

## On open 3-manifolds proper homotopy equivalent to geometrically simply-connected polyhedra <sup>☆</sup>

L. Funar <sup>a</sup>, T.L. Thickstun <sup>b,\*</sup>

<sup>a</sup> Institut Fourier BP 74, University of Grenoble I, 38402 Saint-Martin-d'Hères cedex, France

<sup>b</sup> Department of Mathematics, Southwest Texas State University, San Marcos, TX 78666, USA

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### Abstract

We prove that an open 3-manifold proper homotopy equivalent to a geometrically simply connected polyhedron is simply connected at infinity thereby generalizing the theorem proved by Poénaru (1991). © 2001 Elsevier Science B.V. All rights reserved.

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### 1. Introduction

The immediate antecedent to this paper is [6], the principal theorem of which is the following.

**Theorem 1.1** (V. Poénaru). *If  $U$  is an open simply connected 3-manifold and, for some  $n$ ,  $U \times D^n$  has a handlebody decomposition without 1-handles then  $U$  is geometrically simply connected hence simply connected at infinity.*

**Note.**  $D^n$  denotes the  $n$ -ball; see [6] for the definition of “handlebody decomposition without 1-handles”; all 3-manifolds we consider in the sequel will be orientable unless the contrary is explicitly stated.

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\* Corresponding author.

*E-mail addresses:* funar@fourier.ujf-grenoble.fr (L. Funar), tt04@swt.edu (T.L. Thickstun).

A non-compact polyhedron  $U$  is *simply connected at infinity* (s.c.i.), and we write also  $\pi_1^\infty(U) = 0$ , if given a compactum (i.e., a compact set)  $X \subset U$  there exists another compactum  $Y$  with  $X \subset Y \subset U^3$ , such that any loop in  $U - Y$  is null-homotopic in  $U - X$ . (Some authors call this  $\pi_1$ -triviality at infinity or 1-LC at infinity and reserve the term s.c.i. for the special case in which  $Y$  can be chosen so that, in addition,  $U - Y$  is connected. These notions are equivalent for one ended spaces, such as contractible spaces.)

A non-compact polyhedron  $P$  is *geometrically simply connected* (g.s.c.) if it can be exhausted by compact 1-connected subpolyhedra. It is easily seen that any non-compact polyhedral manifold  $U$  which has a handlebody decomposition with no 1-handles is g.s.c. In addition the projection map  $p: U \times D^n \rightarrow U$  is a proper simple-homotopy equivalence (defined in [7]). In [6] Poénaru hinted at the conjecture which results when the hypothesis of the above-stated theorem is replaced by the (therefore) weaker hypothesis that  $U$  be proper simple-homotopy equivalent to a g.s.c. polyhedron. This conjecture was subsequently established in [2] using the techniques of [6]. The following theorem is an immediate corollary of the principal result of this paper (which is proven using only basic 3-manifold theory). It further generalizes the theorem stated above.

**Theorem 1.2.** *Any open 3-manifold which is proper homotopy equivalent to a geometrically simply connected polyhedron is simply-connected at infinity.*

**Remark 1.1.** If  $M^n$  ( $n > 3$ ) is a compact, contractible  $n$ -manifold with non-simply-connected boundary (e.g., those constructed in [4] and [5]) then  $\text{int}(M^n)$  is easily seen to be g.s.c. but not s.c.i. This demonstrates that the above theorem cannot be extended to include open contractible manifolds of dimension greater than three. Notice that for simply connected open 3-manifolds the g.s.c. and the s.c.i. are equivalent.

**Provisos.** We remain in the polyhedral category throughout and all homology groups are with  $\mathbf{Z}$  coefficients.

## 2. Statement of results

We first require the following definitions.

**Definition 2.1.** A proper map  $f: X \rightarrow Y$  is  $H_3$ -nontrivial if given non-null compacta  $L \subset Y$  and  $K \subset X$  such that  $f(X - K) \subset Y - L$  then  $f_*: H_3(X, X - K) \rightarrow H_3(Y, Y - L)$  is nontrivial (i.e., its image is not a singleton).

