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POLYNOMIAL INVARIANTS OF LINKS SATISFYING CUBIC SKEIN RELATIONS *

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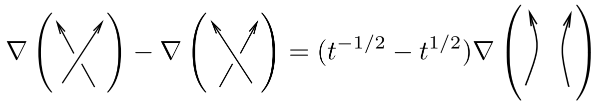
Abstract. The aim of this paper is to define two link invariants satisfying cubic skein relations. In the hierarchy of polynomial invariants determined by explicit skein relations they are the next level of complexity after Jones, HOMFLY, Kauffman and Kuperberg's G_2 quantum invariants. Our method consists of the study of Markov traces on a suitable tower of quotients of cubic Hecke algebras extending Jones approach.

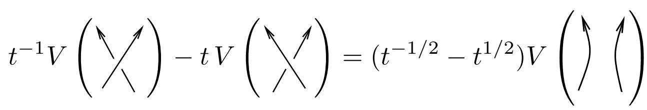
1. Introduction.

1.1. Preliminaries. J.Conway showed that the Alexander polynomial of a knot, when suitably normalized, satisfies the following skein relation:



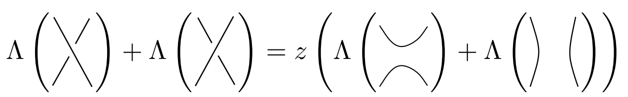
Given a knot diagram, one can always change some of its crossings such that the modified diagram represents the unknot. Therefore, one can use the skein relation for a recursive computation of ∇, although this algorithm is rather time consuming, since it is exponential.

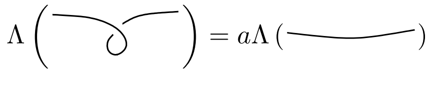
In the mid eighties V.Jones discovered another invariant verifying a different but quite similar skein relation, namely:



which was further generalized to a 2-variable invariant by replacing the factor $(t^{1/2} - t^{-1/2})$ with a new variable x. The latter one was shown to specialize to both Alexander and Jones polynomials.

The Kauffman polynomial is another extension of Jones polynomial which satisfies a skein relation, but this time in the realm of unoriented diagrams. Specifically, the formulas:





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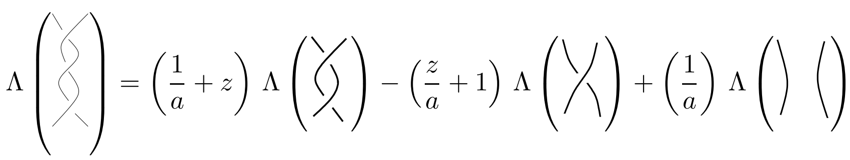
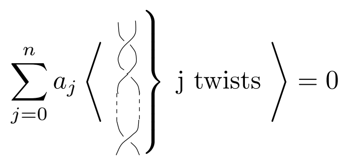
define a regular isotopy invariant of links which can be renormalized, by using the writhe of the oriented diagram, in order to become a link invariant. Remark that some elementary manipulations show that Λ verifies the following cubical skein relation:



It has been recently proved ([12], and Problem 1.59 [18]) that this relation alone is not sufficient for a recursive computation of Λ. Whenever the skein relations and the value of the invariant for the unknot are sufficient to determine its values for all links, the system of skein relations will be said to be complete. Several results concerning the incompleteness of higher degree unoriented skein relations and their skein modules have been obtained by J.Przytycki and his students (see e.g. [12, 28, 29]).

These invariants were generalized to quantum invariants associated to general Lie algebras, super-algebras and their representations. V.Turaev ([33]) identified the HOMFLY and Kauffman polynomials with the invariants obtained from the series A_n and B_n, C_n, D_n respectively. G.Kuperberg ([19]) defined the G_2-quantum invariant of knots by means of skein relations, by making use of trivalent graphs diagrams and exploited further these ideas in [20], for spiders of rank 2 Lie algebras. The skein relations satisfied by the quantum invariants coming from simple Lie algebras were approached also via weight systems and the Kontsevich integral in [22, 23], for the classical series, and in [1, 2] for the case of the Lie algebra g_2 of G_2.

Notice that any link invariant coming from some R-matrix R verifies a skein relation of the type:



which can be derived from the polynomial equation satisfied by the R-matrix R.

Let us mention that the skein relations are somewhat related to the representation theory of the Hopf algebra associated to the R-matrix R. In particular, there are no other known invariants given by means of a complete family of skein relations, but those from above. Moreover, one expects that the quantum invariants associated to other Lie (super) algebras or by cabling the previous ones should satisfy skein relations of degree at least 4, as already the Kuperberg G_2-invariant does.

This makes the search for an explicit set of complete skein relations, in which at least one relation is cubical, particularly difficult and interesting. This problem was first considered in [13] and solved in a particular case. The aim of this paper is to complete the result of [13] by constructing a deformation of the previously constructed quotients of the cubic Hecke algebras and of the Markov traces supported by these algebras. We obtain in this way two link invariants, denoted by $I_{(\alpha,\beta)}$ and $I^{(z,\delta)}$, which are recursively computable and uniquely determined by two skein relations. Explicit computations show that $I_{(\alpha,\beta)}$ detects the chirality of the knots with number crossing at most 10 where HOMFLY, Kauffman and their 2-cablings fail. On the other hand, as HOMFLY, Kauffman and their 2-cablings ([24, 27]), it seems that our invariants do not distinguish between mutant knots. We recall that some mutant knots can be distinguished by the 3-cablings of the HOMFLY polynomial (see [25]).

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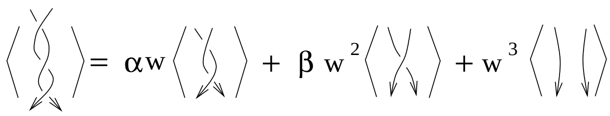
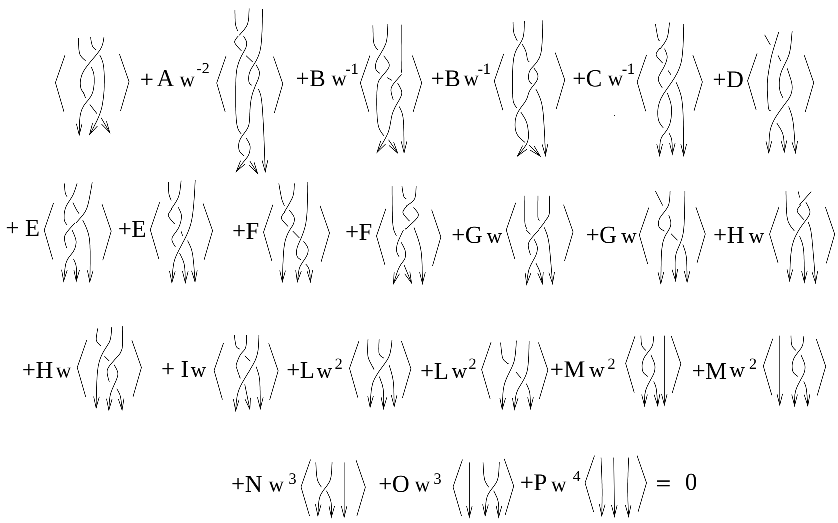
1.2. The main result. The aim of this paper is to define two link invariants by means of a complete set of skein relations. More precisely we will prove the following Theorem (see section 5):

THEOREM 1.1. *There exist a link invariant $I_{(\alpha,\beta)}$ which is uniquely determined by the two skein relations shown in (1) and (2) and its value for the unknot, which is traditionally 1.*



(1)



(2)

The invariant takes values in:

$$\frac{\mathbb{Z}[\alpha,\,\beta,\,(\beta^2-2\alpha)^{\pm\epsilon/2},\,(\alpha^2+2\beta)^{\pm\epsilon/2}]}{(H_{(\alpha,\,\beta)})}$$

where $\epsilon - 1 \in \{0,1\}$ is the number of link components modulo 2, and $H_{(\alpha,\,\beta)}$ is the following polynomial:

$$\begin{aligned} H_{(\alpha,\,\beta)} =& 8\alpha^6 - 8\alpha^5\beta^2 + 2\alpha^4\beta^4 + 36\alpha^4\beta - 34\alpha^3\beta^3 + 17\alpha^3 + \\ & 8\alpha^2\beta^5 + 32\alpha^2\beta^2 - 36\alpha\beta^4 + 38\alpha\beta + 8\beta^6 - 17\beta^3 + 8 \end{aligned}$$

Here (Q) denotes the ideal generated by the element Q in the algebra under consideration. The values of the polynomials $A, B, C, \ldots, P$ appearing in the skein relations for $I_{(\alpha,\beta)}$ are given in the table below:

$w = ((\alpha^2+2\beta)/(2\alpha-\beta^2))^{1/2}$	$A = (\beta^2-\alpha)$
$B = (\alpha^2-\alpha\beta^2-\beta)$	$C = (\alpha^2-\alpha\beta^2)$
$D = (1+2\alpha\beta+\alpha^2\beta^2-\alpha^3)$	$E = (1+\alpha\beta+\alpha^2\beta^2-\alpha^3)$
$F = (1+2\alpha\beta-\beta^3)$	$G = (\alpha\beta^3-2\alpha-2\alpha^2\beta)$
$H = (\alpha\beta^3-2\alpha-2\alpha^2\beta+\beta^2)$	$I = (\alpha^4-\alpha^3\beta^2-2\alpha^2\beta-3\alpha)$
$L = (2\alpha^3\beta+3\alpha^2-\alpha^2\beta^3-\alpha\beta^2)$	$M = (\beta^4-2\beta-3\alpha\beta^2+\alpha^2)$
$N = (1+4\alpha\beta+3\alpha^2\beta^2-\alpha^3-\alpha\beta^4-\beta^3)$	$O = (1+3\alpha\beta+3\alpha^2\beta^2-\alpha^3-\alpha\beta^4)$
$P = (3\beta^2-\beta^5-2\alpha-3\alpha^2\beta+4\alpha\beta^3)$	

Table 1

Furthermore there exists a second link invariant $I^{(z,\delta)}$, which is determined by the skein relations (1) and (2), but with another set of coefficients. Specifically, $I^{(z,\delta)}$ takes values in:

$$\frac{\mathbb{Z}[z^{\pm\epsilon/2}, \delta^{\pm\epsilon/2}]}{(P^{(z,\delta)})}$$

where ϵ is as above and $P^{(z,\delta)}$ is the following polynomial:

$$P^{(z,\delta)} = z^{23} + z^{18}\delta - 2z^{16}\delta^2 - z^{14}\delta^3 - 2z^9\delta^4 + 2z^7\delta^5 + \delta^6 z^5 + \delta^7$$

The values of the rational functions $A, B, C, \ldots, P$ corresponding to the skein relations of $I^{(z,\delta)}$ are obtained as follows: set first $w = \left(-\frac{z^3}{\delta}\right)^{1/2}$ and substitute further $\alpha = -\frac{z^7+\delta^2}{z^4\delta}$ and $\beta = \frac{\delta-z^2}{z^3}$ in the other entries of table 1.

1.3. Properties of the invariants. The following summarize the main features of these invariants (see section 6):

(1) they distinguish all knots with number crossing at most 10 that have the same HOMFLY polynomial, and thus they are independent from HOMFLY. However, like HOMFLY and Kauffman polynomials, they seem to not distinguish among mutants knots. In fact, they do not distinguish between the Kinoshita-Terasaka knot and the Conway knot, which are the simplest non-equivalent mutant knots.

(2) $I_{(\alpha,\beta)} = I_{(-\beta,-\alpha)}$ for amphicheiral knots, and $I_{(\alpha,\beta)}$ detects the chirality of all those knots with number crossing at most 10, whose HOMFLY, Kauffman polynomials as well as the 2-cabling of HOMFLY fail to detect.

(3) $I_{(\alpha,\beta)}$ and $I^{(z,\delta)}$ have a *cubical* behaviour.

Let us explain briefly what we meant by *cubical behaviour.*

Definition 1.1. A Laurent polynomial $\sum_{j\in\mathbb{Z}} c_j a^j$ is a (n,k)-polynomial (for $n, k \in \mathbb{Z}$) if $c_j = 0$ for $j \neq k$ (modulo n).

Remark 1.1.

(1) The HOMFLY polynomial $V(l,m)$ can be written as $\sum_{k\in\mathbb{Z}} R_k(l)m^k$ and respectively as $\sum_{k\in\mathbb{Z}} S_k(m)l^k$, where $R_k(l)$ and $S_k(m)$ are $(2,k)$-Laurent polynomials fulfilling $R_{2k+1}(l) = S_{2k+1}(m) = 0$.

(2) The Kauffman polynomial can be written as $\sum_{k\in\mathbb{Z}} U_k(l)m^k$ and respectively as $\sum_{k\in\mathbb{Z}} T_k(m)l^k$, where $U_k(l)$ and $T_k(m)$ are $(2, k+1)$-Laurent polynomials.

In this respect the HOMFLY and Kauffman polynomials have a quadratic behaviour.

Proposition 1.1. *$I_{(\alpha,\beta)}$ and $I^{(z,\delta)}$ have a cubical behaviour, i.e. for each link L there exists some $l \in \{0, 1, 2\}$ so that:*

$$I_{(\alpha,\beta)}(L) = \frac{\sum_{k\in\mathbb{Z}_+} P_k(\beta)\alpha^k}{\sum_{k\in\mathbb{Z}_+} Q_k(\beta)\alpha^k} = \frac{\sum_{k\in\mathbb{Z}_+} M_k(\alpha)\beta^k}{\sum_{k\in\mathbb{Z}_+} N_k(\alpha)\beta^k}$$

where P_k, Q_k, M_k, N_k are $(3, k+l)$-polynomials, and

$$I^{(z,\delta)}(L) = \sum_{k\in\mathbb{Z}} H_k(\delta)z^k = \sum_{k\in\mathbb{Z}} G_k(z)\delta^k$$

where H_k, G_k are $(3, k)$-Laurent polynomials.

1.4. Comments. There are three link invariants coming from Markov traces on cubic Hecke algebras, presently known. First, for each quadratic factor P_i of the cubic polynomial Q one has a Markov trace which factors through the usual Hecke algebra $H(P_i, n)$, yielding a re-parameterized HOMFLY invariant. Then there is the Kauffman polynomial and the invariant $I_{(\alpha,\beta)}$ (or $I^{(z,\delta)}$) introduced in the present paper. It would be very interesting to find whether there exists some relationship between them. The explicit computations below show that the new invariants are independent on HOMFLY, Kauffman and their 2-cablings.

Further, one expects that our invariants belong to a family of genuine two-parameter invariants, as expressed in the following:

Conjecture 1.1. *There exists a Markov trace on $H(Q, n)$ taking values in an algebraic extension of $\mathbb{Z}[\alpha, \beta]$, which lifts the Markov trace underlying $I_{(\alpha,\beta)}$.*

In other words, the non-determinacy $H_{(\alpha,\beta)}$ in $I_{(\alpha,\beta)}$ can be removed. Notice that the polynomials $H_{(\alpha,\beta)}$ and $P^{(z,\delta)}$ define irreducible planar algebraic curves which are not rational. In particular, one cannot express explicitly the invariants as one variable polynomials.

1.5. Cubic Hecke algebras. The form of the first skein relation (1) explains the appearance of cubic quotients of braid group algebras $\mathbb{C}[B_n]$. Recall that the braid group B_n on n strands is given by the presentation:

$$B_n = \langle b_1, \ldots, b_{n-1} \,|\, b_ib_j = b_jb_i,\ |i-j| > 1 \ \text{ and } \ b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1},\ i < n-1 \rangle$$

Furthermore we define the cubic Hecke algebra by analogy with the usual (i.e. quadratic) Hecke algebra (see [9]), as follows:

$$H(Q, n) = \mathbb{C}[B_n]/(Q(b_j)\,;\ j = 1, \ldots, n-1)$$

where $Q(b_j) = b_j^3 - \alpha\, b_j^2 - \beta\, b_j - 1$ is a cubic polynomial, which will be fixed through out this paper.

Our purpose is to construct Markov traces on the tower of cubic Hecke algebras since these will eventually lead to link invariants. This method was pioneered by V.Jones ([16]) and A.Ocneanu, who applied it to the case of usual Hecke algebras and obtained the celebrated HOMFLY polynomial. Later on several authors (see [14, 15, 21, 26]) employed more sophisticated algebraic and combinatorial tools in searching for Markov traces on other Iwahori-Hecke algebras, for instance those of type B which are leading to invariants for links in a solid torus.

The cubic Hecke algebras are particular cases of the generic cyclotomic Hecke algebras, introduced by M.Broué and G.Malle (see [6]) and studied in [7, 8], in connection with braid group representations. Recall the following results concerning the structure of the cyclotomic Hecke algebras with $Q(0) \neq 0$ (according to [6, 7, 8, 10] and [11], p.148-149):

(1) $\dim_{\mathbb{C}} H(Q,3) = 24$ and $H(Q,3)$ is isomorphic to the group algebra of the binary tetrahedral group $\langle 2,3,3\rangle$ of order 24, i.e. the linear group $SL(2,\mathbb{Z}/3\mathbb{Z})$.
(2) $\dim_{\mathbb{C}} H(Q,4) = 648$ and $H(Q,4)$ is the group algebra of the finite group G_{25} from the Shepard-Todd classification (see [32]).
(3) $H(Q,5)$ is the cyclotomic Hecke algebra of the group G_{32}, whose order is 155520. It is conjectured that this algebra is free of finite dimension which would imply (by using the Tits deformation theorem) that it is isomorphic to the group algebra of G_{32}.
(4) $\dim_{\mathbb{C}} H(Q,n) = \infty$ for $n \geq 6$.

