram

$$\begin{tikzcd} \Gamma \arrow[r, "\phi"] \arrow[d, "\pi"'] & G \arrow[d] \\ \Gamma_{ab} \arrow[r, "\phi_{ab}"] & G_{ab}, \end{tikzcd} \tag{8.1}$$

where $\phi$ is surjective and the subscript ab means abelianization. Then it follows immediately that $\pi(\ker\phi) = \ker\phi_{ab}$. Since $H_1(\partial W^4) = 0$, and the strong compressibility is symmetric, the result follows. ■

Now, let $n = n(i)$ be as in the conclusion of Lemma 8.7. We consider a maximal set of disjoint nonparallel essential 2-spheres (which is uniquely defined up to isotopy) and pull back these spheres up to time $i$ to get a collection of planar surfaces, whose union is a 2-dimensional polyhedron $\Sigma_i \subset M^3_i$.

**Lemma 8.12.** If $\iota$ denotes the map induced by the inclusion $\pi_1(M^3_i - \Sigma_i) \to \pi_1(M^3_i)$, then the restriction

$$\phi_i : \iota\big(\pi_1\big(M^3_i - \Sigma_i\big)\big) \cap \ker(\alpha) \longrightarrow \pi_1\big(\partial W^4\big) \tag{8.2}$$

is surjective. □

Proof. The pullbacks in $M^3_i$ of spheres $S^2_m \subset M^3_{i+j}$ are planar surfaces with boundary components being the loci of future surgeries. Further, after compressing the spheres $S^2_m$ of $M_{i+j}$ (hence arriving at $M^3_{i+j+k}$), we have a surjection $\phi_{i+j+k}$, thus the map $\pi_1(M^3_{i+j} - \cup S^2_m) \to \pi_1(\partial W^4)$ is also surjective. This means that there exist curves in the complement of the planar surfaces in $M^3_i$ mapping to every element of $\pi_1(\partial W^4)$. Moreover, by the above lemma, we have such curves that are homologically trivial in $M^3_{i+j}$, and hence in $M^3_i$ as all surgery curves are null-homologous. ■

**Lemma 8.13.** The homomorphism $i_* : H_1(M^3_i - \Sigma_i) \to H_1(M^3_i)$, induced by inclusion, is the zero map. □

Proof. If the lemma does not hold, then there exists a curve $\gamma \subset M^3_i - \Sigma_i$ that represents a nontrivial element of $H_1(M^3_i)$. Modifying by a homologically trivial element if necessary, we may assume that $\gamma \in \ker(\phi_i)$. By the previous lemma, $\gamma$ persists. The group $\pi_1(K^4_{i+n} - \mathrm{int}(K^4_i))$ is the quotient of $\pi_1(M^3_i)$ by the relations generated by the surgery curves, which are homologically trivial. In particular, $H_1(K^4_{i+n} - \mathrm{int}(K^4_i)) = H_1(M^3_i)$. Then the class of $\gamma \in H_1(K^4_{i+n} - \mathrm{int}(K^4_i))$ is nonzero since its image in $H_1(M^3_i)$ is nonzero by hypothesis. This gives the required contradiction. ■

We are now in a position to prove the structure theorem. The image of the immersion $\psi_i$ is obtained from the polyhedron $\Sigma_i$ by *stitching together* several planar surfaces along the boundary knots. These knots will be the seams of the surfaces. It is clear by construction that we will have all the desired properties as soon as we show that there are enough planar surfaces to be stitched together to represent all the homology.

To see this, we consider the reduced homology exact sequence of the pair $(M^3_i, M^3_i - \Sigma_i)$ and use the fact that $M^3_i - \Sigma$ is connected, since $M^3_{i+n}$ is, as well as Lemma 8.13. Thus, we have the exact sequence

$$\cdots \longrightarrow H_1\left(M^3_i - \Sigma_i\right) \longrightarrow H_1\left(M^3_i\right) \longrightarrow H_1\left(M^3_i, M^3_i - \Sigma_i\right) \longrightarrow \tilde{H}_0\left(M^3_i - \Sigma_i\right) \tag{8.3}$$

