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# TQFT AND WHITEHEAD'S MANIFOLD

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ABSTRACT
The aim of this note is to give two applications of Topological Quantum Field Theories to the topology of open manifolds. The invariants we derived may be used to test if an open manifold is simply connected at infinity (as we did for Whitehead's manifold in case of the $sl_2(\mathbf{C})$-TQFT in level 3) or to construct examples of non-homeomorphic contractible open 3-manifolds.

*Keywords*: TQFT, Whitehead manifold, invariants at infinity.

## 1. Introduction

The aim of this note is to give two examples showing how Topological Quantum Field Theories (abbrev. TQFT) could be used to give informations about the topology of open 3-manifolds. Specifically we derive – in a straightforward way – some invariants for open manifolds starting from topological invariants of compact manifolds. This kind of invariants, when computed for contractible manifolds which are simply connected at infinity, are trivial. The main question which we address here is *to what extent these invariants are really able to detect wild ends*? Our examples suggest a positive answer. We compute one such invariant at infinity (associated to the $sl_2(\mathbf{C})$-TQFT in level 3) for the classical Whitehead manifold and find it is not trivial, hence in this case the TQFT test for the simple connectedness at infinity is effective. The other example is an uncountable family of open contractible manifolds which are pairwise non-homeomorphic. Examples of this kind have already been given by Myers and Brown [9, 4]. Further developments accreditating the idea that TQFT may share some light on the topology of open 3-manifold are pursued in a further paper.

The plan of this note is the following: in the first section we discuss the general TQFT invariants at infinity $Z_\infty(W)$ for an open 3-manifold $W$. In order to preserve the self-contained character of this paper we outline the definition of link and 3-manifold invariants of Witten and Reshetikhin–Turaev [14, 10] based on the quantum $sl_2(\mathbf{C})$ at roots of unity. According to a general result each multiplicative invariant for closed 3-manifolds extends canonically to a TQFT ([6]). We follow the surgical approach used by Turaev in [12] to get the explicit description of the TQFT for cobordisms. In the last section we use these results to compute effectively the TQFT invariant at infinity for the Whitehead manifold.

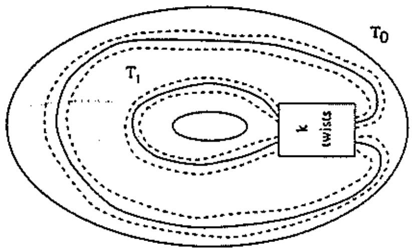


Fig. 1. The inclusion of tori.

Recall ([13]) this is defined as follows: Let $T_1 \hookrightarrow T_0$ be the embedding of the solid tori from Fig. 1 with $k = 1$. Here a twist means a full right-hand twist so that $T_1$ be homotopically trivial embedded. There exists a homeomorphism $h$ of $S^3$ so that $h(T_1) = T_0$. Consider the open manifold $Wh = \bigcup_{n\geq 0} h^n(T_0)$. Then $Wh$ is the typical example of a contractible open 3-manifold which is not homeomorphic to $\mathbf{R}^3$. The precise reason is that $Wh$ is not simply connected at infinity (i.e. not every compact may be engulfed in a compact simply-connected submanifold). We shall consider also the twisted Whitehead manifolds $Wh(k)$ obtained from the inclusion of tori from Fig. 1 for general $k$.

A refined version of this construction would be the alternating order of the cobordisms with different $k$. Assume that we consider only $k \in \{1, 2\}$. We have an open manifold $Wh(\mathrm{r})$ associated to each sequence r consisting in 1 or 2. For each $k$ we have a homeomorphism $h_k$ defined like $h$. We set therefore

$$Wh(\mathrm{r}) = \bigcup_n h_{r_n} \circ h_{r_{n-1}} \circ \ldots h_{r_1}(T_0).$$

All these manifolds are contractible. We know that [4] some contractible open manifolds (of genus 1 at infinity, and which embeds into compacts 3-manifolds) have a sort of uniqueness of prime decompositions into unknotted pairs. The manifolds we considered have such a property and therefore one might associate a topological invariant: it is the cofinal class of the sequence of linear maps associated to the prime cobordisms by an arbitrary TQFT (see for details the further section). Now our first result may be stated as follows:

**Theorem 1.** *The manifolds $Wh(\mathrm{r})$ and $Wh(\mathrm{r}')$ have distinct (cofinal) invariants obtained from the $sl_2(\mathbf{C})$-TQFT in level 2, unless the sequences r and r′ are cofinally equal. In particular they are pairwise non-homeomorphic.*

Now to each TQFT functor $Z$ we associate a map $Z_\infty$ from the category of open manifolds to that of vector spaces which, roughly speaking, would correspond to the space assigned by $Z$ to the ends of the manifold (see Definition 2.1). The second application is:

**Theorem 2.** *For $k \neq -1(mod\ 5)$, $k > 0$ the invariants at infinity of $Wh(k)$ associated to $sl_2(\mathbf{C})$-TQFT in level 3 are:*

$$(Z_{sl_2(\mathbf{C}),3})_\infty(Wh(k)) \cong \mathbf{C}^2 .$$

*In particular these open contractible manifolds are not simply-connected at infinity.*

In fact any open 3-manifold $W$ which is simply-connected at infinity must satisfy $\dim Z_\infty(W) \leq 1$ for all reduced TQFT (see Proposition 2.2).

## 2. Invariants at Infinity from TQFT

### 2.1. *On TQFT*

Recall [2] that a TQFT in dimension 3 is a functor $Z$ from the category of oriented cobordisms into that of hermitian vector spaces. This means that to a compact surface $S$ we associate an hermitian vector space $Z(S)$ depending only on the topological type of $S$. The *quantum* character of the theory is reflected in the rules

$$Z(\cup_i S_i) = \bigotimes_i Z(S_i),\ \ Z(\emptyset) = \mathbf{C},$$

which make the difference with the usual functors encountered in the algebraic topology.

Furthermore to an oriented cobordism $M$ so that $\partial M$ is split into two disjoint manifold $S$ and $T$ (the *incoming* and the *outgoing* boundaries respectively, which are not necessary those given by the orientation) we have assigned a morphism $Z(M) : Z(S) \longrightarrow Z(T)$ satisfying the natural compatibility relations between composition of morphisms and cobordisms. This is usually called an anomaly-free TQFT (see [12]). The main examples yet constructed have an anomaly from a certain group of roots of unity $\Gamma \subset U(1)$. This means that the invariant associated to the composition of cobordisms $M$ and $N$ may be expressed as

$$Z(M \circ N) = \gamma Z(M) \circ Z(N),\ \text{with}\ \gamma \in \Gamma .$$

The usual way to deal with this ambiguity is to work with framed 3-manifolds [1] or $p_1$-structures [3]. However the presence of an anomaly will be irrelevant for the construction of invariants at infinity.

The examples we consider in this paper are *reduced* TQFT, namely they satisfy the additional condition

$$Z(S^2) \cong \mathbf{C} .$$

All the TQFT from quantum groups or quasi-quantum groups are reduced and we may restrict ourselves to the study of reduced TQFT by the results of [6].

### 2.2. *Open 3-manifolds*

Consider first $Z$ is an anomaly-free TQFT. Let $W$ be an open 3-manifold without boundary. We choose an ascending sequence of submanifolds $\{K_n\}$ fulfilling

$$K_n \subset \mathrm{int}(K_{n+1}),\ W = \cup_n K_n .$$

Then $V_i = \mathrm{cl}(K_{i+1} - K_i)$ are oriented cobordism from $\partial K_i$ to $\partial K_{i+1}$. Here int and cl state for the interior and the closure respectively. We get a sequence of linear maps

$$Z(V_i) : Z(\partial K_i) \longrightarrow Z(\partial K_{i+1}),$$

which represent an inductive system of vector spaces. We define $Z_\infty(W)$ be simply the inductive limit of this system.

**Definition-Lemma 2.1.** *The vector space $Z_\infty(W)$ is the topological invariant at infinity associated to the TQFT functor $Z$ and the open 3-manifold $W$.*

In fact it is simply to check the independence of $Z_\infty(W)$ on the choice of the exhaustion or the parametrizations of the intermediary boundaries $\partial K_i$. □

Remark that $Z_\infty(W)$ depends only on the structure at infinity of $W$: if $W'$ is another manifold so that $W$ and $W'$ are homeomorphic outside some compacts then the associated spaces $Z_\infty(W)$ and $Z_\infty(W')$ are isomorphic.