Thus a direct definition of the trace on $H(Q,n)$ for $n \geq 6$ is highly a nontrivial matter, because it would involve in particular, the explicit solution of the conjugacy problem in these algebras. In order to circumvent these difficulties one introduces a tower of smaller quotients $K_n(\alpha,\beta)$ by adding one more relation to $H(Q,3)$, as follows:

$$b_2 b_1^2 b_2 + R_0 = 0$$

where

$$\begin{aligned} R_0 = {} & A\ b_1^2\, b_2^2\, b_1^2 + B\ b_1\, b_2^2\, b_1^2 + B\ b_1^2\, b_2^2\, b_1 + C\ b_1^2\, b_2\, b_1^2 + D\ b_1\, b_2^2\ b_1 + E\ b_1\, b_2\, b_1^2 + \\ & E\ b_1^2\, b_2\, b_1 + F\ b_2^2\, b_1^2 + F\ b_1^2\, b_2^2 + G\ b_2\, b_1^2 + G\ b_1^2\, b_2 + H\ b_2^2\, b_1 + H\ b_1\, b_2^2 + \\ & I\ b_1\, b_2\, b_1 + L\ b_2\, b_1 + L\ b_1\, b_2 + M\ b_1^2 + M\ b_2^2 + N\ b_1 + O\ b_2 + P \end{aligned}$$

and $A, B, \ldots, P$ are the functions from table 1.

Remark 1.2. The main feature of these quotients is the fact that the algebras $K_n(\alpha,\beta)$ are finite dimensional for all values of n. Moreover, these algebras do not collapse for large n, thus yielding an interesting tower of algebras.

Remark 1.3. Let us explain the heuristics behind that choice for the additional relation. For generic Q the algebra $H(Q,3)$ is semi-simple and decomposes as $\mathbb{C}^3 \oplus M_2^{\oplus 3} \oplus M_3$, where M_m is the algebra of $m \times m$ matrices. The quadratic Hecke algebra $H_q(3) = \mathbb{C}[B_3]/(b_i^2 + (1-q)b_i - q)$ arises as a quotient of $H(Q,3)$ by killing the factor $\mathbb{C} \oplus M_2^{\oplus 2} \oplus M_3$. It is known that Jones and HOMFLY polynomials can be derived from the unique Markov trace on the homogeneous tower $\cup_{n=1}^{\infty} H_q(n)$. In a similar way, the rank 3 Birman-Wenzl algebra ([5]) - which supports an unique Markov trace inducing the Kauffman polynomial - is the quotient of $H(Q,3)$ by the factor $\mathbb{C} \oplus M_2^2$. In our case we introduced the extra relation above which kills precisely the central factor $\mathbb{C}^3$ of $H(Q,3)$.

The geometric interpretation of these relations is now obvious: the first skein relation (1) is the cubical relation corresponding to taking the quotient $H(Q,n)$ while the main skein relation (2) defines the smaller quotient algebras $K_n(\alpha,\beta)$.

Our main theorem is a consequence of the more technical result below (see sections 2, 3 and 4).

THEOREM 1.2. *There are precisely four values of* $(z, \bar{z})$ *(formal expressions in* α *and* β*) for which there exists a Markov trace* $\mathcal{T}$ *on the tower* $\cup_{n=1}^{\infty} K_n(\alpha, \beta)$ *with parameters* $(z, \bar{z})$ *i.e. verifying the following conditions:*
(1) $\mathcal{T}(xy) = \mathcal{T}(yx)$ *for all* $x, y \in K_n(\alpha, \beta)$ *and all* n,
(2) $\mathcal{T}(xb_{n-1}) = z\mathcal{T}(x)$ *for all* $x \in K_n(\alpha, \beta)$ *and all* n,
(3) $\mathcal{T}(xb_{n-1}^{-1}) = \bar{z}\mathcal{T}(x)$ *for all* $x \in K_n(\alpha, \beta)$ *and all* n.
The first pair $(z, \bar{z})$ *is given by:*

$$z = \frac{2\alpha - \beta^2}{\alpha\beta + 4}, \; \bar{z} = -\frac{\alpha^2 + 2\beta}{\alpha\beta + 4}$$

and the corresponding trace takes values as follows:

$$\mathcal{T}_{\alpha,\beta} : \cup_{n=1}^{\infty} K_n(\alpha, \beta) \to \frac{\mathbb{Z}[\alpha, \, \beta, \, (\alpha\beta + 4)^{-1}]}{(H_{(\alpha,\,\beta)})}$$

The other three solutions are not rational functions on α *and* β*, but nevertheless one can express* α*,* β *and* $\bar{z}$ *as rational functions of* z *and* δ*, where* $\delta = z^2(\beta z + 1)$*. Specifically, we have a Markov trace:*

$$\mathcal{T}^{(z,\,\delta)} : \cup_{n=1}^{\infty} K_n(\alpha, \, \beta) \to \frac{\mathbb{Z}[z^{\pm 1}, \delta^{\pm 1}]}{(P^{(z,\,\delta)})}$$

where:

$$\beta = \frac{\delta - z^2}{z^3}, \; \alpha = -\frac{z^7 + \delta^2}{z^4\delta} \quad \text{and } \bar{z} = -\frac{z^4}{\delta}$$

REMARK 1.4. For particular values of $(\alpha, \beta) \in \mathbb{C}$ one might find that the indeterminacy ideal for the respective Markov traces is smaller than the specialization of the ideal above. A specific example is the $\mathbb{Z}/6\mathbb{Z}$-valued invariant, corresponding to the values $\alpha = \beta = 0$ in [13], which is a specialization of the invariant $I^{z,\delta}$ for $z^3 = -1$ and $\delta = z^2$. We can refine the general Markov trace in order to restrict to a $\mathbb{Z}/3\mathbb{Z}$-valued trace (see section 6), but this refinement does not survive the deformation process.

There is a natural way to convert a Markov trace $\mathcal{T}$ into a link invariant, by setting:

$$I(x) = \left(\frac{1}{z\bar{z}}\right)^{\frac{n-1}{2}} \left(\frac{\bar{z}}{z}\right)^{\frac{e(x)}{2}} \mathcal{T}(x)$$

where $x \in B_n$ is a braid representative of the link L and $e(x)$ is the exponent sum of x.

Therefore we derive two invariants $I_{(\alpha,\beta)}$ and $I^{(z,\,\delta)}$ from the previous Markov traces, which satisfy the claimed skein relations.

1.6. Outline of the proof. We will prove by recurrence on n that a Markov trace on $K_n(\alpha, \beta)$ extends to a Markov trace on $K_{n+1}(\alpha, \beta)$. Since there is a nice system of generators for $K_{n+1}(\alpha, \beta)$ constructed inductively starting from a generators system for $K_n(\alpha, \beta)$, such an extension, whenever it exists, it must be unique. This is a consequence of the special form of the skein relation (2). However, the most difficult step is to prove that the canonical combinatorial extension from $K_n(\alpha, \beta)$ to $K_{n+1}(\alpha, \beta)$ is indeed a well-defined linear functional, which moreover satisfies the condition of trace commutativity.

The method of proof is greatly inspired by [3]. One defines a graph whose vertices are linear combinations on elements of the free group in $n-1$ letters. The edges correspond to pairs of elements which differ by exactly one relation, from the set of relations which present the algebras $K_n(\alpha, \beta)$.

Some of these edges will be given an orientation. The first problem is whether each connected component of this graph has a minimal element for this orientation. We have to understand further whether different descending paths issued from the same vertex will eventually abut on the same element. Notice that whenever there is an unique minimal element in each component one is able to derive a basis for the module $K_n(\alpha, \beta)$.

In order to achieve the existence of minimal elements in each component one has to add a number of extra edges to our former graph. These new edges correspond to other relations satisfied in $K_n(\alpha, \beta)$.

Let us consider the lexicographic order on the letters generating the free group on $n-1$ letters. We want to use the relations in the algebra $K_n(\alpha, \beta)$ as transformations which replace a word by a linear combination of smaller ones. Using recursively this procedure the initial word is simplified until it reaches a normal form, where no more simplifications are possible.

The simplification procedure is encoded in the oriented paths of the graph: each relation used as above is an oriented edge of our graph. Specifically, these are given by the following monomial substitutions:

$$\text{(C0)(j)}: \ ab_j^3 b \to \alpha\, ab_j^2 b + \beta\, ab_j b + ab \tag{3}$$

$$\text{(C1)(j)}: \ ab_{j+1} b_j b_{j+1} b \to ab_j b_{j+1} b_j b \tag{4}$$

$$\text{(C2)(j)}: \ ab_{j+1} b_j^2 b_{j+1} b \to aS_j b \tag{5}$$

$$\text{(C12)(j)}: \ ab_{j+1} b_j^2 b_{j+1}^2 b \to aC_j b \tag{6}$$

$$\text{(C21)(j)}: \ ab_{j+1}^2 b_j^2 b_{j+1} b \to aD_j b \tag{7}$$

where $E_{j+1} = \alpha b_{j+1}^2 + \beta b_{j+1} + 1$, $S_j = b_{j+1} b_j^2 b_{j+1} - R_0(j)$, $C_j = b_j^2 b_{j+1}^2 b_j + \alpha(b_{j+1} b_j^2 b_{j+1} - b_j b_{j+1}^2 b_j) + \beta(b_{j+1} b_j^2 - b_{j+1}^2 b_j)$ and $D_j = b_j b_{j+1}^2 b_j^2 + \alpha(b_{j+1}^2 b_j^2 b_{j+1}^2 - b_j b_{j+1}^2 b_j) + \beta(b_j^2 b_{j+1} - b_j b_{j+1}^2)$, $j \in \{0, \ldots, n-2\}$. Here $R_0(j)$ is the result of translating the indices of all letters in R_0 by $j-1$ units.

Several edges of our graph will remain unoriented. The reason is that the respective relations are not compatible with the lexicographic order. They correspond to the following monomial substitutions:

$$(\mathrm{P_{ij}}): \ ab_ib_jb \to ab_jb_ib, \quad \text{whenever} \quad |i-j|>1 \tag{8}$$

The transformations (3)-(8) will be called reduction or simplification transformations.

Remark that we introduced some extra relations, namely (5) and (6), which are not among the relations of the given presentation of $K_n(\alpha, \beta)$, but which are nevertheless satisfied in $K_n(\alpha, \beta)$. These new relations make the reduction process ambiguous. The reason for introducing them is to insure the existence of descending paths towards some minimal elements even in the case when the graph might contain closed oriented loops.

The next step consists of checking the existence and uniqueness of minimal elements in this semi-oriented graph by means of so-called Pentagon Lemma (see section 2). One notices that one cannot always find a unique minimal element by using directed paths issued from a fixed vertex. Furthermore we shall enlarge our graph to a tower of graphs modeling not one particular algebra $K_n(\alpha, \beta)$ for fixed n, but the set of linear functionals defined on the whole tower $\cup_{n=2}^{\infty} K_n(\alpha, \beta)$ and satisfying certain compatibility conditions, which relate the values taken on $K_n(\alpha, \beta)$ to those on $K_{n+1}(\alpha, \beta)$. The main feature of the tower is that now one can simplify further the minimal elements by recurrence on the level n, until one abuts on $K_0(\alpha, \beta)$. Here the Colored Pentagon Lemma (see section 3) can be applied and the uniqueness of the minimal elements in the tower of graphs is reduced to finitely many algebraic conditions. We will find actually that the main obstructions lie in $K_4(\alpha, \beta)$, as it might be inferred from the study of quadratic Hecke algebras. From a different perspective, we actually proved that a certain linear functional on the tower $\cup_{n=2}^{\infty} K_n(\alpha, \beta)$ is well-defined.

Eventually one has to verify whether the linear functional obtained above satisfies the commutativity conditions for being a Markov trace. One proves that there is only one obstruction to the commutativity, which lies also in $K_4(\alpha, \beta)$.

Summarizing, there are two types of obstructions to the existence of a Markov traces:

- CPC obstructions, coming from the Colored Pentagon Condition, and
- commutativity obstructions.

These algebraic obstructions are polynomials with integer coefficients in the variables α and β, and have been computed by using a computer code and formal calculus. The output of these computations is a set of explicit polynomials, which belong to the principal ideal generated by $H_{(\alpha,\beta)}$. Furthermore, the functional defined above is indeed a Markov trace, when restricting its values to the quotient by this principal ideal.

2. Markov traces on $K_n(\alpha, \beta)$.

2.1. The cubic Hecke algebra $H(Q, 3)$ revisited. The generalized Hecke algebras $H(P, 3)$ could be considered for polynomials P of higher degree by using the same definition as in the cubic case. One notices however that $\dim_{\mathbb{C}} H(P, 3) = \infty$ as soon as the degree of P is at least 6.

Remark 2.1. The structure of the algebras $H(P, n)$ is well-known in the classical case (see [9]) when P is quadratic. They are finite dimensional semi-simple algebras of dimension $n!$, isomorphic (for generic P) to the group algebra of the permutation group on n elements. There is no general theory for higher degree polynomials P, due to their considerable complexity.

In the particular case of cubic Q and $n = 3$ it was shown in [13] the following:

PROPOSITION 2.1. *If Q is a cubic polynomial with $Q(0) \neq 0$ then $\dim_{\mathbb{C}} H(Q,3) = 24$. A convenient base of the vector space $H(Q,3)$ is:*

$e_1 = 1$, $e_2 = b_1$, $e_3 = b_1^2$, $e_4 = b_2$, $e_5 = b_2^2$, $e_6 = b_1b_2$, $e_7 = b_2b_1$, $e_8 = b_1^2b_2$, $e_9 = b_2b_1^2$, $e_{10} = b_1b_2^2$, $e_{11} = b_2^2b_1$, $e_{12} = b_1^2b_2^2$, $e_{13} = b_2^2b_1^2$, $e_{14} = b_1b_2b_1$, $e_{15} = b_1^2b_2b_1$, $e_{16} = b_1b_2b_1^2$, $e_{17} = b_1b_2^2b_1^2$, $e_{18} = b_1^2b_2b_1^2$, $e_{19} = b_1^2b_2^2b_1$, $e_{20} = b_1b_2^2b_1$, $e_{21} = b_1^2b_2^2b_1^2$, $e_{22} = b_2b_1^2b_2$, $e_{23} = b_2b_1^2b_2b_1 = b_1b_2b_1^2b_2$, $e_{24} = b_2b_1^2b_2b_1^2 = b_1b_2b_1^2b_2b_1 = b_1^2b_2b_1^2b_2$

PROPOSITION 2.2. *$H(Q,3)$ is a semi-simple algebra which decomposes generically as $\mathbb{C}^3 \oplus M_2^{\oplus 3} \oplus M_3$, where M_n is the algebra of $n \times n$ matrices. The morphism into $H(Q,3) \to \mathbb{C}^3$ is obtained via the abelianization map. Each one of the three projections $H(Q,3) \to M_2$ factors through the projection $H(Q,3) \to H(P_i,3) = \mathbb{C}^2 \oplus M_2$ onto the quadratic Hecke algebra $H(P_i)$ defined by one divisor P_i of Q.*

Proof. This follows by a direct computation, making use of the following identities ([13]):

$$b_{j+1}b_j^2b_{j+1}b_j = b_jb_{j+1}b_j^2b_{j+1}$$

$$b_{j+1}^2b_j^2b_{j+1} = b_jb_{j+1}^2b_j^2 + \alpha(b_{j+1}b_j^2b_{j+1} - b_jb_{j+1}^2b_j) + \beta(b_j^2b_{j+1} - b_jb_{j+1}^2)$$

$$b_{j+1}b_j^2b_{j+1}^2 = b_j^2b_{j+1}^2b_j + \alpha(b_{j+1}b_j^2b_{j+1} - b_jb_{j+1}^2b_j) + \beta(b_{j+1}b_j^2 - b_{j+1}^2b_j)$$

□

2.2. The algebras $K_n(\alpha,\beta)$. The tower $\cup_{k=1}^{\infty} P(k)$ of quotients of $\cup_{k=1}^{\infty} H(Q,k)$ is homogeneous if any identity $F(b_i, b_{i+1}, \ldots, b_j) = 0$ which holds in $P(j+1)$, remains valid under the translation of indices i.e. $F(b_{i+k}, b_{i+k+1}, \ldots, b_{j+k}) = 0$, for all $k \in \mathbb{Z}$ such that $k \geq 1 - i$. If one seeks for Markov traces on towers of quotients of $\mathbb{C}[B_n]$ it is convenient to restrict ourselves to the study of homogeneous quotients.