**Definition 2.2.** Given noncompact polyhedra  $X$  and  $Y$  we say  $Y$  is  $H_3$ -semi-dominated by  $X$  if there exists an  $H_3$ -nontrivial proper map  $f: X \rightarrow Y$ .

**Definition 2.3.** An open connected 3-manifold  $U^3$  is *simple-ended* if it has an exhaustion  $\{M_i\}_{i=1}^\infty$  by compact 3-submanifolds where, for all  $i$ , the genus of  $\partial M_i$  is zero.

**Remark 2.1.**

- (1) If  $U$  is an open orientable 3-manifold and  $Y \subset U$  a non-null compactum then  $H_3(U, U - Y)$  is nontrivial.
- (2) If  $f, g : X \rightarrow Y$  are properly homotopic maps where  $X$  and  $Y$  are open, connected 3-manifolds and  $f$  is  $H_3$ -nontrivial then  $g$  is  $H_3$ -nontrivial.

**Proof.** (1) Since we are in the polyhedral category  $Y$  is a compact polyhedron and so has a regular neighborhood  $N$ . Since  $N - Y$  deformation retracts onto  $\partial N$  we have that  $H_3(U, U - Y)$  is isomorphic to  $H_3(U, U - \text{int } N)$ . Taking a triangulation of the pair  $(U, N)$  and using the orientation of  $U$  we see that  $N$  is a relative cycle representing a nontrivial element of  $H_3(U, U - \text{int } N)$ . Moreover, this is a free Abelian group with basis elements represented by the components of  $N$ . This proves the first remark.

(2) Suppose that  $K, L$  are compacta in  $X, Y$ , respectively, with  $g(X - K) \subset Y - L$ . Then  $K \supset g^{-1}(L)$ . Let  $F : X \times I \rightarrow Y$  be a proper homotopy with  $F(x, 0) = f(x)$  and  $F(x, 1) = g(x)$ . Since  $F^{-1}(L)$  is compact, there is a connected compactum  $M$  of  $X$  such that  $M \times I \supset F^{-1}(L) \cup (K \times \{1\})$ . Let  $N$  be a regular neighborhood of  $M$  in  $X$ . Then  $N$  represents a generator of the infinite cyclic group  $H_3(X, X - \text{int } N)$  which is isomorphic to  $H_3(X, X - M)$ . Now  $f^{-1}(L) \subset M$  implies that  $f(X - M) \subset Y - L$ . Since  $f$  is  $H_3$ -nontrivial we have that  $f(N)$  represents a nontrivial element of  $H_3(Y, Y - L)$ . Since  $K \subset \text{int } N$  we have that  $N$  represents an element of  $H_3(X, X - K)$ . Since  $F^{-1}(L) \subset \text{int } N \times I$  we have that  $g(N)$  is homologous to  $f(N)$  in  $(Y, Y - L)$ , and so  $g$  is  $H_3$ -nontrivial.  $\square$

Our main result is the following.

**Theorem 2.1.** *An open, connected, orientable 3-manifold  $U$  is  $H_3$ -semi-dominated by a g.s.c. polyhedron if and only if  $U$  is the connect-sum of a 1-connected, simply connected at infinity open 3-manifold and a closed, orientable 3-manifold with finite fundamental group.*