Also if $Z$ is a TQFT with anomaly this time, then the maps $Z(V_i)$ are defined up to the multiplication by some scalar from $\Gamma$. Nevertheless the space $Z_\infty(W)$ is well determined. It is only the hermitian structure which is lost when we pass to the limit $Z_\infty(W)$.

### 2.3. *h-1-connected manifolds*

The open 3-manifold $W$ is *h-1-connected at infinity* if each compact $K \subset W$ may be engulfed in a compact submanifold $Y \subset W$ with $H_1(Y) = 0$. This is a condition slightly weaker than the simple connectedness at infinity.

**Proposition 2.2.** *If $W$ is h-1-connected at infinity then* $\dim Z_\infty(W) \leq 1$ *for any reduced TQFT.*

**Proof.** It suffices to observe that a compact 3-manifold $Y$ with $H_1(Y) = 0$ has the boundary $\partial Y$ a union of spheres $S^2$, from an Euler characteristic argument. If $\{K_n\}$ is an exhaustion of $W$ like in the introduction then there exists compact submanifold $Y_n$ with $H_1(Y_n) = 0$ and a function $r(n) > n$ so that

$$K_n \subset \mathrm{int}(Y_n) \quad \text{and} \quad Y_n \subset \mathrm{int}(K_{r(n)}) .$$

Then the map $Z(\mathrm{cl}(K_{r(n)} - K_n))$ factors through $Z(\partial Y_n) \cong \mathbf{C}$ (because the TQFT is reduced) hence the rank of the limit is at most 1. □

### 2.4. *The Whitehead manifold*

Consider now $Z$ be the level 3 $sl_2(\mathbf{C})$-TQFT of Witten and Reshetikhin–Turaev (see the next section for complete definitions). Theorem 2 stated in introduction asserts that $Z_\infty(Wh(k)) \cong \mathbf{C}^2$.

We defer for the proof in Sec. 5. We already notice that $Wh(k)$ has periodic ends and we may choose $K_n = \bigcup_{0\leq j\leq n} h^j(T_0)$. Then all intermediary cobordisms $V_n$ are homeomorphic to $X_k = \mathrm{cl}(T_0 - T_1)$. It suffices therefore to compute the linear map $Z(X_k) : Z(\partial T_1) \longrightarrow Z(\partial T_0)$. For the TQFT we are working with, the space associated to a torus $Z(S^1 \times S^1)$ is $\mathbf{C}^4$ (see the further section). So the statement of the theorem is equivalent to the fact that the $\min_n \dim\ker Z(X_k)^n = 2$.

### 2.5. *Prime unknotted decompositions and cofinal invariants*

We review first some topological aspects of Whitehead type manifolds after Brown [4]. Several definitions and notations are needed. Firstly $(A, B)$ is an *unknotted pair* if $B \subset A$ are unknotted solid tori and there exists an embedding of $A$ in $S^3$ so that both $A$ and $B$ are unknotted in the ambient space. The unwrapping number $n(A, B)$ is the minimal number of points the core of $B$ meets an meridian disk of $A$. An unknotted pair is *trivial* provided that $n(A, B) < 2$: so $B$ is contained in a ball in $A$ (if $n(A, B) = 0$) or the boundary tori of $A$ and $B$ are parallel (if $n(A, B) = 1$). Notice that

$$n(A, C) = n(A, B)n(B, C)$$

if we have $C \subset B \subset A$. A similar product formula holds for the *winding* number $w(A, B)$ defined as the class of the core of $B$ in $\pi_1(A) \cong \mathbf{Z}$.

