Arch. Math. 72 (1999) 81–85
0003-889X/99/020081-05 $ 2.50/0

Sonderdruck aus
Archiv der Mathematik

On discrete solvgroups and Poénaru's condition

By

LOUIS FUNAR

Abstract. We prove that a class of discrete cocompact solvgroups do not fulfill Poénaru's condition $P(2)$ (see [3, 4]), improving the result of [2]. This corresponds to the high concavity of balls in the Cayley graphs of such groups.

1. Introduction. The motivation of this note lies in a series of papers of Poénaru [3, 4, 5] which investigated the simple connectivity at infinity of the universal covering space $\tilde{M}$ of a closed 3-manifold M whose fundamental group $\pi_1(M)$ is infinite. It is a long standing conjecture that every such manifold M has the universal covering homeomorphic to $\mathbb{R}^3$. The principal obstruction in proving this conjecture is the existence of open contractible 3-manifolds which are not simply connected at infinity, like Whitehead's original example (see [8]). The main result of [4] states that, for those manifolds M whose fundamental groups satisfy the condition $P(2)$ below, the universal covering is simply connected at infinity (hence homeomorphic to $\mathbb{R}^3$ if irreducible).

Definition 1. A finitely generated group G is said to *verify Poénaru's condition* $P(n)$(for $n \geqq 2$) if there exist a system of generators B, and a function $f : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ fulfilling:
(1) for all constants $c > 0$ we have:

$$\lim_{k \to \infty} k - cf(k) = \infty,$$

(2) any two elements x, y which are sitting on the sphere of radius k and center 1 of the Cayley graph $\Gamma(G, B)$ (which is endowed with the natural distance function d) and which are at distance at most n, can be joined by a path lying inside the ball of radius k of $\Gamma(G, B)$, whose length is less than $f(k)$.

Remark 2. If G satisfies the condition $AC(n)$ from [2] then it satisfies also the condition $P(n)$.

If G is an almost convex group (see [1]) then G satifies $P(n)$ for all n, since the almost convexity corresponds in the above definition to the case when the function $f(k)$ is a constant depending on n. It is already known ([2]) that solvgroups are not almost convex.

Poénaru's condition was stated initially in [3, 4] in the case when the function $f(k)$ is $k^{1-\varepsilon}$, for some $\varepsilon > 0$. It is simple however to see that his main result (with the same proof) is valid for any choice of $f(k)$ fulfilling (1).

Mathematics Subject Classification (1991): 20F32, 57M10.

Observe that the ball of radius k in the Cayley graph of a group not satisfying $P(n)$ is highly concave. In fact we could take always $f(k) = 2k$ violating therefore (1) but not (2).

The purpose of this note is to get a class of groups which do not satisfy $P(k)$, improving the main result of [2]. We don't know however if the condition $P(k)$ is independent on the choice of the system of generators like the almost convexity, the hyperbolicity etc.

We remember first some useful facts concerning the group SOL (see [6]). As a topological space SOL is $\mathbb{R}^3$, and the group law is given by the following formula:

$$(a,b,c)(x,y,z) = (a + xe^{-c}, b + ye^{-c}, c + z).$$

The left invariant metric on this Lie group is given by

$$ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2.$$

A discrete cocompact *solvgroup* is a discrete subgroup of SOL with compact quotient. The action of this subgroup on SOL is given by means of the identification of SOL with the connected component of the identity in the isometry group Isom(SOL) of SOL. It is known that SOL ⊂ Isom(SOL) is a normal subgroup of index 8 and its isotropy group at the origin is the dihedral group D_4.

The SOL geometry (i.e. modelled on the group SOL) is one of the eight geometries existing in dimension 3 (see [6, 7]). According to Thurston's geometrization conjecture each closed 3-manifold may be split along tori into pieces which have one of the eight possible homogeneous geometric structures. It is natural to ask therefore which are the corresponding discrete groups which satisfy the condition $P(2)$ and are cocompact subgroups in one of the eight Lie groups. Hyperbolic groups for instance are almost convex hence they do satisfy $P(2)$. The only geometry for which the answer is not known is SOL, and this explains why the source of counterexamples we get is the set of solvgroups.

