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CUBULATIONS, IMMERSIONS, MAPPABILITY AND A PROBLEM OF HABEGGER

BY LOUIS FUNAR

ABSTRACT. – The aim of this paper (inspired from a problem of Habegger) is to describe the set of cubical decompositions of compact manifolds mod out by a set of combinatorial moves analogous to the bistellar moves considered by Pachner, which we call bubble moves. One constructs a surjection from this set onto the the bordism group of codimension-one immersions in the manifold. The connected sums of manifolds and immersions induce multiplicative structures which are respected by this surjection. We prove that those cubulations which map combinatorially into the standard decomposition of $\mathbf{R}^n$ for large enough n (called mappable), are equivalent. Finally we classify the cubulations of the 2-sphere. © Elsevier, Paris

RÉSUMÉ. – Dans cet article on décrit les classes d'équivalence des cubications d'une variété compacte modulo des mouvements analogues aux mouvements de Pachner. On construit une surjection de l'ensemble des classes d'équivalence dans le groupe des bordismes d'immersions de codimension 1 dans la variété. Les sommes connexes des variétés et des immersions induisent des structures multiplicatives respectées par cette surjection. On prouve que les cubications qui admettent une application combinatoire dans la cubication standard de $\mathbf{R}^n$, pour n assez grand, sont équivalentes. On donne une classification des cubications de la sphére S^2. © Elsevier, Paris

1. Introduction and statement of results

1.1. Outline. Stellar moves were first considered by Alexander ([Al]) who proved that they can relate any two triangulations of a polyhedron. Alexander's moves were refined to a finite set of local (bistellar) moves which still act transitively on the triangulations of manifolds, according to Pachner ([P1, P]). Using Pachner's result Turaev and Viro proved that certain state-sums associated to a triangulation yield topological invariants of 3-manifolds (see [Tu]). Recall that a bistellar move (in dimension n) excises B and replaces it with B', where B and B' are complementary balls, subcomplexes in the boundary of the standard $(n+1)$-simplex. For a nice exposition of Pachner's result and various extensions, see [Li].

The Turaev-Viro invariants carry less information than the Reshetikhin-Turaev invariants, which are defined using Dehn surgery presentations instead of triangulations. In fact the latter have a strong 4-dimensional flavor, as explained by the theory of shadows developed by Turaev (see [Tu]). This motivates the study of state-sums based on cubulations, as an alternative way to get intrinsic invariants possibly containing more information (e.g. the phase factor). A *cubical complex* is a complex K consisting of Euclidean cubes, such that the intersection of two cubes is a finite union of cubes from K, once a cube is in K all its faces are still in K, and no identifications of faces of the same cube are allowed.

A *cubulation* of a manifold is specified by a cubical complex PL homeomorphic to the manifold. In order to apply the state-sum machinery to these decompositions we need an analogue of Pachner's theorem. Specifically, N.Habegger asked (see problem 5.13 from R.Kirby's list ([Kirby])) the following:

Problem 1. *Suppose M and N are PL-homeomorphic cubulated n-manifolds. Are they related by the following set of moves: excise B and replace it by B', where B and B' are complementary balls (union of n-cubes) in the boundary of the standard $(n+1)$-cube?*

These moves will be called *bubble* moves in the sequel. Among them, those for which B or B' does not contain parallel (when viewed in the $n+1$-cube) faces are called *np-bubble* moves. There are $n+1$ distinct np-bubble moves b_k, $k=1,2,...,n+1$ and their inverses, where the support B of b_k is the union of exactly k cubes. For $n=2$ there is one bubble move which is not a np-bubble (see Figure 1). Set $C(M)$ for the set of cubulations of a closed manifold M, $CBB(M)$ for the equivalence classes of cubulations mod np-bubble moves and $CB(M)$ for the equivalence classes of cubulations mod bubble moves. The answer to Habegger's question, as it states, is negative because the triangle and the square are not bubble equivalent. In fact, for $n=1$ the move b_1 divides an edge into three edges and so $CB(\mathrm{S}^1) = CBB(\mathrm{S}^1) = \mathbf{Z}/2\mathbf{Z}$. Therefore a complete answer would rather consist of a description of $CB(M)$. Another way is to avoid the difficulties of a direct approach by looking for a sufficiently large class of cubulations having an intrinsic characterization and within which the cubulations are equivalent. The aim of this paper is to formulate some partial solutions along these lines.

For instance, one associates to each cubulation C of M, a codimension-one normal crossings immersion φ_C in M. In this way one obtains a surjective map from the set

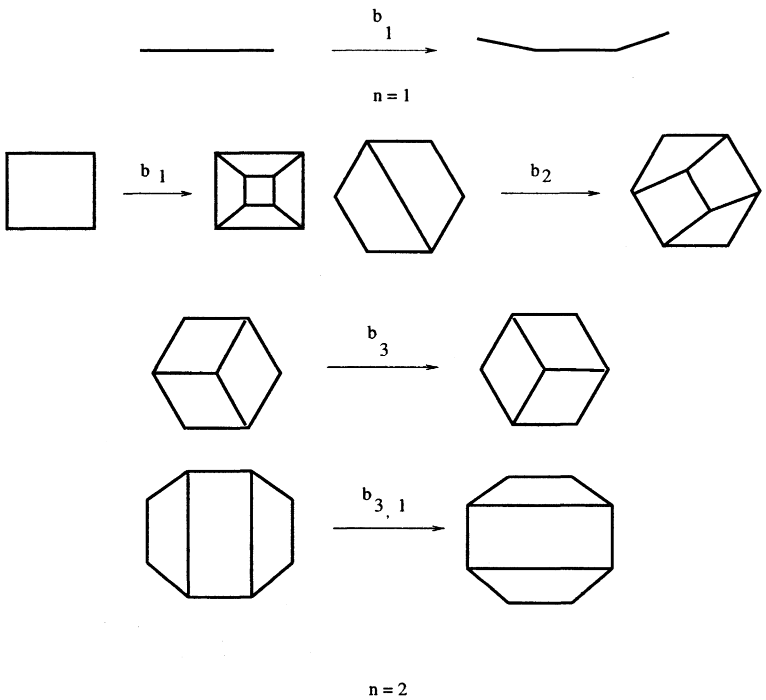

Fig. 1. Bubble moves for $n=1$ and $n=2$.

of (marked) cubulations mod bubble moves to the bordism set of immersions. The latter has a homotopical description via the Pontryagin-Thom construction. We conjecture that this surjection is a bijection. On the other hand let us restrict to cubulations which can be combinatorially mapped into the standard cubulation of some Euclidean space (called mappable). One can approximate an ambient isotopy between two cubulations by some cubical sub-complexes of the standard cubulation. The path of cubical approximations is locally constant except for a finite number of critical values of the parameter, when a jump described by a bubble move occurs. As a consequence two mappable cubulations are bubble equivalent. We prove that the connected sum of cubulations mod bubble moves is well-defined, and this is compatible with the composition map for immersions. Finally we consider the case of $CB(\mathrm{S}^2)$ and show by a direct combinatorial proof that $CB(\mathrm{S}^2) = \mathbf{Z}/2\mathbf{Z}$.

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1.2. Elementary obstructions. We outline the combinatorial approach in higher dimensions from [Fun]. For a cubulation $x \in C(M)$ of the n-manifold M the component $f_i(x)$ of the f-vector $\mathbf{f}(x)$ counts the number of i-dimensional cubes in x. The orbit of the f-vector $\mathbf{f}$ under bubble moves has the form $\mathbf{f} + \Lambda(n) \subset \mathbf{Z}^{n+1}$, where $\Lambda(n)$ is a lattice. Therefore we have an induced map $CB(M) \longrightarrow \mathbf{Z}^{n+1}/\Lambda(n)$ taking values in a finite Abelian group.

Proposition 1.1. *There exist nonzero even numbers $a_i(n) \in \mathbf{Z}_+$ such that the projection $\mathbf{Z}^{n+1}/\Lambda(n) \longrightarrow \prod_{i=0}^{n} \mathbf{Z}/a_i(n)\mathbf{Z}$ is surjective. The greatest such numbers $a_i(n)$ verify $a_n(n) = 2$, $a_{n-1}(n) = 2n$, $a_{n-2}(n) = 2$, $a_0(n) = 2$, $a_1(n) = 3 + (-1)^n, (n > 2)$.*

See [Fun] for the proof. Let fb be the class of $\mathbf{f}$ in $\prod_{i=0}^{n} \mathbf{Z}/a_i(n)\mathbf{Z}$ and $fb^{(2)}$ be the reduced elements modulo $(2, 2, 2, ..., 2, 2n, 2)$. Notice that $\Lambda(n)$ is not a product lattice in general. For instance, when $n = 3$ there is an additional invariant $f_0 + f_1 \in \mathbf{Z}/4\mathbf{Z}$.