which gives the exact sequence

$$0 \longrightarrow H_1\left(M^3_i\right) \longrightarrow H_1\left(M^3_i, M^3_i - \Sigma_i\right) \longrightarrow 0 \tag{8.4}$$

which, together with an application of Alexander duality, gives $H_1(M^3_i) \cong H_1(M^3_i, M^3_i - \Sigma_i) \cong H^2(\Sigma_i)$. Further, as the isomorphisms $H_1(M^3_i) \cong H^2(M^3_i)$ and $H_1(M^3_i, M^3_i - \Sigma_i) \cong H^2(\Sigma_i)$, given respectively by Poincaré and Alexander dualities, are obtained by taking cup products with the fundamental class, the diagram

$$\begin{tikzcd} H^2\left(M^3_i\right) \arrow[r] \arrow[d] & H^2\left(\Sigma_i\right) \arrow[d] \\ H_1\left(M^3_i\right) \arrow[r] & H_1\left(M^3_i, M^3_i - \Sigma_i\right) \end{tikzcd} \tag{8.5}$$

commutes.

Thus the inclusion of $\Sigma_i$ in $M_i^3$ gives an isomorphism $H^2(M_i^3) \cong H^2(\Sigma_i)$. Since $H_2(M_i^3)$ and $H_2(\Sigma_i)$ have no torsion, the cap product induces perfect pairings $H_2(M_i^3) \times H^2(M_i^3) \to \mathbb{Z}$ and $H_2(\Sigma_i) \times H^2(\Sigma_i) \to \mathbb{Z}$. Therefore, by duality, the map $H_2(\Sigma_i) \to H_2(M_i^3)$ induced by inclusion is also an isomorphism.

Now take a basis for $H^2(\Sigma_i)$. Each element of this basis can be looked at as an integral linear combination of the planar surfaces (as in cellular homology) with trivial boundary. We obtain a surface corresponding to each such homology class by taking copies of the planar surfaces, with the number and orientation determined by the coefficient. Since the homology classes are cycles, these planar surfaces can be glued together at the boundaries to form closed, oriented, immersed surfaces. Without loss of generality, we can assume these to be connected.

Remark 8.14. By doing surgeries on the seams of $\Sigma_i \subset M_i^3$, some new homology is created (the homology of $M_{i+1}^3$). One constructs naturally surfaces representing the homology of $M_{i+1}^3$, as follows. One considers *generalized* Seifert surfaces in $M_i^3$ of the loci of the surgeries, which are surfaces which might have boundary components along other seams. Then one caps off the boundaries by using the cores of the 2-handles which are added and pushes the closed surfaces into $M_{i+1}^3$. Notice that we can consider also some Seifert surface whose boundary components are seams in some $M_{i+n}^3$ for $n > 1$.

### 8.3 On Casson finiteness

Suppose we do have a 4-manifold $(W^4, \partial W^4)$ with a g.s.c. handle decomposition of its interior. Since there may be infinitely many handles, we cannot use Casson's argument. However, we note that we can use Casson's argument if we can show that

(i) $(W^4, \partial W^4)$ has a (finite) handle decomposition without 1-handles;

(ii) some $(Z^4, \partial Z^4)$ has a handle decomposition without 1-handles, where $Z^4$ is compact, contractible with $\pi_1(\partial Z^4) = \pi_1(\partial W^4)$;

(iii) some $(Z^4, \partial Z^4)$ has a handle decomposition without 1-handles, where $Z^4$ is compact, contractible, and there is a surjection $\pi_1(\partial Z^4) \twoheadrightarrow \pi_1(\partial W^4)$ (by Proposition 7.5).

Thus, we can apply Casson's argument if we show finiteness, or some weak form of finiteness such as the latter statements above.

We now assume that the handle decomposition is as in the conclusion of Theorem 8.8. We will change our measures of time so that passing from $M_i^3$ to $M_{i+1}^3$ consists of performing all the surgeries required to compress the surfaces, splitting along the 2-spheres, and also performing the necessary surgeries off the surface.

In $M_i^3$, we have a collection of embedded surfaces representing all the homology of $M_i^3$. We see that we have Casson finiteness in a special case.

