SIMPLE HOMOTOPY TYPE AND OPEN 3-MANIFOLDS

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The main result of this paper is that a contractible open 3-manifold W^3, which has the same simple homotopy type as a geometrically simply connected simplicial complex P, is simply connected at infinity. This is obtained as a consequence of the fact that W^3 is simply connected at infinity provided that it has a geometrically simply connected enlargement. The latter is a generalization of a theorem of Poénaru [7].

AMS 1991 Subject Classification: 57 M 50, 57 M 10, 57 M 30.

Key words and phrases: Enlargement, geometric simple connectivity, simple connectivity at infinity, simple proper homotopy, Φ/Ψ-theory, Dehn-type lemma, π_1^∞, Hilbert cube.

1. INTRODUCTION

We start with the following definitions:

Definition 1.1. An enlargement (with strongly connected 3-skeleton) of a smooth k-dimensional manifold M^k is a locally finite simplicial complex X which fits into a commutative diagram

$$\begin{array}{ccc} M^k & \overset{i}{\hookrightarrow} & X \\ & \mathrm{id}\searrow & \downarrow \pi \\ & & M^k \end{array}$$

where

1. i is a proper PL embedding with respect to the (unique) PL structure on M^k compatible with the DIFF structure;

2. π is a proper PL map;

3. the 3-skeleton ske^3X of X is strongly connected, i.e. for any two 3-simplices σ and τ of X there exists a sequence of 3-simplices $\sigma = \sigma_1, \sigma_2, ..., \sigma_n = \tau$ such that σ_j and σ_{j+1} have a common 2-dimensional face, for all $j = 1, 2, ..., n-1$.

The simplest examples of enlargements are the regular neighborhoods of embeddings in Euclidean spaces. In the sequel, we will consider that all the enlargements have a strongly connected 3-skeleton unless the contrary is explicitly stated.

Definition 1.2. An open contractible 3-manifold W^3 is said to be simply connected at infinity (s.c.i.), and we also write $\pi_1^\infty(W) = 0$, if for any compact set

K_1 there exists another compact set K_2 with $K_1 \subset K_2 \subset W^3$, such that any loop in $W^3 - K_2$ is null-homotopic in $W^3 - K_1$.

Definition 1.3. A locally finite simplicial complex P is said to be weak geometrically simply connected (w.g.s.c.) if there exists an exhaustion $Z_0 \subset Z_1 \subset \subset Z_2 \subset ...Z_n \subset ...$ of P by finite sub-complexes with all Z_n being connected and simply connected.

First, we prove the following result.

THEOREM 1.1. *Let W^3 be an open contractible 3-manifold, and X^n a finite dimensional enlargement of W^3. If X^n is weak geometrically simply connected (w.g.s.c.), then the manifold W^3 is simply connected at infinity (s.c.i.).*

Actually, we will prove a stronger statement: under the same hypotheses, there exists an exhaustion $Z_0 \subset Z_1 \subset Z_2 \subset ...Z_n \subset ...$ of W^3 by compact connected and simply connected sub-manifolds Z_n. In dimension 3, this condition implies that $\pi_1^\infty(W^3) = 0$.

It is well-known that the simple connectedness at infinity is invariant under a proper homotopy, without any dimension restriction. However, our result is purely 3-dimensional, and is not a consequence of the previous remark, as it may seem at the first glance. The g.s.c does not imply (in other dimension than 3) the simple connectivity at infinity. For instance, from "$W^n \times D^k$ is w.g.s.c." we cannot deduce a priori that "$W^n \times D^k$ is s.c.i.", in order to conclude that W^n also is. Here D^k is the closed k-ball. Moreover, this is not true if the dimension of W^n is $n > 3$.

Moreover the condition w.g.s.c. is a consequence of the s.c.i. In [18], p.350 and [19] the (partially known) relation between the (usual) connectivity and the geometric connectivity is discussed. Further, in [11] it is proved that, if W^n is s.c.i. open, simply connected and its dimension is $n \geq 5$, then W is w.g.s.c. The authors conjectured that the s.c.i. condition is necessary. Observe that this would imply both theorems presented here.

However, the last conjecture cannot be extended to more general non-compact manifolds with boundary, like the products $W \times D^k$ with closed k-balls. There exist manifolds W^n in every dimension $n \geq 4$ (e.g. the Poénaru-Mazur manifolds, see [5, 3]), such that $W \times D^k$ is w.g.s.c. for some k, but $W \times D^k$ (and henceforth W) is not s.c.i., so our result cannot be extended to higher dimensions. Examples of such W are the interiors of compact contractible manifolds, whose boundaries have nontrivial fundamental group. Also, in dimension 4, there is an obstruction (due to A.Casson) for the geometric simple connectivity (still in the compact case): if the fundamental group of the boundary has a nontrivial representation in a Lie group, then the manifold is not w.g.s.c. In particular, the Poénaru-Mazur manifolds are not w.g.s.c.

Remark 1.1. 1. Consider $X^n = W^3 \times D^{n-3}$, D^k being the closed k-ball, or, more generally, that X is a proper codimension 0 sub-manifold of $W^3 \times D^{n-3}$, which engulfs the zero-section (i.e. $W^3 \times 0 \subset X^n \subset W^3 \times D^{n-3}$). Then Poénaru's result ([7]) states that, if X^n has no 1-handles, then W^3 is simply connected at infinity. In fact, the condition to have no 1-handles implies that, after triangulating the manifold and taking a sufficiently fine subdivision, we obtain a w.g.s.c. simplicial complex (see [7], PL-lemma, p. 441). Thus Corollary 1.5 (see below) can be viewed as an extension of this result.

If we had worked in a DIFF context,by considering only those enlargements which are manifolds, then the "w.g.s.c." condition in Definition 1.1 should be replaced by "without 1-handles". [We recall what is meant by "to have no 1-handles" for a non-compact manifold with boundary (according to [7, 10]): there exists a *proper* smooth function $f : X \to [0, \infty)$ whose critical points are in $\mathrm{int}(X)$, they are non-degenerate and each of them has index different from 1. Furthermore, the restriction of f to the boundary ∂X has only non-degenerate critical points: those meaningful (or non-fake) points $c \in \partial X$, for which the inclusion $f^{-1}(-\infty, f(c) - \varepsilon] \subset f^{-1}(-\infty, f(c) + \varepsilon]$ is not a homotopy equivalence, must also have the index different from 1.]

2. In the case where it is not required that π verifies condition (2) in Definition 1.1, we can take $X^n = W^3 \times \mathrm{int}(D^{n-3})$ and complete the diagram above in an obvious manner. From the results of Mazur [4], for large n, the manifold $W^3 \times \mathrm{int}(D^{n-3})$ has no 1-handles, since its homeomorphism class depends only on the homotopy type of W^3. However, there are many examples where W^3 is not simply connected at infinity, as the Whitehead-type manifolds (see [15]). Thus the properness is an essential condition for the validity of Theorem 1.1.