A natural problem is to compute the image $fb(CB(M))$ for given M. Some partial results for the mod 2 reductions $fb^{(2)}(CB(M))$ are known. This is equivalent to characterizing those f-vectors mod 2 which can be realized by cubulations of the manifold M. There are constraints for the existence of a simplicial polyhedron with a given f-vector and fixed topological type. For convex simplicial polytopes one has McMullen's conditions (see [Mcm, B1, B2, BL, S, Mcm2]). The complete characterization of the f-vectors of simplicial polytopes (and PL-spheres) was obtained in [S2]. The analogous problem of the realization of f-vectors by cubical polytopes has also been considered in some recent papers, for example [BB, BC, He, J] and references therein. The new feature is that, unlike for the simplicial case, there are parity restrictions on the f-vectors (see [BB]). The relationship between cubical PL n-spheres and the immersions was described in the following result of Babson and Chan (see [BC]):

Proposition 1.2. – *There exists a cubical n-sphere K with given $f_i \pmod 2$ if and only if there exists a codimension 1 normal crossings immersion $\varphi : M \longrightarrow \mathrm{S}^n$ such that*

$f_i(K) = \chi(X_i(M, \varphi)) \pmod 2$, *where* χ *denotes the Euler characteristics, and* $X_i(M, \varphi)$ *is the set of* i*-tuple points.*

There is a wide literature on immersions, and especially on the function θ_n counting the number of multiple n-points mod 2, which was considered first by Freedman ([F]). Earlier Banchoff [B] has proved that the number of triple points of a closed surface S immersed in $\mathbf{R}^3$ is $\chi(S) \pmod 2$. There is an induced homomorphism $\theta_n : B_n \longrightarrow \mathbf{Z}/2\mathbf{Z}$ on the Abelian group B_n of bordism classes of immersions of $(n-1)$-manifolds in S^n. Now θ_n is surjective (i.e. nontrivial) if and only if $fb^{(2)}_{n-1}(\mathrm{S}^n) = \mathbf{Z}/2\mathbf{Z}$. From the results concerning the function θ_n obtained in [F, E1, E2, E3, H, Kos, KS, L, C1, C2, C3] we deduce that the f-vectors of a n-sphere have the following properties (see also [BC]):

1. For $n = 2$ we have $f_0 = f_2 \pmod 2$ and $f_1 = 0 \pmod 2$ and thus $fb^{(2)}(CB(\mathrm{S}^2)) = fb^{(2)}(CBB(\mathrm{S}^2)) = \mathbf{Z}/2\mathbf{Z}$.
2. For $n = 3$, $f_0 = f_1 = 0 \pmod 2$, $f_2 = f_3 \pmod 2$. The Boy immersion $j : \mathbf{R}P^2 \longrightarrow \mathrm{S}^3$ has a single triple point and so there exists a PL 3-sphere with an odd number of facets. Therefore $fb^{(2)}(CB(\mathrm{S}^3)) = fb^{(2)}(CBB(\mathrm{S}^3)) = \mathbf{Z}/2\mathbf{Z}$.
3. The characterization of $fb^{(2)}_{n-1}(\mathrm{S}^n)$ is reduced to a homotopy problem: $fb^{(2)}_{n-1}(\mathrm{S}^n) = \mathbf{Z}/2\mathbf{Z}$ if and only if
   (a) either n is $1, 3, 4$ or 7.
   (b) or else $n = 2^a - 2$, with $a \in \mathbf{Z}_+$, and there exists a framed n-manifold with Kervaire invariant 1. The latter is known to be true for $n = 2, 6, 14, 30, 62$.
4. If we consider only edge-orientable cubulations (see [He]) then $fb^{(2)}_{n-1}(\mathrm{S}^n)$ is known. The edge-orientability is equivalent to the orientability of the manifold immersed in S^n and the restriction of the map θ_n to the subgroup of oriented bordism classes was computed in [F]. In particular $f_{n-1} = 0 \pmod 2$ if $n \neq 1, 2, 4$.

1.3. The 2-dimensional case. To a surface cubulation we can associate a set of immersed circles K_i obtained from the union of arcs joining the opposite sides in each square. The cubulation is *simple* if the circles K_i are individually embedded in the respective surface. Simple is equivalent to mappable for the cubulations of S^2 (see below). A cubulation is called *semi-simple* if each image circle $\varphi(K_i)$ has an even number of double points, which form cancelling pairs. Two double points form a *cancelling pair* if they are connected by two distinct and disjoint arcs.

Theorem 1.3. *The np-bubble moves act transitively on the set of simple cubulations of* S^2. *The orbit of the standard cubulation is the set of semi-simple cubulations. The map* $fb^{(2)} = f_0 \pmod 2$ *is an isomorphism between* $CB(\mathrm{S}^2)$ *and* $\mathbf{Z}/2\mathbf{Z}$.

1.4. Bordisms of immersions. Let us consider the set $\mathcal{I}(M)$ (respectively $\mathcal{I}^+(M)$ in the orientable case) of bordisms of codimension 1 nc-immersions (i.e. normal crossings) in the manifold M. Two nc-immersions $f_i : N_i \longrightarrow M$ of the $(n-1)$-manifolds N_i are *bordant* if there exists a proper nc-immersion $f : N \longrightarrow M \times [0, 1]$ of some cobordism N between N_1 and N_2, such that the restriction of f to N_i is isotopic to f_i. Using general position arguments one may get rid of the nc-assumption.

A *marked* cubulation is a cubulation C of the manifold M, endowed with a PL-homeomorphism $|C| \longrightarrow M$ of its subjacent space $|C|$, considered up to isotopy. If a bubble move is performed on C, then there is a natural marking induced for the bubbled cubulation. Thus it makes sense to consider the set $\widetilde{CB}(M)$ of marked cubulations mod bubble moves. We associate to each marked cubulation C a codimension 1 nc-immersion $\varphi_C : N_C \longrightarrow M$ (the cubical complex N_C was called the derivative complex in [BC]). Each cube is divided into 2^n equal cubes by n hyperplanes which we call sections. When gluing together cubes in a cubical complex the sections are glued accordingly. The union of the hyperplane sections form the image of a codimension 1 nc-immersion. In the differentiable case one uses a suitable smoothing when gluing the faces. If the cubulation C is edge-orientable (see [He]), and if M is oriented, then N_C is an oriented manifold.

Theorem 1.4. *The map $C \to \varphi_C$ induces a surjection $I : \widetilde{CB}(M) \longrightarrow \mathcal{I}(M)$.*

Theorem 1.3 says that the map I is injective for $M = \mathrm{S}^2$. We conjecture that I is bijective. In particular $\widetilde{CB}(M)$ would depend only on the homotopy type of M and the functor $\widetilde{CB}$ which associates to M the set $\widetilde{CB}(M)$ would be (homotopically) representable. Notice that $CB(M) = \widetilde{CB}(M)/\mathcal{M}(M)$, where $\mathcal{M}(M)$ is the mapping class group of M, i.e. the group of homeomorphisms of M up to isotopy. Using the classical Pontryagin-Thom construction (see e.g. [V]) it follows that $\mathcal{I}(M) = [M_c, \Omega^\infty \mathrm{S}^\infty \mathbf{RP}^\infty]$, and $\mathcal{I}^+(M) = [M_c, \Omega^\infty \mathrm{S}^\infty \mathrm{S}^1]$, where M_c is the one point compactification, Ω denotes the loop space, S the reduced suspension and the brackets denote the set of the homotopy classes of maps. Moreover $\mathcal{I}^+(M) = \pi^1(M_c)$ is the first cohomotopy group $\pi^1(M_c)$. The cohomotopy groups of spheres can be computed: $\mathcal{I}(\mathrm{S}^n) = \pi_n^s(\mathbf{RP}^\infty)$, where $\pi_n^s(\mathbf{RP}^\infty)$ is the n-th stable homotopy group, and $\mathcal{I}^+(\mathrm{S}^n) = \pi_n^s(\mathrm{S}^1) = \pi_{n-1}^s$. It is known that $\pi_1^s(\mathbf{RP}^\infty) = \pi_2^s(\mathbf{RP}^\infty) = \mathbf{Z}/2\mathbf{Z}$, $\pi_3^s(\mathbf{RP}^\infty) = \mathbf{Z}/8\mathbf{Z}$, and a few values of the stable stems are tabulated below:

n	0	1	2	3	4	5	6	7	8
π_n^s	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/24$	0	0	$\mathbf{Z}/2$	$\mathbf{Z}/240$	$\mathbf{Z}/2 \oplus \mathbf{Z}/2$