**Theorem 8.15.** If there exists $i$ so that the immersion $\psi_i : \cup_j F_j^2(i) \to M_i^3$ is actually an embedding and $\phi_i(\psi_i(F_j^2(i))) = \{1\} \subset \pi_1(\partial W^4)$, then $\pi_1(\partial W^4)$ violates the Kervaire conjecture. □

Proof. Let $k$ be the rank of $H_1(M_i^3)$ and $P_j$, $1 \le j \le k$, the fundamental groups of the surfaces. Since the surfaces are disjoint, $\pi_1(M_i^3)$ is obtained by HNN extensions from the fundamental group $G$ of the complement of the surfaces. Thus, if $\psi_j$ are the gluing maps, we have

$$\pi_1\left(M_i^3\right) = \left\langle G, t_1, \ldots, t_k;\ t_j x t_j^{-1} = \psi_j(x)\ \forall x \in P_j \right\rangle. \tag{8.6}$$

Now, since $\phi_i(P_j) = 1$ and $\phi_i(G) = \pi_1(\partial W^4)$, $\pi_1(M_i^3)$ surjects onto $\langle \pi_1(\partial W^4), t_1, \ldots, t_n \rangle$, the group obtained by adding $k$ generators to $\pi_1(\partial W^4)$. But $M_i^3$ is obtained by using $n$ 2-handles and $n-k$ 3-handles. Thus, as in Casson's theorem, $\pi_1(M_i^3)$ is killed by adding $n-k$ generators and $n$ relations. This implies that $\pi_1(\partial W^4)$ is killed by adding $n$ generators and $n$ relations. ■

**Theorem 8.16.** There exists always an immersion as in the structure theorem with the additional property that $\phi_i(\psi_i(F_j^2(i))) = \{1\} \subset \pi_1(\partial W^4)$ holds true. □

Proof. By construction, the images of the seams (which are roughly speaking half the generators of the fundamental group) are null-homotopic. If the fundamental groups of the generalized Seifert surfaces from the previous remark map to the trivial group, then after doing surgery on the seams we obtain surfaces $F_j^2(i+1)$ representing homology with trivial $\pi_1$ images by $\phi_{i+1}$. Thus, it suffices to show that we will obtain this condition for a choice of Seifert surfaces for all seams, at some time in the future.

Fix $i$ large enough so that $M_i^3$ is in the collar $\partial W^4 \times [0, \infty)$.

**Lemma 8.17.** There exists some $n' = n'(i)$ such that, whenever a 2-sphere immersed in $\mathrm{int}(W^4) - K_{i+n'}^4$ bounds a 3-ball immersed in the collar, it actually bounds a 3-ball immersed in $\mathrm{int}(W^4) - K_i^4$. □

Proof. Choose $n'$ large enough so that a small collar $\partial W^4 \times [x, y] \subset K_{i+n'}^4 - \mathrm{int}(K_i^4)$. Then use the horizontal flow to send $\partial W^4 \times [0, y]$ into $\partial W^4 \times [x, y]$ by preserving the right-hand side boundary. This yields a ball in the complement of $K_i^4$. ■

We need the following analogue, for $\pi_2$ instead of $\pi_1$, of Lemma 8.7.

**Lemma 8.18.** There exists $k = k(i)$ such that any immersed 2-sphere in $K_i^4$ (resp., in the intersection of $K_i^4$ with the collar) which bounds a 3-ball in $\mathrm{int}(W^4)$ (resp., in the collar) does so in $K_{i+k}^4$ (resp., in the collar). □

Proof. The same trick we used in the proof of Lemma 8.17 applies. ■

Choose now $n$ large enough such that $K_{i+n}^4 - K_{i+n'}^4$ contains a nontrivial collar $\partial W^4 \times [x, y]$, and $n > k(i + n')$ provided by Lemma 8.7. Consider a surgery curve $\gamma \subset M_j^3$ for some $i + n' \le j < i + n$.