3. Theorem 1.1 implies that, whenever W^3 is not simply connected at infinity, $X^n = W^3 \times D^{n-3}$ must have 1-handles. Further, the existence of at least one 1-handle implies the existence of an infinite number of such 1-handles. Assume now, that a sequence of 2-handles $b_1, b_2, ..., b_k, ...$ are recurrently attached to X^n, in order to kill all 1-handles. Since W^3 is contractible, we can slide the 2-handles to have their attaching circles $\gamma_1, \gamma_2, ..., \gamma_k, ...$ on $W^3 \times \partial D^{n-3}$. We claim that the union of these circles cannot be a closed subset of ∂X^n. Suppose the contrary holds: then the manifold Y^n, obtained by surgery on these circles, would be without 1-handles. Meantime X^n embeds in Y^n and the projection $X^n \to W^3$ extends to a proper map $Y^n \to W^3$. Therefore Y would be a w.g.s.c. enlargement of W^3 so that $\pi_1^\infty(W^3) = 0$, which is a contradiction. Thus, the union of these circles must have a non-void set of accumulation points, say Σ_X.

4. Our w.g.s.c. property is the same as the property P for a triangulation of a manifold considered by Poénaru [7].

5. Remark that any manifold (or union of manifolds of dimensions greater than 3) satisfying the first two conditions from Definition 1.1 is automatically an enlargement.

6. Our result is in some sense sharp: with the given method we obtain the most general conditions on X ensuring the simple connectivity at infinity for W^3.

Concerning the first remark above, we have the following result.

COROLLARY 1.1. *If W^3 has a finite dimensional enlargement which is a non-compact manifold with boundary and without 1-handles, then W^3 is simply connected at infinity.*

The *proof* is reminiscent of Whitehead's (Smooth) Hauptvermutung (see [16]): any two triangulations compatible with the same DIFF structure on X have isomorphic subdivisions (the PL structure subjacent to the DIFF structure is uniquely defined). Then we may use the same proof as for Theorem 1.1. □

Intuitively, the third remark above says that the set of accumulation points Σ_X is the obstruction for W^3 to be simply connected at infinity. Moreover, this set has many similarities with the limit sets arising in the collapsible representations for open 3-manifolds (see [9, 10]).

As an application of Theorem 1.1 we get

COROLLARY 1.2. *If $\pi_1^\infty(W^3) \neq 0$, then the set $\pi(\Sigma_X) = \Sigma_W \subset W^3$ is larger than a tame Cantor subset of W^3.*

Proof. First $W^3 - \Sigma_W$ is s.c.i. because it has a enlargement without 1-handles. If C is a tame Cantor set, then we would have $\pi_1^\infty(W^3) = \pi_1^\infty(W^3 - C)$ (see [9], p. 13, Lemma 1.1), which leads to a contradiction, and the claim follows. □

Notice that in [9,10] a regularization theorem is obtained: the limit sets associated to some collapsible representations of $W^3 \times D^n$ are unions of a tame Cantor set with a codimension 1 stratified proper sub-manifold. A similar result should hold for Σ_X.

The next result in the paper gives a uniform answer to two questions. On one hand, there is the guess expressed by Poénaru ([7], Remark C, p. 432). This author claimed that it might be possible to have a connection between the simple homotopy type and π_1^∞ in dimension 3. On the other hand, one can ask whether the result presented in [7] can be naturally generalized to the infinite dimensional case: $W^3 \times Q$ must have 1-handles unless $\pi_1^\infty(W^3) = 0$, where Q is the Hilbert cube (see [1]).

In fact both problems can be reduced to the same one. Let us first explain the meaning of *without 1-handles* in an infinite dimensional context. Recall that any locally finite simplicial (or CW) complex Y has the property that $Y \times Q$ is a Q-manifold ([1], p. 54). Thus, a Q-manifold without 1-handles is a Q-manifold having a triangulation $Y \times Q$, where Y is a w.g.s.c. simplicial complex. (A triangulation of the Q-manifold Z is defined as a homeomorphism $Y \times Q \longrightarrow Z$, where Y is a locally finite simplicial complex.) Further, the second question can be reformulated as follows (as was pointed to me by Frank Quinn): if $W^3 \times Q$ is homeomorphic to $Y \times Q$, where Y is a w.g.s.c. complex, then $\pi_1^{\infty}(W^3) = 0$.

The simple homotopy theory was defined and first used by Whitehead [17], in the context of finite complexes, and then it was generalized by Siebenmann [13] for infinite complexes as follows.

Definition 1.4. Two locally finite simplicial complexes P and R have the same infinite simple homotopy type if there exists a finite sequence of infinitely many simultaneous and disjoint Whitehead moves, which allow to pass from P to R. For each element of the sequence, the (simultaneous) moves are all either expansions, or collapses.

Now, the so-called stabilization lemma from [1] asserts that the locally finite simplicial complexes P and R are (infinite) simply homotopy equivalent if and only if the Q-manifolds $R \times Q$ and $P \times Q$ are homeomorphic.

Thus the previous question could be stated differently: if W^3 is simple homotopy equivalent to a w.g.s.c. simplicial complex P, then W^3 is simply connected at infinity.

We can state now the second result, which answers in the affirmative this question.

THEOREM 1.2. *The open 3-manifold W^3 is simply connected at infinity if and only if there exists an infinite simple (proper) homotopy equivalence between W^3 and a locally finite simplicial complex P which is weak geometrically simply connected.*

Remark 1.2. The main result is valid in the case when P is a CW-complex, for an appropriate definition of the simple homotopy equivalence, with essentially the same proof.

Notice that a proper homotopy equivalence is simple if and only if Siebenmann's obstructions σ_∞ and τ_∞ vanish (see [13]).

Observe that one half of the theorem is trivially valid, since W^3 can be triangulated and, if $\pi_1^{\infty}(W^3) = 0$, then the associated simplicial complex is w.g.s.c. The difficult part is to prove the converse: if we have an infinite simple homotopy equivalence between a weak geometrically simply connected simplicial complex and an open 3-manifold, then the manifold is simply connected at infinity.

Acknowledgements. I am indebted to Valentin Poénaru, who had the patience to introduce me to his theory during my stay at Orsay between 1990-1994, for his suggestions, remarks and continuous encouragements. I wish to thank Ross Geoghegan, Frank Quinn and Larry Siebenmann for helpful conversations and advice. Part of this work has been done during the special semester on low dimensional topology, when the author was visiting the Center Emile Borel whose hospitality is kindly acknowledged.

2. THE PLAN OF THE PROOF

The idea of the proof of Theorem 1.1 emerged from the series of papers [7, 8, 9, 10]. In this paper we fully exploit the technique introduced there. The main arguments are contained in the following three lemmas. In order to make this paper self-contained, we added an appendix on the Φ/Ψ-theory (developed in [6]). We will use the following notation: if $h : A \longrightarrow B$ is a map and $n \in \mathbf{Z}_+$, we will denote by $M_n(h) \subset A$ the set of $x \in A$ for wich card $(f^{-1}(f(x)) \geq n$. We also write $M^2(h) \subset A \times A$ for the set of pairs $(x, y) \in A \times A$ with $x \neq y$ and $h(x) = h(y)$.

We have the following Dehn-type result.

LEMMA 2.1. *Let X^3, M^3 be two simply-connected manifolds, K a connected compact set, such that X^3 is compact, connected with non-void boundary and M^3 is closed without boundary. Assume we have a commutative diagram*

$$\begin{array}{ccc} K & \overset{g}{\hookrightarrow} & \mathrm{int}(X^3) \subset X^3 \\ & f \searrow & \downarrow F \\ & & M^3 \end{array}$$

where

1. *f and g are embeddings;*
2. *F is a smooth generic immersion;*
3. *$gK \cap M_2(F) = \emptyset$.*

Then fK can be engulfed in a smooth connected and simply connected submanifold Y^3 of M^3.