**Remark 2.2.**

- (1) Theorem 1.2 is a corollary of Theorem 2.1. In fact, let  $f : P \rightarrow U$  be a proper homotopy equivalence with proper homotopy inverse  $g : U \rightarrow P$ . Suppose  $K, L$  are compacta in  $P, U$ , respectively, such that  $f(P - K) \subset U - L$ . Let  $M = g^{-1}(K)$ . Then  $g(U - M) \subset P - K$ . Since  $f \circ g$  is proper homotopic to  $id_U$  and  $id_U$  is  $H_3$ -nontrivial, we have by Remark 2.1(2) that  $(f \circ g)_*$  is nontrivial and hence  $f_*$  is nontrivial.
- (2) The proof of the “if” part of the theorem is very brief. Just observe that if  $U$  satisfies the hypothesis then the universal covering of  $U$  is 1-connected and simply connected at infinity (hence g.s.c. – see (3) below) and the covering projection is proper and has non-zero degree (hence is  $H_3$ -nontrivial). (In the sequel when we refer to the hypothesis or conclusion of the theorem we will mean the “only if” part.)
- (3) By the methods of [9] the class of 1-connected, simply connected at infinity open 3-manifolds is equal to each of the following two classes of open 3-manifolds. Those which can be constructed as follows: delete a tame, 0-dimensional, compact

subspace from  $S^3$ , denote the result by  $U$  and replace each element of a pairwise disjoint, proper family of 3-balls in  $U$  by a homotopy 3-ball. Those each of which has an exhaustion  $\{M_i\}_{i=1}^\infty$  by compact 3-submanifolds where, for each  $i$ ,  $M_i$  is 1-connected (and hence the genus of  $\partial M_i$  is 0) and each component of  $M_{i+1} - M_i$  is homeomorphic to a space obtained by taking finitely many pairwise disjoint 3-balls in  $S^3$ , replacing one by a homotopy 3-ball and deleting the interiors of the rest.

To establish the theorem we will demonstrate the following three propositions.

**Proposition 2.1.** *If  $U$  is as in the hypothesis of the theorem then  $U$  is simple-ended.*

**Proposition 2.2.** *If  $U$  is as in the hypothesis of the theorem then  $\pi_1(U)$  is a torsion group.*

**Proposition 2.3.** *If  $U$  is an open, connected simple-ended 3-manifold such that  $\pi_1(U)$  is a torsion group then  $U$  is as in the conclusion of the theorem.*

### 3. Proof of Proposition 2.1

**Lemma 3.1.** *Suppose the following:  $U$  is an open, orientable 3-manifold;  $K$  is a compact, connected 3-submanifold of  $U$  such that each component of  $cl(U - K)$  is noncompact and has connected boundary; and  $f : (M, \partial M) \rightarrow (U, U - K)$  is a map of a compact connected 3-manifold with boundary such that  $\partial M$  has genus zero and  $f_* : H_3(M, \partial M) \rightarrow H_3(U, U - K)$  is nontrivial. Then there exists a compact, connected 3-submanifold  $N$  of  $U$  such that  $K$  is in  $N$  and the genus of  $\partial N$  is zero.*

**Proof.** It will suffice to find, for each component  $V$  of  $cl(U - K)$  a compact, connected 3-submanifold  $N(V)$  of  $V$  such that  $\partial V$  is in  $N(V)$  and the genus of  $\partial N(V) - \partial V$  is zero. So let  $V$  be such a component. We assume  $f$  is transverse to  $\partial V$  and denote the intersection of  $f^{-1}(V)$  and  $\partial M$  by  $C$ . The orientation of  $U$  determines an orientation of  $V$ , which in turn determines an orientation of  $\partial V$ . Similarly the orientation of  $M$  determines an orientation of the 3-dimensional submanifold  $f^{-1}(V)$ , which in turn determines an orientation of  $f^{-1}(\partial V) \subset \partial f^{-1}(V)$ .

From the hypothesis on  $f_*$  we derive that  $(f|_{f^{-1}(\partial V)})_*$  carries the homology class of  $f^{-1}(\partial V)$ , as determined by the orientation above, to a nontrivial element of  $H_2(\partial V)$ . In fact, consider the commutative diagram

$$\begin{array}{ccc} H_3(M, \partial M) & \xrightarrow{\partial_M} & H_2(f^{-1}(\partial V)) \\ f_* \downarrow & & \downarrow (f|_{f^{-1}(\partial V)})_* \\ H_3(U, U - int K) & \xrightarrow{\partial_U} & H_2(\partial V) \end{array} ,$$