We define $K_n(\alpha,\beta)$ as the homogeneous quotient $H(Q,n)/I_n$, where I_n is the two-sided ideal generated by:

$b_jb_{j-1}^2b_j + (\beta^2 - \alpha)b_{j-1}^2b_j^2b_{j-1}^2 + (\alpha^2 - \alpha\beta^2 - \beta)b_{j-1}b_j^2b_{j-1}^2 + (\alpha^2 - \alpha\beta^2 - \beta)b_{j-1}^2b_j^2 b_{j-1} + (\alpha^2 - \alpha\beta^2)b_{j-1}^2b_jb_{j-1}^2 + (1 + 2\alpha\beta + \alpha^2\beta^2 - \alpha^3)b_{j-1}b_j^2b_{j-1} + (1 + \alpha\beta + \alpha^2\beta^2 - \alpha^3)b_{j-1}b_jb_{j-1}^2 + (1 + \alpha\beta + \alpha^2\beta^2 - \alpha^3)b_{j-1}^2b_jb_{j-1} + (1 + 2\alpha\beta - \beta^3)b_j^2b_{j-1}^2 + (1 + 2\alpha\beta - \beta^3)b_{j-1}^2b_j^2 + (\alpha\beta^3 - 2\alpha - 2\alpha^2\beta)b_jb_{j-1}^2 + (\alpha\beta^3 - 2\alpha - 2\alpha^2\beta)b_{j-1}^2b_j + (\alpha\beta^3 - 2\alpha - 2\alpha^2\beta + \beta^2)b_j^2b_{j-1} + (\alpha\beta^3 - 2\alpha - 2\alpha^2\beta + \beta^2)b_{j-1}b_j^2 + (\alpha^4 - \alpha^3\beta^2 - 2\alpha^2\beta - 3\alpha)b_{j-1}b_jb_{j-1} + (2\alpha^3\beta + 3\alpha^2 - \alpha^2\beta^3 - \alpha\beta^2)b_jb_{j-1} + (2\alpha^3\beta + 3\alpha^2 - \alpha^2\beta^3 - \alpha\beta^2)b_{j-1}b_j + (\beta^4 - 2\beta - 3\alpha\beta^2 + \alpha^2)b_{j-1}^2 + (\beta^4 - 2\beta - 3\alpha\beta^2 + \alpha^2)b_j^2 + (1 + 4\alpha\beta + 3\alpha^2\beta^2 - \alpha^3 - \alpha\beta^4 - \beta^3)b_{j-1} + (1 + 3\alpha\beta + 3\alpha^2\beta^2 - \alpha^3 - \alpha\beta^4)b_j + 3\beta^2 - \beta^5 - 2\alpha - 3\alpha^2\beta + 4\alpha\beta^3$
for $j \in \{1, \ldots, n-1\}$.

PROPOSITION 2.3. *Under the identification $H(Q,3) \cong \mathbb{C}^3 \oplus M_2^{\oplus 3} \oplus M_3$, the quotient $K_3(\alpha,\beta)$ corresponds to $M_2^{\oplus 3} \oplus M_3$.*

Proof. In fact, it suffices to show that the ideal I_3 is a vector space of dimension 3. Let I be the span of R_0, R_1, R_2, where:

$R_0 = b_2b_1^2b_2 + (\beta^2 - \alpha)\, b_1^2b_2^2b_1^2 + (\alpha^2 - \alpha\beta^2 - \beta)b_1b_2^2b_1^2 + (\alpha^2 - \alpha\beta^2 - \beta)b_1^2b_2^2b_1 + (\alpha^2 - \alpha\beta^2)b_1^2b_2b_1^2 + (1 + 2\alpha\beta + \alpha^2\beta^2 - \alpha^3)b_1b_2^2b_1 + (1 + \alpha\beta + \alpha^2\beta^2 - \alpha^3)b_1b_2b_1^2 +$

$(1+\alpha\beta+\alpha^2\beta^2-\alpha^3)b_1^2b_2b_1$ $+(1+2\alpha\beta-\beta^3)b_2^2b_1^2+(1+2\alpha\beta-\beta^3)b_1^2b_2^2+$ $(\alpha\beta^3-2\alpha-2\alpha^2\beta)b_2b_1^2$ $+(\alpha\beta^3-2\alpha-2\alpha^2\beta)b_1^2b_2+(\alpha\beta^3-2\alpha-2\alpha^2\beta+\beta^2)b_2^2b_1$ $+(\alpha\beta^3-2\alpha-2\alpha^2\beta+\beta^2)b_1b_2^2+$ $(\alpha^4-\alpha^3\beta^2-2\alpha^2\beta-3\alpha)b_1b_2b_1+(2\alpha^3\beta+3\alpha^2-\alpha^2\beta^3-\alpha\beta^2)b_2b_1+$ $(2\alpha^3\beta+3\alpha^2-\alpha^2\beta^3-\alpha\beta^2)b_1b_2+$ $(\beta^4-2\beta-3\alpha\beta^2+\alpha^2)b_1^2+(\beta^4-2\beta-3\alpha\beta^2+\alpha^2)b_2^2$ $+(1+4\alpha\beta+3\alpha^2\beta^2-\alpha^3-\alpha\beta^4-\beta^3)b_1$ $+(1+3\alpha\beta+3\alpha^2\beta^2-\alpha^3-\alpha\beta^4)b_2+3\beta^2-\beta^5-2\alpha-3\alpha^2\beta+4\alpha\beta^3$

$R_1=b_1R_0=b_1b_2b_1^2b_2-\beta b_1^2b_2^2b_1^2+(1+\alpha\beta)b_1b_2^2b_1^2+(1+\alpha\beta)b_1^2b_2^2b_1^2$ $+(1+\alpha\beta)b_1^2b_2b_1^2$ $(-\alpha^2\beta-2\alpha)b_1b_2^2b_1+(-\alpha^2\beta-2\alpha)b_1b_2b_1^2+$ $(-\alpha^2\beta-2\alpha)b_1^2b_2b_1+$ $(\beta^2-\alpha)b_2^2b_1^2+(\beta^2-\alpha)b_1^2b_2^2+$ $(\alpha^2-\alpha\beta^2)b_2b_1^2+$ $(\alpha^2-\alpha\beta^2)b_1^2b_2+(\alpha^2-\alpha\beta^2-\beta)b_2^2b_1+$ $(\alpha^2-\alpha\beta^2-\beta)b_1b_2^2+$ $(\alpha^3\beta+\beta+3\alpha^2)b_1b_2b_1+(1+\alpha\beta+\alpha^2\beta^2-\alpha^3)b_2b_1+$ $(1+\alpha\beta+\alpha^2\beta^2-\alpha^3)b_1b_2+$ $(1+2\alpha\beta-\beta^3)b_1^2+(1+2\alpha\beta-\beta^3)b_2^2+(\alpha\beta^3-2\alpha-2\alpha^2\beta+\beta^2)b_1$ $+(\alpha\beta^3-2\alpha-2\alpha^2\beta)b_2$ $+\beta^4-2\beta-3\alpha\beta^2+\alpha^2$

$R_2=b_1R_1=b_1^2b_2b_1^2b_2+$ $b_1^2b_2^2b_1^2-\alpha b_1b_2^2b_1^2-\alpha b_1^2b_2^2b_1^2$ $-\alpha b_1^2b_2b_1^2+$ $\alpha^2b_1b_2^2b_1+(\alpha^2+\beta)b_1b_2b_1^2+(\alpha^2+\beta)b_1^2b_2b_1+$ $(-\beta)b_2^2b_1^2+(-\beta)b_1^2b_2^2+(1+\alpha\beta)b_2b_1^2+$ $(1+\alpha\beta)b_1^2b_2+(1+\alpha\beta)b_2^2b_1+$ $(1+\alpha\beta)b_1b_2^2+$ $(-\alpha^3\beta-\alpha\beta+1)b_1b_2b_1+(-\alpha^2\beta-2\alpha)b_2b_1+$ $(-\alpha^2\beta-2\alpha)b_1b_2+$ $(\beta^2-\alpha)b_1^2+(\beta^2-\alpha)b_2^2+(-\alpha\beta^2+\alpha^2-\beta)b_1+$ $(-\alpha\beta^2+\alpha^2)b_2+1+2\alpha\beta-\beta^3$

LEMMA 2.1. *There is an isomorphism of vector spaces $I\cong I_3$.*

Proof. Remark first that the following identities hold true in $H(Q,3)$:

$$b_1R_0=R_0b_1=R_1,\ b_1R_1=R_1b_1=R_2,\ b_1R_2=R_2b_1=R_0+\beta R_1+\alpha R_2$$

Then, by direct computation, we obtain that:

$$b_2R_0=R_0b_2=R_1,\ b_2R_1=R_1b_2=R_2,\ b_2R_2=R_2b_2=R_0+\beta R_1+\alpha R_2$$

From these relations we derive that $xR_0y\in I$ for all $x,y\in H(Q,3)$, and hence $I_3\subset I$. The other inclusion is immediate. □

The proposition is then a consequence of the previous lemma. □

2.3. Uniqueness of the Markov trace on the tower $\cup_{n=1}^{\infty}K_n(\alpha,\beta)$. From now on we will work with the group ring $\mathbb{Z}[\alpha,\beta][B_n]$ instead of $\mathbb{C}[B_n]$.

DEFINITION 2.1. Let $z,\bar{z}\in\mathbb{Z}(\alpha,\beta)$ be rational functions in the variables α and β, and R a $\mathbb{Z}[\alpha,\beta,z,\bar{z}]$-module. The linear functional $\mathcal{T}:\cup_{n=1}^{\infty}K_n(\alpha,\beta)\to R$ is said to be an admissible functional (with parameters z and $\bar{z}$) on $\cup_{n=1}^{\infty}K_n(\alpha,\beta)$ if the following conditions are fulfilled:

$$\mathcal{T}(xb_ny)=z\mathcal{T}(xy)\quad\text{for all }\ x,y\in K_n(\alpha,\beta)$$

$$\mathcal{T}(xb_n^{-1}y)=\bar{z}\mathcal{T}(xy)\quad\text{for all }\ x,y\in K_n(\alpha,\beta)$$

An admissible functional $\mathcal{T}$ is a Markov trace if it satisfies the following trace condition:

$$\mathcal{T}(ab)=\mathcal{T}(ba)\ \text{ for any }\ a,b\in K_n(\alpha,\beta)$$

REMARK 2.2. The tower of quadratic Hecke algebras admits an unique Markov trace ([16]). Similarly, the tower of Birman-Wenzl algebras ([5]) admits an unique Markov trace.

DEFINITION 2.2. The admissible functional $\mathcal{T}$ is multiplicative if $\mathcal{T}(xb_n^k) = \mathcal{T}(x)\mathcal{T}(b_n^k)$ holds for all $x \in H(Q,n)$ and $k \in \mathbb{Z}$.

REMARK 2.3. The Markov trace on the quadratic Hecke algebras is multiplicative, and hence $\mathcal{T}(xy) = \mathcal{T}(x)\mathcal{T}(y)$ for any $x \in H(Q,n)$ and $y \in \langle 1, b_n, b_{n+1}, \ldots, b_{n+k}\rangle$. However, one cannot expect that this property holds true for Markov traces on arbitrary higher degree Hecke algebras.

PROPOSITION 2.4. *The admissible functionals on the tower of cubic Hecke algebras are multiplicative. In particular:*

$$\mathcal{T}(ab_n^2 b) = t\mathcal{T}(ab) \quad \text{for all } a, b \in H(Q,n)$$

where $t = \alpha z + \beta + \bar{z}$.

Proof. One uses the identity $b_n^2 = \alpha b_n + \beta + b_n^{-1}$ for proving the multiplicativity for $k = 2$, and then continue by recurrence for all k. □

One can state now the unique extension property of Markov traces.

PROPOSITION 2.5. *For fixed* (z, t) *there exists at most one Markov trace on* $K_n(\alpha, \beta)$ *with parameters* (z, t).

Proof. Define recursively the modules L_n as follows:

$$L_2 = H(Q,2)$$

$$L_3 = \mathbb{C}\langle b_1^i b_2^j b_1^k \mid \text{where } i,j,k \in \{0,1,2\}\rangle$$

$$L_{n+1} = \mathbb{C}\langle ab_n^{\varepsilon} b \mid \text{where } a, b \text{ are elements of the basis of } L_n, \text{ and } \varepsilon \in \{1,2\}\rangle \oplus L_n$$

LEMMA 2.2. *The natural projection* $\pi : L_n \to K_n(\alpha, \beta)$ *is surjective.*

Proof. For $n = 2$ it is clear. For $n = 3$ we know that $b_2 b_1^2 b_2,\ b_1 b_2 b_1^2 b_2,\ b_1^2 b_2 b_1^2 b_2 \in \pi(L_3)$, from the exact form of the relations R_0, R_1, R_2, generating the ideal I_3. We shall use a recurrence on n and assume that the claim holds true for n.

Consider now $w \in K_{n+1}(\alpha, \beta)$ represented by a word in the b_i's having only positive exponents. We assume that the degree of the word in the variable b_n is minimal among all linear combinations of words (with positive exponents) representing w.

(1) If this degree is less or equal to 1 then there is nothing to prove.
(2) If the degree is 2 then either $w = ub_n^2 v$, $u, v \in K_n(\alpha, \beta)$ so using the induction hypothesis we are done, or else $w = ub_n z b_n v$, where $u, z, v \in K_n(\alpha, \beta)$. Therefore $z = xb_{n-1}^{\varepsilon} y$ where $x, y \in K_{n-1}(\alpha, \beta)$ by the induction hypothesis and $\varepsilon \in \{0, 1, 2\}$.
   (a) If $\varepsilon = 0$ then w can be reduced to $uzb_n^2 v$.
   (b) If $\varepsilon = 1$ then $w = ub_n x b_{n-1} y b_n v = uxb_{n-1} b_n b_{n-1} yv$ hence the degree of w can be lowered by one, which contradicts our minimality assumption.
   (c) If $\varepsilon = 2$ then $w = uxb_n b_{n-1}^2 b_n yv$. One derives that:

$$b_n b_{n-1}^2 b_n \in \mathbb{C}\langle b_{n-1}^i b_n^j b_{n-1}^k,\ i,j,k \in \{0,1,2\}\rangle$$

   hence we reduced the problem to the case when w is a word of type $u' b_n^2 v'$.

(3) If the degree of w is at least 3 we will contradict the minimality assumption. In fact, in this situation w will contain either a sub-word $w' = b_n^a u b_n^b$, with $u \in K_n(\alpha, \beta)$ and $a + b \geq 3$, or else a sub-word $w'' = b_n u b_n v b_n$, with $u, v \in K_n(\alpha, \beta)$.
   (a) In the first case using the induction we can write $u = x b_{n-1}^{\varepsilon} y$, with $x, y \in K_{n-1}(\alpha, \beta)$.
      (i) Furthermore, if $\varepsilon = 0$ then $w' = b_n^{a+b} xy = \alpha b_n^{a+b-1} xy + \beta b_n^{a+b-2} xy + b_n^{a+b-3} xy$, and hence the degree of w can be lowered by one.
      (ii) If $\varepsilon = 1$ then $w' = b_n^{a-1} x b_n b_{n-1} b_n y b_n^{b-1} = b_n^{a-1} x b_{n-1} b_n b_{n-1} y b_n^{b-1}$, and again its degree can be reduced by one unit.
      (iii) If $\varepsilon = 2$ then either a or b is equal 2. Assume that $a = 2$. We can therefore write:

$$w' = x b_n^2 b_{n-1}^2 b_n y b_n^{b-1} = x b_{n-1} b_n^2 b_{n-1}^2 y b_n^{b-1} + \alpha(b_n b_{n-1}^2 b_n - b_{n-1} b_n^2 b_{n-1}) y b_n^{b-1} + \beta(b_{n-1}^2 b_n - b_{n-1} b_n^2) y b_n^{b-1}$$

      contradicting again the minimality of the degree of w.
   (b) In the second case we can write also $u = x b_{n-1}^{\varepsilon} y$, $v = r b_{n-1}^{\delta} s$ with $x, y, r, s \in K_{n-1}(\alpha, \beta)$.
      (i) If ε or δ equals 1 then, after some obvious commutations the word w" contains the sub-word $b_n b_{n-1} b_n$ which can be replaced by $b_{n-1} b_n b_{n-1}$ and hence diminishing its degree.
      (ii) If $\varepsilon = \delta = 2$ then $w" = x b_n b_{n-1}^2 b_n y r b_{n-1}^2 b_n s$. We use the homogeneity to replace $b_n b_{n-1}^2 b_n$ by a sum of elements of type $b_{n-1}^i b_n^j b_{n-1}^k$. Each term of the expression of w" which comes from a factor which has the exponent $j < 2$, has diminished its degree. The remaining terms are $x b_{n-1}^i b_n^2 b_{n-1}^k y r b_{n-1}^2 b_n s$, so they contains a sub-word $b_n^2 u b_n$ whose degree we already know that it can be reduced as above. This proves our claim.