Let us introduce now the set $\mathcal{C}(M)$ of bordisms of cubulations of the manifold M. The cubulations C_1 and C_2 are *bordant* if there exists a cubulation C of $M \times [0,1]$ whose restrictions on the boundaries are the C_i. The identity induces a map $CB(M) \longrightarrow \mathcal{C}(M)$. The question on the existence of an inverse arrow is similar to Wall's theorem about the existence of formal deformations between simple homotopy equivalent n-complexes through $(n+1)$-complexes (for $n \neq 2$). Remark that any two cubulations become bordant when suitably subdivided. Consider some cubulation of the sphere S^n which is bubble equivalent to the standard one. We can view the bubble moves as the result of gluing and deleting $(n+1)$-cubes (after some thickening) to the given cubulation. It follows that any such cubulation bounds, i.e. it is the boundary of a cubulation of the $(n+1)$-ball. For instance a polygon bounds iff it has an even number of edges. For $n = 2, 3$ it might be true that the boundary of a ball cubulation is bubble equivalent to the standard one, but the result cannot be extended to $n \geq 4$. This is analogous to the existence of non-shellable triangulations of the ball for $n \geq 3$ (see [Li2]). We define a *shuffling* to be a sequence of moves where shellings (i.e. adding iteratively cells, each intersecting the union of the previous ones along a ball) alternate with inverse shellings. An equivalent statement of Pachner's theorem is that all triangulations can be shuffled. In the case of cubulations the first obstruction for shuffling is that the cubulation bounds. However there exist cubulations which bound but cannot be shuffled for $n = 4$. Consider for example the connected sum $x \sharp x$, where $I(x)$ is the generator of the third stable stem. We will prove below that the connected sum of cubulations makes $CB(\mathrm{S}^n)$ a monoid. But $x \sharp x$ bounds and if it can be shuffled then x should have order 2. This is impossible because I is a monoid homomorphism and the bordism group is $\mathcal{I}^+(\mathrm{S}^4) = \mathbf{Z}/24\mathbf{Z}$.

1.5. Embeddable and mappable cubulations. Cubical complexes, as objects of study from a topological point of view, were also considered by Novikov ([novikov], p.42) which asked whether a cubical complex of dimension n embeds in (or can be mapped to) the n-skeleton of the standard cubic lattice of some dimension N. These are called *embeddable* and respectively *mappable* cubulations. By the standard cubic lattice (or the standard cubical decomposition) is meant the usual partition of $\mathbf{R}^N$ into cubes with vertices in $\mathbf{Z}^N$. Several results were obtained in [DSS1, DSS2, DSS3, DSS, Kara].

Theorem 1.5. *The mappable cubulations of a PL manifold M are bubble equivalent.*

Let us say that a cubulation is *simple* if no path in which consecutive points correspond to edges which are opposite sides of some square of the cubulation, contains two orthogonal edges from the same cube. The cubulation is *standard* if any two of its cubes are either disjoint or have exactly one common face. An immediate observation is that embeddable cubulations are standard and simple and mappable cubulations are simple. On the other hand the simplicity is very close to the mappability, at least for manifolds with small fundamental group. We have for instance the following results of Karalashvili ([Kara]) and Dolbilin, Shtanko and Shtogrin ([DSS3]):

1. The double (i.e. the result of dividing each k-dimensional cube in 2^k equal cubes) of a simple cubulation is mappable.
2. A simple cubulation of a manifold M satisfying $\mathrm{H}^1(M, \mathbf{Z}/2\mathbf{Z}) = 0$ is mappable.
3. From a simplicial decomposition S one constructs a cubulation $C(S)$ by dividing each n-simplex into $n+1$ cubes. Then the cubical decomposition $C(S)$ is embeddable.

In particular cubulations coming from triangulations are bubble equivalent. Also the simple cubulations of the sphere are equivalent. Notice that the set of simple (or mappable) cubulations is not closed to arbitrary bubble moves. In general the simplicity is not preserved by the move b_2.

1.6. Multiplicative structures. The connected sum of the manifolds M and N is denoted by $M\sharp N$. When appropriately extended to cubulations $\sharp$ depends on various choices, but after passing to bubble equivalence classes these ambiguities disappear.

Theorem 1.6. *There exists a map $CB(M) \times CB(N) \longrightarrow CB(M\sharp N)$ induced by the connected sum of cubulations.*

As a consequence $\sharp$ induces also a composition map $\widetilde{CB}(M)\times\widetilde{CB}(N) \longrightarrow \widetilde{CB}(M\sharp N)$. On the other hand there is a natural composition map on the sets of bordisms of immersions, by using the connected sum away from the immersions. We prove that the map I is functorial:

Theorem 1.7. *We have a commutative diagram*

$$\begin{array}{ccccc}
\widetilde{CB}(M) & \times & \widetilde{CB}(N) & \longrightarrow & \widetilde{CB}(M\sharp N) \\
I\downarrow & & & & \downarrow I \\
\mathcal{I}(M) & \times & \mathcal{I}(N) & \longrightarrow & \mathcal{I}(M\sharp N)
\end{array}$$

We believe that the monoid $CB(\mathrm{S}^n)$ is actually a group. Notice that $\mathcal{I}(M)$ has a group structure for any M (induced from the cohomotopy group structure), but we don't know whether this can be lifted to $\widetilde{CB}(M)$.

2. The proof of theorem 1.4

Let K be the derivative complex (having the connected components K_i) associated to a cubulation C of the manifold M and let $\varphi : K \longrightarrow M$ be the associated immersion. In order to rule out some pathologies we restrict here to the *combinatorial* cubulations, meaning that the star of each vertex (or the link) is a PL ball (respectively a PL sphere) of the right dimension.

Let us first show that we have an induced map $I : \widetilde{CB}(M) \longrightarrow \mathcal{I}(M)$. Consider the local picture of a bubble move, viewed in the boundary of the $(n+1)$-cube. The set of sections on the boundary are intersections of the hyperplane sections of the $(n+1)$-cube with the faces. Let B and B' be the n-balls interchanged by a bubble move. Then the union of hyperplane sections in the $(n+1)$-cube yields a bordism between the immersions φ_B and $\varphi_{B'}$ in the $(n+1)$-ball. Thus immersions associated to equivalent cubulations are cobordant.

The immersion φ is *pseudo-spine* if the closures of the connected components of the complementary $M - \mathrm{Im}\,(\varphi)$ are balls, and the image of φ is connected. The immersion is called *admissible* if each connected component L of the set $X_k(M, \varphi)$ of k-tuple points is a PL ball and we have $\mathrm{cl}\,(L) \cap X_{k+1}(M, \varphi) \neq \emptyset$ (for all $k \leq n-1$, where cl denotes the closure).

Observe that any nc-immersion is cobordant to an admissible pseudo-spine immersion. In fact consider a set of small (bounding) spheres embedded in M, transverse to the immersion φ. If they are sufficiently small they cut the connected components of the complementary $M - \mathrm{Im}\,(\varphi)$ into balls. We add sufficiently many of them so that all connected components of the strata $X_k(M, \varphi)$ are divided into balls by the additional spheres. The immersion φ' whose image consists of the union of $\mathrm{Im}\,(\varphi)$ with the spheres is cobordant to φ. In fact let us choose some balls in $M \times [0,1]$ bounded by the spheres in $M = M \times \{1\}$. The required cobordism is obtained by putting the balls in standard position with respect to $\mathrm{Im}\,(\varphi) \times [0,1]$.

Furthermore we want to associate a cubulation C to the immersion φ such that $I(C) = \varphi$. For an admissible pseudo-spine immersion one takes the corresponding cell decomposition of the manifold, then the dual decomposition is just the cubulation we are looking for. The pseudo-spine condition implies that the dual decomposition has the structure of a cubical complex and the admissibility is required for the cubulation comes endowed with a natural marking $\mid C \mid \rightarrow M$.