Now a (non-trivial) unknotted pair $(A, B)$ *factors* if there is some solid torus $C$, $B \subset C \subset A$ so that both of $(A, C)$ and $(C, B)$ are non-trivial unknotted pairs. Otherwise the pair $(A, B)$ is *prime*. Since $n$ takes only integer values and is multiplicative it follows that any non-trivial pair has a finite prime factorization. Fortunately in the context of unknotted pairs we have also the uniqueness of the prime factorization (due to Brown): *if $B \subset E_1 \subset E_2 \subset \ldots \subset E_k \subset A$, and $B \subset E'_1 \subset E'_2 \subset \ldots \subset E'_m \subset A$, are two prime factorizations of the unknotted non-trivial pair $(A, B)$ then $k = m$ and there exists an isotopy of $A$ rel $B$ carrying each $E_j$ into $E'_j$.*

Consider now an irreducible contractible open 3-manifold $W$ of genus 1 at infinity. It is an ascending union of solid tori $T_n$ which sit homotopically trivial each one in the forthcoming. If additionally $W$ embeds in some compact 3-manifold then Brown, using results of Haken, proved that one may choose the exhaustion so that $(T_{n+1}, T_n)$ are unknotted pairs. If all pairs $(T_{n+1}, T_n)$ are prime the sequence is called a *prime decomposition* of $W$. Now an easy extension of the previous uniqueness result is: *for two prime decompositions $\{T_n\}$ and $\{T'_n\}$ of an open manifold as before, there is a homeomorphism $h$ and integers $k, m$ so that $h(T_{n+k}) = T'_{n+m}$ for all positive integers $n$.* The integers $k, m$ cannot be eliminated since a prime decomposition can be extended by adding arbitrary terms at the beginning. An immediate application of this description is:

**Corollary 2.3.** *For any TQFT $Z$, and prime decomposition $\{T_n\}$ of the irreducible open contractible 3-manifold $W$ embedding in a compact 3-manifold the cofinal class of the sequence of linear maps*

$$Z_{cf}(W) = ...Z(\partial T_n) \stackrel{Z(\mathrm{cl}(T_{n+1}-T_n))}{\longrightarrow} Z(\partial T_{n+1}) \stackrel{Z(\mathrm{cl}(T_{n+2}-T_{n+1}))}{\longrightarrow} Z(\partial T_{n+2})...$$

*is a topological invariant of $W$, which we call the cofinal invariant induced by $Z$.*

Notice that choosing specific basis for all $Z(\partial T_n)$ we need to take care that the matrices $Z(\mathrm{cl}(T_{n+1}-T_n))$ give not raise to an invariant cofinal sequence of matrices: we need to take into account the various changes of basis. As for example we can choose Jordan normal forms for the matrices, etc.

Remark now that the manifolds $Wh(\mathbf{r})$ we wish to test fit into this description, because the pairs $(T_0, T_1)$ are unknotted and prime. Therefore it suffice to compute the cofinal sequence associated to a particular TQFT. We do that for the $sl_2(\mathbf{C})$-TQFT in level 2 in Sec. 4.

## 3. The $sl_2(\mathbf{C})$-TQFT

We fix some integer $r > 2$; usually $r-2$ is called the level of the $sl_2(\mathbf{C})$ theory. The description we outline follows from [10, 7, 12].

### 3.1. *Framed tangles*

Recall that a tangle $T$ is a 1-manifold properly embedded in the unit cube $I^3$ in $\mathbf{R}^3$ with $\partial T \subset \{\frac{1}{2}\} \times I \times \partial I$, considered up to isotopy rel boundary. If $\partial_- T = T \cap I^2 \times \{0\}$ and $\partial_+ T = T \cap I^2 \times \{1\}$ then $T$ is a $(m, n)$-*tangle* provided that $\mid \partial_- T \mid = m$ and $\mid \partial_+ T \mid = n$. Thus a link is a (0,0)-tangle and a general tangle consists of a link with a collection of proper arcs. We assume the tangles are oriented and transverse to $I^2 \times \partial I$.

A *framed tangle* is a tangle equipped with a framing of its normal bundle (up to isotopy) which is standard on the boundary. It is equivalent to a ribbon tangle from [10] if we think the tangle is thickened to a ribbon in the direction of the second vector of the framing. Anyway framings may be specified by integers assigned to the components of $T$.

Also one studies tangles using generic projections, called *diagrams*, onto $\{0\} \times I^2$ having only ordinary double points. We assume the framing considered is the blackboard one, in which the second vector is parallel to $\{0\} \times I^2$, and further coincides with the 0-framing by eventually adding the necessary number of kinks.