□

Eventually recall that the Markov traces $\mathcal{T}$ on $\cup_{n=1}^{\infty} H(Q, n)$ are multiplicative, and hence they satisfy: $\mathcal{T}(x b_n^{\varepsilon} y) = \mathcal{T}(b_n^{\varepsilon}) \mathcal{T}(yx)$. Therefore there is a unique extension of $\mathcal{T}$ from $K_n(\alpha, \beta)$ to $K_{n+1}(\alpha, \beta)$. This ends the proof of our proposition. □

Proposition 2.6. *The admissible functionals on the tower of algebras* $\cup_{n=1}^{\infty} K_n(\alpha, \beta)$ *satisfy the identities:*

$$\mathcal{T}(xuv) = \mathcal{T}(u)\mathcal{T}(xv) \quad \text{for } x, v \in H(Q, m) \quad \text{and } u \in \langle 1, b_m, b_{m+1}, \ldots, b_{m+k} \rangle$$

Proof. For $k = 0$ this is equivalent to the multiplicativity of the admissible functional. We will use a recurrence on k and assume that the claim holds true for k. By lemma 2.2 one can reduce the element u in $K_{m+k+1}(\alpha, \beta)$ to a (non-necessarily unique) normal form $u = u_1 b_{m+k}^{\varepsilon} u_2$, where $u_j \in \langle 1, b_m, b_{m+1}, \ldots, b_{m+k} \rangle$, $j \in \{1, 2\}$ and $\varepsilon \in \{0, 1, 2\}$. The multiplicativity of the admissible functionals implies that:

$$\mathcal{T}(xuv) = \mathcal{T}(b_{m+k}^{\varepsilon}) \mathcal{T}(x u_1 u_2 v)$$

By the recurrence hypothesis one knows that:

$$\mathcal{T}(x u_1 u_2 v) = \mathcal{T}(u_1 u_2) \mathcal{T}(xv)$$

and since:

$$\mathcal{T}(u) = \mathcal{T}(b_{m+k}^{\varepsilon})\mathcal{T}(u_1u_2)$$

one derives our claim. □

3. CPC Obstructions.

3.1. The pentagonal condition. The following is an immediate consequence of lemma 2.2:

LEMMA 3.1. *There is a surjection of* $(K_n(\alpha,\beta), K_n(\alpha,\beta))$*-bimodules:*

$$K_n(\alpha,\beta) \oplus K_n(\alpha,\beta) \otimes_{K_{n-1}(\alpha,\beta)} K_n(\alpha,\beta) \oplus K_n(\alpha,\beta) \otimes_{K_{n-1}(\alpha,\beta)} K_n(\alpha,\beta) \longrightarrow K_{n+1}(\alpha,\beta)$$

given by:

$$x \oplus y \otimes z \oplus u \otimes v \to x + yb_nz + ub_n^2v$$

REMARK 3.1. In particular, the admissible functionals on the tower $\cup_{n=1}^{\infty} K_n(\alpha,\beta)$ are unique up to the choice of $\mathcal{T}(1) \in R$.

Now, we want to use the transformations (3)-(7) to simplify the positive words from $K_n(\alpha,\beta)$, so that the degree of b_{n-1} becomes as small as possible. According to the previous lemma every word in $K_n(\alpha,\beta)$ can be written as a linear combination of words of the form $x_i b_{n-1}^{\varepsilon_i} y_i$, with $\varepsilon_i \in \{0,1,2\}$ and $x_i, y_i \in K_{n-1}(\alpha,\beta)$. Unfortunately, one needs to use in both directions the transformations P_{ij} from (8): $b_ib_j \leftrightarrow b_jb_i$, for $|i-j| > 1$.

REMARK 3.2. The linear combination we obtained above is a kind of *normal form* for the word with which we started. It could happen that this normal form is not unique since we may perform again permutations of type (8) among some of its letters. However, if any two such normal forms were equivalent under the transformations (8), then we would obtain an almost canonical description of the basis of $K_n(\alpha,\beta)$. This assumption is equivalent to saying that the surjection from lemma 3.1 is an isomorphism. Unfortunately, this is not the case. However, one can describe the obstructions to the uniqueness for this almost canonical form, as follows.

We return now to the module of the admissible functionals on the whole tower of algebras $\cup_{n=1}^{\infty} K_n(\alpha,\beta)$. The conditions satisfied by admissible functionals enable us to add a new type of simplifications, by means of the following formulas:

$$ab_{n-1}b \to z\,ab, \text{ and respectively } ab_{n-1}^2b \to t\,ab, \text{ where } a,b \in K_{n-1}(\alpha,\beta) \qquad (9)$$

This way we can reduce a word from $K_n(\alpha,\beta)$ to a linear combination of words from $K_{n-1}(\alpha,\beta)$. Assume that we are using repeatedly the transformations (9). Then we will eventually reduce the initial word to a linear combinations of words in $K_0(\alpha,\beta)$, thus to an element of R. Remark that this element is actually the value that the admissible functional takes on the initial word. Our main task is to understand whether the final reduction is independent on the way we chose to make the simplifications. When this happens to be true then we obtain that the functional which associates to each element of $K_n(\alpha,\beta)$ its final reduction is a well-defined admissible functional.

However, we will encounter below some obstructions to the uniqueness, which fortunately we can treat explicitly.

One formalizes this procedure at follows. Let Γ be a semi-oriented graph. This means that some of its edges are oriented while the remaining ones are left unoriented. We write $v \to w$ if there is an oriented edge from v to w. A path $v_1 v_2 \cdots v_n$ in Γ is called a *semi-oriented path* if, for each j, one has either $v_j \to v_{j+1}$ or else $v_j v_{j+1}$ is an unoriented edge of Γ. If all edges of the path are unoriented then we say that its endpoints are *(weakly) equivalent.*

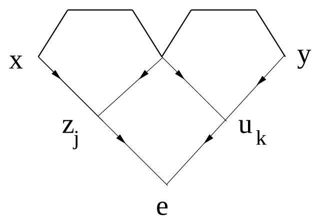
DEFINITION 3.1. The sequence of vertices $[v_0, v_1, \ldots, v_{n+1}]$ is an *open pentagon configuration* in Γ (abbreviated o.p.c.) if $v_1 \to v_0$, $v_1 v_2 \cdots v_{n-1}$ is an unoriented path and $v_n \to v_{n+1}$.

DEFINITION 3.2. The semi-oriented graph Γ verifies the pentagon condition (abbreviated PC) if for any open pentagon configuration $[v_0, v_1, \ldots, v_{n+1}]$ there exist semi-oriented paths $v_0 x_1 x_2 \cdots x_m e$ and $v_{n+1} y_1 y_2 \cdots y_p e$ having the same endpoint.

Given a graph like above one has a binary relation induced as follows: we set $x \le y$ if there exists an semi-oriented path from y to x in Γ. Of course $\le$ is not always a partial order relation. A necessary and sufficient condition for $\le$ to be a partial order is that Γ contains no closed semi-oriented closed loops. One says that x is *minimal* if $y \le x$ implies that y is weakly equivalent to x.

LEMMA 3.2. *Suppose that the (PC) holds. If a connected component C of the graph Γ has a minimal element then this is unique up to weak equivalence.*

Proof. Consider two minimal elements x and y which lie in C. Then there exists some path $x x_0 x_1 \cdots x_n y$ joining them. Since x is minimal the closest oriented edge - if it exists - must be in-going; and the same is true for y. If this path is not unoriented, then the minimality implies that there are at least two oriented edges. Therefore one can find a sequence of open pentagon configurations lining on the path which joins x to y. We apply then the (PC) iteratively, whenever we see one such o.p.c., or one o.p.c. appears at the next stage, as in the figure below:



When this process stops, we find two semi-oriented paths $x z_1 z_2 \cdots z_p e$ and $y u_1 u_2 \cdots u_s e$ having the same endpoint e. So $e \le x$ and $e \le y$. From minimality both these paths must be unoriented, and thus x and y are weakly equivalent. ☐

REMARK 3.3. The existence of minimal elements is not *a priori* granted, without additional conditions. If $\le$ had been a partial order with descending chain condition, then the existence of minimal elements would be standard. We will show that in the present case, of the graph modeling the admissible functionals on the tower $\cup_{n=1}^{\infty} K_n(\alpha, \beta)$, such minimal elements exist, though as $\le$ is not a partial order.

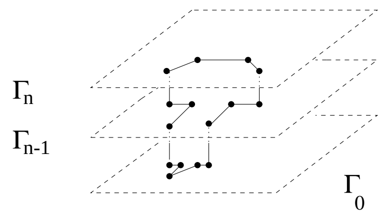
3.2. The colored tower of graphs Γ_n^*. Suppose now that we have a family of semi-oriented graphs Γ_n as follows. Each graph Γ_n has a distinguished subset of vertices V_n^0 whose elements are minimal elements in their connected respective components. Assume also that each connected component of the Γ_n admits at least one minimal element. Further, we suppose that each vertex from V_n^0 has exactly one outgoing edge which joins it to a vertex of Γ_{n-1}. We color the edges connecting graphs Γ_n and Γ_{n-1} in red. Set Γ_n^* for the union of all Γ_j, with $j \leq n$ to which we add all red edges connecting graphs Γ_k and Γ_{k+1}, for $k \leq n-1$. We can have an intuitive view of Γ_n^* by looking at the Γ_n as graphs lying on different floors which are connected by vertical red edges pointing downwards.

DEFINITION 3.3. The graph Γ_n^* is *coherent* if any connected component of Γ_n has an unique minimal element within Γ_n^*, up to weak equivalence.

REMARK 3.4. A minimal element should belong to Γ_0.

We state now the *colored* version of the Pentagon Lemma for this type of graphs.

DEFINITION 3.4. We say that Γ_n verifies the colored pentagon condition (CPC) if, for any open pentagon configuration $[v_0, v_1, \ldots, v_{m+1}]$ in Γ_n, there exist bicolored semi-oriented paths (in Γ_n^*) from v_0 and v_{m+1} having the same endpoint. In addition, if xy is an unoriented edge in Γ_n with $x, y \in V_n^0$ then there exist semi-oriented paths in Γ_n^* starting with red edges and having the same endpoint, as in the figure below:



LEMMA 3.3. *Suppose that Γ_{n-1}^* is coherent and the (CPC) condition is fulfilled. Then Γ_n^* is coherent.*

Proof. The proof is similar to that of Pentagon Lemma. □

Now, we are ready to define the sequence of semi-oriented graphs Γ_n, which models the admissible functionals on $\cup_{n=1}^{\infty} K_n(\alpha, \beta)$.

DEFINITION 3.5. The vertices of Γ_n are the elements of the ring algebra $\mathbb{Z}[\alpha, \beta, z, \bar{z}][F_n]$, where F_n is the free monoid F_{n-1} generated by $n-1$ letters $\{b_1, b_2, \ldots, b_{n-1}\}$. The vertices of Γ_0 are the elements of $\mathbb{Z}[\alpha, \beta, z, \bar{z}]$. Two vertices $v = \sum_i \alpha_i \ x_i$ and $w = \sum_i \beta_i \ y_i$, where $\alpha_i, \beta_i \in \mathbb{Z}[\alpha, \beta, z, \bar{z}]$ and $x_i, y_i \in F_n$, are related by an oriented edge if exactly one monomial x_i of v is changed by means of a reduction transformation among the rules (3)-(7). An unoriented edge between v and w corresponds to a simplification transformation (8) of one monomial x_i from the previous expression of v.

REMARK 3.5. The use of (C12) and (C21) is somewhat ambiguous since we can always use (C2) for a sub-word of the given word. Their role is to break in some sense the closed oriented loops in Γ_n, as we shall see below.

Consider now the following sets of words in the b_i's:

$$W_0 = \{1\}$$

$$W_{n+1} = W_n \cup W_n b_{n+1} Z_n \cup W_n b_{n+1}^2 Z_n$$

where:

$$Z_n = \{b_n^{i_0} b_{n-1}^{i_1} \cdots b_{n-p}^{i_p} | \text{ where the indices } i_1, i_2, \ldots, i_p \in \{1,2\}, \\ \text{and } p \in \{0,1,\ldots,n-1\}\}.$$

Let V_n^0 be the set of vertices corresponding to elements of the $\mathbb{Z}[\alpha,\beta,z,\bar{z}]$-module generated by W_n. This completes the definition of the tower of graphs Γ_n. We have the following result:

PROPOSITION 3.1. *Each connected component of* Γ_n *has a minimal element in* V_n^0*, not necessarily unique.*

Proof. We use an induction on n. For $n = 0$ the claim is obvious. Let now w be a word in the b_i's having only positive exponents.

(1) If its degree in b_n is zero or one, then we apply the induction hypothesis and we are done.

(2) If the degree in b_n is 2 and w contains the sub-word b_n^2, then again we are able to apply the induction hypothesis.

(3) By using (C0) several times one can also suppose that no exponents greater than 2 occur in w.

   (a) If the degree of b_n is 2 then $w = xb_nyb_nz$ with $x,y,z \in F_{n-1}$. The induction hypothesis applied to y implies that $w \geq xb_nab_{n-1}^{\varepsilon}bz$ with $a,b \in F_{n-1}$. Then several transforms of type (P_{nj}) and (Cε) will do the job.

   (b) Consider now that the degree in b_n is at least 3. Then w contains a sub-word which has either the form $b_n^{\alpha}xb_n^{\beta}$ with $3 \leq \alpha+\beta \leq 4$, or else one of the type $b_nxb_nyb_n$. The second case reduce to the first one as above. In the first case assume that $x \geq ab_{n-1}^{\varepsilon}b$ for some $a,b \in F_{n-2}$. Then several applications of (P_{nj}) lead us to consider the sub-word $b_n^{\alpha}b_{n-1}^{\varepsilon}b_n^{\beta}$.

      (i) If $\varepsilon = 1$ we use two times (C1) and we are done.

      (ii) Otherwise use either (C$\alpha\beta$) and then (C1) if $\alpha \neq \beta$ or else both (C12) and (C21) and then (C1), if $\alpha = \beta = 2$.

This proves that every vertex descends to V_n^0. But these vertices have not outgoing edges, as can be easily seen. When we use the unoriented edges some new vertices have to be added. But it is easy to see that these new vertices do not have outgoing edges either. Since any vertex has a semi-oriented path ending in V_n^0 our claim follows. $\square$

REMARK 3.6. The moves (C12) and (C21) are really necessary for the conclusion of proposition 3.1 hold true. For instance look at the case $\alpha = \beta = 0$. From $b_{j+1}b_j^2b_{j+1}^2$ only (C2) can be applied; its reduction is a linear combination containing the factor $b_{j+1}^2b_j^2b_{j+1}$. If we continue, then we shall find at each stage one of these two monomials. Moreover, after making all possible reductions at the second stage, we recover the word $b_{j+1}b_j^2b_{j+1}^2$. Therefore there exist closed oriented loops in the graph. In particular the connected component of $b_{j+1}b_j^2b_{j+1}^2$ has no minimal element, unless we enlarge the graph by adding the extra edges associated to (C12) and (C21). For general α,β a similar argument holds and it can be checked by a computer program. If one does not use (C12) or (C21) then the reduction process for $b_{j+1}b_j^2b_{j+1}^2$ yields at the sixth stage a sum of words generating an oriented loop.

We are able now to define the bicolored graph $\Gamma_n^*(H)$, where the non-uniqueness of the reduction process is measured by means of an ideal $H \subset R$.

DEFINITION 3.6. Consider a minimal vertex of Γ_n which can therefore be written as the linear combination: $v = \sum_{i,k} \lambda_{i,k}(x_{i,k} b_n^k y_{i,k})$, where $k \in \{0, 1, 2\}$, $x_{i,k}, y_{i,k}$ are words from F_{n-1} and $\lambda_{i,k}$ are scalars. Then we join v by an oriented red edge to the vertex of Γ_{n-1} which corresponds to the linear combination:

$$w = \sum_i \lambda_{i,0}(x_{(i,0)} y_{i,0)}) + \sum_i z\lambda_{i,1}(x_{i,1} y_{i,1}) + \sum_i t\lambda_{i,2}(x_{i,2} y_{i,2})$$

Finally, the level zero graph $\Gamma_0(H)$ is the graph having the vertices corresponding to the module R. Two vertices of $\Gamma_0(H)$ are connected by an unoriented edge if the corresponding elements lie in the same coset of R/H, where H is a given ideal of R.

REMARK 3.7. The submodule H is necessary because going on different descending paths, we might obtain different elements of R.

4. The coherence conditions for $\Gamma_n^*(H)$.

4.1. General considerations. The purpose of this section is to reduce the coherence test for $\Gamma_n^*(H)$ to finitely many algebraic checks.

We test the coherence conditions for each $\Gamma_n^*(H)$ by recurrence on n. Notice that for $n \in \{1, 2\}$ there are no non-trivial requirements for H.

The coherence test for Γ_n (fixed n) amounts to checking that all open pentagon configurations, which are infinitely many, verify (PC). Moreover the open pentagon configurations themselves can be organized in a pattern which has the additional structure of an algebra, in fact a planar algebra. We will not make use directly of this algebra structure in the sequel. However, it can be inferred from it that it is enough to verify the (PC) only for those o.p.c. which generate this algebra. A detailed analysis of these generators reduces then the test problem to an explicit infinite family of o.p.c. At this point we notice that the (PC) might not hold for all o.p.c. in this family. Now, one enlarges Γ_n to the tower of colored graphs Γ_n^* and look for the weaker (CPC) condition for the last one. Eventually, we show that the (CPC) for these graphs can be reduced to finitely many checks.

The o.p.c. $[w_0, w_1, \ldots, w_{m+1}]$ is said to be *irreducible* if none of the vertices $w_1, w_2, \ldots, w_m$ has an outgoing edge (except the obvious one for w_1 and w_m).