For the *proof* see [7], p. 433 439.

LEMMA 2.2. *There exists a triangulation τ_W of W^3 and a subdivision τ_X of X^n such that*

1. *$i : \tau_W \hookrightarrow \tau_X$ is a simplicial embedding, identifying τ_W to a sub-complex of τ_X;*
2. *τ_X is w.g.s.c;*
3. *there is some subdivision θ of the 3-dimensional skeleton of τ_X and a map $\lambda : \theta \longrightarrow \tau_W$ such that λ is proper simplicial and non-degenerate, and $\lambda \circ i = \mathrm{id}$.*

This lemma does not use the strong connectivity of the 3-skeleton, but only the first two conditions from Definition 1.1. Notice that it implies that θ is a enlargement of τ_W, but only when the natural projection map is replaced by λ (so that all the maps become simplicial). The proof will be given in the next section.

It follows that θ is w.g.s.c. from [7], Lemma 5.1: thus, there exists a sequence of finite simply connected sub-complexes $Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset \theta$ exhausting θ. Set $\lambda^\infty = \lambda\,|_\theta$, $\lambda^j = \lambda\,|_{Z_j}$, and also $\Psi_j = \Psi(\lambda^j)$, $\Phi_j = \Phi(\lambda^j)$, $j = 1, 2, \ldots, \infty$. The equivalence relations Φ and Ψ were introduced in [6] and all the definitions are included in the appendix. Recall that for $j < \infty$ we have $\Phi_j = \Phi_\infty |_{Z_j}$, but in general we only have the inclusion $\Psi_j \subset \Psi_\infty |_{Z_j}$.

LEMMA 2.3. *The equality* $\Phi_\infty(\lambda) = \Psi_\infty(\lambda)$ *holds.*

From now on the *proof* of Theorem 1.1 is standard. For the sake of completeness we outline it below. The conclusion of Proposition B from [7] remains true in our situation, so that for any k, there exists some number $N(k) > k$ such that

$$\Psi_{N(k)} |_{Z_k} = \Phi_k.$$

Fix further a connected compact $K \subset W^3$. Then there is some m for which $\lambda^m Z_m \supset K$ holds, and therefore we can find some (sufficiently large) n satisfying $(\lambda^\infty)^{-1}(\lambda^m Z_m) \subset Z_n$ (both assertions follow from a compactness argument).

If $(x_1, x_2) \in M^2(\lambda^\infty)$ and $x_1 \in i(K)$, then necessarily $x_2 \in Z_n$. Furthermore, we have the diagram of maps

$$i(K) \subset Z_n/\Phi_n = Z_n/\Psi_N(n) \subset Z_{N(n)}/\Psi_{N(n)} \xrightarrow{\lambda^{N(n)}} W^3.$$

Since the map $\lambda^{N(n)}$ is an immersion and no double point of $\lambda^{N(n)}$ can involve Z_n (as a consequence of the relation $\Psi_{N(k)} |_{Z_k} = \Phi_k$, which was previously obtained), we deduce that

$$K \cap M_2(\lambda^{N(n)}) = \emptyset.$$

From Lemma 3.1 of [7] we have $\pi_1(Z_{N(n)}/\Psi_{N(n)}) = 0$. Therefore the diagram

$$\begin{array}{ccc} K & \overset{g}{\hookrightarrow} & Z_{N(n)}/\Psi_{N(n)} \\ & f \searrow & \downarrow \lambda^{N(n)} \\ & & W^3 \end{array}$$

has all the properties required in the Dehn-type lemma except that $Z_{N(n)}/\Psi_{N(n)}$ is a simplicial complex. But as already noticed in [7], p. 444, we may replace it by a smooth regular neighborhood of $Z_{N(n)}/\Psi_{N(n)}$, generically immersed in W^3. Thus the compact K can be engulfed in a simply connected compact sub-manifold of W^3. Once we know this for any connected compact, it follows automatically for any compact subset of W^3. Therefore W^3 is simply connected at infinity, as claimed by the theorem. □

3. THE PROOF OF LEMMA 2.2

By performing a suitable subdivision of the initial triangulations τ_X^0, τ_W^0 of X and W, we may suppose that $i : \tau_W^0 \hookrightarrow \tau_X^0$ is a simplicial embedding. Furthermore, we can subdivide again $\tau_X^1 < \tau_X^0, \tau_W^1 < \tau_W^0$, in order to make $\pi : \tau_X^1 \longrightarrow \tau_W^1$ simplicial. This can be done in a relative context, so that $(\tau_X^1, \tau_W^1) < (\tau_X^0, \tau_W^0)$, because $\pi \mid_{\tau_W^0} = \mathrm{id}$, if τ_W^0 is identified with its image in τ_X^0. This is a standard argument (see [12]). Eventually, we obtain the simplicial mappings $\pi : \tau_X^1 \longrightarrow \tau_W^1$ and $i : \tau_W^1 \hookrightarrow \tau_X^1$. Remark that τ_X^1 is w.g.s.c. by Lemma 5.1 from [7]. It remains to prove that π can be replaced by another map f which is simplicial and whose restriction to the 3-skeleton of some subdivision τ_X^2 is non-degenerate. The image of the latter is some subdivision $\tau_W^2 <' \tau_W^1$.

Before proceeding we make a simple remark, which will be freely used in the sequel: if $f : \sigma^n \longrightarrow \sigma^k$ is a surjective simplicial map between two simplices of dimension $n \geq k$, then for some k-face δ^k of σ^n, the map $f \mid_{\delta^k}: \delta_k \longrightarrow \sigma^k$ is an isomorphism.

Another remark is that W^3 has a non-complete flat Riemannian structure: thus τ_W^1 can be realized as an affine triangulation of W^3 because the geodesics are unique.

Denote $\mathrm{ske}^3 \tau_X^1$ by t and τ_W^1 by τ, for simplicity. We have given a simplicial map $\pi : t \to \tau$, but it is possible that some simplices of t be collapsed via π. We outline below the method to change π into another map which *flatten* t, but it does not collapse 3-dimensional simplices. Intuitively, imagine that we have given a specified floor τ_W^1 in a high dimensional building τ_X^1. The 3-dimensional structure (the union of walls) of the building corresponds to t. We flatten this structure by a generic compression map onto the specified floor. If this procedure is carefully carried out, we obtain a new partition of the floor, in terms of which the compression map would be a non-degenerate cellular map. In fact, once every wall is slightly pushed from the vertical, its horizontal projection cannot completely disappear, and generically it is 3-dimensional.

The main technical point consists in replacing the simplicial complexes by cellulations in the sense of Siebenmann [14]. Here the term cellulation corresponds to the term *cellulation régulière* of a polyhedron used in [14]. Let us give first the definition, according to Siebenmann:

Definition 3.1. A cellulation of a metric space X is given by a locally finite covering by compact cells fulfilling the three conditions below.

1. Each cell has a linear structure induced from the identification (by a homeomorphism) with the convex hull of a finite set of points in an Euclidean space. In particular, all cells are convex with respect to this linear structure.