where  $\partial_M$  comes from the relative Mayer–Vietoris sequence of  $(M, \partial M) = (f^{-1}(V), C) \cup (M - f^{-1}(V), \partial M - C)$  and  $\partial_U$  comes from the relative Mayer–Vietoris sequence of  $(U, U - int K) = (V, V) \cup (\overline{U - V}, \overline{U - (V \cup K)})$ . A chase through the zig-zag lemma

shows that  $\partial_M$  takes the orientation class of  $(M, \partial M)$  to the class of  $f^{-1}(\partial V)$  and  $\partial_U$  takes the generator of the infinite cyclic group  $H_3(U, U - \text{int } K)$  to the class of  $\partial V$ . The proof is completed by noting that inclusion induces an isomorphism of  $H_3(U, U - \text{int } K)$  with  $H_3(U, U - K)$  (since  $U - \text{int } K$  and  $U - K$  both deformation retract onto  $U$  minus the interior of a regular neighborhood of  $K$ ).

Note that  $f|_C : C \rightarrow V$  and  $f|_{f^{-1}(\partial V)} : f^{-1}(\partial V) \rightarrow V$  are homologous and hence  $f|_C$  is homologous (in  $V$ ) to a nonzero multiple of  $\partial V$ .

Applying the prime factorization theorem for compact 3-manifolds to a regular neighborhood of  $f(C)$  in  $\text{int}(V)$  we conclude the existence of an embedding  $R$  in  $\text{int}(V)$  where  $R$  is a closed, oriented surface of genus zero such that  $R$  is homologous to  $f|_C$  in  $\text{int}(V)$  (recall that the prime factorization and the sphere theorem imply that  $\pi_2$  of a compact 3-manifold is generated, as a  $\pi_1$ -module, by a finite family of pairwise disjoint embedded 2-spheres, and by the Hurewicz theorem  $H_2$  is isomorphic to  $\pi_2$  modulo the action of  $\pi_1$ ). Now let  $W$  be a regular neighborhood of  $R$  in  $\text{int}(V)$  and denote by  $N(V)$  that component of  $\text{cl}(V - W)$  containing  $\partial V$ . It remains only to show that  $N(V)$  is compact. Suppose otherwise. Then there exists a proper ray in  $V$  extending from  $\partial V$  to infinity and avoiding  $R$ . Such a ray has intersection number one with  $\partial V$  but intersection number zero with  $R$ . This contradicts the fact that  $R$  is homologous in  $V$  to a multiple of  $\partial V$ .  $\square$

**Definition 3.1.** An *admissible pair* is a map  $f : (X, Y) \rightarrow (M, M - L)$  and a subspace  $K \subset L$  satisfying the following conditions:

- (1)  $(X, Y)$  is a pair of compact simplicial complexes and  $X$  is simply connected.
- (2)  $M$  is a compact orientable 3-manifold and  $K$  and  $L$  are compact 3-submanifolds with  $K \subset \text{int}(L)$ ,  $L \subset \text{int}(M)$  and  $L$  connected.
- (3) The map  $f$  is simplicial and  $M$  is an (abstract) regular neighborhood of  $f(X)$ .
- (4) Only one component of  $X - f^{-1}(\partial L)$  has image under  $f$  meeting  $K$ .
- (5) The map  $f_* : H_3(X, Y) \rightarrow H_3(M, M - L)$  is nontrivial.

We will refer to the  $M$  above as the *target* of the admissible pair.

**Notation.** If  $X \subset P$  then  $\partial X$  denotes the frontier of  $X$  in  $P$ .

**Lemma 3.2.** If  $f : P \rightarrow U$  is the map of the theorem and  $K$  is a compact 3-submanifold of  $U$  then we can choose  $X \subset P$  and a compact 3-submanifold  $L$  of  $U$  such that the pair  $\{f|_X : (X, \partial X) \rightarrow (M, M - L), K\}$  is admissible where  $M$  is a regular neighborhood of  $f(X)$  in  $U$ .