3. Embeddable and mappable cubulations

3.1. Mappable cubulations are equivalent. The proof of Theorem 1.5 goes as follows: we show first that embeddable cubulations are equivalent and then that a mappable cubulation is equivalent to an embeddable one.

For the first step consider N sufficiently large so that both cubulations P and Q embed into the standard cubulation $\mathbf{R}_c^N$ of the Euclidean space and such that there exists an ambient isotopy carrying P into Q. The image of P during the isotopy is denoted by P_t. We define an approximation P_t^{st} of the manifold P_t, which is a cubulated sub-manifold of $\mathbf{R}_c^N$, which is sufficiently close to P_t, and when t varies the family P_t^{st} is either locally constant or changes around a "critical value" by a bubble move. We realize an arbitrarily fine approximation by taking the cubical structure be $\mathbf{R}_c^N[\varepsilon]$, based on cubes whose edges are of length ε, for small ε. A way to do that is to divide each cube of the lattice into 2^N equal cubes. Then the initial cubulations P and Q are replaced by some iterated doublings, say $2^m P$ and $2^m Q$. It remains to prove that $2^m P$ is equivalent to P.

Proposition 3.1. *Let P and Q be two cubulations of a PL manifold which are embedded in the standard cubical lattice $\mathbf{R}_c^N$. Then there exists m arbitrarily large, such that $2^m P$ and $2^m Q$ are bubble equivalent.*

Proposition 3.2. *If P is embeddable, then for big enough m the iterated doubling $2^m P$ is bubble equivalent to P.*

Notice that the analogous statement for np-bubble moves is false in general.

Proposition 3.3. *Let P be a mappable cubulation. Then P is np-bubble equivalent to an embeddable cubulation.*

3.2. The proof of Proposition 3.1. Consider a cubulation $X \subset \mathbf{R}_c^N[\varepsilon]$, of codimension at least 1. Let us denote by $\Lambda[\varepsilon]$ the union of all hyperplanes defining the cubulation $\mathbf{R}_c^N[\varepsilon]$, which can be written as a disjoint union of the strata $\Lambda[\varepsilon]^{(m)}$ consisting of all open codimension m cubes. Set C for the cube given by the equations $\{|z_j| \leq 1, j = 1, N\}$. Let $\overline{C}$ be the $(N-1)$-complex obtained from ∂C by adding the hyperplanes $\{z_j = 0\}$. Denote by W the star of the origin in $\overline{C}$ i.e. the union of cells having the form $W_{k,\mu} = \{z_k = 0, \mu_j z_j \geq 0, |z_j| \leq 1, \forall j\}$, where $\mu_j \in \{-1, 1\}$, $\forall j$.

Definition 3.4. *The disk D is a standard model in C if D is properly embedded in C (and transverse to ∂C), D is contained in W and the origin lies in* $\operatorname{int}(D)$.

Definition 3.5. *Let C be a N-cube of the cubulation $\mathbf{R}_c^N[\varepsilon]$. We say that X is standard with respect to C if the following conditions are fulfilled:*

1. *X is transversal to $\Lambda[\varepsilon] \cap C$.*
2. *There exists an isotopy supported on* $\operatorname{int}(C) \cup (\Lambda[\varepsilon]^{(0)} \cap C)$, *if the codimension of X is at least 2, and respectively* $\operatorname{int}(C) \cup (\Lambda[\varepsilon]^{(0)} \cup \Lambda[\varepsilon]^{(1)} \cap C)$, *if the codimension is precisely 1, which transforms $X \cap C$ in a standard model.*

Finally X is standard (or in standard position) with respect to $\mathbf{R}_c^N[\varepsilon]$ (or $\Lambda[\varepsilon]$) if X is standard w.r.t. all cubes.

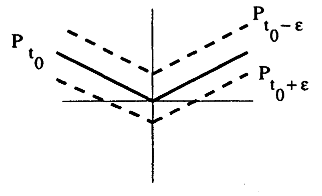
Observe first that a plane L transverse to the boundary of C is in standard position w.r.t. C. In fact using a recurrence argument the intersection of L with any face $F \subset \partial C$ can be put in standard position by means of an isotopy. The union of standard models for $L \cap F$ over the faces is a PL sphere and the cone centered at the origin on it is isotopic to $L \cap C$ and hence it is a standard model.

Consider a submanifold X which is standard w.r.t. the lattice Λ. Then there is an isotopy transforming X into $X^{st}(\Lambda)$, where $X^{st}(\Lambda)$ intersects each cube along a standard model. In fact this can be done in each cube that X intersects and the standard models for different cubes have disjoint interiors. It suffices to check the compatibility of the boundary gluings: if X cuts two adjacent cubes C and C' then the standard models of $X \cap C$ and $X \cap C'$ can be glued together. First the neighborhood of the common face is determined by the standard model of X inside the face. Hence we have to prove the uniqueness of the standard model for $X \cap C \cap C'$, which follows by a recurrence argument on the dimension.

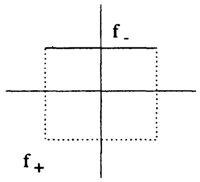
Therefore the cubical complex $X^{st}(\Lambda)$ is uniquely determined by X and Λ, and it will be called *the standard model* of X w.r.t. Λ.

Let us consider the cubulations P and Q of a n-manifold M, embedded in $\mathbf{R}_c^N$. There exists an isotopy with compact support $\varphi : \mathbf{R}^N \times [0,1] \longrightarrow \mathbf{R}^N$, with φ_0 being the identity and $\varphi_1(P) = Q$. For big enough N one can choose the isotopy $\varphi\,|_{P\times[0,1]}$ to be an embedding. We assume that the cubulations P and Q are embedded in the standard cubulation given by the affine lattice $\tilde{\Lambda} = \Lambda + (\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$, which has the origin translated into $(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$. Notice that a cubulation $P \subset \tilde{\Lambda}$ is automatically in standard position w.r.t. Λ. Moreover the intersection of P with each cube of Λ is a standard model and so $P^{st}(\Lambda) = P$. We obtained a submanifold $Y = \varphi(P \times [0,1])$ of $\mathbf{R}^N$, whose boundary ∂Y is in standard position w.r.t. the lattice Λ. We claim that for big enough m there exists an isotopy carrying Y into X, such that ∂X and ∂Y are isometric and X is in standard position w.r.t. the lattice $\Lambda[2^{-m}]$. There exists a subdivision $\Lambda[2^{-m}]$ (for large m), such that Y becomes standard w.r.t. $\Lambda[2^{-m}]$, after a small isotopy which is identity near the boundary. We translate Y into the lattice whose origin is at $(2^{-m-1}, 2^{-m-1}, ..., 2^{-m-1})$. Then P and Q transform into $2^m P$ and $2^m Q$, and they are in standard position w.r.t. $\Lambda[2^{-m}]$. The last condition is an open condition, so we can keep Y fixed near the boundary during the isotopy. Let us denote by Z the tube describing an isotopy between $2^m P$ and $2^m Q$, which is in standard position w.r.t. $\Lambda[2^{-m}]$. There exist topologically trivial tubes Z_1 between P and $2^m P$, and respectively Z_2 between Q and $2^m Q$, which are in a standard positions w.r.t. $\Lambda[2^{-m}]$. Then set $X = Z_1 \cup Z \cup Z_2$.