2. The formal interiors of the convex cells form a partition of the space X. (We recall that the formal interior of a convex compact subset D of a vector space is the set of all x with the property that for each line l which pass through x, the segment $l \cap D$ contains x in interior. Next, the formal boundary is the complementary of the formal interior.)

3. For any convex cell D, its formal boundary ∂D is an union of a finite number of cells d_i and the inclusions $d_i \hookrightarrow D$ are linear.

The natural transformations for cellulations corresponding to subdivisions of triangulations are the *bisections* . By definition a bisection replaces one cell D by 3 cells D_0, D_+, and D_-, where D_0 is linear of codimension 1 in D (a hyperplane section in D) and cuts D into two non-void pieces D_- and D_+. The inverse operation is called a *coupling*. The closure of the cell D_0 is called the *support* of the bisection. We write $X << Y$ if X can be obtained by bisections from Y. The number of bisections is finite, if the underlying spaces are compact, or there may be an infinity of bisections whose union of supports is without accumulation points, in the non-compact case. Such an infinite family of bisections will be called *proper*.

There are two reasons to prefer cellulations and bisections to simplicial complexes and subdivisions.

LEMMA 3.1. *Given a cellulation X and a sub-complex Y then any subdivision of Y (by bisections) induces canonically a subdivision (by bisections) of X which does not touch any other cell of X which is not a cell of Y.*

LEMMA 3.2. *Let K be a polyhedron (i.e. a metric space with a maximal family of cellulations) and D_1, D_2 two cellulations of K. Then there exists a common refinement of both cellulations using a proper family of bisections: $D_2 >> D << D_1$.*

In the compact case these two facts are proved in [14]. The same proof works as well in the non-compact case, when restricting ourselves to proper family of bisections.

Now, in the context of cellulations, we can define the w.g.s.c. property analogous by to the simplicial complexes case. Specifically a cellulation is w.g.s.c. if it admits an exhaustion by simply connected. connected and finite cellular subcomplexes.

The next lemma is the natural extension to cellulations of Lemma 5.1 from [7].

LEMMA 3.3. *A cellulation is w.g.s.c. if and only if its 2-dimensional skeleton is w.g.s.c.. In particular, if X is w.g.s.c., then the 3-dimensional skeleton* $\mathrm{ske}^3 X$ *is w.g.s.c.. If $Y << X$, then Y is w.g.s.c. if and only if X is also w.g.s.c..*

The *proof* is obvious. $\square$

We come back now to the proof of Lemma 2.2. It is known that π is proper. This means that, for any (closed) 3-simplex $\sigma \subset \tau$, the preimage $\pi^{-1}(\sigma) = \cup_i \sigma_i$ is a finite union of 3-simplices of t. We choose an arbitrary simplex σ at the beginning. Among the preimage simplices σ_i there is one, which we denote by $i(\sigma)$, such that the restriction of π to $i(\sigma)$ is an isomorphism on the image. Set $V(\sigma)$ for the union of the set of vertices of all σ_i which do not appear as vertices of $i(\sigma)$, and order them arbitrarily: $V(\sigma) = \{v_i, i \geq 4\}$. Afterwards we label $\{v_1, ..., v_4\}$ the vertices of $i(\sigma)$.

Step I: Choose some set of points (in a generic position) $A^0(\sigma) \subset \mathrm{int}(\sigma) \subset W^3$ which are in one-to-one correspondence with $V(\sigma)$. We denote the points of $A^0(\sigma)$ as $\{x_i, i \geq 4\}$ and the vertices of σ by $\{x_1, ..., x_4\}$. We suppose that the projection of v_i is x_i for $i = 1, 2, 3, 4$. Consider now the set $A^k(\sigma)$ of those k-dimensional simplices whose vertices are from $A^0(\sigma)$, which are realized as affine simplices in W^3, and are related to the simplices from $\pi^{-1}(\sigma)$ as

$$A^k(\sigma) = \{[x_{i_0}, x_{i_1}, ..., x_{i_k}] \subset \sigma : [v_{i_0}, v_{i_1}, ..., v_{i_k}] \text{ is a simplex in } \pi^{-1}(\sigma)\}.$$

Here $[y_0, y_1, ..., y_k]$ denotes the simplex with vertices y_i. Notice that the affine simplex $[x_{i_0}, x_{i_1}, ..., x_{i_k}]$ is uniquely determined by its vertices in W^3, because W^3 has an affine structure.

Example 3.1. Consider $\sigma = [x_1, x_2, x_3, x_4]$, $\pi^{-1}(\sigma) = \partial[v_1, v_2, v_3, v_4, v_5]$ and a projection map π which sends v_i into x_i, for $i = 1, 2, 3, 4$. Then it is easy to see that

$$A^0(\sigma) = \{x_5\}, \text{ for an arbitrary point } x_5 \text{ in the interior of } \sigma,$$

$$A^1(\sigma) = \{[x_i, x_5], i = 1, ..., 4\},$$

$$A^2(\sigma) = \{[x_i, x_j, x_5], i \neq j = 1, ..., 4\},$$

$$A^3(\sigma) = \{[x_i, x_j, x_k, x_5], i \neq j \neq k = 1, ..., 4\}.$$

Remark that $A^*(\sigma)$ would be a simplicial complex if some cells hadn't overlapped.

Example 3.2. We give a 2-dimensional picture, since it is easier to draw it. Figure 1 shows $A^1(\sigma)$, where σ is a 2-simplex and $\pi^{-1}(\sigma) = \mathrm{ske}^2\Delta^4$ (Δ^n being the standard n-simplex). It is clear that the associated graph of edges is not planar (it is the complete graph K_5), so there are some new intersection points between edges, like the vertex x_6. Using a transversality argument, in a 3-dimensional picture we can reduce ourselves to the case where the edges in $A^1(\sigma)$ are disjoint, but we may have new intersection points between the 2-dimensional faces and edges.

We assume now that all simplices in τ are sufficiently small to be convex with respect to the affine structure. Let $A_1^*(\sigma)$ be the closure of $A^*(\sigma) \cup \{\sigma\} \cup \emptyset$ with respect to the intersection operation. This means that

1. once σ_1 and σ_2 are in $A_1^*(\sigma)$, their intersection $\sigma_1 \cap \sigma_2$ belongs to $A_1^*(\sigma)$, too;

2. $A^*(\sigma) \cup \{\sigma\} \cup \emptyset$ is a subset of $A_1^*(\sigma)$;

3. $A_1^*(\sigma)$ is the smallest collection fulfilling the two previous conditions.