It is simply to check that every tangle diagram may be factored into the elementary tangles from Fig. 2. There are well-known Reidemester moves describing the local moves necessary and sufficient to obtain two framed tangle diagrams one from the other if they are coming from the same framed tangle. For the sake

I (density) R (right hand twist) L (left hand twist) A (annihilation) C (creation)

Fig. 2. Elementary tangles.

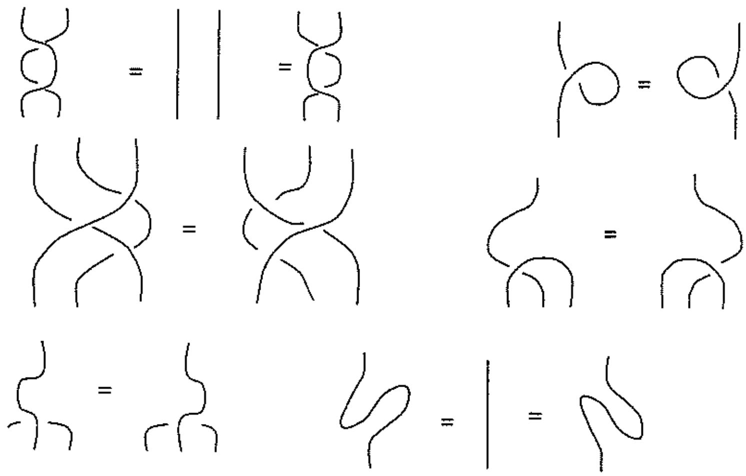

Fig. 3. Reidemester moves for tangles.

of completeness we pictured them in Fig. 3 (see [10]). Notice the orientations are arbitrary and the framing is the blackboard one.

### 3.2. *Quasi-triangular Hopf algebras*

We discuss the quantum $sl_2(\mathbf{C})$ which is the main example of a ribbon Hopf algebra. Recall $sl_2(\mathbf{C})$ is 3-dimensional as vector space and the Lie bracket is given (in terms of preferred generators) by:

$$[H,X]=2X,\ [H,Y]=-2Y,\ [X,Y]=H\,.$$

The *universal enveloping algebra* $U = U(sl_2(\mathbf{C}))$ is the associative algebra over $\mathbf{C}$ generated by $X,Y,H$ and the relations from above. Notice that there exists an unique $k$-dimensional irreducible $sl_2(\mathbf{C})$-module $V^k$, for each integer $k$, which has also an $U$-module structure. Also $U$ is a Hopf algebra when endowed with the comultiplication $\Delta : U \longrightarrow U \otimes U$ given by $\Delta(u) = u \otimes 1 + 1 \otimes u$, antipode $s : U \longrightarrow U$ given by $s(u) = -u$, and counit $\varepsilon : U \longrightarrow \mathbf{C}$ determined by $\varepsilon(u) = 0$, for all $u$ which are Lie polynomials in $X,Y,H$.

Now the *quantized universal enveloping algebra* $U_h = U_h(sl_2\mathbf{C}))$ is defined as $U[[h]]$ (the formal series in $h$) with the same relations as $U$ excepting for $[X,Y] = H$

which is replaced by

$$[X,Y]=[H]=\frac{e^{\frac{h}{2}H}-e^{-\frac{h}{2}H}}{e^{\frac{h}{2}}-e^{-\frac{h}{2}}} \tag{1}$$

If $K=e^{\frac{h}{4}H}$ we have the relations

$$\begin{aligned} KX &= e^{\frac{h}{2}}XK \\ KY &= e^{-\frac{h}{2}}YK \\ [X,Y] &= \frac{K^2-K^{-2}}{e^{\frac{h}{2}}-e^{-\frac{h}{2}}}. \end{aligned} \tag{2}$$