Therefore we derived a PL cylinder $X \subset \Lambda[2^{-m}]$ which interpolates between P and Q. Notice that the cubical structure of P in $\Lambda[2^{-m}]$ is that of $2^m P$ in Λ. In general one cannot shell the boundary from P to Q. The tube X carries a PL foliation by submanifolds $P_t = \varphi_t(P)$, where φ states for the isotopy carried by X. The leaf P_t does not contain *flat directions* in the cube U if P_t does not contain any segment parallel to some vector in ∂U. Using a small isotopy one can get rid of flat directions in all leaves P_t. Furthermore there exists some m such that either P_t is standard w.r.t. $U \cap \Lambda[2^{-m}]$, or else P_t contains vertices of the lattice $\Lambda[2^{-m}]$. The first alternative would hold if we are allowed to move slightly P_t, using an arbitrary small isotopy (in order to achieve the transversality). On the other hand the leaf is not transverse iff it contains vertices from $\Lambda[2^{-m}]$, because there are no flat directions. Further "to be in standard position" is an open condition and so one can choose the constant m such that for each t either the leaf P_t is in standard position w.r.t. the cube $U \cap \Lambda[2^{-m}]$, or else P_t contains vertices from $U \cap \Lambda[2^{-m}]$. The set of those exceptional t for which the second alternative holds is finite because each critical leaf contains at least one vertex from $U \cap \Lambda[2^{-m}]$ and the different leaves are disjoint. Observe that we can change the isotopy P_t such that no exceptional leaf contains more than one



The leaves around a critical value



Bubble move supported on c

Fig. 2. The jump of a standard model at a critical value.

vertex from $\Lambda[2^{-m}]$, while keeping all the other properties obtained up to now. Moreover, there is such an m which is convenient for all cubes that X intersects. If one replaces a leaf P_t by the standard model $P^{st}(\Lambda[2^{-m}])$ then we get a family of cubulations embedded in $\mathbf{R}_c^N[2^{-m}]$. This family should be locally constant, until t reaches an exceptional value t_0 (where the standard model cannot be defined). Set U for the cube of size 2^{-m+1} centered at the (exceptional) vertex. The intermediary set $P_{[t_0-\varepsilon,t_0+\varepsilon]}\cap U$, for $\varepsilon << 2^{-m}$, is a trivial cobordism properly embedded in U. Using a small isotopy one can change $P_{t_0}\cap U$ into a union of planes passing through the vertex. If dir is the set of the $2n$ directions of the coordinate axes around the vertex lying in X we put $\mathrm{dir}_- = \{x \in \mathrm{dir}; P_{t_0-\varepsilon}\cap\ x \neq \emptyset\}$, , $\mathrm{dir}_+ = \{x \in \mathrm{dir}; P_{t_0+\varepsilon}\cap\ x \neq \emptyset\}$. Each such direction is dual to a face of a cube c in the dual lattice where the standard models $P_t^{st}(\Lambda[2^{-m}])$ live. We have $\mathrm{dir}_-\cup\mathrm{dir}_+ = \mathrm{dir}$, and $\mathrm{dir}_-\cap\mathrm{dir}_+ = \emptyset$ since P_{t_0} separates the directions which are cut by $P_{t_0-\varepsilon}$ from the directions cut by $P_{t_0+\varepsilon}$. Let f_- and f_+ be the union of faces of c duals to the directions in dir_- and dir_+ respectively. Then f_- and f_+ are PL balls because $P_{t_0-\varepsilon}\cap U$ is a ball, as well as $P_{t_0+\varepsilon}\cap U$. We need only to see that both are non-void. If f_- is empty then $P_{t_0-\varepsilon}$ would be contained in a half cube $U_0 \subset U$ of the lattice $\Lambda[2^{-m}]$. Then $P_{t_0+\varepsilon}\cap U_0$ will be a cylinder, and thus it cannot be standard, contradicting our hypothesis. Therefore the standard model $P^{st}_{t_0+\varepsilon}(\Lambda[2^{-m}])$ is obtained from $P^{st}_{t_0-\varepsilon}(\Lambda[2^{-m}])$ by means of the bubble move $f_- \longrightarrow f_+$ having the support on c. This proves Proposition 3.1.

3.3. The proof of Proposition 3.2. Actually a stronger statement concerning sub-complexes of the standard lattice $\mathbf{R}_c^N$ is true. We will ask also the bubble moves which pass from one cubulation to the other to be *embedded* in $\mathbf{R}_c^N$. This means that each bubble move which exchanges the balls B and B' has the property that the cube bounded by $B\cup B'$ is contained in the skeleton of $\mathbf{R}_c^N$.

Two isotopic lattice knots (or graphs) in $\mathbf{R}^3$ are bubble equivalent, by means of bubble moves which can be realized on the lattice of $\mathbf{R}^3$ and which avoid self-crossings. Consider now a lattice d-manifold (or complex) $M\subset\mathbf{R}_c^N$, and a preferred coordinate axis defining a height function $h:\mathbf{R}^N\longrightarrow\mathbf{R}$. The preimage of the open interval $h^{-1}((n,n+1))$, for $n\in\mathbf{Z}$ is an open PL cylinder because an open n-cell $e\subset h^{-1}((n,n+1))$ must be vertical with respect to h. This means that $e = f\times(n,n+1)$, where f is a $(n-1)$-cell whose projection $h(f)$ is a single point. A horizontal cell is one whose image under h is a point. Therefore we derive the sub-complexes $A_n^+\subset h^{-1}(n)$, $A_n^-\subset h^{-1}(n)$, with $A_n^+\cong A_{n+1}^-$, such that $A_n^+\cup A_n^-\cup\{\text{horizontal cells}\} = h^{-1}(n)$ and

$$\mathrm{cl}\,(h^{-1}((n,n+1)))\cap(h^{-1}(n)\cup h^{-1}(n+1)) = A_n^+\cup A_n^-\cup\{\text{vertical cells}\}.$$

Let $H(n)$ be the union of interiors of the horizontal cells in $h^{-1}(n)$. Then one can decompose the sets A_n as follows: $A_n^+ = (h^{-1}(n)-H(n))\cup Z_n$, $A_n^- = (h^{-1}(n)-H(n))\cup CZ_n$, where $Z_n\cap CZ_n=\emptyset$, and $Z_n\cup CZ_n = \partial(\mathrm{cl}\,(H(n))$. Using a recurrence argument we assume that A_n^- and $2^kA_n^-$ are bubble equivalent for some k, by means of the sequence of embedded bubble moves X_i. Then the same sequences of bubble moves transforms A_n^+ into $2^kA_n^+$. Since $A_{n+1}^-\cong A_n^+$, there exist some cone constructions over the bubble moves X_i, which are realized in the $(N+1)$-dimensional lattice, and relate $A_n^+\times[0,1]$ to $2^kA_n^+\times[0,1]$, as follows. If the bubble move X_i touches only $A_n^+ - Z_n$, then consider the usual cone of X_i, which is also a bubble move in one more dimension. If the bubble move X_i touches Z_n (or CZ_n, on the other side) then construct an extension with one more dimension for X_i, by using the horizontal flat. The slices $A_n^+\times[0,1]$, and

$h^{-1}((n, n+1))$ can now be glued back, and we obtain a bubble equivalence between M and a $(2^k, 2^k, ..., 2^k, 1)$-dilatation, meaning that the dilatation acts trivially in the direction of the chosen axis. The same procedure works for any other coordinate axis. Then the product of all such dilatations is a homothety of factor a power of 2, hence the claim follows.

3.4. The proof of Proposition 3.3. A mappable cubulation is not embeddable for two reasons: either it is non-standard or else the map to the lattice is not injective. Both accidents can be resolved using np-bubble moves. The cubulation C is k-*standard* if its k-skeleton is standard, i.e. two cubes of dimensions at most k are either disjoint or else they have exactly one common face. Assume for the moment that the n-dimensional cubulation C is $(n-1)$-standard. Let $f : C \longrightarrow \mathbf{R}_c^N$ be a combinatorial map, locally an isometry on each cube. The singularities of the map f are therefore either *foldings* or *double points*. A *double point singularity* is when two disjoint cubes x and y have the same image $f(x) = f(y)$. The codimension of the cubes is called *the defect* of the double points. If the defect is positive then $x \subset u \cup v$, $y \subset u' \cup v'$, where u and v (respectively u' and v') are top dimensional cubes having a common face. A *folding* corresponds to a pair of cubes having a common face and the same image under f. In order to get rid of double points of positive defect one performs a b_1 followed by a b_2-move on u and v respectively, in an additional dimension. If the defect of the double point is zero then we can settle using a cubical Whitney trick, as follows. Consider C' be obtained by a b_1-move on y. Perform a b_1-move on $f(y)$ in an additional dimension. Then $D' = b_1(f(C))$ embeds into $\mathbf{R}_c^{N+1}$, since the interiors of the new cubes do not intersect the cubes of $f(C)$. There exists an extension $f' : C' \longrightarrow D'$ of $f\mid_{C-\{y\}}$ sending $b_1(y)$ onto $b_1(f(y))$, which is combinatorial and has less singularities than f. Iterate this procedure until an embeddable cubulation is obtained. In order to solve a folding of two cubes one uses a b_1-move over one folding cube, and on its image. The folding is replaced then by a double point singularity.