Set also $A_{11}^*(\sigma)$ for the closure of $A_1^*(\sigma)$ with respect to the face-boundary operator ∂, which is extended canonically to convex cells. (The face-boundary operator associates to a cell c the collection of faces of the boundary and it should be not confused with the algebraic sums arising in the chain complexes.) Roughly

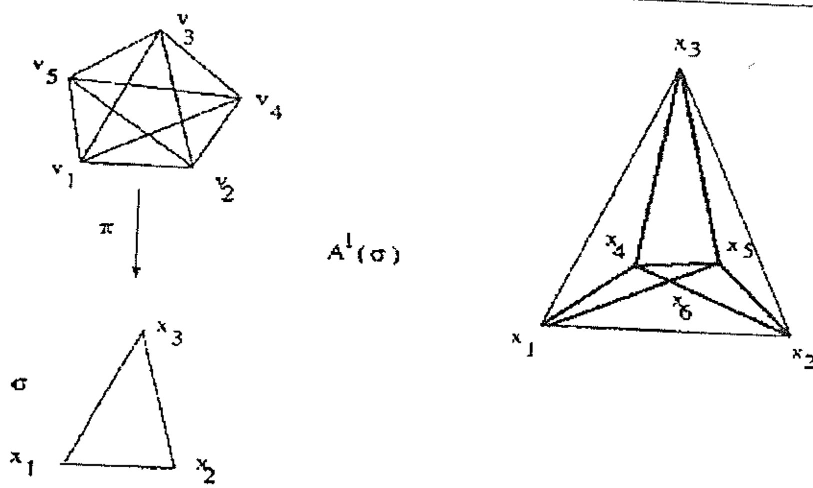


Fig. 1 – $A^*(\sigma)$.

speaking, this closure is intended to be the smallest set X of cells having the property that, once a cell is in X, then all the faces of its boundary belong to X. The complex $A^*_{11}(\sigma)$ is closed with respect to intersection and ∂, but generally speaking, it is not a cellulation of σ in the sense of Siebenmann, as we defined above. The reason is that we do not necessarily obtained a partition of σ. In fact, we need to refine further this collection of cells to a new collection $A^*_2(\sigma)$ with the property that the formal interiors of the cells form a partition. This may be done in a canonical way: set c_i for a maximal set of open cells (formal interiors) in $A^*_{11}(\sigma)$, all of them being contained inside some other cell c. Then remove c and add $c - \cup_i c_i$ as a new cell. When this procedure cannot be applied anymore, it means that we arrived at a partition of σ into smaller cells. However, we introduced this way some cells which are no more convex cells.

Consider now the map $f : \pi^{-1}(\sigma) \longrightarrow \sigma$, which is the extension by linearity of the map defined on vertices by $f(v_i) = x_i$ for all i. Set $C^*(\sigma) = f^{-1}(A^*_2(\sigma))$. Then $C^*(\sigma)$ is a cellular complex and f is a non-degenerate cellular map.

Example 3.3. Typically, f has singularities. Figure 2 shows a folding map f which maps two triangles, having a common edge, on the plane. In the plane the two triangles overlap on a smaller triangle which is doubly covered.

Step II: We pass now to a global picture, from the simplex σ to the whole complex τ. We choose an enumeration of all 3-simplices of τ, say $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_k, \ldots$ with the property that, for each compact K, there exists some integer $m = m(K)$ such that $\cup_{i=1}^{m} \sigma_i \supset K$. We built up to now, inductively, the global complex associated to this exhaustion.

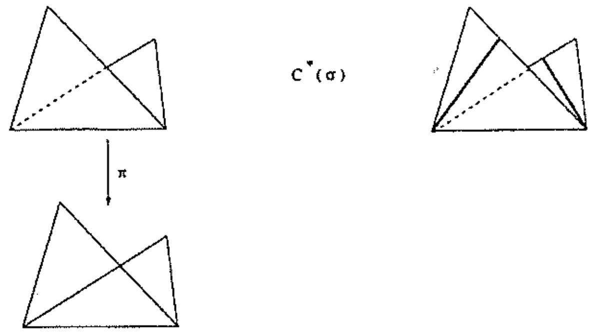


Fig. 2 – Pulling back A_2^*.

1. At the beginning (for σ_1) we start with the cell decomposition $A_2^*(\sigma_1)$ defined above.

2. Assume we constructed refinements of $\pi^{-1}(\cup_{i=1}^n \sigma_i)$ and of $\cup_{i=1}^n \sigma_i$, and a cellular map f replacing the projection π, between the refined cell complexes. We look now for the new vertices which appear in $\pi^{-1}(\sigma_{n+1})$. There are some of the vertices of $\pi^{-1}(\sigma_{n+1})$, namely those which are also vertices of $\pi^{-1}(\cup_{i=1}^n \sigma_i)$, which have been taken already into account at the previous stages of the construction. In fact, the vertex v has been considered (this means that, at an earlier stage, a value $x = f(v) \in \cup_{i=1}^n \sigma_i$ has been associated to v) at the kth step, where k is the smallest integer such that v is the vertex of $\pi^{-1}(\cup_{i=1}^k \sigma_i)$. Define therefore $V^0(\sigma_{n+1})$ as the set of vertices of $\pi^{-1}(\cup_{i=1}^{n+1} \sigma_i)$ which have not been considered before. Choose, as at the first step, a set of points $A^0(\sigma_{n+1})$ inside the simplex σ_{n+1} so that the vertices of $i(\sigma_{n+1})$ are in one-to-one correspondence with the vertices of σ_{n+1} and the other points are lying in the interior of σ_{n+1}. We assume that the vertices in $V^0(\sigma_{n+1}) - i(\sigma_{n+1})$ are in bijection with the interior points. The restriction of f to vertices can be naturally extended now from $\pi^{-1}(\cup_{i=1}^n \sigma_i)$ to $\pi^{-1}(\cup_{i=1}^{n+1} \sigma_i)$, say $f(v_i) = x_i$, for all i. Remark that this procedure is highly non canonical but it is well enough for our purposes. The global complex B_0^*, which depends on the enumeration we chose, is therefore given by

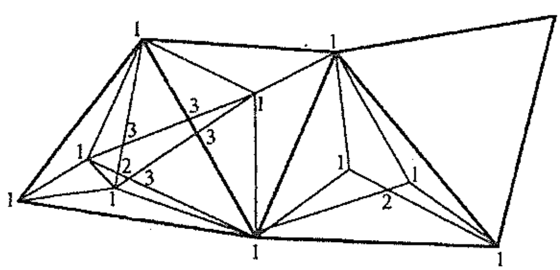
$$B_0^k = \{[x_{i_0}, x_{i_1}, \ldots, x_{i_k}] : x_{i_j} \in \cup_{j=1}^{n+1} A^0(\sigma_j)$$

$$\text{such that } [v_{i_0}, v_{i_1}, \ldots, v_{i_k}] \text{ is a simplex in } t.\}$$

Here all the simplices in W^3 are affine. Remark we have specified only the first generation vertices from A^0, not from A_2^0. Consider now the closure B_1^* of B_0^* with respect to the intersection and let B_2^* be the closure of B_1^* with respect to the face-boundary operator. An easy remark is that B_2^* is closed also for the intersection.

We saw before how to refine B_2^* by adding the complementary of unions of cells (and removing the cells which contain them), in such a way that the formal interiors form a partition: if $x_i \subset y$ are k-cells in B_2^k then we want that the complementary $\mathrm{cl}(y - \cup_i x_i)$ be also an union of cells. In the first step we considered such maximal families $\{x_i\}$ inside a fixed cell y, added the complementary, as a new cell, and removed the cell y from our collection. But some of the new cells arising this way, are not convex. Observe that all of them are polyhedra whose edges are geodesics and the faces are flats in W^3. A polyhedron with geodesic edges, and affine faces in an affine manifold can be partitioned into convex polyhedra, possibly introducing new vertices, as intersection among flats spanned by the vertices. These can be lifted upside, and the initial triangulation can be refined to include the new vertices. In the last situation, the downside cells are now convex. Thus we may suppose, without loss of generality, that there are no vertices to add and the partition has convex components. We obtained another complex, say B_3^*, which is closed to intersection, to the face-boundary operator and is made of convex cells. Now the map f extends to B_3 in the obvious way.