**Proof.** Let  $L_0$  be a regular neighborhood of  $K$  in  $U$ . The hypothesis implies the existence of some  $X_0$  with  $f^{-1}(L_0) \subset X_0 \subset P$ , such that  $f_* : H_3(P, P - X_0) \rightarrow H_3(U, U - L_0)$  is nontrivial. Moreover once such an  $X_0$  is chosen then larger  $X$  with  $\text{int}(X) \supset X_0$  are also convenient for this purpose. Moreover  $f$  is proper implies the existence of an  $X_1 \supset X_0$  such that  $f(\partial X_1) \cap L_0 = \emptyset$ . Using excision on both sides above we find that the map  $(f|_{X_1})_* : H_3(X_1, \partial X_1) \rightarrow H_3(M, M - L_0)$  is nontrivial, where  $M$  denotes a regular

neighborhood of  $f(X_1)$  in  $U$ . Denote the set  $X_1$  with these properties by  $X(L_0)$ . Note that  $X(L_0)$  is defined for any compact  $L_0$  engulfing  $K$ . If  $X - f^{-1}(\partial L_0)$  has at least two components each of which has image meeting  $K$  then  $f^{-1}(K) \cap (X(L_0) - f^{-1}(\partial L_0))$  is not connected. Since  $f$  is simplicial the latter is a simplicial subcomplex of  $P$  and so it cannot be path connected. In particular there exist points  $x, y \in f^{-1}(K)$  such that any path connecting them in  $X(L_0)$  should meet  $f^{-1}(\partial L_0)$ . But  $f^{-1}(K)$  is a compact (thus finite) simplicial complex because  $f$  is proper hence it has a finite number of components. Consider some arcs joining these components in  $P$ . The union of these arcs with the components of  $f^{-1}(K)$  is contained in some compact subset  $K' \subset P$ . Consider now  $L$  large enough such that  $f^{-1}(\partial L) \cap K' = \emptyset$ , and  $K' \subset \text{int}(f^{-1}(L))$ . Then  $X(L) \supset f^{-1}(L) \supset K'$ , and we claim that this  $X$  fulfills all conditions needed. If  $x, y$  are two points in  $f^{-1}(K)$  then there exists a path connecting them inside  $K'$  and so there exists a path inside  $X(L) - f^{-1}(\partial L)$ .  $\square$

**Lemma 3.3.** *Given an open, connected 3-manifold  $U$  and compactum  $X$  in  $U$  there exists a compact 3-submanifold  $K$  of  $U$  containing  $X$  such that each component of  $\text{cl}(U - K)$  is noncompact and has connected boundary.*

**Proof.** Let  $M$  be a compact 3-submanifold of  $U$  containing  $X$ . Let  $N$  be the union of  $M$  and all compact components of  $\text{cl}(U - M)$ . To obtain  $K$  from  $N$  add 1-handles to  $N$  (in  $\text{cl}(U - N)$ ) which connect different components of  $\partial N$  which are in the same component of  $\text{cl}(U - N)$ .  $\square$

The proof of Proposition 2.1 will proceed by applying the tower construction to the admissible pair of Lemma 3.2 (where  $K$  is also chosen to satisfy the conclusion of Lemma 3.3) to obtain a map satisfying the hypothesis of Lemma 3.1. It is convenient to state first the following definition.

**Definition 3.2.** A *reduction of the admissible pair*  $\{f_0 : (X, Y) \rightarrow (M_0, M_0 - L_0), K_0\}$  is a second admissible pair  $\{f_1 : (X, Y) \rightarrow (M_1, M_1 - L_1), K_1\}$  such that there exists a map  $p : M_1 \rightarrow M_0$ , the “projection map” of the reduction, satisfying the following conditions:

- (1)  $p \circ f_1 = f_0$ .
- (2)  $p(K_1) = K_0$ ,  $p(L_1) = L_0$ , and the maps  $p|_{L_1} : L_1 \rightarrow L_0$ ,  $p|_{K_1} : K_1 \rightarrow K_0$  are boundary preserving.
- (3)  $p|_{L_1} : L_1 \rightarrow L_0$  has non-zero degree.
- (4) The complexity of  $f_1$  is strictly less than the complexity of  $f_0$  (where the complexity of a simplicial map  $g$  with compact domain is the number of simplexes  $s$  in the domain for which  $g^{-1}(g(s)) \neq s$ ).
- (5) The image under  $p$  of only one component of  $M_1 - \partial L_1$  meets  $K_0$ . Observe that, by (2), the image of  $L_1$  must meet  $K_0$ .