It remains to see how we can use np-bubble moves in order to assume the $(n-1)$-standardness. Using $(n-1)$-dimensional np-bubble moves $b_i(n-1)$ one can transform $ske^{n-1}(M)$ into a standard complex. It suffices to observe that the action of the n-dimensional np-bubble move $b_i(n)$ in one more dimension agrees with that of $b_i(n-1)$ on the $(n-1)$-skeleton. This completes the proof of the Theorem 1.5.

Corollary 3.6. *The simple cubulations of a manifold M satisfying* $\mathrm{H}^1(M, \mathbf{Z}/2\mathbf{Z}) = 0$ *are bubble equivalent.*

3.5. np-bubble equivalence and mappability. We want to find out whether a cubulation which is bubble equivalent to a mappable one is mappable itself. In general the answer is negative, and we have to restrict to np-bubble moves. The set of embeddable cubulations is not stable under np-bubbles either. In fact an embeddable cubulation may become non-standard, after performing some b_k^{-1} moves. We will show that this is the only accident which can happen. More precisely, we say that M and N are *standard np-bubble* equivalent if there exists a chain of np-bubble moves joining M and N among standard cubulations. Also the simplicity is preserved by all np-bubble moves but b_2, hence the mappability cannot be np-bubble invariant. Two cubulations are *simply np-bubble* equivalent if they are np-bubble equivalent among simple cubulations. Let M be a mappable cubulation and $X \subset M$ be the support of a b_2-move. The move is *rigid* with respect to M if there is a combinatorial map $f : M \longrightarrow \mathbf{R}_c^N$ for which $f(X)$ is the union of two orthogonal n-cubes. Otherwise, either $f(X)$ consists of a single cube (f is a *folding*) or $f(X)$ is the union of two cubes lying in the same n-plane (f is *flexible*). The following result is a coarse converse of the Theorem 1.5:

Proposition 3.7.

1. *Let M be mappable and N be a cubulation which is np-bubble equivalent to M using only rigid b_2-moves. Then N is mappable.*
2. *If M is embeddable then $b_2(M)$ is mappable.*
3. *If M is embeddable and N is standard np-bubble equivalent to M then N is embeddable.*
4. *The class of mappable cubulations is closed to simple np-bubble equivalences.*

Proof. Let $f : M \longrightarrow \mathbf{R}_c^N$ be a combinatorial map into the standard cubulation and $D_k^+ \subset M$ be the support of a b_k move (i.e. the union of k-cubes). Since f is non-degenerate on each cube either f is an embedding on D_k^+ or else f is a folding and $k = 2$. Also f is always an embedding on the support D_k^- of b_k^{-1}. If the move is rigid (for $k = 2$) or f is an embedding on the support then there exists a cube $C \subset \mathbf{R}_c^{N+1}$, such that $C \cap \mathbf{R}_c^N = f(D_k)$. In fact there is an unique embedding of D_k^+ ($k \neq 2$) and respectively D_k^- into $\mathbf{R}_c^{N+1}$, up to isometry. The map f extends then to a map $\tilde{f}$ over $b_k(M)$ using the cube C. This completes the proof of the first part of the Proposition.

Let us introduce some notations and definitions from [DSS], for the sake of completeness. Two edges e and e' of a cubic complex Q are said to be *equivalent* if there exists a sequence of edges joining them, in which any two successive edges e_i, e_{i+1} are opposite sides of some square in Q. An edge equivalence class is called *simple* if all the edges in it belonging to a single cube in Q are parallel. An equivalence class is called *orientable* if all the edges in it can be oriented such that whenever two equivalent edges are parallel to each other, their orientations are parallel. We consider a partition F of the edge equivalence classes into certain families of classes $F_1, ..., F_N$ such that each class has a fixed orientation and two perpendicular edges are members of equivalence classes that belong to different families. Let $\gamma = (e_1, e_2, ..., e_k)$ be an oriented edge path in the cubulation Q. Let $\mathrm{sgn}\,(e_i)$ be +1 if the direction of travel of γ on the edge e_i coincides with the orientation of e_i, and -1 otherwise. Consider then the following formal sum, taking values in the free $\mathbf{Z}$-module generated by the symbols F_j:

$$D_F(\gamma) = \sum_{i=1}^{k} \mathrm{sgn}\,(e_i) F(e_i) \in \mathbf{Z} < F_1, F_2, ..., F_N >$$

where $F(e)$ is the family to which the edge e belongs. We can state now (see [DSS], p. 305-306):

Proposition 3.8. 1. *A simple cubulation Q maps into the skeleton of $\mathbf{R}_c^N$, where N is the affine dimension of the image, if and only if there exist orientations of the equivalence classes and a partition F of these classes into N families such that for any closed edge path γ in Q, we have $D_F(\gamma) = 0$.*

2. *A simple and standard cubulation Q embeds into the skeleton of $\mathbf{R}_c^N$, where N is the affine dimension of the image, if and only if there exist orientations of the equivalence classes and a partition F of these classes into N families such that, for an edge path γ in Q, we have $D_F(\gamma) = 0$ if and only if the path γ is closed.*

We are now ready to prove the second part:

Proof. According to Proposition 3.8 the cubulation M (which is mappable, hence simple) admits a partition F of the equivalence classes of edges such that the development map D_F vanishes on all closed curves. The orientations of the edge equivalence classes of M naturally induce orientations for $b_2(M)$. Further any loop in $b_2(M)$ can be deformed to a loop in M, and hence $b_2(M)$ is mappable from the previous criterion, provided that $b_2(M)$ is simple. Simplicity is not preserved by b_2, in general, but we asked the cubulation to be an embeddable. If $b_2(M)$ is not simple, then the support of the b_2-move should consist of two *twin* cells. This means that we have two cells e and e' which have a common face f, and the two layers parallel to f containing e and e' coincide. But the layers of an embeddable cubulation cannot be self-tangent (see [DSS3]), hence $b_2(M)$ is simple. □

The third and fourth statements in Proposition 3.7 follow in the same way.

Remark that the collection of non-trivial homotopy classes of immersions $\varphi(K_i)$ (of the connected components K_i of the derivative complex K) is invariant to np-bubble moves. In particular $CBB(M)$ is infinite if the manifold M has non-trivial topology. If we consider two homothetic cubulations X and λX, then in general they cannot be np-bubble equivalent, because any non-trivial homotopy class appears λ times more in the latter. Thus the np-bubble equivalence may be interesting only for PL-spheres. We believe that any two simple cubulations of the sphere are np-bubble equivalent, as it happens for S^2.

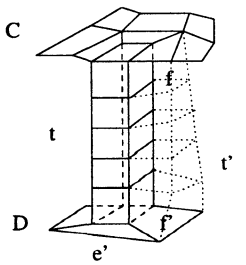


Fig. 3. The developing map.

4. Multiplicative structures

4.1. The composition of cubulations. Assume the manifolds we consider in the sequel are connected. We wish to prove that the connected sum of cubulations induces a monoid structure on $CB(S^n)$. We believe that $CBB(S^n)$ inherits also a monoid structure. Consider the cubulations $C \in C(M)$ and $D \in C(N)$ of the manifolds M and N, and choose two cells $e \subset C$, $e' \subset D$. A cell is always a top dimensional cube in this section. Let t be a length $l \geq 3$ cubical cylinder made up from $e \times [0, l]$ by removing the interiors of $e \times \{0\}$ and $e \times \{l\}$. We define therefore the map "connected sum of cubulations" $c_{e,e',t} : C(M) \times C(N) \longrightarrow C(M \sharp N)$, by

$$c_{e,e',t}(C, D) = C - \mathrm{int}\,(e)_{\partial e \cong \partial e \times \{0\} \subset \partial t} \cup_{\partial e' \cong \partial e' \times \{1\} \subset \partial t} D - \mathrm{int}\,(e').$$

with the obvious identifications of the boundaries.

We make some remarks before we proceed to prove the Theorem 1.6. In the course of the proof, we will freely use the fact that the tube t can be changed into another tube while it remains mappable. The gluing of the tube t requires some self-identification of the boundary ∂t. We fix an arbitrary identification $\partial e \times \{0\} \longrightarrow \partial t$. Then the other boundary of the tube can be glued to $D - \mathrm{int}\,(e')$ in 2^n ways corresponding to the elements of the symmetry group D_n of the n-cube. The relative twist $tw \in D_n$ measures the difference between two gluings on e'. Notice that there is no canonical choice in gluing e'. All tubes of length $l \geq 3$ are bubble equivalent rel boundary so that their length has not to be specified. Then the connected sum cubulation depends on the choice of e, e' and of the relative twist.