LEMMA 4.1.
(1) *In order to verify (PC) it suffices to restrict to irreducible configurations.*
(2) *It suffices to verify (PC) only for words from F_n.*
(3) *Let $[w_0, w_1, \ldots, w_{m+1}]$ be an o.p.c. and $w'_j = Aw_jB$, for $j \in \{0, \ldots, m+1\}$, where A, B are two arbitrary words. If (PC) holds for $[w_0, w_1, \ldots, w_{m+1}]$, then it holds for $[w'_0, w'_1, \ldots, w'_{m+1}]$.*
(4) *Suppose that (PC) holds for the two o.p.c. $[w_0, w_1, \ldots, w_{m+1}]$ and $[y_0, y_1, \ldots, y_{k+1}]$. Then for all A, B, C the (PC) is valid also for the following mixed o.p.c.:*
$[Aw_0By_1C, Aw_1By_1C, \ldots, Aw_mBy_1C, Aw_mBy_2C, \ldots, Aw_mBy_{k+1}C]$.
More generally, if one keeps fixed the endpoints of the o.p.c., then we can mix the unoriented edges of each subjacent o.p.c. following an arbitrary pattern. Specifically, let $(i_s, j_s) \in \{0, 1, \ldots, m+1\} \times \{0, 1, \ldots, k+1\}, s \in \{1, \ldots, p\}$ such that: $i_0 = 0 < i_1 \leq i_2 \leq \cdots \leq i_p, j_p = k+1 > j_{p-1} \geq \cdots \geq 0$ and $i_{s+1} - i_s + j_{s+1} - j_s = 1$, for all s. Then the o.p.c. $[Aw_{i_0}By_{j_0}C, Aw_{i_1}By_{j_1}C, \ldots, Aw_{i_p}By_{j_p}C]$ fulfills the (PC).

Proof. 1) First, any o.p.c. can be decomposed into irreducible ones. Further, if each irreducible component satisfies the (PC) then their composition verifies, too.

2) The reduction transformations acting on different monomials of a linear combination commute with each other.

3) Obvious.

4) The simplification transformations for w_m and y_1 commute with each other. □

From now on we can restrict ourselves to analyze only those o.p.c. $[w_0, w_1, \dots, w_{m+1}]$ which are irreducible.

4.2. Resolving the diamonds. We consider first the case when the top line is trivial i.e. $m = 1$ and so the pentagon degenerates into a diamond.

Lemma 4.2. *If the top line is trivial then the (PC) holds.*

Proof. By using lemma 4.1 there are only finitely many words w on the top line, to check. Furthermore $w = abc$, where $ab, bc \in \{b_{j+1}^3, b_{j+1}b_jb_{j+1}, b_{j+1}b_j^2b_{j+1}, b_{j+1}^2b_j^2b_{j+1}, b_{j+1}b_j^2b_{j+1}^2\}_{j\in\{1,\dots,n-2\}}$. The number of cases to study can be easily reduced, since:

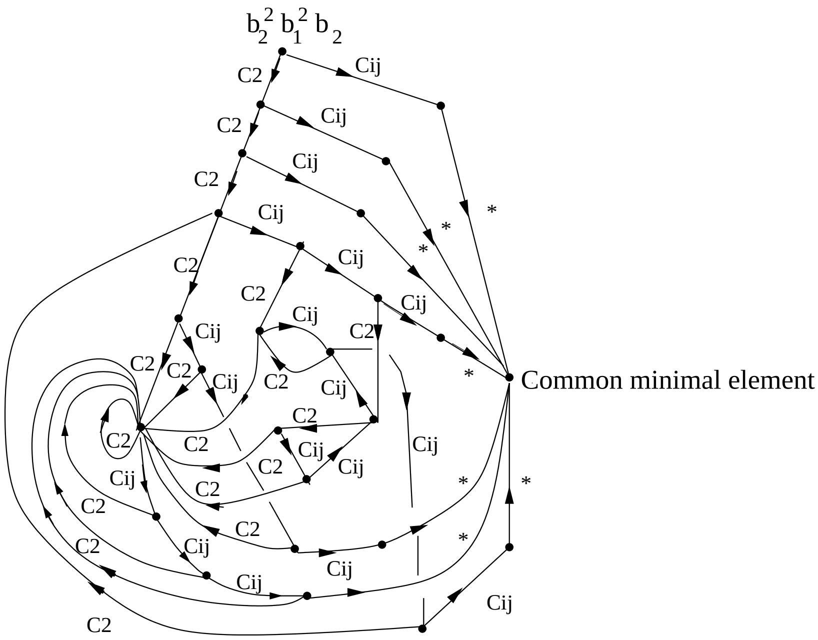
(1) If b is the empty word, then the (PC) holds;
(2) By homogeneity it suffices to consider $j = 1$;
(3) Let $w^* = w_r \cdots w_1$ denote the reversed word associated to $w = w_1 \cdots w_r$. If the (PC) holds for w, then it also holds for w^*;
(4) Several cases, as $b_{j+1}^3 b_j b_{j+1}$, can be easily tested at hand.

The nontrivial situations are those when a (C12)-move (and then a (C2)-move) can be applied. It suffices therefore to check the case of $b_{j+1}^2b_j^2b_{j+1}$, since $b_{j+1}b_j^2b_{j+1}^2$ is its reversed and the remaining $b_{j+1}^{\varepsilon_1}b_j^2b_{j+1}^{\varepsilon_2}$ ($\varepsilon_i \in \{2,3\}$) are consequences of these two. Then we have the situation depicted in the diagram:

$$b_2S_1 \longleftarrow b_2^2b_1^2b_2 \longrightarrow C_1$$

where S_1, C_1 are those from (5-6). If we apply (C12) and (C21) whenever it is possible on b_2S_1, then after a long computation we find a common minimal element for b_2S_1 and C_1. □

Remark 4.1. We used a computer code in order to obtain the complete oriented graph associated to the reductions of $b_2^2b_1^2b_2$:



Its vertices are linear combinations in words in b_1 and b_2. The edges are labeled by the corresponding reduction. When there are no sub-words $b_2^2b_1^2b_2$ or $b_2b_1^2b_2^2$ in the factors of a vertex, its reduction is unique; we marked then the respective edges by an asterix. The label (Cij) stands for the convenient one among (C12) and (C21). As we already noticed in Remark 3.6, if we apply six times the simplification procedure without the use of (Cij)'s then we find a closed loop.

4.3. The diagrams associated to o.p.c. We will be concerned henceforth with the o.p.c. having nontrivial top lines. By Lemma 4.2 we can suppose that w_1 and w_m have each one exactly one outgoing edge. Moreover, an o.p.c. is determined by the following data:

(1) The word $w = w_1$. Assume that w has length k.
(2) The sequence $w_1, \ldots, w_m$, which is encoded in a permutation $\sigma \in S_k$, with a specified decomposition into transpositions.
(3) The two reduction transformations which simplify w and respectively w_m. These should also determine uniquely the blocks of letters in w and w_m to which the transformations apply.

Set T_j for the transposition which interchanges the letters on the positions j and $j+1$. Let $P(w)$ denote the set of those permutations which can be realized on the top line of an o.p.c. having its left upper corner labeled w. Permutations from $P(w)$ will be called permitted permutations. One can characterize them as follows. Let $e_w : \{1, 2, \ldots, l\} \longrightarrow \{1, 2, \ldots, n-1\}$ denote the evaluation map:

$$e_w(j) = \text{ the index of the letter lying on the } j\text{-th position in } w$$

Recall that the index of b_j is j. Consider $\sigma \in P(w)$. Then the permutation $T_j\sigma$ is also permitted if and only if the following inequality holds true:

$$|e_{\sigma(w)}(j) - e_{\sigma(w)}(j+1)| > 1$$

DEFINITION 4.1. Two permitted permutations σ and σ' from $P(w)$, together with their specific decomposition into transpositions, are said to be equivalent if the (PC) holds true or fails for their associated o.p.c., simultaneously.

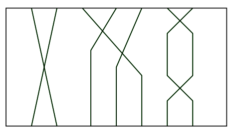
LEMMA 4.3.
(1) *Suppose that* $\sigma_1 T_j T_i \sigma_2 \in P(w)$, $|i-j| > 1$. *Then* $\sigma_1 T_i T_j \sigma_2 \in P(w)$ *and these two permutations are equivalent.*
(2) *Suppose that* $\sigma_1 T_{i+1} T_i T_{i+1} \sigma_2 \in P(w)$. *Then* $\sigma_1 T_i T_{i+1} T_i \sigma_2 \in P(w)$ *and these two permutations are equivalent. The converse is still true.*
(3) *If* $\sigma_1 T_i T_i \sigma_2 \in P(w)$ *then* $\sigma_1\sigma_2$ *is permitted and equivalent to the previous one.*

Proof. The existence in the first case is equivalent to $|e_{\sigma_2(w)}(j) - e_{\sigma_2(w)}(j+1)| > 1$ and $|e_{\sigma(w)}(i) - e_{\sigma(w)}(i+1)| > 1$, so it is symmetric. In the second case also it is equivalent to $|e_{\sigma_2(w)}(j+\varepsilon_1) - e_{\sigma_2(w)}(j+\varepsilon_2)| > 1$ for all $\varepsilon_j \in \{0,1,2\}$, so it is again symmetric. The equivalence is trivial. □

COROLLARY 4.1. *Two different decompositions into transpositions of the permutation σ lead to equivalent o.p.c.*

We will use a graphical representation for the decomposition of σ into transpositions, similar to the braid pictures (see picture below). We specify on the top and bottom lines of the rectangle the values of the respective evaluation maps. Further, the diagram is made of arcs which connect the points on the top to the points on the bottom having the same indices; these arcs will be called trajectories, or strands in the sequel. We denote by $e(w)$ the vector $(e_w(j))_{j=1,\dots,k}$, which can be seen as a word in the free group (monoid) on $n-1$ letters.

e(w)



e(σ (w))

This picture will be called a *diagram* of the respective o.p.c. Notice that the strands in a diagram inherit a labeling by the common indices of their endpoints. There is also a natural orientation on them, going from the top to the bottom.

The reduction blocks are sets of consecutive endpoints of strands (from three to five) in the upper and lower lines of a given diagram, corresponding to the sub-words on which the simplification transformations acts. We call them accordingly, the top and the bottom block.

We will draw below the incomplete diagram consisting only of those trajectories of the six (to ten) elements which enter in the two reduction blocks.

EXAMPLE 4.1. Suppose for instance that the reductions consist of two transformations of type (C0). This implies that $e(w) = xiiiy$ and $e(\sigma(w)) = x'jjjy'$.
(1) Assume that $i = j$. Then the trajectories of the i's can be assumed to be disjoint since the transposition which invert the letters in the couple ii has trivial effect when looking at the word w and its transformations. Thus the possible trajectories of these six strands fit into the four cases, according to the number of strands connecting the upper and lower blocks, which might be 0, 1, 2 or 3.
(2) Further, if $i \neq j$ we have again two sub-cases.

(a) If $|i - j| = 1$ then the trajectories labeled i must be disjoint from those labeled j, and hence there is only one obvious combinatorics.
(b) If $|i - j| \neq 1$ then there are sixteen diagrams up to isotopy (see [13] for a list).

REMARK 4.2. One can describe all configurations of the strands involved in a pair of reduction transforms (C1)-(C0), (C2)-(C0), (C12)-(C0), (C21)-(C0) (see [13] for an exhaustive list), similar to that from the example above.

DEFINITION 4.2. A diagram is called *interactive* if there is at least one strand connecting the upper and lower blocks.

LEMMA 4.4. *The (PC) holds true for the o.p.c. associated to non-interactive diagrams.*

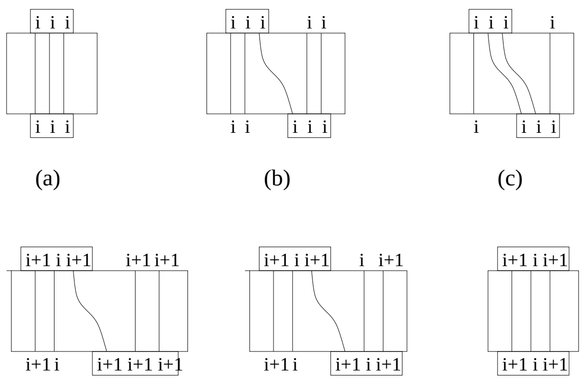
Proof. We call the strands which come or arrive to the reduction blocks essential strands.

(1) If the essential arcs coming from the top block are disjoint from those arriving in the bottom block then $w = xy$, $\sigma(w) = xy'$, where the first block is contained in x and the second one in y'. These two reductions commute with each other.
(2) If there is an essential strand labeled i which intersects some essential strand of the other block, then it will intersect all of them. In particular b_i commutes with all letters of the reduction block. Moreover, a simple verification shows that, if b_i commutes with all letters of the monomial from the left hand side of one formula among (3)-(7), then it will commute with the elements from the right hand side of the same formula. This shows that the commutations depicted in the diagram can be realized after the first reduction transformation (of the upper block). This implies our claim.

□

Therefore, it remains to understand the interactive configurations.

LEMMA 4.5. *It suffices to check the (PC) for those interactive configurations whose essential strands are as following:*



(a) (b) (c)

(d) (e) (f)

REMARK 4.3. We represented in the picture above each block as a sequence of three letters, but some of the letters are allowed to have exponent 2, and thus to represent two letters in a genuine diagram. Moreover, in this situation we require that the two strands coming from two consecutive letters labeled by the same index be parallel, and thus to arrive on two consecutive positions on the bottom line. Therefore, the couple of 2-strands can be identified with one strand in the picture above.

Proof. There are no restrictions arising from the above identification of two parallel strands because their labels are the same. This means that any permutation involving one of the two strands is also allowable for the second one, as well. Thus, we can always get such a normal form for the respective interactive configuration. □

4.4. Solving the o.p.c. associated to non-interactive diagrams. The (PC) is verified in the cases (a),(b),(c),(d) and (f) by direct computation of the first step of their respective simplifications. The only relations needed are the consistency of relations defining the algebra $K_3(\alpha,\beta)$. We skip the details.

Let us check a sub-case of (d), corresponding to the pair of transformations (Cε)-C(0), where $\varepsilon \in \{0,1,2\}$. The monomial to be reduced has the form $w = b_{i+1}b_i^{\varepsilon}b_{i+1}xb_{i+1}^2$, which is weakly equivalent in the o.p.c. to $w' = b_{i+1}b_i^{\varepsilon}xb_{i+1}^3$. We write below $\sim$ for the weak equivalence of words. Notice that all letters of x should commute with b_{i+1} because the respective strands will cross each other. Thus we may suppose that x lies in F_{i-1}. Therefore: $x \to x_0 b_{i-1}^{j_1}b_{i-2}^{j_2}\cdots b_{i-p}^{j_p}$, with $x_0 \in F_{i-2}$. Again, we can restrict ourselves to the situation when $x_0 = 1$. Consider now the case $\varepsilon = 2$ because the other cases are trivially verified. Set $q = b_{i-2}^{j_2}\cdots b_{i-p}^{j_p}$. We have then the following reduction transformations:

$$S_j b_{i-1}^{j_1}b_{i+1}^2 q \longleftarrow w \sim w' \longrightarrow b_{i+1}b_i^2 E_j b_{i-1}^{j_1} q$$

where S_j, E_j as above. From the lemmas 4.1 and 4.2 it follows that the (PC) holds for:

$$S_j b_{i+1}^2 b_{i-1}^{j_1} q \longleftarrow b_{i+1}b_i^2 b_{i+1}^3 b_{i-1}^{j_1} q \longrightarrow b_{i+1}b_i^2 E_j b_{i-1}^{j_1} q$$

Since $S_j b_{i-1}^{j_1} b_{i+1}^2 q$ is weakly equivalent to $S_j b_{i+1}^2 b_{i-1}^{j_1} q$, we are done.

All remaining cases but (e) follow by similar computations. However, for the diagrams of type (e) the situation is different. Using the commutation rules as above one must preserve the term $b_{i-1}^{j_1}$. So, we have to check the configurations where the word w is given by:

$$w = x b_{i+1}^{\alpha} b_i^{\varepsilon} b_{i+1}^{\beta} b_{i-1}^{\mu} b_i^{\delta} b_{i+1}^{\gamma} b_{i-2}^{j_2}\cdots b_{i-p}^{j_p}, \text{ where } x \in F_{i-1}$$

At this point one cannot prove that the (PC) holds for these o.p.c.

Remark 4.4. In fact the (PC) might not hold since the surjection of lemma 3.1 might have a nontrivial kernel in rank $n = 3$.

Summarizing what we obtained until now, we proved that these are the only o.p.c. that could possibly not verify (PC). Moreover, we will check whether the weaker condition (CPC) is valid for these o.p.c. The explicit computation of the minimal elements will show that these are well-defined only for the graph $\Gamma_n^*(H)$, for a suitable ideal H. Let us explain how to find the generators for the ideal H.