**Lemma 3.4.** *If  $f_1$  is a reduction of  $f_0$  and  $f_2$  is a reduction of  $f_1$  then  $f_2$  is a reduction of  $f_0$ .*

**Proof.** This is obvious.  $\square$

**Lemma 3.5.** Any admissible pair with non-simply connected target has a reduction with simply-connected target.

**Proof.** It will suffice to show that any admissible pair with non-simply connected target has a reduction. Because then, by iteration (which could occur at most finitely many times by condition (4)) and applying Lemma 3.3 we obtain a reduction with simply connected target.

Let  $\{f|_X : (X, Y) \rightarrow (M, M - L), K\}$  be the admissible pair. Let  $\tilde{M}$  be the universal covering of  $M$  and  $p$  be the covering projection. Since  $X$  is simply connected there exists a lift  $\tilde{f} : X \rightarrow \tilde{M}$  of  $f$ . We will show that  $\{f_1 : (X, Y) \rightarrow (M_1, M_1 - L_1), K_1\}$  is a reduction, where

- (1)  $f_1 = \tilde{f}$ .
- (2)  $M_1$  is a regular neighborhood of  $\tilde{f}(X)$ .
- (3)  $L_1$  is the only component of  $p^{-1}(L)$  which is contained in  $f_1(X) = \tilde{f}(X)$ .
- (4)  $K_1 = p^{-1}(K_0) \cap L_1$ .
- (5) The covering map restricted to  $M_1$  is the map  $p$  from the Definition 3.2.

Let us show that  $f_1$  is well-defined and admissible.

*$f_1$  is well-defined:* First we show that  $f_1(Y) \subset M_1 - L_1$ . We know that  $f(Y) \subset M - L$  since  $f$  is admissible, and so  $f_1(Y) = M_1 \cap p^{-1}(f(Y)) \subset M_1 \cap p^{-1}(M - L) = M_1 - p^{-1}(L) \subset M_1 - L_1$ .

In order to have a consistent definition of  $L_1$  we must show that there exists one and only one component of  $p^{-1}(L)$  contained in  $f_1(X)$ .

First we prove the existence. Since  $f$  is nontrivial on  $H_3$  (the condition (5) for  $f$ ) we derive that  $\tilde{f}_* : H_3(X, Y) \rightarrow H_3(\tilde{M}, \tilde{M} - p^{-1}(L))$  is nontrivial. In particular the Abelian group  $H_3(\tilde{M}, \tilde{M} - p^{-1}(L))$  is non-zero. Since this group is freely generated by an equivariant regular neighborhood of the compact components of  $p^{-1}(L)$ , there exists at least one such (the deck transformations act transitively on the components of  $p^{-1}(L)$  and so every component is compact). Since  $L$  is connected  $H_3(M, M - L)$  is generated by the orientation class of a regular neighborhood of  $L$  and  $\tilde{f}_*$  nontrivial implies that  $L$  is in  $f(X)$ .