We remark that every compact 3-dimensional solvmanifold (i.e. a quotient SOL/G by a discrete cocompact subgroup) has a finite covering (of degree at most 4 from above) which is a torus fibration over a circle with hyperbolic characteristic map. Therefore this covering will have the fundamental group presented as follows:

$$G_A = \langle a, b, t \mid ab = ba, tat^{-1} = a^{A_{11}}b^{A_{12}}, tbt^{-1} = a^{A_{21}}b^{A_{22}} \rangle,$$

where A is a 2-by-2 hyperbolic matrix (i.e. it has no eigenvalues of module 1, and its determinant is 1).

We are able now to state the result of this note:

Theorem 3. 1. *A discrete cocompact solvgroup G does not satisfy $P(n)$ for some large n depending on G and the system of generators B.*

2. *The class of groups of the form G_A does not satisfy $P(2)$ with respect to the natural system of generators $\{a, b, t\}$.*

3. *If G satisfies $P(n)$ for some $n > 2$ then G satisfies also $P(m)$ for all $m \leqq n$.*

Acknowledgements. This note was prepared during my stay at University of Paris-Sud. I am indebted to my thesis advisor Valentin Poénaru who proposed this problem, and to Corrado Tanasi, for the discussions we had on this subject.

2. The proof of the theorem. We observe first that the estimations given in [2] suffice to obtain the first claim with only minor modifications. Also the 3-rd claim follows on the lines developped in [1] for the k-almost convexity, word by word. We shall give the detailed proof of the second assertion, which contains also the ingredients for the two other.

The Cayley graph $\Gamma(G_A, B)$ may be effectively realized in $\mathbb{R}^3$ as follows: we identify the subgroup $\langle a, b\rangle$ (which is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$) with the standard lattice in the horizontal plane $\{z=0\}$. In coordinates the element $a^i b^j$ corresponds to the vector of coordinates $(i, j, 0)$. Further the coset $t^k\langle a, b\rangle$ will be identified with a lattice in the horizontal plane $\{z=k\}$. This is carried as follows: the element $t^k a^i b^j$ is identified with the vector of coordinates (x, y, k) where x and y are determined by the equation in G_A

$$t^k a^i b^j = a^x b^y.$$

We join the vertex $t^k a^i b^j$ to the vertex $t^k a^{i+1} b^j$ by an edge labeled a and by an edge labeled b to the vertex $t^k a^i b^{j+1}$.

Then the lattice in $\{z=k\}$ corresponds under the vertical projection to the lattice $t^k(\mathbb{Z}\oplus\mathbb{Z})t^{-k}$ in the plane $\{z=0\}$. We join also the vertex sitting in the slice $\{z=k+1\}$ to that vertex directly above it in $\{z=k\}$ by an edge labeled t. The resulting graph is the Cayley graph of G_A with respect to the generators $\{a, b, t\}$. If $\alpha = t^k a^i b^j$ corresponds to the vertex (x, y, k) we say that (x, y, k) are the coordinates of α. Set also $|(x, y, k)| = \max(|x|, |y|)$. Observe first that the coordinates may be easily computed in terms of the matrix A:

$$(x, y)^T = A^k(i, j),$$

where T denotes the transposition of vectors.

Consider the following elements:

$$u_k = t^k a^{i_0} b^{j_0} t^{-k}, \quad v_k = t^k a^{i_1} b^{j_1} t^{-k}, \quad \alpha_k = t u_k v_k, \quad \beta_k = t^{-1} u_k v_k.$$

We make the following remarks:

- u_k and v_k commute with each other.
- Since $t^{-1}\alpha_k = t\beta_k$ the distance between α_k and β_k in the Cayley graph is $d(\alpha_k, \beta_k) \leqq 2$.
- $d(1, \alpha_k) \leqq 4k + i_0 + i_1 + j_0 + j_1 + 1 = M$, and $d(1, \beta_k) \leqq M$.
- Consider the function $\sigma : G \longrightarrow \mathbb{Z}$ defined by $\sigma(w) = \sum$ exponents of t in the word $w =$ the last coordinate of w.
  Then σ is well-defined since $\sigma(\textit{relation}) = 0$ for all relations in the presentation of G_A. Also $\sigma(\alpha_k) = 1$, and $\sigma(\beta_k) = -1$.
- The coordinates $(x_k, y_k, 1)$ of α_k are given by

$$(x_k, y_k)^T = A^k(i_0, j_0) + A^{-k}(i_1, j_1).$$

- Any path in the graph between α_k and β_k intersects $\sigma^{-1}(0)$.