Proposition 4.1. *The (CPC) is verified in $\Gamma_n^*(H)$ if and only if it is verified for the following pairs of elements:*

$$b_3^{\xi}\, b_2^{\epsilon}\, b_1^{\nu}\, b_3^{\mu}\, b_2^{\delta}\, b_3^{\gamma} \text{ and } b_3^{\xi}\, b_2^{\epsilon}\, b_3^{\mu}\, b_1^{\nu}\, b_2^{\delta}\, b_3^{\gamma} \quad \text{for } \xi,\epsilon,\mu,\nu,\delta,\gamma \in \{1,2\}$$

Proof. The only thing one needs to know is that:

LEMMA 4.6. *It suffices to consider the words w as above with $x = 1$ and $p = 1$.*

Proof. The proposition 2.6 shows that any admissible functional $\mathcal{T}$ on $\cup_{n=1}^{\infty} K_n(\alpha, \beta)$ satisfies:

$$\mathcal{T}(xuv) = \mathcal{T}(u)\mathcal{T}(xv) \text{ for } x, v \in H(Q, m) \text{ and } u \in \langle 1, b_m, b_{m+1}, \ldots, b_{m+k}\rangle$$

In the same way one shows that in the simplification process the minimal element in $R = \Gamma_0$ associated to the word xuv must be the product of the minimal elements associated to the two words u and xv. This proves the claim. □

This shows that the cases left unverified can be reduced to those which we claimed above. □

Thus, the obstructions to the existence of the Markov trace come out from these couples. In section 5 we study these obstructions and we find the ideal H in R containing them.

5. The computation of obstructions.

5.1. The algorithm. As we have not yet proved that the trace is well-defined we have to specify the choices made in the computation of the minimal element associated to a given word. Moreover, after the verification of the (CPC) and commutativity obstructions it will follow *a posteriori* that all descending paths in $\Gamma_n^*(H)$ will eventually lead to the same element.

Here is the algorithm which was used for computing the values of the minimal element in the particular situation of the proposition 4.1. Moreover, it can be used for any element of the braid group. Notice that the algorithm for reducing the elements of B_n uses recurrently the algorithms for the previous stages when simplifying elements of B_{n-1}.

- The input is a word w is an element of B_n.
- Step 1: use the cubical relations (3) until we find a linear combination of words having all exponents within $\{0, 1, 2\}$. We identify each word as an element of F_{n-1}. Further, one writes each word in the form: $w = x_1 b_{n-1}^{\varepsilon_1} x_2 \cdots x_p b_{n-1}^{\varepsilon_p} x_{p+1}$, where $x_i \in F_{n-2}$.
- Step 2: if some x_j, for $j \in \{2, \ldots, p\}$ are actually in F_{n-2} then bring together the two letters $b_{n-1}^{\varepsilon_j}$ and $b_{n-1}^{\varepsilon_{j+1}}$ by moving the latter to the left, using the permutations (8).
- Step 3: perform the steps 1 and 2 until the output is the same as the output.
- Step 4: if $p \geq 2$ then start reducing sub-words, starting from left to the right. The first sub-word is then $b_{n-1}^{\varepsilon_1} x_2 b_{n-1}^{\varepsilon_2}$. Using the recurrence hypothesis, reduce x_2 to a normal form in $K_{n-1}(\alpha, \beta)$, and therefore write $x_2 = y_2 b_{n-2}^{\delta_2} z_2$, where $y_2, z_2 \in F_{n-3}$. Further bring as close as possible the two letters $b_{n-1}^{\varepsilon_1}$ and $b_{n-1}\varepsilon_2$, by means of permutations (8) and obtain the equivalent sub-word $y_2 b_{n-1}^{\varepsilon_1} b_{n-2}^{\delta_2} b_{n-1}^{\varepsilon_2} z_2$.
- Step 5: use the simplification moves C(1), C(12), C(2) or C(21), according to the values of exponents until we reach an element where the letter b_{n-1} occurs only once, possibly with exponent 2. Consider the new instance of the word w by concatenating with the complementary sub-words, left untouched.
- Step 6: keep repeating the transformations from Step 4 until w has a normal form with $p = 1$.

- Step 7: simplify w by using (9) and keep track of the polynomial coefficients. If $n = 2$ then stop and send as output the coefficients. Otherwise go to the step 1.

REMARK 5.1. It is important to notice that the normal form for elements in $K_3(\alpha, \beta)$ is unique, and hence the step 4 lead us to a well-defined element for $n = 4$. For $n \geq 5$, the (CPC) obstructions being verified it follows that the output will be independent on the element we chose for the normal form at the step 4.

5.2. The CPC obstructions for n=4. It was pointed out in section 4 that the coherence of $\Gamma_n^*(H)$ depends only on the following couples:

$$b_3^\xi\, b_2^\epsilon\, b_1^\nu\, b_3^\mu\, b_2^\delta\, b_3^\gamma \text{ et } b_3^\xi\, b_2^\epsilon\, b_3^\mu\, b_1^\nu\, b_2^\delta\, b_3^\gamma \quad \xi, \epsilon, \mu, \nu, \delta, \gamma = 1 \text{ or } 2$$

Furthermore, if a linear functional $\mathcal{T}$ is admissible then it should verify $\mathcal{T}(w) = \mathcal{T}(w^*)$, where w^* is the reversal of the word w. One can therefore reduce ourselves to the study of the following 24 couples:

- $(1.i)$: $b_3\, b_2\, P_i\, b_2^2\, b_3$ and $b_3\, b_2\, P_i'\, b_2^2\, b_3$
- $(2.i)$: $b_3\, b_2\, P_i\, b_2\, b_3^2$ and $b_3\, b_2\, P_i'\, b_2\, b_3^2$
- $(3.i)$: $b_3\, b_2^2\, P_i\, b_2\, b_3^2$ and $b_3\, b_2^2\, P_i'\, b_2\, b_3^2$
- $(4.i)$: $b_3^2\, b_2^2\, P_i\, b_2^2\, b_3$ and $b_3^2\, b_2^2\, P_i'\, b_2^2\, b_3$
- $(5.i)$: $b_3^2\, b_2\, P_i\, b_2^2\, b_3^2$ and $b_3^2\, b_2\, P_i'\, b_2^2\, b_3^2$
- $(6.i)$: $b_3^2\, b_2^2\, P_i\, b_2\, b_3$ and $b_3^2\, b_2^2\, P_i'\, b_2\, b_3$

where $P_1 = b_1\, b_3$, $P_2 = b_1^2\, b_3$, $P_3 = b_1\, b_3^2$, $P_4 = b_1^2\, b_3^2, P_1' = b_3\, b_1$, $P_2' = b_3\, b_1^2$, $P_3' = b_3^2\, b_1$, $P_4' = b_3^2\, b_1^2$.

From now on we denote the difference between the minimal elements associated to the left hand side and the right hand side by the corresponding label (s, i). For general α, β the computation based on the algorithm from above is very long and and we needed to be computer-assisted. For more information about the code, see the remark 7.2.

One finds 15 different polynomials from these CPC obstructions, and the following identities among them: $(5.2) = -\alpha(3.2)$, $(6.2) = \alpha(1.2)$, $(1.4) = -\alpha(1.2)$. Thus, we must consider the couples $(1,2)$, $(2,4)$, $(3,2)$, $(3,3)$, $(3,4)$, $(4,1)$, $(4,2)$, $(4,3)$, $(4,4)$, $(5,3)$, $(5,4)$, $(6,4)$.

However the 4-variables polynomials we found above should be evaluated at specific values of the parameters $(z, \bar{z})$, which are compatible with the commutativity requirements for a Markov trace. We postpone then the calculation of obstructions until the next section where we find which are the convenient values for the parameters, as functions on (α, β).

5.3. Commutativity obstructions. We are concerned in this section with the commutativity constraints imposed for an admissible functional to be a Markov trace:

$$\mathcal{T}(ab) = \mathcal{T}(ba) \text{ for all } a,\, b$$

LEMMA 5.1. *An admissible linear functional on $K_3(\alpha, \beta)$ satisfies the trace conditions above if and only if the the values of (z, t) are given either by the type (I) rational parameters:*

$$z = \frac{-\beta^2 + 2\alpha}{\alpha\beta + 4}, \quad t = \frac{\alpha^2 + 2\beta}{\alpha\beta + 4} \tag{10}$$

or else by the type (II) parameters:

$$t = \frac{2\alpha z - 2z^2 + \beta}{2 + \beta z}, \quad \textit{where } z \textit{ verifies } (\alpha\beta + 1)z^3 + (\alpha + \beta^2)z^2 + 2\beta z + 1 = 0 \quad (11)$$

Proof. A trace $\mathcal{T}$ defined on $K_3(\alpha, \beta)$ should satisfy the following identities:

$$\mathcal{T}(b_2\, b_1^2\, b_2) = \mathcal{T}(b_1^2\, b_2^2),\ \mathcal{T}(b_1\, b_2\, b_1^2\, b_2) = \mathcal{T}(b_2\, b_1\, b_2\, b_1^2)$$

These are equivalent to:

$$\mathcal{T}(R_0) = \mathcal{T}(R_1) = 0$$

Remark that these are also sufficient conditions for an admissible functional be actually a trace on $K_3(\alpha, \beta)$. Moreover, the equations above can be expressed in the following algebraic form:

$$\begin{aligned}
0 = {} & (-\beta^3 + 3\alpha\beta + 4)t^2 + (3\alpha^2 - 7\alpha\beta^2 - 6\beta + 2\beta^4)t + \\
& (3\beta^2 - \beta^5 - 2\alpha - 3\alpha^2\beta + 4\alpha\beta^3) + (2\alpha\beta^3 + \beta^2 - 6\alpha^2\beta - 10\alpha)zt + \\
& (-3\alpha^3 + 7\alpha^2\beta^2 + 9\alpha\beta + 4 - \beta^3 - 2\alpha\beta^4)z + (3\alpha^3\beta + 7\alpha^2 - \alpha^2\beta^3 - \alpha\beta^2 + 2\beta)z^2 \\
0 = {} & (\beta^2 - 2\alpha)t^2 + (4 + 5\alpha\beta - 2\beta^3)t + (\beta^4 - 2\beta - 3\alpha\beta^2 + \alpha^2) + \\
& (2\beta + 5\alpha^2 - 2\alpha\beta^2)zt + (\beta^2 + 2\alpha\beta^3 - 5\alpha^2\beta - 6\alpha)z + (4 + \alpha^2\beta^2 + \alpha\beta - 2\alpha^3)z^2
\end{aligned}$$

The solutions of these equations are those claimed above. □

Consider now the following polynomials in α and β:

$$\begin{aligned}
L = {} & 3\alpha\beta^4 + 5\alpha^2\beta^5 - 2\alpha\beta + 2\alpha^4\beta - 7\alpha^3\beta^3 - 7\alpha^2\beta^2 - \alpha\beta^7 + \alpha^3 + (13\alpha^3\beta^2 - 10\alpha^2\beta^4 \\
& + 13\alpha^2\beta - 6\alpha\beta^3 - 2\alpha^4 + 3\alpha + 2\alpha\beta^6)t + (-6\alpha^3\beta - \alpha\beta^5 - 6\alpha^2 + 3\alpha\beta^2 + 5\alpha^2\beta^3)t^2 \\
& + (-16\alpha^4\beta^2 - 5\alpha\beta^2 - 2\alpha^2 + 3\alpha^5 + 2\alpha\beta^5 - 13\alpha^3\beta + 11\alpha^3\beta^4 - 2\alpha^2\beta^6)z + (-2\alpha\beta^4 \\
& + 15\alpha^4\beta + 2\alpha^2\beta^5 - 11\alpha^3\beta^3 + 15\alpha^3 + 6\alpha\beta)zt + (-3\alpha - \alpha^3\beta^5 + 6\alpha^4\beta^3 - 3\alpha^3\beta^2 + \\
& 2\alpha^2\beta^4 - 9\alpha^5\beta - 9\alpha^2\beta - 10\alpha^4)z^2
\end{aligned}$$

$$\begin{aligned}
M = {} & \alpha - \alpha^4 + 6\alpha^2\beta - 2\alpha^5\beta - 2\alpha\beta^3 + 7\alpha^4\beta^3 + 11\alpha^3\beta^2 + \alpha\beta^6 - 7\alpha^2\beta^4 - 5\alpha^3\beta^5 + \\
& \alpha^2\beta^7 + (-21\alpha^3\beta - 2\alpha^2\beta^6 + 2\alpha\beta^2 + 14\alpha^2\beta^3 - 13\alpha^4\beta^2 - 7\alpha^2 + 10\alpha^3\beta^4 - 2\alpha\beta^5 + \\
& 2\alpha^5)t + (-7\alpha^2\beta^2 + 6\alpha^4\beta + 10\alpha^3 + \alpha\beta^4 + \alpha^2\beta^5 - 5\alpha^3\beta^3)t^2 + (-3\alpha^6 + 2\alpha^3\beta^6 + \\
& 5\alpha\beta + 11\alpha^2\beta^2 + 16\alpha^5\beta^2 + 8\alpha^3 + 25\alpha^4\beta - 11\alpha^4\beta^4 - 4\alpha\beta^4 - 10\alpha^3\beta^3)z + (11\alpha^4\beta^3 \\
& - 14\alpha^2\beta + 10\alpha^3\beta^2 - \alpha + 4\alpha\beta^3 - 15\alpha^5\beta - 27\alpha^4 - 2\alpha^3\beta^5)zt + (4\alpha\beta^2 - 4\alpha^2\beta^3 + \\
& \alpha^4\beta^5 + 19\alpha^5 - \alpha^3\beta^4 + 4\alpha^2 - 3\alpha^4\beta^2 + 21\alpha^3\beta - 6\alpha^5\beta^3 + 9\alpha^6\beta)z^2
\end{aligned}$$

$$\begin{aligned}
N = {} & 12\alpha^2\beta^3 + \alpha\beta^8 - 6\alpha^2\beta^6 - 2\alpha^2 + 3\alpha\beta^2 + 11\alpha^3\beta^4 - 4\beta^5\alpha - 6\alpha^4\beta^2 - 7\alpha^3\beta + \\
& (-21\alpha^3\beta^3 + 7\alpha\beta^4 + 5\alpha^3 + 10\alpha^4\beta - 2\alpha\beta^7 - 2\alpha\beta - 17\alpha^2\beta^2 + 12\alpha^2\beta^5)t + (-4\alpha^4 \\
& + 10\alpha^3\beta^2 - 3\alpha + \alpha\beta^6 + 5\alpha^2\beta - 6\alpha^2\beta^4 - 3\alpha\beta^3)t^2 + (3\alpha + 3\alpha\beta^3 + 2\alpha^2\beta^7 + \\
& 16\alpha^3\beta^2 - 2\alpha\beta^6 - 7\alpha^4 - 13\alpha^5\beta + 5\alpha^2\beta - 13\alpha^3\beta^5 + 25\alpha^4\beta^3)z + (\alpha^2 - 12\alpha^3\beta + \\
& 10\alpha^5 + 13\alpha^3\beta^4 - \alpha^2\beta^3 - 2\alpha^2\beta^6 + 2\alpha\beta^5 - 24\alpha^4\beta^2 - 5\alpha\beta^2)zt + (5\alpha^3 + 4\alpha^3\beta^3 + \\
& 14\alpha^5\beta^2 + 8\alpha^4\beta + 7\alpha^2\beta^2 + \alpha^3\beta^6 + 5\alpha\beta - 2\alpha^2\beta^5 - 6\alpha^6 - 7\alpha^4\beta^4)z^2
\end{aligned}$$

PROPOSITION 5.1. *Consider an admissible functional $\mathcal{T}$ defined on the whole tower of algebras $\cup_{n=1}^{\infty} K_n(\alpha,\beta)$. Suppose that $\mathcal{T}$ is a trace on $K_3(\alpha,\beta)$. Then $\mathcal{T}$ is a Markov trace on the tower $\cup_{n=1}^{\infty} K_n(\alpha,\beta)$ if the equations:*

$$L = M = N = 0$$

are satisfied.

Proof. We will prove that the commutativity constraints are verified by induction on n. The claim is true for $n = 3$, and now we suppose that it holds for all algebras $K_m(\alpha,\beta)$, for $m \leq n$. In order to prove the claim for $K_{n+1}(\alpha,\beta)$ it suffices to consider $b \in \{b_1, \ldots, b_n\}$ and a belonging to some system of generators of $K_{n+1}(\alpha,\beta)$, as a module. In particular we will choose the set of generators W_n from section 3.2.

For $b = b_i$, $i < n$ the claim is obvious. It remains to check whenever $\mathcal{T}(ab_n) = \mathcal{T}(b_n a)$ holds true. There are three cases to consider:

i) $a \in K_n(\alpha,\beta)$;
ii) $a = xb_n y$, with $x, y \in K_n(\alpha,\beta)$;
iii) $a = xb_n^2 y$, with $x, y \in K_n(\alpha,\beta)$.

which will be discussed in combination with the following six sub-cases:

(1) $x \in K_{n-1}(\alpha,\beta)$, and $y \in K_{n-1}(\alpha,\beta)$,
(2) $x \in K_{n-1}(\alpha,\beta)$, and $y = ub_{n-1}v$, $u, v \in K_{n-1}(\alpha,\beta)$,
(3) $x \in K_{n-1}(\alpha,\beta)$, and $y = ub_{n-1}^2 v$, $u, v \in K_{n-1}(\alpha,\beta)$,
(4) $x = rb_{n-1}s$, $r, s \in K_{n-1}(\alpha,\beta)$, $y = ub_{n-1}v$, $u, v \in K_{n-1}(\alpha,\beta)$,
(5) $x = rb_{n-1}s$, $r, s \in K_{n-1}(\alpha,\beta)$, $y = ub_{n-1}^2 v$, $u, v \in K_{n-1}(\alpha,\beta)$,
(6) $x = rb_{n-1}^2 s$, $r, s \in K_{n-1}(\alpha,\beta)$, $y = ub_{n-1}^2 v$, $u, v \in K_{n-1}(\alpha,\beta)$.