Observe now that  $p(M_1 \cap p^{-1}(L)) = L$  since  $p(M_1) = M$ , and also  $p(M_1 - M_1 \cap p^{-1}(L)) = M - L$ . Then since  $f$  is  $H_3$  nontrivial the map  $\tilde{f}_* : H_3(X, Y) \rightarrow H_3(M_1, M_1 - M_1 \cap p^{-1}(L))$  should be nontrivial. The same argument used above shows that  $\tilde{f}(X) \supset M_1 \cap p^{-1}(L)$ . Let  $L_1$  be a component of  $p^{-1}(L)$  which meets  $\tilde{f}(X)$ . Suppose that  $L_1$  is not completely contained in  $\tilde{f}(X)$ . Since  $L_1$  is connected then  $L_1 \cap (M_1 - \tilde{f}(X)) \neq \emptyset$ , or in other words the regular neighborhood of  $\tilde{f}(X)$  meets a larger subset of  $L_1$  than the image  $\tilde{f}(X)$ . This contradicts the fact that  $M_1 \cap L_1 \subset \tilde{f}(X)$ . Thus any component of  $p^{-1}(L)$  meeting  $\tilde{f}(X)$  is entirely contained in  $\tilde{f}(X)$ . At least one component has non-void intersection with the image because  $p(\tilde{f}(X) \cap p^{-1}(L)) = p(M_1 \cap p^{-1}(L)) = L$ .

Suppose now that there are two components,  $L_1$  and  $L'_1$  meeting  $\tilde{f}(X)$ . The two components are then disjoint and contained in  $\tilde{f}(X)$ . Furthermore there are two

components of  $M_1 - p^{-1}(\partial L)$ , namely  $int(L_1)$  and  $int(L'_1)$  whose images under  $p$  meet  $K$ . However there is only one component, say  $\xi$ , of  $X - f^{-1}(\partial L)$  whose image by  $f$  meets  $K$ . Then  $\tilde{f}(\xi) \subset M_1 - \partial L_1 \cup \partial L'_1$  since  $\tilde{f}(\xi)$  avoids  $p^{-1}(\partial L)$  and  $\tilde{f}(\xi) \supset int(L_1) \cup int(L'_1) \supset K_1 \cup K'_1$  because  $p(f(\xi))$  meets  $K$ . This is a contradiction as  $\tilde{f}(\xi)$  must be connected.

$f_1$  is admissible: Conditions (1)–(3) from Definition 3.1 are immediate. The condition (5) for  $f_1$  is satisfied since  $f = p \circ f_1$  is  $H_3$ -nontrivial. Finally (4) is implied by  $f_1^{-1}(\partial L_1) = f^{-1}(\partial L)$ .

$f_1$  is a reduction of  $f$ : With the exception of (3) and (5) we leave them to the reader. Condition (3) follows from the fact that  $p|_{L_1} : L_1 \rightarrow L$  is a covering map, and it is well known that any covering map from one compact, connected, orientable 3-manifold to another has non-zero degree. An easy argument is the following. Triangulate and orient the base space. Lift the triangulation and the orientation of each 3-simplex to the covering space. If this does not give an orientation of the covering space, then there will be a 2-simplex which is a face of two 3-simplexes which are mapped to the same 3-simplex in the base space, contradicting the fact that the map is a covering map. It then follows that a fundamental cycle for the covering space is sent to  $n$  times the fundamental cycle for the base space, where  $n$  is the number of sheets of the covering.

We saw before that  $int(L_1)$  is the only component of  $M_1 - p^{-1}(\partial L)$  whose image meets  $K$  hence establishing (5).  $\square$

**Proof of Proposition 2.1.** Begin with the admissible pair of Lemma 3.2 where  $K$  also satisfies the conclusion of Lemma 3.3. Apply Lemma 3.5 to that admissible pair. Note that the projection map of that reduction satisfies the hypothesis of Lemma 3.1 whose application then completes the proof.  $\square$

**4. Proof of Proposition 2.2**

**Lemma 4.1.** *Suppose the diagram*

$$\begin{array}{ccc}
 M & \xrightarrow{\varphi} & X \\
 h \downarrow & & \downarrow \psi \\
 N & \subset & U
 \end{array}$$

*is commutative and satisfies the following conditions:*

- (1)  $M$  and  $N$  are compact, connected, orientable 3-manifolds with  $h(\partial M) \subset \partial N$ .
- (2)  $h$  has non-zero degree.
- (3)  $X$  is simply-connected.