This completes the induction step, and so we obtain a cellulation of W^3. But this cellulation can further be refined to a simplicial decomposition B^*. According to Fact 2 stated above, τ^1_{W} and B^* have a common refinement obtained by bisections. This proves that B^* is again w.g.s.c. by Lemma 3.1. Now the map f gets a map $f : D^* = f^{-1}(B^*) \longrightarrow B^*$ which is simplicial. The pull-back complex D^* is a cellular complex, finer than t, and we will show that it is a simplicial complex.



1 points of the first generation
2 points of the second generation
3 points of the third generation

Fig. 3 The vertices.

Example 3.4. The vertices we added to our initial triangulations are therefore of 3 generations, as shown in the Figure 3. These corresponds to A^0, to $A_1^0 = A_2^0$, and B_1^0. An example of how f looks like is given in Figure 4: here τ is the union of 2 simplices of dimension 2, and t is the 2-skeleton of $\partial[v_1, v_2, x_3, v_5] \cup \partial[v_4, v_2, x_3, v_6] \cup \cup\, \partial[v_6, v_2, x_3, v_5]$. There are two new vertices of first generation figured in A^* and a new vertex x_7 when we pass to B^*. The preimage cellular complex and the modified map f corresponds to the cone over the subdivision of the edge $[v_5, v_6]$.

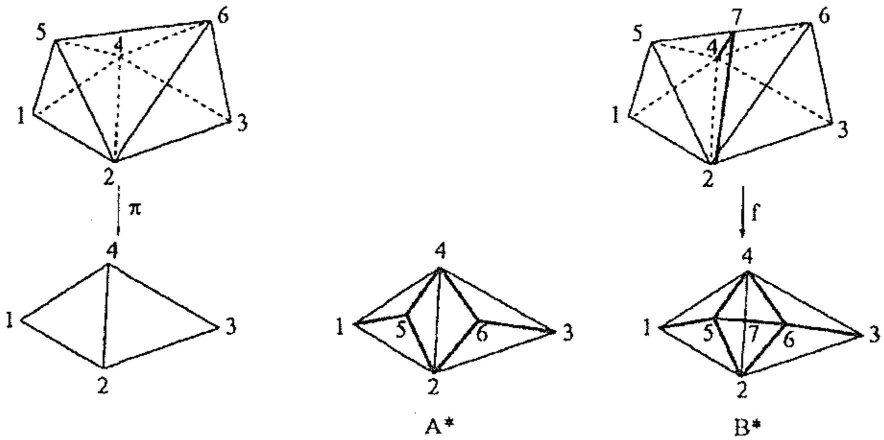


Fig. 4 - The general picture.

LEMMA 3.4. *The map f is non-degenerate and simplicial.*

Proof. These features were achieved directly by construction. It suffices to understand how D^* is obtained from t, and that D^* is indeed a simplicial complex. The new vertices in $D^0 - t^0$ come from intersections points of 2-flats in W. For a generic choice of $A^0(\sigma_j)$ the 2-flats are in general position, and there are only 1-dimensional intersections. Since f was made cellular, the local model around a singularity of f is exactly the folding from Figure 5 (the 3-dimensional analog of Figure

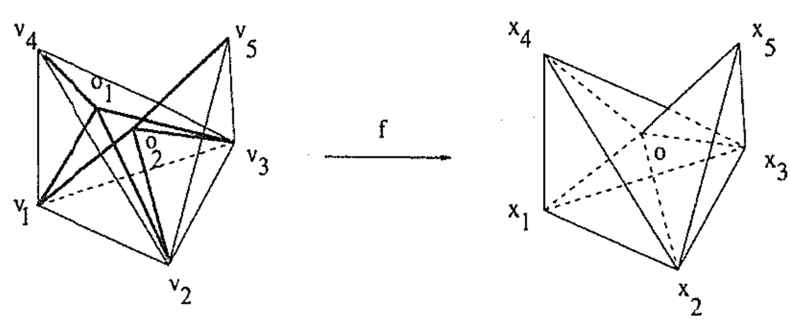


Fig. 5 The local picture around a singularity.

2). Let's explain it: we have two simplices σ_1^3 and σ_2^3 in D^* having a common 2-face, which are projected down by f onto the union of two 3-simplices $f\sigma_1^3$ and $f\sigma_2^3$. The last two have a common face and $f\sigma_1^3 \cap f\sigma_2^3 = \sigma$ is a simplex with a new vertex which appears o which is the intersection of three singular lines ox_i, $i = 1,3$. The double point o has two preimages, $o_i \in \sigma_i^3$ and we have also the preimages of double lines which are $o_i v_j$, $i = 1,2$ and $j = 1,3$. We must add to our decomposition the edge ox_4, and this yields a decomposition of $f\sigma_1^3$ into 3 simplices while $f\sigma_2^3$ is cut into two tetrahedra. The preimage decomposition is a decomposition into tetrahedra of $\sigma_1^3 \cup \sigma_2^3$, which is the pull-back of the partition into tetrahedra from downside and it is not necessary to add any other edges or vertices.

Further, f is locally an étale map around a non-singular point. It follows that D^* is in fact a simplicial complex finer than t, and f is non-degenerate and simplicial, as we wanted. □

Lemma 3.5. *The induced map $f : X \longrightarrow W^3$ is proper.*

Proof. Observe first that all objects $\bigcup_{\sigma^3 \in \tau} C^*(\sigma)$, $\bigcup_{\sigma^3 \in \tau} A_2^*(\sigma)$ are locally finite. Therefore the new vertices of $\bigcup_{j, \sigma_j^3 \in \tau} A_2^0(\sigma_j)$ are not accumulating in W^3 except at infinity. Since everything takes place in some small convex region we deduce that the edges in $\bigcup_{\sigma^3 \in \tau} A_2^1(\sigma)$ (which are viewed as geodesics in W^3) do not have accumulating points either. Notice that the geodesics are unique in the flat structure of W^3. Since the simplices are affine we deduce that no k-simplex (from those whose vertices are in $\bigcup_{j, \sigma_j^3 \in \tau} A_2^0(\sigma_j)$) has accumulating points.

It remains to look at the edges introduced at the second step. Assume that in the induction process, when we pass from the stage n to $n+1$ we have to add some new edge. The image by π of such an edge $e \in B_2^1$ is either one edge of τ or a vertex of τ. The second case corresponds to the following situation: we have two vertices v_1 and v_2 having the same image x by π. These two may appear either at the same stage of the enumeration (so that their perturbed images by f will belong to the same simplex), or in different places. But then the images are sitting inside two simplices, say σ_1 and σ_2, having the vertex x in common. The first case leads to the following situation: the images are sitting in σ_1 and σ_2, such that there exists a 1-dimensional simplex e having one endpoint on σ_1 and the other one on σ_2. The other edges in $B_3^1 - B_2^1$ were added inside a convex cell, in order to complete the partition into a partition with smaller convex cells.