**Lemma 8.19.** There exists a generalized Seifert surface for $\gamma$ in $M_i^3$, so that the other boundary components are surgery curves from $M_m^3$, with $m \le i + n$, and whose fundamental group maps to the trivial group under $\phi_i$. □

Proof. The curve $\gamma$ bounds a disc in the 2-handle attached to it. Further, as it can be pulled back, say along an annulus, to time $M_{i+n'}^3$, and then dies by $M_{i+n}^3$, it bounds another disc consisting of the annulus and the disc by which it dies. These discs together form an immersed 2-sphere. Consider the class $\nu \in \pi_2(\partial W^4)$ of this 2-sphere by using the projection of the collar on $\partial W^4$. We can realize the element $\nu$ by an immersed 2-sphere in a small collar $\partial W^4 \times [x, y] \subset K_{i+n}^4 - K_i^4$. Therefore, by modifying the initial 2-sphere by this sphere (which is far from $\gamma$) in the small collar, one finds an immersed 2-sphere whose image in $\pi_2(\partial W^4)$ is trivial. Since $i$ was large enough, $K_{i+n}^4 - K_i^4$ is a subset of a larger collar $\partial W^4 \times [0, z]$. Then the 2-sphere we constructed bounds a 3-ball in $\partial W^4 \times [0, z]$, and so, by Lemma 8.17, it also does so in $X^4 = K_{i+n}^4 - K_i^4$. Let $\mu : S^2 \to X^4$ denote this immersion realizing a trivial element of $\pi_2(X)$. Then $\mu$ lifts to a map $\tilde{\mu} : S^2 \to \tilde{X}^4$, where $\tilde{X}^4$ is the universal covering space of $X^4$. Since $\mu$ is null-homotopic, the homology class of $[\tilde{\mu}] = 0 \in H_2(\tilde{X}^4)$ is trivial when interpreting $\tilde{\mu}$ as a 2-cycle in $\tilde{X}^4$.

The homology of $\tilde{X}^4$ is computed from the $\pi_1(X^4)$-equivariant complex associated to the handle decomposition, whose generators in degree $d$ are the $d$-handles attached to $K_i^4$ in order to get $K_{i+n}^4$. Therefore, one has then the following relation in this differential complex:

$$[\tilde{\mu}] = \sum_j c_j \partial \left[h_j^3\right], \quad c_j \in \mathbb{Z}. \tag{8.7}$$

The action of the algebraic boundary operator $\partial$ on the element $[h_j^3]$ can be described geometrically as the class of the 2-cycle which represents the attachment 2-sphere $\partial^+ h_j^3$ of the 3-handle $h_j^3$. Consequently, the previous formula can be rewritten as

$$\tilde{\mu} = \sum_j c_j \partial^+ h_j^3 + \sum_k d_k L_k, \quad c_j, d_k \in \mathbb{Z}, \tag{8.8}$$

where $L_k$ are closed surfaces (actually, these are closed 2-cycles, but they can be represented by surfaces by the well-known results of R. Thom) with the property that

$$[L_k] \cdot [\delta_m^2] = 0, \quad \forall k, m \tag{8.9}$$

($\delta_m^2$ denotes the core of the 2-handle $h_m^2$). We compute explicitly the boundary operator on the 3-handles in terms of the surfaces we have in the 2-complex $\Sigma_i$. Set

$$\partial[h_j^3] = \sum_k m_{jk}[h_k^2]. \tag{8.10}$$

Then the coefficient $m_{jk}$ is the number of times the boundary $\partial^+ h_j^3$ runs over the core of $h_k^2$. But the 2-sphere $\partial^+ h_j^3$, when pulled back in $M_i^3$, is a planar surface in $M_i^3$ whose boundary circles (i.e., at seams) are capped off by the core disks $\delta_k^2$ of the 2-handles $h_k^2$. Therefore, the number $m_{jk}$ is the number of times the seam $\partial^+ h_k^2$ appears in the planar surface which is a pullback of $\partial^+ h_j^3$. In particular, the coefficient of a 2-handle vanishes in a 3-cycle only if the boundaries of the planar surface glue together to close up at the corresponding surgery locus.