Proof. One can assume that C and D are standard cubulations. We want to define a *developing map* $S : \{\text{paths in } D\} \longrightarrow \{\text{cells in } C\}$. The map S depends on the particular data e, e', t, tw we choose. A *path* in D is a sequence of cells starting at e', consecutive cells having a common face. It suffices to define the map S for the trivial path, and then to use a recurrence on the length of the path. If the path is trivial, consisting of the cell e', we define $S(e') = e$. Further choose a cell f' having a common face u' with e'. Let u be the face of e which is opposite to u' using the tube t. Then the face f neighbouring e and intersecting it along u is by definition $S(\gamma)$, where γ is the path (e', f').

Let t' be a tube isometric to t which is glued on C and D along f and f' respectively, such that $\mathrm{cl}\,(t \cup t' - t \cap t')$ is a cylinder on the union of two neighbouring cells. Here $t \cap t' = e \cap e' \times [0, l]$ is the common face of the tubes. This condition determines uniquely the gluing twist tw' of t'. Let Y denotes the cubulation obtained by gluing $\mathrm{cl}\,(t \cup t' - t \cap t')$ to C and D, along $e \cup e'$ and $f \cup f'$ respectively. Then Y is obtained from $c_{e,e',t}(C, D)$ by a sequence of b_2-moves (along the tube t') and a final b_3-move. The same procedure transforms $c_{f,f',t'}(C, D)$ into Y, hence $c_{e,e',t}(C, D)$ and $c_{f,f',t'}(C, D)$ are equivalent. Using a recurrence one proves that $c_{e,e',t}(C, D)$ and $c_{S(\gamma),\gamma(m),t_\gamma}(C, D)$ are equivalent for any path γ, where t_γ is a tube isometric to t whose gluing twist is that induced by γ.

Set $O(e) = \{f \text{ such that } c_{e,e',t}(C, D) \text{ is equivalent to } c_{e,f',t'}(C, D) \text{ for a suitable twist}\}$. We have to prove that, possibly using bubble moves on the initial cubulations, we have $O(e) = C$. For any loop γ based at e one knows that $S(\gamma) \in O(e)$. Consider a loop γ of length m and assume that the last segment of the curve $\gamma\,|_{[m-m_0,m]}$, for $m > m_0 \geq 3$ is *straight*. A curve is *straight* if it consists of a sequence of cells e_i, each cell e_i having a common face f_i with the preceding cell e_{i-1}, so that the faces f_i and f_{i+1} are opposite faces in e_i. Then a straight curve $\gamma\,|_{[m-m_0,m]}$ defines a strip Σ consisting of the cells of the maximal straight extension of $\gamma\,|_{[m-m_0,m]}$. Set $f' = S(\gamma)$. The image of the strip Σ under the developing map is also a strip $\Sigma' \subset C$. When a basepoint cell e and a preferred direction are fixed, a discrete flow is defined on the strip Σ. The action of $k \in \mathbf{Z}$ on the cell v is the cell (k, v) which is k steps forward in the given direction, starting from v. Notice that there is also a flow defined on the other strip Σ'.

We claim that $(2\mathbf{Z}, f') \subset O(e)$ (after using bubble moves). Let $u = \gamma(r)$, $r \in [m - m_0, m]$, be a cell of $\Sigma \subset D$, located between e and $\gamma(m - m_0)$. One performs a b_1-move on u and set γ' for the natural extension of γ to $D' = b_1(D)$. We can express $S(\gamma')$ in terms of the flow on the strip Σ' as $(2, f')$ because γ' is also straight and its length was increased by 2. Thus $(2\mathbf{Z}_+, f') \subset O(e)$ holds, but the strip is finite hence the $\mathbf{Z}$-action has cyclic orbits, implying that $(2\mathbf{Z}, f') = (2\mathbf{Z}_+, f') \subset O(e)$. A *bicoloring* of C associates a color $c(e) \in \{0, 1\}$ to each cell e such that adjacent cells have different colors. For any two cells f and f' there exists a system of strips allowing to pass from

f to f' using only the action of $2\mathbf{Z}$ on these strips, except for the case when C admits bicolorings and $c(f) \neq c(f')$. However any bicolored cubulation can be transformed into one having no bicolorings using one b_1-move. This shows that $O(e) = C$.

Recall that given a path γ between e and f one associates a twist for the corresponding tube over f. Set tw_0 for the twist over f associated to a straight path $\gamma = (e, f, g)$. Let us perform a b_1-move over f. The curve γ has several lifts γ_j (not necessarily straight) relating e and g through the cells of $b_1(f) - f$. It is simple to check that the set of twists over f induced by these paths is all of $D_n tw_0$. Therefore the connected sum does not depend on the choice of the twist. This proves the Theorem 1.6. □

4.2. The compatibility with the bordism composition The composition law for immersions is the disjoint union of immersions inside the connected sum of manifolds, where the latter is made away from the immersions. The immersion associated to the connected sum of cubulations is obtained from the initial immersions by some surgery which involves only local data. We will prove that this surgery can be also realized by a local relative cobordism of the associated immersions. Consider the manifolds M and N with their respective cubulations X and Y. The connected sum cubulation $X\sharp Y$ is obtained by removing the cells e from X and f from Y. One uses b_1-moves on e and f in order to reduce the cubulated piping tube to a boundary identification $\partial e = \partial f$. The bordism classes of φ_X and φ_Y are preserved under these transformations. We consider that the connected sum of manifolds is done by means of a piping tube which is close to the images $\mathrm{Im}\,(\varphi_e)$ and $\mathrm{Im}\,(\varphi_f)$. Both $\mathrm{Im}\,(\varphi_e)$ and $\mathrm{Im}\,(\varphi_f)$ have as local models the set of coordinate hyperplanes around the origin in $\mathbf{R}^n$. The surgery which changes $I(X\sharp Y)$ into $I(X)\sharp I(Y)$ excises $\mathrm{Im}\,(\varphi_e)$ and $\mathrm{Im}\,(\varphi_f)$ and replaces them by a cylinder with the same boundary. One can realize the surgery on the piping tube, after pushing the local models through it. A small isotopy moves them outside of a longitude and then the configurations embed in the ball obtained from the tube by cutting it along the longitude. It suffices therefore to show that the surgery can be realized by a cobordism in the ball which is a product on the boundary. For $n = 2$ the different slices of the cobordism are given in the Figure 4. Assume that the local models living in a ball have the corresponding hyperplane sections parallel to each other. Let us consider two parallel hyperplane sections u and u' of the respective local models $\mathrm{Im}\,(\varphi_e)$ and $\mathrm{Im}\,(\varphi_f)$. Then one constructs the neighborhood of $\varphi_{X\sharp Y}$ around the piping tube by removing the interiors of u and u' and gluing back the cylinder of boundary $\partial u \cup \partial u'$. This transformation can be realized also by a cobordism of immersions since it is represented by the local picture around a critical point of index 1, where the images of the immersions are the non-critical levels before and respectively after passing through the critical value. The local picture can be made transversal with respect to the other coordinate hyperplanes of the immersions which were left untouched. One composes the cobordisms associated to all n pairs of parallel hyperplane sections. The restrictions on the boundary of these cobordism are products and so we can glue the composition with the trivial cobordism outside the local pictures. Thus we derived a cobordism between $\varphi_{X\sharp Y}$ and $\varphi_X\sharp\varphi_Y$.

Notice that, in general, the map $CBB(M)\times CBB(N) \longrightarrow CBB(M\sharp N)$ should depend on the length of the tube t. The case when M and N are spheres could be different however.

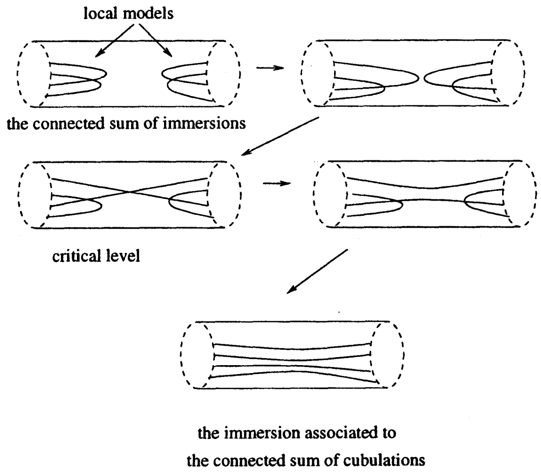


Fig. 4. A relative cobordism for $n = 2$.