The cases (*,i), (1,ii) and (1,iii) are trivially verified by an immediate calculation. Furthermore one obtains:

$$\begin{array}{rlcl}
(2,\text{ii}) & \mathcal{T}(b_n x b_n u b_{n-1} v) & = & tz\mathcal{T}(xuv) = \mathcal{T}(xb_n u b_{n-1} v b_n) \\
(2,\text{iii}) & \mathcal{T}(b_n x b_n^2 u b_{n-1} v) & = & (\alpha t + \beta z + 1)\mathcal{T}(xub_{n-1}v) = (\alpha t + \beta z + 1)z\mathcal{T}(xuv) \\
 & & = & \mathcal{T}(xub_{n-1}b_n b_{n-1}^2 v) = \mathcal{T}(xb_n^2 u b_{n-1} v b_n). \\
(2,\text{iii}) & \mathcal{T}(b_n x b_n^2 u b_{n-1} v) & = & (\alpha t + \beta z + 1)\mathcal{T}(xub_{n-1}v) = (\alpha t + \beta z + 1)z\mathcal{T}(xuv) \\
 & & = & \mathcal{T}(xub_{n-1}b_n b_{n-1}^2 v) = \mathcal{T}(xb_n^2 u b_{n-1} v b_n). \\
(3,\text{ii}) & \mathcal{T}(b_n x b_n u b_{n-1}^2 v) & = & t^2\mathcal{T}(xuv) = \mathcal{T}(b_n^2 b_{n-1}^2)\mathcal{T}(xuv) = \mathcal{T}(b_n b_{n-1}^2 b_n)\mathcal{T}(xuv) \\
 & & = & \mathcal{T}(xub_n b_{n-1}^2 b_n v) = \mathcal{T}(xb_n u b_{n-1}^2 v b_n) \\
(3,\text{iii}) & \mathcal{T}(b_n x b_n^2 u b_{n-1}^2 v) & = & (\alpha t + \beta z + 1)\mathcal{T}(xub_{n-1}^2 v) = (\alpha t + \beta z + 1)t\mathcal{T}(xuv) \\
 & & = & \mathcal{T}(xuv)\mathcal{T}(b_n^2 b_{n-1}^2 b_n) = \mathcal{T}(xb_n^2 u b_{n-1}^2 v b_n)
\end{array}$$

For the remaining cases, we need also to know more precisely the form of su. Specifically, let us write $su = pb_{n-2}^{\varepsilon} q$ with $p, q \in K_{n-2}(\alpha,\beta)$ and $\varepsilon \in \{0, 1, 2\}$. We can show by a direct computation that the equalities hold also for (4,ii), (4,iii), (6,ii) and (6,iii). Moreover, using Maple we have found that in the cases (5, ii) and (5, iii), for $su = pb_{n-2}^2 q$, there are two additional identities, which are not consequences of those in the previous lemma. The difference $\mathcal{T}(ab) - \mathcal{T}(ba)$ is expressed as a linear combination with polynomial coefficients in $\mathcal{T}(rpb_{n-2}^2 qv), \mathcal{T}(rpb_{n-2}qv)$ and $\mathcal{T}(rpqv)$. For arbitrary elements r, p, q, v as above the three traces above seem to be unrelated. A sufficient condition for the commutativity to hold is that the coefficients in front of these terms vanish. We derive therefore the following obstructions:

(5,ii) $L\mathcal{T}(rpb_{n-2}^2 qv) + M\mathcal{T}(rpb_{n-2}qv) + N\mathcal{T}(rpqv) = 0$
(5,iii) $-\alpha(L\mathcal{T}(rpb_{n-2}^2 qv) + M\mathcal{T}(rpb_{n-2}qv) + N\mathcal{T}(rpqv)) = 0$

Furthermore the vanishing of L, M and N insures the commutativity of the admissible functional. $\square$

Remark 5.2. It seems that the conditions stated in the proposition 5.1 are also necessary for the existence of a Markov trace extension. Nevertheless, the ideal defined by all obstructions could not be made strictly smaller, even if we could get rid of the equations $L = M = N = 0$, because of the (CPC) obstructions.

6. The existence of Markov traces.

6.1. Statements.

Theorem 6.1. *There exists an unique Markov trace:*

$$\mathcal{T}_{(\alpha,\,\beta)} : \cup_{n=1}^{\infty} K_n(\alpha,\,\beta) \to \frac{\mathbb{Z}[\alpha,\,\beta,\,(4+\alpha\beta)^{-1}]}{(H_{(\alpha,\,\beta)})}$$

with type (I) parameters: $z = (2\alpha - \beta^2)/(\alpha\beta + 4)$ *and* $\bar{z} = -(\alpha^2 + 2\beta)/(\alpha\beta + 4)$, *where:*

$$\begin{aligned} H_{(\alpha\,,\beta)} =& 8\alpha^6 - 8\alpha^5\beta^2 + 2\alpha^4\beta^4 + 36\alpha^4\beta - 34\alpha^3\beta^3 + 17\alpha^3 + 8\alpha^2\beta^5 + 32\alpha^2\beta^2 - 36\alpha\beta^4 \\ &+ 38\alpha\beta + 8\beta^6 - 17\beta^3 + 8. \end{aligned}$$

It is more convenient now to put $\delta = z^2(\beta z+1)$, so that the obstructions associated to the type (II) parameters become Laurent polynomials in z and δ.

Theorem 6.2. *Set* $\alpha = -(z^7+\delta^2)/(z^4\delta)$, $\beta = (\delta-z^2)/z^3$ *and* $\bar{z} = -z^2/(\beta z+1) = -z^4/\delta$. *There exists an unique Markov trace with parameters* $(z, \bar{z})$*:*

$$\mathcal{T}^{(z,\,\delta)} : \cup_{n=1}^{\infty} K_n(\alpha,\,\beta) \to \frac{\mathbb{Z}[z^{\pm 1}, \delta^{\pm 1}]}{(P^{(z,\,\delta)})}$$

where $P^{(z,\,\delta)} = z^{23} + z^{18}\delta - 2z^{16}\delta^2 - z^{14}\delta^3 - 2z^9\delta^4 + 2z^7\delta^5 + \delta^6 z^5 + \delta^7$.

6.2. Proof of Theorem 6.1. Notice that the parameters z, t have to satisfy the conditions:

$$\mathcal{T}(R_0) = \mathcal{T}(R_1) = 0$$

because the Markov trace vanishes on the ideal I_3 of relations defining $K_3(\alpha, \beta)$. In particular (z, t) are either of type (I) or type (II) parameters from (10)-(11).

For (z, t) as in (10) we derive that $\bar{z} = -t$. Set $u := 1/(\alpha\beta + 4)$, $z_0 := 2\alpha - \beta^2$ and $t_0 := \alpha^2 + 2\beta =: -\bar{z}_0$.

6.2.1. The commutativity obstructions. The equations $L = M = N = 0$ are equivalent to:

- $u^2\beta H_{(\alpha,\,\beta)} = 0$
- $-u^2(\alpha\beta + 2)H_{(\alpha,\,\beta)} = 0$
- $u^2(\alpha - \beta^2)H_{(\alpha,\,\beta)} = 0$

6.2.2. CPC obstructions.

- (1.2): $-u^3\alpha(\alpha - \beta^2)H_{(\alpha\,,\beta)}W$
- (2.4): $u^3(\alpha - \beta^2)(\alpha^2 + \beta)H_{(\alpha\,,\beta)}W$
- (3.2): $u^3(-\alpha^2\beta^2 + 2 + \alpha\beta + \alpha^3)H_{(\alpha,\,\beta)}W$
- (3.3): $u^3(\alpha\beta + 2)H_{(\alpha,\beta)}W$
- (3.4): $u^3\alpha\beta(\alpha - \beta^2)H_{(\alpha,\,\beta)}W$
- (4.1): $-u^3(\alpha - \beta^2)(\alpha^2 + \beta)H_{(\alpha,\,\beta)}W$

- (4.2): $u^3\alpha(\alpha^3 + 2 + 2\alpha\beta - \alpha^2\beta^2 - \beta^3)H_{(\alpha,\,\beta)}W$
- (4.3): $u^3\alpha(\alpha^3 - \alpha^2\beta^2 - 2 - \beta^3)H_{(\alpha,\,\beta)}W$
- (4.4): trivial
- (5.3): $-u^3(\beta^2 + 2\alpha + 2\alpha^2\beta)H_{(\alpha,\,\beta)}W$
- (5.4): $u^3\alpha(-\alpha^3\beta^2 - \beta^2 - \alpha^2\beta + \alpha^4)H_{(\alpha,\,\beta)}W$
- (6.4): $-u^3\alpha(\beta + 2\alpha^2)(\alpha - \beta^2)H_{(\alpha,\,\beta)}W$

where $W = (\alpha + 2 - \beta)(\alpha^2 - 2\alpha + 4 + \alpha\beta + 2\beta + \beta^2) = \alpha^3 + 8 - \beta^3 + 6\alpha\beta$.

Consequently, the simplification algorithm defines an admissible functional $\mathcal{T}_{(\alpha,\beta)}$ on the tower of algebras $\cup_{n=1}^{\infty} K_n(\alpha,\beta)$ with values in $\frac{\mathbb{Z}[\alpha,\,\beta,\,(4+\alpha\beta)^{-1}]}{(H_{(\alpha,\,\beta)}}$), because the (CPC) obstructions vanish. Moreover $\mathcal{T}_{(\alpha,\beta)}$ is a trace on $K_3(\alpha,\beta)$ since the parameters verify (10) and it is a Markov trace on the whole tower because the commutativity obstruction vanish, too.

6.3. Proof of Theorem 6.2. Consider now the situation of type (II) parameters. We will express all obstructions as rational functions on z and β.

6.3.1. The commutativity obstructions. The equations $L = M = N = 0$ are equivalent to:

- $-ZB_1/(z^7(z\beta + 1)^4) = 0$
- $-ZB_2/(z^9(z\beta + 1)^5) = 0$
- $ZB_3/(z^7(z\beta + 1)^5) = 0$

6.3.2. The CPC obstructions.

- (1.2): $-ZB_4B_5B_6/(z^{13}(z\beta + 1)^8)$
- (2.4): $-ZB_4B_6B_7/(z^{15}(z\beta + 1)^9)$
- (3.2): $ZB_4B_8/(z^{15}(z\beta + 1)^9)$
- (3.3): $-ZB_4B_9/(z^{11}(z\beta + 1)^7)$
- (3.4): $ZB_4B_5B_6\beta/(z^{13}(z\beta + 1)^8)$
- (4.1): $ZB_4B_6B_7/(z^{15}(z\beta + 1)^9)$
- (4.2): $ZB_4B_5B_{10}/(z^{17}(z\beta + 1)^{10})$
- (4.3): $ZB_4B_5B_{11}/(z^{17}(z\beta + 1)^{10})$
- (4.4): trivial
- (5.3): $-ZB_4B_{12}/(z^{13}(z\beta + 1)^8)$
- (5.4): $-ZB_4B_5B_{13}/(z^{19}(z\beta + 1)^{11})$
- (6.4): $-ZB_4B_5B_6B_{14}/(z^{17}(z\beta + 1)^{10})$

where $Z, B_1, \dots, B_{14}$ are the following polynomials in z, β:

(1) $Z = 1+7z\beta+21z^2\beta^2+z^3+35z^3\beta^3+35z^4\beta^4+21z^5\beta^5+7z^6\beta^6+z^7\beta^7+z^9\beta^6+8z^8\beta^5 + 23z^7\beta^4 + 32z^6\beta^3 + 23z^5\beta^2 + 8z^4\beta - 2z^6 + z^9 - z^9\beta^3 - 5z^8\beta^2 - 6z^7\beta$

(2) $B_1 = 3z^3 + z^4\beta + 1 + z\beta$

(3) $B_2 = 5z^3 + 10z^4\beta + 6z^5\beta^2 + z^6\beta^3 + 4z^6 + 2z^7\beta + 1 + 3z\beta + 3z^2\beta^2 + z^3\beta^3$

(4) $B_3 = \beta + 2z\beta^2 + 4z^3\beta + 5z^4\beta^2 + z^5\beta^3 + z^2\beta^3 - 2z^5$

(5) $B_4 = (z\beta + z^2\beta + 1 + z - z^2)(z\beta + 1 + 2z^3)(z^4\beta^2 - z^3\beta^2 + z^2\beta^2 + 1 + 2z\beta - z - 2z^2\beta + 2z^2 + 3z^3\beta + z^3 + z^4\beta + z^4)$

(6) $B_5 = 1 + z^3 + z^2\beta^2 + 2z\beta$

(7) $B_6 = z^3\beta^3 + 1 + 2z\beta + 2z^2\beta^2 + z^3$

(8) $B_7 = 1 + 4z\beta + 6z^2\beta^2 + 2z^3 + 4z^3\beta^3 + z^4\beta^4 + z^6\beta^3 + 4z^5\beta^2 + 5z^4\beta + z^6)$

(9) $B_8 = z^2\beta^3 + \beta + 2z\beta^2 - 2z^2 - z^3\beta$

(10) $B_9 = 1 + 6z\beta + 16z^2\beta^2 + 3z^3 + 25z^3\beta^3 + 25z^4\beta^4 + 16z^5\beta^5 + 6z^6\beta^6 + z^7\beta^7 + 3z^8\beta^5 + 13z^7\beta^4 + 24z^6\beta^3 + 24z^5\beta^2 + 13z^4\beta + z^7\beta + z^6 + z^9$

(11) $B_{10} = 1 + 6z\beta + 16z^2\beta^2 + 3z^3 + 25z^3\beta^3 + 25z^4\beta^4 + 16z^5\beta^5 + 6z^6\beta^6 + z^7\beta^7 + z^9\beta^6+7z^8\beta^5+20z^7\beta^4+31z^6\beta^3+28z^5\beta^2+14z^4\beta+z^6+z^9+z^9\beta^3+2z^8\beta^2+2z^7\beta$

(12) $B_{11} = 6z\beta+16z^2\beta^2+3z^3+10z^8\beta^2+5z^8\beta^5+z^7\beta^7+z^9\beta^6+12z^7\beta+12z^7\beta^4+19z^6\beta^3+20z^5\beta^2+12z^4\beta+6z^6\beta^6+3z^9\beta^3+5z^6+z^9+1+25z^3\beta^3+25z^4\beta^4+16z^5\beta^5$

(13) $B_{12} = 2\beta+4z^5\beta^3-2z^5+2z^4\beta^5+8z\beta^2+12z^2\beta^3-2z^2+8z^3\beta^4+3z^4\beta^2-2z^3\beta+z^6\beta^4$

(14) $B_{13} = 1+8z\beta+29z^2\beta^2+63z^3\beta^3+80z^6\beta^3+29z^7\beta^7+13z^9\beta^6+17z^9\beta^3+91z^4\beta^4+57z^5\beta^2+23z^4\beta+4z^3+6z^6+4z^9+91z^5\beta^5+63z^6\beta^6+39z^8\beta^5+70z^7\beta^4+30z^8\beta^2+22z^7\beta+z^{12}+z^9\beta^9-z^{12}\beta^6+z^{10}\beta^4+2z^{10}\beta^7+8z^8\beta^8-3z^{11}\beta^5+3z^{11}\beta^2+7z^{10}\beta$

(15) $B_{14} = 2+8z\beta+12z^2\beta^2+4z^3+8z^3\beta^3+2z^4\beta^4+z^6\beta^3+6z^5\beta^2+9z^4\beta+2z^6$

Now the proof follows as above, after noticing that $Z(z,\beta) = P^{(z,\,\delta)}(z,\delta)$.

6.3.3. Corollaries. COROLLARY 6.1. *There exist unique Markov traces:*

$$\mathcal{T} : \cup_{n=1}^{\infty} K_n(0,\,2\lambda) \to \frac{\mathbb{Z}[\lambda]}{(8\lambda^6-17\lambda^3+1)}$$

with parameters $z=-\lambda^2$, $t=\lambda$ and $\bar{z}=-\lambda$, and respectively:

$$\mathcal{T} : \cup_{n=1}^{\infty} K_n(-2\lambda,\,0) \to \frac{\mathbb{Z}[\lambda]}{(8\lambda^6-17\lambda^3+1)}$$

with parameters $z=-\lambda$, $t=\lambda^2$ and $\bar{z}=-\lambda^2$.

We have a similar situation for the other three solutions. In fact for $\alpha=0$, we derive $z=-(t-\beta)^2$, where t satisfies $(t^3-4\beta t^2+5\beta^2 t+1-2\beta^3)=0$. In particular $\bar{z}^3-\beta\bar{z}^2+1=0$ because $\bar{z}=t-\beta$.