We claim that the new edges cannot be too long: in fact, by the triangle inequality, the length of a new edge in a compact ball R is at most 3 times the longest (old) edge in that ball. We used compacts because all the choices we made were local, and the upper bound on the edge length is uniform (in [7] the initial triangulation is chosen with simplices which become smaller and smaller when the distance from a fixed point goes to infinity). This argument shows that the new edges do not have accumulating points, except at infinity. For a generic choice of $A^0(\sigma_j)$, the affine k-simplices are in general position. Since the edges are not accumulating somewhere, the k-simplices are not accumulating either. This proves that f is proper. □

4. THE PROOF OF LEMMA 2.3

We consider now the central object in this section, namely the canonical diagram

$$\begin{array}{ccc} \theta & \longrightarrow & \theta/\Psi_\infty \\ & \lambda^\infty \searrow & \downarrow \overline{\lambda^\infty} \\ & & W^3 \end{array}$$

The map $\overline{\lambda^\infty}$, which we obtained after factorization, is known to be an immersion by the definition of Ψ.

LEMMA 4.1. *The map $\overline{\lambda^\infty}$ is a simplicial isomorphism between θ/Ψ_∞ and τ_W.*

Proof. Consider the sub-complex $i(\tau_W) \subset \theta$. There is an induced map $\iota : i(\tau_W)/\Psi_\infty \longrightarrow \theta/\Psi_\infty$ and we have the commutative diagram

$$\begin{array}{ccccc} i(\tau_W)/\Psi_\infty & \xrightarrow{\iota} & \theta/\Psi_\infty & \xrightarrow{\overline{\lambda^\infty}} & \tau_W \\ \uparrow & & \uparrow & \nearrow \lambda^\infty & \\ i(\tau_W) & \hookrightarrow & \theta & & \\ \uparrow & \nearrow i & & & \\ \tau_W & & & & \end{array}$$

To complete the proof we need the next three lemmas.

LEMMA 4.2. *The map $\alpha = \overline{\lambda^\infty} \circ \iota : i(\tau_W)/\Psi_\infty \longrightarrow \tau_W$ is a simplicial isomorphism.*

Proof. In fact the map α is

- *surjective* since $\alpha(i(\tau_W)/\Psi_\infty) = \overline{\lambda^\infty} \circ i(\tau_W) = \lambda^\infty \circ i(\tau_W) = \tau_W$,
- *simplicial* as a composition of simplicial maps,
- an *immersion* because $\Psi_\infty(\lambda^\infty \circ i) \subset \Psi_\infty$; this may be rephrased by saying that, once we kill all the singularities, then a fortiori the singularities lying in $i(\tau_W)$ are killed,
- *injective* because the composition $\alpha \circ \beta =$id, where β is the vertical map in the diagram going from τ_W to $i(\tau_W)/\Psi_\infty$. □

LEMMA 4.3. *Consider the simplicial complex (or cellulation) τ which has a strongly connected 3-skeleton. Assume that we pass from τ to another complex τ' by using one of the following transformations:*

1. *by subdivisions (or respectively, by a proper family of bisections);*
2. *we replace τ by* $\mathrm{ske}^3\tau$*;*
3. *assume that f is a non-degenerate simplicial map, and $\tau' = \tau/\Psi(f)$.*

Then τ' has a strongly connected 3-skeleton, too.

Proof. Obvious. □

LEMMA 4.4. *The map $\iota : i(\tau_W)/\Psi_\infty \longrightarrow \theta/\Psi_\infty$ is surjective.*

Proof. Assume the contrary holds. Then, for some 3-simplex $\sigma \subset \theta/\Psi_\infty$ we will have $\mathrm{int}(\sigma)\cap\mathrm{Im}(\iota) = \emptyset$. But we know that θ is strongly connected, hence θ/Ψ_∞ is strongly connected, so that any two 3-simplices can be joined by a continuous chain of 3-simplices. This follows from the previous claim. Notice that this is the only place where the third condition in the definition of the enlargement is used. It follows that there exists some σ with $\mathrm{int}(\sigma) \cap \mathrm{Im}(\iota) = \emptyset \neq \sigma \cap \mathrm{Im}(\iota)$. But we have seen above that $\overline{\lambda^\infty}(\iota \circ i(\tau_W)/\Psi_\infty) = \tau_W$, so that any point $z \in \partial\sigma \cap \mathrm{Im}(\iota)$ would be singular for $\overline{\lambda^\infty}$. But this is a contradiction because $\overline{\lambda^\infty}$ is an immersion. □

Now, ι is obviously injective hence $\overline{\lambda^\infty}$ is injective so that it is an isomorphism. This ends the proof of Lemma 4.1. □

The final argument is by now standard (see [7]). We have two bijections $\theta/\Phi_\infty \overset{\sim}{\to} \tau_W$ (by definition, the quotient by Φ_∞ is the image) and $\theta/\Psi_\infty \overset{\sim}{\to} \tau_W$. But we also have an inclusion among the two relations, which induces a map $\theta/\Psi_\infty \overset{\sim}{\to} \theta/\Phi_\infty$, hence $\Phi_\infty = \Psi_\infty$. □

5. THE PROOF OF THEOREM 1.2

The simple homotopy type was introduced by Whitehead [17] and represents a refinement of the usual homotopy theory for finite complexes. Basically, two finite simplicial complexes have the same simple homotopy type if, when they are embedded in an Euclidean space of sufficiently high dimension, their regular neighborhoods are PL-homeomorphic. Another way to get the simple homotopy is via Whitehead moves: we say that Y is obtained from the sub-complex X by an elementary expansion if $\mathrm{int}(Y - X)$ is one simplex whose closure intersects X along a disk which can be a face, or a connected union of several faces. We denote this by $X \nearrow Y$. The inverse operation, from Y to X, is denoted $Y \searrow X$ and is called an elementary collapse. Now, by definition, X and Y have the same *simple homotopy type* if there exists a sequence of elementary moves $X = X_0, X_1, \ldots, X_k = Y$, such that for each j we have either $X_j \nearrow X_{j+1}$ or $X_j \searrow X_{j+1}$.

The obstruction for two homotopy equivalent complexes to be simply homotopy equivalent was formulated by Milnor in algebraic terms, via the Whitehead group associated to the fundamental group. This notion was extended by Siebenmann [13] to locally finite complexes, as follows: an elementary collapse of the locally finite complexes Y onto X is a set of an infinite number of disjoint collapses. This means that we have pairwise disjoint sub-complexes $\{Z_j\}$ of Y such that $Y = X \bigcup_{i=1}^{\infty} Z_i$, and each $Z_i \searrow Z_i \cap X$ is a finite sequence of elementary collapses. The inverse move is called an expansion. Now, two locally finite complexes have the same *infinite simple homotopy type* if there exists a finite sequence of elementary collapses and expansions which transforms one simplicial complex into the other. Observe that the infinite simple homotopy equivalence is finer than the proper homotopy equivalence. The obstructions that two proper homotopy equivalent complexes be infinite simple homotopy equivalent are algebraic, too, and were described in [13].

Theorem 1.2 is a consequence of the following two lemmas.