Notice that there is a Hopf algebra structure on $U_h$ as a module over $\mathbf{C}[[h]]$. Following [10] we consider $A$ be the quotient of $U_h$ obtained by setting

$$h=\frac{2\pi\sqrt{-1}}{r},\ X^r=Y^r=0,\ K^{4r}=1.$$

Then $A$ is a finite dimensional algebra over the complex numbers with generators $X,Y,K,K^{-1}$ and the relations stated above. As in the case of $U$ there are unique $A$-modules $V^k$ in each dimension $k$ but $V^k$ is irreducible only if $k\leq r$. Also $A$ acquires a Hopf algebra structure from $U_h$ and so tensor products and duals of $A$-modules are still $A$-modules. Moreover the following Clebsch–Gordon rules remain valid

$$V^k\otimes V^l=\bigoplus_{p=|l-k|+1;p+k+l=\mathrm{odd}} V^p,\ \text{if } k+l\leq r+1 \tag{3}$$

This $A$ is a *quasi-triangular* Hopf algebra (see [5]): there exists an invertible element $R\in A\otimes A$ satisfying

$$\begin{aligned} R\Delta(u)R^{-1} &= \check{\Delta}(u),\ u\in A, \\ (\Delta\otimes 1)(R) &= R_{13}R_{23}, \\ (1\otimes\Delta)(R) &= R_{13}R_{12}, \end{aligned}$$

where $\check{\Delta}=P\Delta$, $P$ is the permutation endomorphism of $A\otimes A$, $P(u\otimes v)=v\otimes u$, $R_{12}=R\otimes 1$, $R_{23}=1\otimes R$, $R_{13}=(P\otimes 1)R_{23}$.

Specifically $R$ may be given by

$$R=\frac{1}{4r}\sum_{0\leq n,a,b}^{n<r;a,b<4r}\frac{e^{\frac{h}{2}}-e^{-\frac{h}{2}}}{[n!]}t^{ab+(b-a+1)n}\left(X^nK^a+Y^nK^b\right) \tag{4}$$

where $t=e^{-\frac{2\pi\sqrt{-1}}{4r}}$.

Remark that $R$ may be viewed as acting on tensor products of two $A$-modules $V\otimes W$. We set $\check{R}:V\otimes W\longrightarrow W\otimes V$ be the *flip R-matrix* $\check{R}=P\circ R$.

### 3.3. *Colored framed tangle operators*

Assume the quasi-triangular Hopf algebra $A$ is fixed. A coloring of a tangle $T$ is the assignment of an $A$-module to each of its components. This way a coloring of $\partial T$ is induced: if $s$ is an arc colored by $V$ then assign $V$ to the endpoint of $s$ where is oriented down, and the module $V^*$ to the other one. Tensoring from left to the right the modules associated to the bottom (or upper) endpoints we get the boundary $A$-modules assigned to $\partial_- T$ and $\partial_+ T$, which we denote $T_-$ and $T_+$ respectively. By convention the empty product is $\mathbf{C}$.

We have two composition laws on tangles $\circ$ and $\otimes$ illustrated in Fig. 4.

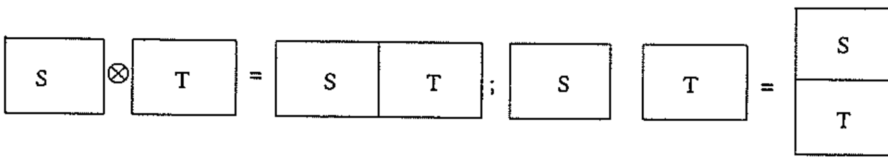


Fig. 4. Composition laws for tangles.

**Theorem 3.1.** ([10, 7]) *There exist uniquely $A$-linear operators $J_T : T_- \longrightarrow T_+$ assigned to each colored tangle which satisfy*

$$J_{S \circ T} = J_S \circ J_T \,,$$

$$J_{S \otimes T} = J_S \otimes J_T \,,$$

*and for elementary tangles are defined by*

$$J_I = 1, \;\; J_R = \check{R}, \;\; J_L = \check{R}^{-1}, \;\; J_{CR} = E, \;\; J_{CL} = \check{E}, \;\; JAR = N, \;\; J_{AL} = \check{N} \,,$$

*where $R$ is the right-hand twist (the orientation points down) tangle, $L$ is the left-hand twist, $CR$ (respectively $AR$) is the creation (annihilation) tangle with the sense of the orientation from left to the right, $CL$ (respectively $AL$) have opposite orientation than $CR$ and $AR$ respectively,*

$$E(f \otimes x) = f(x), \;\; \check{E}(x \otimes f) = f(K^2 x) \,,$$

$$N(1) = \sum_i e_i \otimes e^i, \;\; \{e_i\} \textit{ is an arbitrary basis and } \{e^i\} \textit{ its dual} \,,$$

$$\check{N}(1) = \sum_i e^i \otimes K^{-2} e_i \,.$$

Notice that $J_K$ is just a scalar if $K$ is a colored link.