COROLLARY 6.2. *There exist unique Markov traces:*

$$\mathcal{T} : \cup_{n=1}^{\infty} K_n\left(0,\,\frac{\lambda^3+1}{\lambda^2}\right) \to \frac{\mathbb{Z}[\lambda^{\pm 1}]}{(\lambda^9-2\lambda^6+\lambda^3+1)}$$

with parameters $z=-\lambda^2$, $\bar{z}=\lambda$ and $t=\dfrac{2\lambda^3+1}{\lambda^2}$, and respectively:

$$\mathcal{T} : \cup_{n=1}^{\infty} K_n\left(-\frac{\lambda^3+1}{\lambda^2},\,0\right) \to \frac{\mathbb{Z}[\lambda^{\pm 1}]}{(\lambda^9-2\lambda^6+\lambda^3+1)}$$

with parameters $z=\lambda$, $\bar{z}=-\lambda^2$ and $t=-\dfrac{2\lambda^3+1}{\lambda^2}$.

7. The invariants.

7.1. The definition of $I_{(\alpha,\,\beta)}$. As in section 5.2 we set $z=(2\alpha-\beta^2)/(\alpha\beta+4)$, $t=(\alpha^2+2\beta)/(\alpha\beta+4)$, $u:=1/(\alpha\beta+4)$, $z_0:=2\alpha-\beta^2$ and $t_0:=\alpha^2+2\beta=:-\bar{z}_0$ (notice that in this case $\bar{z}=-t$).

DEFINITION 7.1. Let L be an oriented link. We set therefore:

$$I_{(\alpha,\,\beta)}(L) = \left(\frac{1}{z\bar{z}}\right)^{\frac{n-1}{2}} \left(\frac{\bar{z}}{z}\right)^{\frac{e(x)}{2}} \mathcal{T}_{(\alpha,\,\beta)}(x) \in \frac{\mathbb{Z}[\alpha,\,\beta,\,z_0^{\pm\epsilon/2},\bar{z}_0^{\pm\epsilon/2}]}{(H_{(\alpha,\,\beta)})}$$

where $x\in B_n$ is any braid whose closure is isotopic to L. Here $\epsilon-1$ is the number of components of L modulo 2.

LEMMA 7.1. *$I_{(\alpha,\,\beta)}$ is well-defined.*

Proof. Since $b_j^{-1} = b_j^2 - \alpha b_j - \beta$, we can suppose that x is a positive braid. All the elements in $\Gamma_0(H)$ associated to x are polynomials in the variables z, t of degree at most $n-1$. The substitutions $z = uz_0$ and $t = ut_0$ imply that, if $\mathcal{T}_{(\alpha,\,\beta)}(x)$ and $\mathcal{T}_{(\alpha,\,\beta)}(x)'$ are representatives of the trace of x, then $\mathcal{T}_{(\alpha,\,\beta)}(x)' - \mathcal{T}_{(\alpha,\,\beta)}(x) = u^{n-1}G(\alpha,\,\beta)H_{(\alpha,\,\beta)}$, where $G(\alpha,\,\beta)$ is a polynomial in α, β. It follows that:

$$I_{(\alpha,\,\beta)}(L) = \left(\frac{1}{z_0\bar{z}_0}\right)^{\frac{n-1}{2}}\left(\frac{\bar{z}_0}{z_0}\right)^{\frac{e(x)}{2}}\widetilde{\mathcal{T}}_{(\alpha,\,\beta)}(x)$$

where we put:

$$\widetilde{\mathcal{T}}_{(\alpha,\,\beta)}(x) = u^{-n+1}\mathcal{T}_{(\alpha,\,\beta)}(x) \in \frac{\mathbb{Z}[\alpha,\,\beta]}{(H_{(\alpha,\,\beta)})}$$

□

7.2. The cubical behaviour.

PROPOSITION 7.1. *For any link K there exists some $l \in \{0,\,1,\,2\}$ such that:*

$$I_{(\alpha,\,\beta)}(K) = \frac{\sum_{k\in\mathbb{Z}_+} P_k(\beta)\alpha^k}{\sum_{k\in\mathbb{Z}_+} Q_k(\beta)\alpha^k} = \frac{\sum_{k\in\mathbb{Z}_+} M_k(\alpha)\beta^k}{\sum_{k\in\mathbb{Z}_+} N_k(\alpha)\beta^k}$$

where P_k, Q_k, M_k, N_k are $(3, k+l)$-polynomials.

Proof. We will show that M_k, N_k are $(3, k+l)$-polynomials. The other case is analogous. Suppose first that $x \in B_n^+$, where B_n^+ is the set of positive braids and n is such that $x \notin B_{n-1}^+$. Then $e(x) = |x|$ where $|x|$ denotes the length of x. In the process of computing the value of the trace on the word x we make two types of reductions: either one uses the relations from $K_n(\alpha,\,\beta)$, or else one replaces ab_lb by zab (respectively ab_l^2b by tab). In the first alternative the word y is replaced by $\sum_s(\sum_{k\in\mathbb{Z}_+} D_{k,s}(\alpha)\beta^k)y_s$, where the y_s are words from B_n, the coefficients $D_{k,s}(\alpha)$ are $(3, k+e(x)-l_s)$-polynomials, and $l_s = |y_s|$. In the second case the word w is replaced by $zw' + tw$" where $|w'| = |w| - 1$ and $|w"| = |w| - 2$. When we substitute for z and t their values as functions on α and β one finds that:

$$\mathcal{T}_{(\alpha,\,\beta)}(x) = \sum_{k\in\mathbb{Z}_+} u^{s_k}V_k(\alpha)\beta^k$$

where $0 \le s_k \le n-1$ and $V_k(\alpha)$ are $(3, k+e(x))$-polynomials. In particular:

$$\widetilde{\mathcal{T}}_{(\alpha,\,\beta)}(x) = \sum_{k\in\mathbb{Z}_+} u^{s_k-n+1}V_k(\alpha)\beta^k$$

Now $u^{s_k-n+1} = \sum_{k\in\mathbb{Z}_+} Y_k(\alpha)\beta^k$ where $Y_k(\alpha)$ are $(3, k)$-polynomials. Thus, it follows that:

$$\widetilde{\mathcal{T}}_{(\alpha,\,\beta)}(x) = \sum_{k\in\mathbb{Z}_+} L_k(\alpha)\beta^k$$

where $L_k(\alpha)$ are $(3, k+e(x))$-polynomials.

Now, remark that the same reasoning holds true for non necessarily positive $x \in B_n$, by getting rid of the negative exponents in x by making use of the cubic relation. The only difference is that one has to take into account the normalization factor in front of the trace. The claim follows. □

Corollary 7.1. *$I_{(\alpha,0)}(K) = \sum_{i\in\mathbb{Z}_+} a_{3i}\alpha^{3i}$ and, respectively, $I_{(0,\beta)}(K) = \sum_{i\in\mathbb{Z}_+} b_{3i}\beta^{3i}$, where a_{3i}, $b_{3i} \in \mathbb{Z}[\frac{1}{2}]$.*

7.3. Chirality and a few other properties of $I_{(\alpha,\beta)}$.

Lemma 7.2. *Set $x^\dagger \in B_n$ for the word obtained from x by replacing each term b_j^ϵ by the corresponding $b_j^{-\epsilon}$. Then the following identity $\mathcal{T}_{(\alpha,\beta)}(x) = \mathcal{T}_{(-\beta,-\alpha)}(x^\dagger)$ holds true. In particular, if the link K is amphicheiral then the identity $I_{(\alpha,\beta)}(K) = I_{(-\beta,-\alpha)}(K)$ is fulfilled.*

Proof. Let $Q(b_j)^\dagger$ (respectively $R_0^\dagger$) denote the image of $Q(b_j)$ (and respectively R_0) after the substitutions $\alpha \to -\beta$, $\beta \to -\alpha$ and $b_l \to b_l^{-1}$ for $l = 1,\ldots,n-1$. It is easy to check that $Q(b_j)^\dagger = b_j^{-3}Q(b_j) = 0$. By some more involved computations we verified that $R_0^\dagger = R_1 = 0$. Since $H_{(\alpha,\beta)} = H_{(-\beta,-\alpha)}$ the claim follows. □

The following properties have been checked by direct calculation (see the table from the appendix).

(1) $I_{(\alpha,\beta)}$ is independent from HOMFLY and in particular it distinguishes among knots that have the same HOMFLY polynomial. The knots 5.1 and 10.132 have the same HOMFLY polynomial but different $I_{(\alpha,0)}$ and $I_{(0,\beta)}$ invariants. This holds true for the three other couples of prime knots with number crossing ≤ 10 that HOMFLY fails to distinguish, i.e. (8.8, 10.129), (8.16,10.156), and (10.25, 10.56).
(2) $I_{(\alpha,\beta)}$ detects the chirality of those knots with crossing number at most 10, where HOMFLY fails i.e. the knots 9.42, 10.48, 10.71, 10.91, 10.104 and 10.125.
(3) The Kauffman polynomial does not detect the chirality of 9.42 and 10.71 (see [30]). Therefore $I_{(\alpha,\beta)}$ is independent from the Kauffman polynomial.
(4) The 2-cabling of HOMFLY does not detect the chirality of 10.71 (this result was communicated to us by H. R. Morton). Therefore $I_{(\alpha,\beta)}$ is independent from the 2-cabling of HOMFLY. We notice that the 2-cabling of Jones polynomial can be deduced from Dubrovnic polynomial ([34]), which is a variant of Kauffman polynomial ([17]).
(5) $I_{(\alpha,\beta)}$ does not distinguish between the Conway knot (C) and the Kinoshita-Terasaka knot (KT), which form a pair of mutant knots.

7.4. The definition of $I^{(z,\delta)}$.

Definition 7.2. For each oriented link L we define:

$$I^{(z,\delta)}(L) = \left(\frac{1}{z\bar{z}}\right)^{\frac{n-1}{2}} \left(\frac{\bar{z}}{z}\right)^{\frac{e(x)}{2}} \mathcal{T}^{(z,\delta)}(x) \in \frac{\mathbb{Z}[z^{\pm\epsilon/2},\delta^{\pm\epsilon/2}]}{(P^{(z,\delta)})}$$

where $x \in B_n$ is any braid whose closure is isotopic to L and α, β, t, $\bar{z}$ as in Theorem 6.2. Here $\epsilon - 1$ is the number of components modulo 2, $\epsilon \in \{1, 2\}$.

Remark 7.1. This invariant does not detect the amphichirality of knots. Also $I^{(z,\delta)}$ does not distinguish the Conway knot from the Kinoshita-Terasaka knot.

PROPOSITION 7.2. *The invaraint $I^{(z,\delta)}$ can be expressed as follows:*

$$I^{(z,\,\delta)}(K) = \sum_{k\in\mathbb{Z}} H_k(\delta) z^k = \sum_{k\in\mathbb{Z}} G_k(z)\delta^k$$

where H_k, G_k are $(3,k)$-Laurent polynomials.

Proof. The proof is analogous to the proof of Proposition 7.1. □

REMARK 7.2. For evaluating obstructions and traces of braids we used a Delphi code. The input is a word, or a linear combination of words, and we restricted to words representing 5-braids for memory reasons. One transforms first the word to a sum of positive words, by using the cubic relations. Furthermore the transformations $C(j)$ and $C(ij)$ are used in order to reduce the shape of the word as much as possible. When it gets stalked, one allows permutations of the letters. The final result is the value of the trace on the braid element. The program is available at:
http://www-fourier.ujf-grenoble.fr/∼bellinge.html.

8. Appendix. The values of the polynomials for $I_{(\alpha,\,0)}(K)$ and $I_{(0,\,\beta)}(K)$ are displayed below for all knots with no more than 8 crossings. The second column is a braid representative for the knot. The bold entries in the table are the coefficients of α^0 and, respectively β^0. The other entries are the non zero coefficients of α^{3k} and β^{3k} respectively, for $k \in \mathbb{Z}$. For example,

$$I_\alpha(6.2) = [-5 - \frac{19}{4}\alpha^3 - \frac{1}{2}\alpha^6],\ \ I_\beta(6.2) = [-16\beta^{-3} + 19 - 2\beta^3]$$

The entry "A" in the last column means that the knot is amphicheiral.

3.1	b_1^3	**-1** $-1/4$	-8 **2**	
4.1	$b_1b_2^{-1}b_1b_2^{-1}$	8 **10** 1	-8 **10** -1	A
5.1	b_1^5	**0** 7/8 1/8	-24 **4**	
5.2	$b_1^2b_2^2b_1^{-1}b_2$	**2** 17/8 1/4	-8 **2**	
6.1	$b_1^{-1}b_2b_1^{-1}b_3b_2^{-1}b_3b_2$	-8 **-16** -10 -1	**1**	
6.2	$b_1^{-1}b_2b_1^{-1}b_2^3$	**-5** $-19/4$ $-1/2$	-16 **19** -2	
6.3	$b_1^{-1}b_2^2b_1^{-2}b_2$	**-3** $-1/2$	**-3** 1/2	A
7.1	b_1^7	**0** $-5/8$ $-9/16$ $-1/16$	-56 **8**	
7.2	$b_1^{-1}b_3^3b_2b_1^2b_3^{-1}b_2$	**-3** $-11/2$ $-21/8$ $-1/4$	-64 -64 **-6**	
7.3	$b_1^2b_2b_1^{-1}b_2^4$	**-1** $-7/4$ $-19/16$ $-1/8$	-64 48 **-4**	
7.4	$b_1^2b_2b_3^2b_1^{-1}b_2b_3^{-1}b_2$	**0** $-17/8$ $-9/4$ $-1/4$	-64 $+128$ **-78** 8	
7.5	$b_1^4b_2b_1^{-1}b_2^2$	**0** $-9/8$ $-9/8$ $-1/8$	-24 **4**	
7.6	$b_1b_2^{-1}b_1^{-2}b_3b_2^3b_3$	**-4** $-37/8$ $-1/2$	-24 **20** -2	
7.7	$b_1b_3^{-1}b_2b_3^{-1}b_2b_1^{-1}b_2b_3^{-1}b_2$	-8 **-20** $-21/2$ -1	**-19** 37/2 -2	
8.1	$b_1^{-1}b_2b_3b_2^{-1}b_1^{-1}b_4^2b_3b_2b_4^{-1}$	16 **43** 37 12 1	-64 144 **-88** 9	
8.2	$b_1^{-1}b_2^5b_1^{-1}b_2$	**4** 59/8 23/8 1/4	-24 **36** -4	
8.3	$b_1^{-2}b_2^{-1}b_1b_4^2b_3b_4^{-1}b_2^{-1}b_3$	-8 **-8** -1	8 **-8** 1	A
8.4	$b_1^3b_3b_2^{-1}b_3^{-2}b_1b_2^{-1}$	8 **8** 3/4	8 **-24** 19 -2	
8.5	$b_1^3b_2^{-1}b_1^3b_2^{-1}$	**1** 3 19/8 1/4	-24 **36** -4	
8.6	$b_1^{-1}b_2b_1^{-1}b_3^{-1}b_2^3b_3^2$	**5** 21/2 21/4 1/2	**1**	
8.7	$b_1^4b_2^{-2}b_1b_2^{-1}$	**3** 9/4 1/4	16 **-25** 3	
8.8	$b_1^{-1}b_2b_1^2b_3^{-1}b_2^2b_3^{-2}$	**3** 17/4 1/2	16 **-21** 5/2	
8.9	$b_1^{-1}b_2b_1^{-3}b_2^3$	**-7** -9 -1	**-7** 9 -1	A
8.10	$b_1^{-1}b_2^2b_1^{-2}b_2^3$	**1** 2 1/4	8 **-8** 1	
8.11	$b_1^{-1}b_2^2b_3^{-1}b_2b_3^2b_1^{-1}b_2$	8 **21** 147/8 6 1/2	-64 136 **-79** 8	
8.12	$b_1b_2^{-1}b_3b_4^{-1}b_3b_4^{-1}b_2b_1b_3^{-1}b_2^{-1}$	24 **44** 21 2	-24 **44** -21 2	A
8.13	$b_1^2b_2b_3^{-1}b_2b_1^{-1}b_3^{-2}b_2$	8 **12** 21/4 $-1/2$	8 **-28** 39/2 -2	
8.14	$b_1^2b_2^2b_1^{-1}b_3^{-1}b_2b_3^{-1}b_2$	**6** 85/8 21/4 1/2	-8 **18** -2	
8.15	$b_1^2b_2^{-1}b_1b_3^2b_2^2b_3$	**0** $-17/8$ $-9/4$ $-1/4$	64 -32 **4**	
8.16	$b_1^2b_2^{-1}b_1^2b_2^{-1}b_1b_2^{-1}$	**-3** 3/2 1/4	**-7** 1	
8.17	$b_1^{-1}b_2b_1^{-1}b_2^2b_1^{-2}b_2$	**-11** $-19/2$ -1	**-11** 19/2 -1	A
8.18	$b_1b_2^{-1}b_1b_2^{-1}b_1b_2^{-1}b_1b_2^{-1}$	-8 **-16** -10 -1	8 **-16** 10 -1	A
8.19	$b_1b_2b_1b_2b_1b_2^2b_1$	**0** 3/8 1/16	64 -64 **1**	
8.20	$b_1^3b_2b_1^{-3}b_2$	**5** 9/2 1/2	-8 **0**	
8.21	$b_1b_2^{-2}b_1^2b_2^3$	**1** -1 $-1/8$	8 **0**	

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