5. Cubulations of surfaces

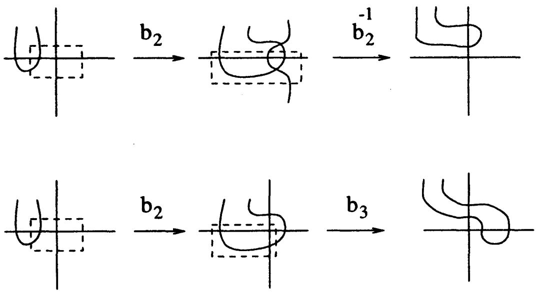
5.1. The simple cubulations of S^2**.** Observe first that the set $CBB(S^2)$ is infinite. In fact the number of components with odd self-intersections of the associated immersion is an invariant, because the only move creating new components is b_1, and each new created component is an embedded circle. Set $\mathrm{ns}\,(C)$ for the sum over the various connected components K_i of the number of self-intersections.

The similarity with the Reidemeister moves in the plane suggests that $CBB(S^2)$ is the set of framed circles in the plane. Unlike the case of Reidemeister moves, the image of the immersion remains connected, so its components cannot be separated using bubble moves. On the other hand the move b_2 can create/annihilate a pair of self-intersections. It is not clear that the singularities can always be paired such that suitable np-bubble moves destroy all pairs of singularities and so each transformed circle has $\mathrm{ns}\,(K_i)\,(\mathrm{mod}\ 2) \in \{0,1\}$ singularities. If this is true it will remain to prove that all configurations of circles among which there are exactly m singular circles are np-bubble equivalent. This would establish an isomorphism between $CBB(S^2)$ and $\mathbf{Z}_+$. We are able to prove this statement for the case $m = 0$ (and $m = 1$):

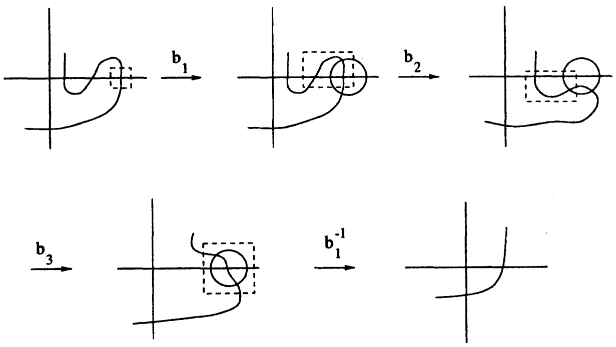
Proposition 5.1. *The simple cubulations of* S^2 *are np-bubble equivalent.*

Proof. Call a disk bounded by two disjoint arcs a *biangle*. A biangle is *tight* if no other arc intersects its interior. Observe that using np-bubble moves one can transform all minimal biangles into tight biangles. We use isotopies of the boundary of the biangle (b_2-moves) reducing the number of squares contained in the disk and b_3-moves.

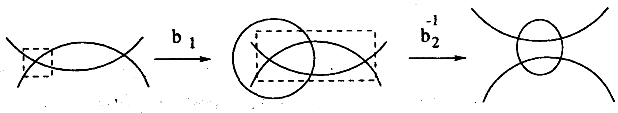
The possible configurations of tight biangles can be rather complicated, even if the cubulation is simple. Let us show that the tight triangles can travel along the edges (sliding). This means that a regular homotopy moving a tight biangle along the arcs can be realized by np-bubble moves. The proof of this claim is contained in the picture below. We marked in a little rectangle the area on which the bubble move acts:



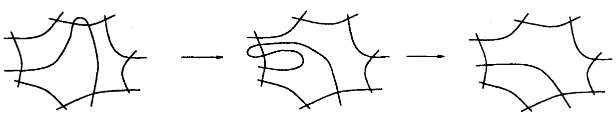
Set X for the union of two transversal arcs. An arc which intersects three times one branch of X and then the other arc can be simplified by a composition of bubble moves which we call a S move:



Now it is easy to obtain that the set of semi-simple cubulations mod np-bubble moves is equal to the set of simple cubulations mod np-bubble moves. In fact any tight biangle coming from the same component can be transformed into tight biangles on different components:



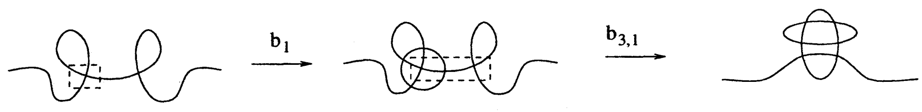
Assume that the cubulation is simple. The union of the circles K_i for $i \geq 2$, divides the sphere into polygonal faces, each of them having at least two vertices. We claim that, either there are no biangles involving an arc from K_1 or else K_1 is a small circle contained in the union of two faces which intersects minimally (i.e. twice) the common edge. Indeed, suppose that there exists a minimal tight biangle. Consider the face which is adjacent to the biangle and not containing it. If K_1 does not satisfy the claim then we can simplify the biangle using slidings and S-moves, as it is shown below:



Thus K_1 can be transformed into a circle which does not support any biangle. Further use of the moves b_2 and b_3 allows us to isotope K_1 into the union of two faces as claimed. One continues the simplification procedure with the other components K_i and the Proposition 5.1 follows. □

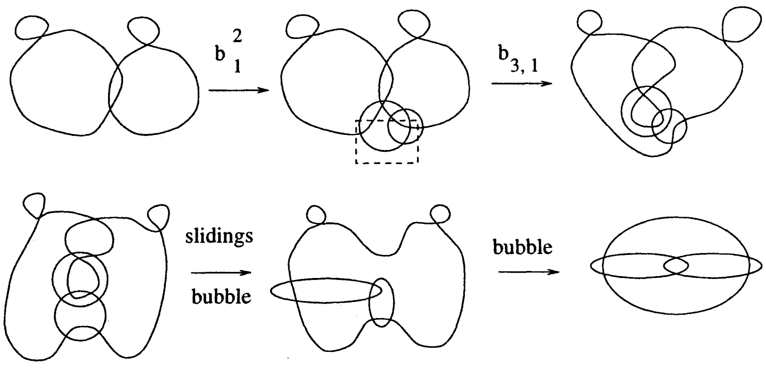
A similar proof works when the circle K_1 is allowed to have one self-crossing. Now K_1 is transformed into a figure-eight contained in the union of two faces which intersects minimally their common edge and forms one biangle.

5.2. The cubulation group $CB(\mathrm{S}^2)$. In order to prove that $CB(\mathrm{S}^2) = \mathbf{Z}/2\mathbf{Z}$ one has to get rid of those self-intersections which are not cancelling pairs. The following figures describe the simplifications obtained with the additional move $b_{3,1}$ for two adjacent self-intersections:



Further, if the two self-intersections are separated by additional arcs then use b_2-moves and slide across these arcs. It follows that any dual graph can be transformed using bubble moves into one satisfying $\mathrm{ns}\,(K_i) \in \{0,1\}$, for each component K_i.

The next step is to show that two components "can be added". Let K_1 and K_2 be two components having non-void intersection. There exists an equivalent configuration in which K_1 and K_2 are replaced by the circle K_1+K_2 verifying $\mathrm{ns}\,(K_1+K_2) = \mathrm{ns}\,(K_1)+\mathrm{ns}\,(K_2)$, the other K_i's ($i \geq 3$) are left unchanged and the additional components have $ns = 0$. It suffices to do that in the case when $\mathrm{ns}\,(K_1) = \mathrm{ns}\,(K_2) = 1$. Then the kinks can be added and transformed in consecutive self-intersections which have already been solved.



Thus, if one adds those components which are not embedded, we obtain a planar graph whose components are embedded, except possibly for one which has $\mathrm{ns}\,(C) \in \{0,1\}$. We have therefore $\mathrm{ns}\,(C) = f_0 \,(\mathrm{mod}\ 2)$. If $f_0 = 0\,(\mathrm{mod}\ 2)$ then the result of the previous section shows that all these configurations are np-bubble equivalent. If $f_0 = 1\,(\mathrm{mod}\ 2)$ it means that C is a figure-eight in the plane. The remark after the proof of Proposition 5.1 completes the proof of the Theorem 1.3.

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