Thus the pullbacks of the surfaces $\sum_j c_j \partial^+ h_j^3$ give a surface $F^2$ in $M_i^3$ with the curve $\gamma$ with which we started as boundary, plus some other curves along which surgery is performed by time $i+n$. As this is in fact a closed cycle in the universal cover, the surface $F^2$ lifts to a surface in $\tilde{X}^4$, with a single boundary component, corresponding to a curve which is not surgered by the time $i+n$. Therefore, the map $\pi_1(F^2) \to \pi_1(X^4) \to \pi_1(\partial W^4)$ factors through $\pi_1(\tilde{X}^4) = 1$, hence the image of $\pi_1(F^2)$ in $\pi_1(\partial W^4)$ is trivial. ■

Thus, after surgering along the curves up to the $M_{i+n}^3$, we do have the required Seifert surfaces to compress to get embedded surfaces with trivial $\pi_1(\partial W^4)$ image. ■

**Proposition 8.20.** If $\gamma_{i,k} \subset M_i^3$ are the surgery curves (i.e., the seams), then $\gamma_{i,k} \in LCS_\infty(\pi_1(M_i^3))$, where $LCS_s(G)$ is the lower central series of the group $G$, $LCS_1(G) = G$, $LCS_{s+1}(G) = [G, LCS_s(G)]$, and $LCS_\infty(G) = \cap_{s=1}^\infty LCS_s(G)$. □

Proof. We will express each surgery locus $\gamma$ as a product of commutators of the form $[\alpha_i, \beta_i]$, with each $\alpha_i$ being conjugate to a surgery locus (possibly $\gamma$ itself). It then follows readily that $\gamma \in LCS_\infty(\pi_1(M_i^3))$, as now if each $\gamma_{i,k} \in LCS_s(\pi_1(M_i^3))$, then each $\gamma_{i,k} \in LCS_{s+1}(\pi_1(M_i^3))$.

Suppose now that $\gamma = \gamma_{i,k}$ is a surgery locus. Then the 0-frame surgery along $\gamma$ creates homology in $M^3_{i+1}$ which, by our structure theorem, is represented by a surface $S^2 = \psi_{i+1}(F^2_j(i+1))$. The pullback of $S^2$ to $i$ gives a surface with boundary along seams, and being compressed to a sphere by the seams, so that the algebraic multiplicity of $\gamma$ is 1 while that of all other seams is 0. In terms of the fundamental group, this translates to the relation that was claimed. ■

Since we have immersed surfaces of the required form, the obstruction we encounter is in making these surfaces disjoint at some finite stage. Note that, for a finite decomposition, we do indeed have disjoint surfaces representing the homology after finitely many surgeries, since we in fact have a family of such spheres.

**Proposition 8.21.** There exists a 2-complex $\Sigma = \cup_{i=1}^{\infty} \Sigma_i$ with intersections along double curves, coming from a handle decomposition as above, where all the seams are trivial in homology, but which does not carry disjoint, embedded surfaces representing all of the homology. □

Proof. For the first stage, take two surfaces of genus 2 and let them intersect transversely along two curves (which we call seams) that are disjoint and homologically independent in each surface. Next, take as Seifert surfaces for these curves once punctured surfaces of genus 2 intersecting in a similar manner, and glue their boundary to the above-mentioned curves of intersection. Repeat this process to obtain the complex.

At the first stage, we cannot have embedded, disjoint surfaces representing the homology as the cup product of the surfaces is nontrivial. As the surfaces are compact, we must terminate at some finite stage. We will prove that if we can have disjoint surfaces at the stage $k+1$, then we do at the stage $k$. This will suffice to give the contradiction.

Now, we know that the complex cannot be embedded in the first stage. Suppose we did have disjoint embedded surfaces $F_1$ and $F_2$ at stage $k+1$. Since these form a basis for the homology, they contain curves on them that are the seams at the first stage with algebraically nonzero multiplicity, that is, the collection of curves representing the seam is not homologically trivial in the intersection of the first stage with the surface. Further, some copy of the first seam must bound a subsurface $F'_i$ in each of the surfaces, for otherwise the surface contains a curve dual to the seam. Since the cup product of such a dual curve with the homology class of the other surface is nontrivial, hence it must intersect the other surface, contradicting the hypothesis that the surfaces are disjoint. Similarly, at the other seam, we get surfaces $F''_i$.