LEMMA 5.1. *If X_1 and X_2 are simplicial complexes which are simple homotopy equivalent (if there are finite, then in the usual sense, if not we use Siebenmann's infinite proper simple homotopy equivalence), then there exists a finite dimensional complex Y such that $Y \searrow X_i$, for $i = 1, 2$.*

Moreover, if X_1 is a manifold, then Y may be chosen to be an enlargement of X_1.

LEMMA 5.2. *If $Y \searrow X$ and X is w.g.s.c., then Y is a w.g.s.c.*

Proof of Theorem 1.2. In fact if W^3 is (infinite proper) simply homotopy equivalent to a locally finite simplicial complex P and P is w.g.s.c., then there is an enlargement of W^3 which collapses on P. By the second lemma this enlargement is a w.g.s.c. and, by Theorem 1.1, W^3 is simply connected at infinity. □

Proof of Lemma 5.1. The first part of this lemma (for finite complexes) was already formulated as Proposition 5.5 in [4], p. 31. Not only Y is finite dimensional, but its dimension is a priori bounded by $\max(\dim X_1 + 1, \dim X_2, 3) + 1$. A stronger result of Cohen [2] states that Y can be taken as the product $X_1 \times B^n$, for $n \geq \dim X_i \geq 3$, and $n \geq 7$, for 2-dimensional complexes.

In the non-compact case we have to notice that in the family of deformations (elementary collapsings or dilations), which allow to pass from X_1 to X_2, everything is proper: only a finite number of deformations touch a given compact, and its transformations. Therefore we can change the order of the expansions and contractions, at each finite stage. This implies that we can use first only dilations (an infinity of them) and further we realize all the collapsings.

A transfinite recurrence provides us with a simplicial complex Y which is the result of all expansions in the sequence which transforms X_1 into X_2. The main property of this complex Y is that it must be properly obtained from X_1. This means that, as in the previous case, a fixed compact of X_1 is touched by only a finite number of expansion cells. Let $Z_1 = Z(X_1)$ be the first floor added, i.e. the union of X_1 with all those cells whose closure touch X_1. Consider next

$Z_2 = Z(Z_1)$, and so on. The properness is equivalent to the fact that, for any compact $K \subset X_1$, there are only a finite number of floors which can be reached: for some fixed $n = n(K)$ we have $Z_j(K) \subset Z_{n(K)}$, for all j. Here K was supposed to be a sub-complex of X_1, and the tower Z_* is built up in the obvious manner. This follows directly from the definition of the infinite proper simple homotopy.

Therefore we obtain a simplicial complex Y such that $Y \searrow X_i$. Since Y can be obtained by an infinite number of expansions from X, then X is automatically PL embedded in Y. On the other hand consider the inverse (projection) map induced by the collapsing. Since every compact K sees only a finite number of floors Z_j, the projection map is proper.

It remains to deal with the third property of the enlargement. First, we remark that $Y \times D^n \searrow X_1 \times D^n \searrow X_1$. Then $Y \times D^n$ has a specific cellulation: one replace each cell D^k of Y by $D^k \times D^n$, which is identified to D^{k+n}. Of course, we have no more a simplicial complex. Moreover, this cellulation has a refinement as a simplicial complex, by dividing each prism $D^k \times D^n$ (both cells are simplices) into simplices. Each collapsing (coming from a cell c) at the Y level is realized by a sequence of collapsings corresponding to the set of simplices in which $c \times D^n$ splits. A simple argument shows now that $Y \times D^n$ has a strongly connected 3-skeleton if $n \geq 3$. So we can choose $Y \times D^n$ to be the wanted enlargement. Remark also that the result of [2] extends to the non-compact case, and Y can be chosen as a product of X_1 with a ball of sufficiently high dimension. □

Remark that the third condition from Definition 1.1 says that the enlargement is no far from being a manifold. The trick used above was suggested by the fact that the product of a locally finite complex with the Hilbert cube is an infinite dimensional manifold (see [1]).

Proof of Lemma 5.2. Let e_i denotes the composition of the first i dilations from the infinite family which constructs Y beginning from X_1. Let K_i be an exhaustion by connected and simple connected compact sub-complexes of X. Then $e_i(K_i)$ is an exhaustion of Y by connected and simple connected sub-complexes, which shows that Y is w.g.s.c.. □

6. APPENDIX: THE Φ/Ψ-THEORY

For the sake of completeness we recall here some of the basic tools of this paper, which were originally introduced and used by Poénaru [6, 7].

Let $f : P \longrightarrow M^3$ be a non-degenerate simplicial map between the locally finite simplicial complex P and the 3-manifold M. The *equivalence relation defined by* f is $\Phi(f) \subset P \times P$ given by

$$(x, y) \in \Phi(f) \text{ iff } fx = fy.$$

It is clear that $P/\Phi(f)$ is just the image fP.

The other relation, $\Psi(f)$ is introduced in order to see whether it is possible to exhaust all singularities of fP by folding maps, and it is also called the equivalence relation which is commanded by the singularities of f. A *folding* map corresponds to the following situation: if $x \in \sigma_1$, and $y \in \sigma_2$ are two points of P lying on the simplices σ_i of same dimension, if $fx = fy$ and $f\sigma_1 = f\sigma_2$ then we first wish to identify $f\sigma_1$ to $f\sigma_2$. When we pass to such a quotient (by a folding) the induced map remains simplicial.

The equivalence relation $\Psi(f) \subset \Phi(f)$ is completely characterized by the following two properties.

- If $\bar{f}$ denotes the induced map $P/\Psi(f) \longrightarrow M^3$, then $\bar{f}$ is an immersion, i.e., it has no singularities. (The point z is said to be singular for f if the restriction of f to the star of z is not immersive. Alternatively, there exist two distinct simplices σ_1 and σ_2 such that $z \in \sigma_1 \cap \sigma_2$ and $f(\sigma_1) = f(\sigma_2)$.)
- There is no equivalence relation $R \subset \Phi(f)$, smaller than $\Psi(f)$ having the first property, i.e., $\Psi(f)$ is the smallest equivalence relation compatible with f which kills all the singularities.

Furthermore, the projection map $\pi : P \longrightarrow P/\Psi(f)$ induces a surjection on fundamental groups $\pi_* : \pi_1(P) \longrightarrow \pi_1(P/\Psi(f))$. In particular, if P is simply connected, then $P/\Psi(f)$ is simply connected, too. Remark that also the strong connectivity of the 3-skeleton is preserved when passing from P to $P/\Psi(f)$.

Roughly speaking, the construction of $P/\Psi(f)$ is given by considering the quotients, obtained recurrently, by all foldings commanded by the singular points of f. In this way all singularities will disappear, one after another, and no new others are created. Specifically, let z be a singular point and σ_i two simplices containing z, having the same dimension and the same image by f. Consider the quotient P' of P, obtained by identifying σ_1 to σ_2. The map f induces a simplicial non-degenerate map $f' : P' \longrightarrow M^3$. If f' is not an immersion, then it has a singular point, say $z' \in P'$, and therefore some simplices σ_i', as above. We next consider the quotient P'' of P' commanded by the singular point z', and so on. If P is a finite simplicial complex, then this process stops when we get an immersion $f^{(n)} : P^{(n)} \longrightarrow M^3$. The quotient $P^{(n)}$ in this case is $P/\Psi(f)$. If P is not finite, then we need a transfinite recurrence to construct the analogous immersion.

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Received June 30, 1998

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