*Then  $e_{\#}(\pi_1(N))$  is a torsion subgroup of  $\pi_1(U)$  (where  $e$  denotes the inclusion  $N \subset U$ ).*

**Proof.** Let  $p : \tilde{N} \rightarrow N$  be the covering map such that  $p_{\#}(\pi_1(\tilde{N})) = h_{\#}(\pi_1(M))$ . Then  $h$  lifts to  $\tilde{h} : M \rightarrow \tilde{N}$ . Since the degree of  $h$  is non-zero  $H_3(\tilde{N}, \partial \tilde{N}) \neq 0$ , and so  $\tilde{N}$  is compact and thus  $p$  is finite sheeted. Hence  $h_{\#}(\pi_1(M))$  has finite index in  $\pi_1(N)$ . (This is a standard



argument.) Thus for any element of  $\pi_1(N)$  some power will be in the image of  $h_{\#}$  and therefore will be trivial in  $\pi_1(U)$ .  $\square$

Now to prove Proposition 2.2 it will suffice to show that if  $N$  is a compact, connected 3-submanifold of  $U$  then  $e_{\#}(\pi_1(N))$  is a torsion subgroup of  $\pi_1(U)$ . Let  $f: X \rightarrow U$  be as in the definition of “ $H_3$ -semi-dominated” (where  $X$  is g.s.c.). By excision we have  $f_*: H_3(Y, \partial Y) \rightarrow H_3(N, \partial N)$  is nontrivial, where  $Y = f^{-1}(N)$ . We can “realize” any element of  $H_3(Y, \partial Y)$  by a map  $g: (M, \partial M) \rightarrow (Y, \partial Y)$ , where  $M$  is a compact orientable 3-manifold (i.e.,  $g_*: H_3(M, \partial M) \rightarrow H_3(Y, \partial Y)$  sends the orientation class of  $M$  to the preassigned element of  $H_3(Y, \partial Y)$  – see [8]). The 3-manifold  $M$  coming from Thom’s theorem might not be connected. But since  $(f \circ g)_*: H_3(M, \partial M) \rightarrow H_3(N, \partial N)$  is nontrivial there will be some component  $M_0$  of  $M$  such that  $((f \circ g)|_{M_0})_*: H_3(M_0, \partial M_0) \rightarrow H_3(N, \partial N)$  is nontrivial. Now apply the lemma with  $h = f \circ (g|_{M_0})$ .  $\square$

### 5. Proof of Proposition 2.3

**Lemma 5.1.** *If  $A$  and  $B$  are groups and  $a \in A$ ,  $b \in B$  are neither the identity then  $a * b$  is not a torsion element in  $A * B$  (the free product of  $A$  and  $B$ ).*

**Proof.** This is a standard fact from combinatorial group theory. Every element of finite order in  $A * B$  is conjugate to an element of  $A$  or of  $B$  (Corollary 4.1.4 of [3]), and every element of  $A * B$  is conjugate to a cyclically reduced word which is unique up to cyclic permutation (Theorem 4.2 of [3]). Since  $a * b$  is cyclically reduced it cannot be conjugate to an element of  $A$  or of  $B$  and hence cannot have finite order.  $\square$

**Corollary 5.1.** *The prime factorization of a closed 3-manifold whose fundamental group is a torsion group can have at most one non-simply connected factor.*

Now to prove Proposition 2.3 we assume that  $U$  is not simply-connected (otherwise we are done) and let  $M$  be a non-simply connected 3-submanifold of  $U$  such that  $\partial M$  has genus zero. We can express  $M$  as a connect-sum of a punctured homotopy 3-ball and a closed orientable 3-manifold  $N$  where  $\pi_1(N)$  is a torsion group. By the above corollary we can assume  $N$  is irreducible. By [1] any orientable, irreducible closed 3-manifold with torsion must have finite fundamental group. It remains only to show that  $U$  can have no other non-simply connected factor but this also follows from the corollary.  $\square$

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