Now we restrict ourselves to colorings by irreducible $A$-modules so the set of colors correspond to $\{1, 2, ..., r\}$. We denote by $\mathbf{k}$ the coloring of $T$ where the $j^{th}$ component is colored with the module of dimension $k_j$. Thus the theorem yields a family of topological invariants $J_{T,\mathbf{k}}$ for colored tangles.

Remark that $J_{T,\mathbf{k}}$ are independent on the various orientations of closed components of $T$. Also from ([7] p. 506) if a color in the vector k is $r$ then the invariant $J_{T,\mathbf{k}} = 0$. So we may assume the colors are from the subset $\{1, 2, ..., r-1\}$.

### 3.4. *Closed 3-manifold invariants*

Let $L$ be a framed link in $S^3$. Recall $L$ determines a 4-manifold $W_L$ obtained by adding 2-handles to the 4-ball $B^4$ along the components of $L$ in $S^3 = \partial B^4$. The manifold $D(L) = \partial W_L$ oriented "outward first" is the result of Dehn surgery on $L$, and any 3-manifold may be obtained this way. We can pass from one surgery link $L$ for $M = D(L)$ to another link $L'$ with $D(L') = M$ by a finite sequence of Kirby moves (blow-ups and handle slidings) or equivalently $m$-strands K-moves (see [7]).

Define for a framed link $L$

$$Z_L = \alpha_L \sum_{\text{coloring } \mathbf{k}} [\dot{k}] J_{L,\mathbf{k}}, \tag{5}$$

where

$$\alpha_L = b^{n_l} c^{\sigma(L)},$$

$$[\mathbf{k}] = \prod [k_i],$$

$$[k] = \frac{e^{\frac{kh}{2}} - e^{-\frac{kh}{2}}}{e^{\frac{h}{2}} - e^{-\frac{h}{2}}} = \frac{\sin \frac{\pi k}{r}}{\sin \frac{\pi}{r}},$$

$$b = \sqrt{\frac{2}{r}} \sin \frac{\pi}{r},$$

$$c = e^{-\frac{6\pi\sqrt{-1}(r-2)}{8r}},$$

$n_L$ is the number of components of $L$, $\sigma_L$ is the signature of the linking matrix of $L$.

The invariance of $Z_L$ to Kirby moves is proved in [10]. Now if $M$ is obtained by Dehn surgery on the framed link $L$ we set $Z_r(M) = Z_L \in \mathbf{C}$ which is a topological invariant for 3-manifolds.

### 3.5. *Cobordisms and TQFT*

For simplicity we restrict ourselves to cobordisms $M$ with connected boundaries $\partial_- M$ (incoming) and $\partial_+ M$ (outgoing).

We define first the spaces associated to closed oriented surfaces. Consider $(\Gamma, \mathbf{i})$ be one of the colored 3-valent graph of genus $g$ from Fig. 5, viewed as a (0,2g)-tangle in $\mathbf{R}^3$.

We associate to this colored graph the space

$$Z(\Gamma_g, \mathbf{i}) = \bigotimes_{l=1}^{g} (V^{i_l} \otimes V^{i_l *}).$$

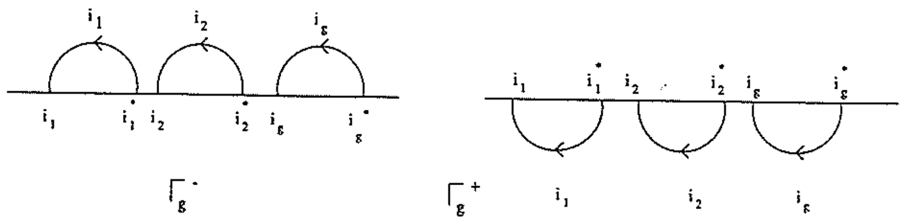


Fig. 5. The standard spine for $\Sigma_g$.