Consider then a path of length N between α_k and β_k, and set $z = a^u b^v$ for the first point where the path touch $\sigma^{-1}(0)$. From the symmetry of the situation we may suppose that the length of the path between α_k and z is less than $N/2$. Set ξ for the point preceding

z on the path. Then the arc connecting α_k and ξ is contained in the set $\{\sigma \geqq 1\}$. Any move on the path is obtained by left multiplication by one of the generators. Then the coordinates of a point $(e, f, *)$ are changed by a left multiplication move into $(e', f', *)$ and we have

$$|(e', f', *)| \geqq |A|^{-1}|(e, f, *)|, \text{ for } |e|, |f| >> 0.$$

Observe now that the coordinates of ξ are necessary $(u, v, 1)$. But ξ is obtained from α_k after a number of $N/2$ moves, so we get

$$r = \max(|u|, |v|) \geqq |A|^{-N/2}|(x_k, y_k, 1)|.$$

We choose now the vectors (i_0, j_0) and (i_1, j_1) be good approximations of the two eigenvectors of A, namely so that

$$A^k(i_0, j_0) \cong |A|^k c_0, A^k(i_1, j_1) \cong |A|^{-k} c_1,$$

where c_0 and c_1 are two distinct vectors in the plane. This is possible since A is an hyperbolic matrix.

Then we derive that for some $\lambda > 0$ we have

$$r \geqq \lambda |A|^{k-\frac{N}{2}}.$$

But the distance between two points in the plane $\{z = 0\}$ may be easily computed and we obtain

$$d(1, z) \geqq r.$$

Suppose now that the number $N = N(k)$ satisfies the condition (1) stated in the introduction, so that

$$\lim_{k \to \infty} k - \frac{N(k)}{2} = \infty.$$

Then for large k we find that each path connecting α_k to β_k has a point z with the property that $d(1, z) > M$, because $|A| > 1$. In other words any path connecting α_k and β_k (which are at distance 2, and contained in the ball of radius M), whose length is less than $N(k)$ need to have points outside the ball of radius M. This proves that G_A does not satisfy $P(2)$, as claimed. □

References

[1] J. W. CANNON, Almost convex groups. Geom. Dedicata **22,** 197–210 (1987).

[2] J. W. CANNON, W. J. FLOYD, M. A. GRAYSON and W. P. THURSTON, Solvgroups are not almost convex. Geom. Dedicata **31,** 291–300 (1989).

[3] V. POÉNARU, Almost convex groups, Lipschitz combing and π_1^∞ for universal covering spaces of closed 3-manifolds. J. Differential Geometry **35,** 103–130 (1992).

[4] V. POÉNARU, Geometry à la Gromov for the fundamental group of a closed 3-manifold M^3 and the simple connectivity at infinity of $\tilde{M}^3$. Topology **33,** 181–196 (1994).

[5] V. POÉNARU and C. TANASI, Hausdorff combing of groups and π_1^∞ for universal covering spaces of closed 3-manifolds. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **20,** 387–414 (1993).

[6] P. SCOTT, The geometries of 3-manifolds. Bull. London Math. Soc. **15,** 401–487 (1983).

[7] W. P. THURSTON, The geometry and topology of 3-manifolds, preprint Princeton, 1983.
[8] J. H. C. WHITEHEAD, A certain open 3-manifold whose group is unity. Quart. Journal Math. **6**, 268–279 (1935).

Eingegangen am 18. 3. 1996

Anschrift des Autors:

Louis Funar
Institut Fourier, BP 74, UFR Mathématiques
Univ. Grenoble I
F-38402 Saint-Martin-d'Hères cedex
France