By deleting the first-stage surfaces and capping off the first-stage seams by attaching discs, we get a complex exactly as before with the $(j+1)$th stage having become

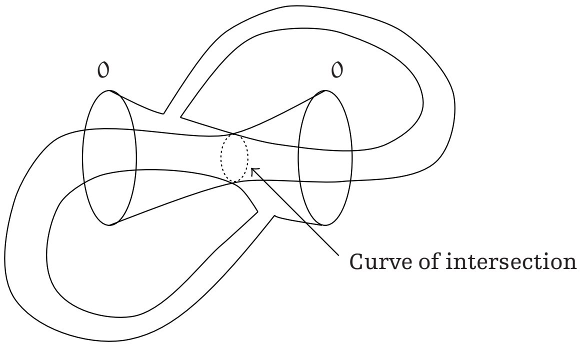


**Figure 8.1** Tori with one intersection curve.

the $j$th stage. Further, the $F_1'$ and $F_2''$ now give disjoint, embedded surfaces representing the homology that are supported by stages up to $k$. This suffices as above to complete the induction argument.

It is easy to construct a handle decomposition corresponding to this complex. Figure 8.1 shows a construction of tori with one curve of intersection. Here we have used the notation of Kirby calculus, with the thickened curves being an unlink along each component of which 0-frame surgery has been performed. It is easy to see that the same construction can give surfaces of genus 2 intersecting in two curves. On attaching the first two 2-handles, the boundary is $(S^2 \times S^1)\#(S^2 \times S^1)$. Since the curves of intersection are unknots, after surgery they bound spheres. Further, it is easy to see by cutting along these that the boundary is $(S^2 \times S^1)\#(S^2 \times S^1)$ after attaching the 2-handles and 3-handles as well. Repeating this process, we obtain our embedding.

Thus we have an infinite handle decomposition satisfying our hypothesis for which this 2-complex is $\Sigma$. ■

## 8.4 A wild example

We will construct an example of an open, contractible 4-manifold that is not tame and that has a handle decomposition without 1-handles.

**Theorem 8.22.** There is a proper handle decomposition of an open, contractible 4-manifold $W^4$ such that $W^4$ is not the interior of a compact 4-manifold. In particular, $W^4$ does not have a finite handle decomposition. □

Proof. We will take a variant of the example in Section 8.5. Namely, we construct an explicit handle decomposition according to a canonical form.

Start with a 0-handle and attach to its boundary three 2-handles along an unlink. The resulting manifold has boundary $(S^2 \times S^1)\#(S^2 \times S^1)\#(S^2 \times S^1)$ obtained by 0-frame surgery about each component of an unlink with three components. We now take as Seifert surfaces for these components surfaces of genus 2, so that each pair intersects in a single curve, so that the curves of intersection form an unlink and are unlinked with the original curves.

Now, attach 2-handles along the curves of intersections, and then 3-handles along the Seifert surfaces compressed to spheres by adding discs in the 2-handles just attached. It is easy to see that the resulting manifold once more has boundary $(S^2 \times S^1)\#(S^2 \times S^1)\#(S^2 \times S^1)$. Thus, we may iterate this process. Further, the generators of the fundamental group at any stage are the commutators of the generators at the previous stage.

Suppose $W^4$ is in fact tame. Then, we may use the results of the previous sections. Now, by construction, no curve dies as only trivial relations have been added. Thus every element in kernel$(\phi_i)$ must fail to persist by some uniform time. In particular, the image of the group after that time in the present (curves that persist beyond that time) must inject under $\phi_i$. But we know that it also surjects. Thus, we must have an isomorphism.

Thus, there is a unique element mapping onto each element of $\pi_1(\partial W^4)$. Hence this element must persist till infinity as we have a surjection at all times. On the other hand, since the limit of the lower central series of the free group is trivial, no nontrivial element persists. This gives a contradiction unless $\pi_1(\partial W^4)$ is trivial.

But there are nontrivial elements that do persist beyond any given time. As no element dies, we again get a contradiction. ■

### 8.5 Further obstructions from gauge theory

To further explore some of the subtleties that one might encounter in trying to construct a handle decomposition without 1-handles for a contractible manifold, given one for its interior, we consider a more general situation. We will consider sequences of 3-manifolds $M_i^3$ that begin with $S^3$. As before, we require that each manifold comes from the previous one by 0-frame surgery about a homologically trivial curve or by splitting along a nonseparating $S^2$ and capping off. Also, we require degree-one maps $f_i$ to a common manifold $N^3$, related as before. We will say that the sequence limits to $N^3$ if *any curve that persists dies* as in Lemma 8.7.

In this situation, our main question generalizes to a *relative version*, namely, given any such sequence $\{M_i^3\}$, with $M_k^3$ an element in the sequence, is there a finite sequence that agrees up to $M_k^3$ with the old sequence and whose final term is $N^3$?

We will show that there is an obstruction to completing certain sequences to finite sequences when $N^3 = S^3$. We do not know whether there are infinite sequences limiting to $N^3$ in this case.

Let $\mathfrak{P}^3$ denote the Poincaré homology sphere. Observe that we cannot pass from this to $S^3$ by 0-frame surgery about homologically trivial curves and capping off nonseparating spheres, for, if we could, $\mathfrak{P}^3$ would bound a manifold with $H^2 = \oplus_k \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, which is impossible as $\mathfrak{P}^3$ has Rochlin invariant 1. On the other hand, for the same reason, $\mathfrak{P}^3$ cannot be part of any sequence of the above form.

Using Donaldson's theorem [5], we have a similar result for the connected sum $\mathfrak{P}^3\#\mathfrak{P}^3$ of $\mathfrak{P}^3$ with itself. The main part of the proof of this lemma is due to R. Gompf (personal communication).

**Lemma 8.23.** There do not exist sequences of 3-manifolds starting with $\mathfrak{P}^3\#\mathfrak{P}^3$ and ending with $S^3$ which use only 0-frame surgery along homologically trivial curves and capping off nonseparating $S^2$'s. □

Proof. If we did have such a sequence of surgeries, then $\mathfrak{P}^3\#\mathfrak{P}^3$ bounds a 4-manifold $M_1^4$ with $H^2 = \oplus_k \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ , with a half-basis formed by embedded spheres. Now glue this to a manifold with form $E_8 \oplus E_8$ which is bounded by $\mathfrak{P}^3\#\mathfrak{P}^3$ to get $M^4$.

We can surger out the disjoint family of $S^2$'s from $M^4$ to get a 4-manifold with form $E_8 \oplus E_8$ and trivial $H_1$. This contradicts Donaldson's theorem. ■

We still do not have a sequence as claimed, for Cassson's argument shows that $\mathfrak{P}^3\#\mathfrak{P}^3$ cannot be part of a sequence. To obtain such a sequence, we will construct a manifold $N^3$ that can be obtained by 0-frame surgery on algebraically unlinked 2-handles from each of $S^3$ and $\mathfrak{P}^3\#\mathfrak{P}^3$. Thus, $N^3$ is part of a sequence. On the other hand, if we had a sequence starting at $N^3$ that terminated at $S^3$, then we would have one starting at $\mathfrak{P}^3\#\mathfrak{P}^3$, which contradicts the above lemma.

To construct $N^3$, take a contractible 4-manifold $K^4$ that bounds $\mathfrak{P}^3\#\mathfrak{P}^3$. By Freedman's theorem [6], this exists, and can moreover be smoothed after taking connected sums with sufficiently many copies of $S^2 \times S^2$. Take a handle decomposition of $K^4$. This may include 1-handles, but these must be boundaries of 2-handles. Hence, by handle slides, we can ensure that each 1-handle is, at the homological level, a boundary of a 2-handle and is not part of the boundary of any other 2-handle. Replacing the 1-handle by a 2-handle does not change the boundary, and changes $H^2(K^4)$ to $H^2(K^4) \oplus (\oplus_k \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right])$. We do this dually with 3-handles too. Sliding 2-handles over the new ones, we can ensure that the attaching maps of the 2-handles have the same algebraic linking (and framing) structure as a disjoint union of Hopf links.

Now take $N^3$ obtained from $S^3$ by attaching half the links so that these are pairwise algebraically unlinked. The manifold $N^3$ has the required properties.

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Louis Funar: Institut Fourier BP 74, Université Joseph Fourier Grenoble 1, 38402 Saint-Martin-d'Hères cedex, France
E-mail address: funar@fourier.ujf-grenoble.fr

Siddhartha Gadgil: Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651, USA
Current address: Statistics & Mathematics Unit, Indian Statistical Institute, Bangalore 560059, India
E-mail address: gadgil@math.sunysb.edu