Then to the closed oriented surface of genus $g$ we assign the space

$$Z(\Sigma_g) = \bigoplus_{\text{coloring k}} Z(\Gamma_g, \mathrm{k}) .$$

We may extend now the Dehn surgery construction to cobordisms using 3-valent graphs. We call $\Gamma$ a special framed graph if it satisfies the conditions:

(i) $\Gamma \cap I^2 \times [0, \frac{1}{10}]$ is the base of a standard 3-valent graph $\Gamma^+_{g+}$. This means that $\Gamma \cap I^2 \times [0, \frac{1}{10}]$ from which the components not touching $I^2 \times \{0\}$ are removed is isomorphic to $\Gamma^+_{g+}$.
(ii) $\Gamma \cap I^2 \times [\frac{9}{10}, 1]$ is the base of the standard 3-valent graph $\Gamma^-_{g-}$.
(iii) $\Gamma \subset I^3$ and the union of the components which do not touch the boundaries form a link $L$.

Now each special framed link $\Gamma$ gives rise to a decorated cobordism $(M, \Sigma_-, \Sigma_+)$ with parametrized surfaces $\Sigma_-, \Sigma_+$ of genera $g_-, g_+$ respectively. The construction goes as follows: we have a regular neighborhood $N(\Gamma^+_{g+}) \subset S^3$ and a homeomorphism $f^+ : H_g \longrightarrow N(\Gamma^+_{g+})$ from the handlebody of genus $g$ (and a similar situation for $\Gamma^-_{g-}$). Cut out open handlebodies $N(\Gamma^+_{g+})$ and $N(\Gamma^-_{g-})$ from $S^3$ to get a compact oriented 3-dimensional cobordism $E$ between the respective surfaces. Now the maps $f^+, f^-$ induce parametrizations of the boundary of $E$. Then surgery on $E$ on the remaining link $L$ produces a compact cobordisms $M = D(\Gamma)$ with parametrized boundaries $\partial_- M \cong \Sigma_{g_-}$ and $\partial_+ M \cong \Sigma_{g_+}$. Again each cobordism $M$ whose boundary is partitioned into two disjoint parts may be obtained this way, and there are generalized Kirby moves for such surgery presentations (see [12] p. 168, [6]).

We set now

$$Z^{\mathrm{i}}_{\mathrm{j}}(\Gamma) = b^{n_L - g_+} c^{\sigma(L)} [j] \sum_{\text{coloring k}} [\mathrm{k}] J_{\Gamma,(\mathrm{i},\mathrm{j},\mathrm{k})} \tag{6}$$

where $\mathrm{i}, \mathrm{j}$ are colorings of $\Gamma^-_{g_-}, \Gamma^+_{g_+}$ respectively, $\mathrm{k}$ is coloring of $L$, and $(\Gamma, \mathrm{i}, \mathrm{j}, \mathrm{k})$ is viewed as a colored tangle giving rise to a map $J_{\Gamma,\mathrm{i},j,k} : Z(\Gamma^-_{g_-}, \mathrm{i}) \longrightarrow Z(\Gamma^+_{g_+}, \mathrm{j})$. We set $Z(\Gamma) : Z(\Sigma_{g_-}) \longrightarrow Z(\Sigma_{g_+})$ for the linear map whose blocks are the matrices $Z^{\mathrm{i}}_{\mathrm{j}}(\Gamma)$.

As $Z(\Gamma)$ is invaried by Kirby moves it follows that the formula $Z_r(M) = Z(\Gamma)$ defines a topological invariant for 3-dimensional oriented cobordisms with parametrized boundaries (i.e. decorated cobordisms).

Moreover we have

$$Z_r(M \circ N) = cZ_r(M) \circ Z_r(N),$$

where $c$ lies in the group of roots of unity generated by $e^{\frac{2\pi\sqrt{-1}}{r}}$. Therefore this data is a TQFT with anomaly which we call the $sl_2(\mathbf{C})$-TQFT at level $r-2$.

## 4. The Proof of Theorem 1

### 4.1. *The Arf invariant*

We have a recurrent method to compute the Arf invariant (see [11]) of a proper link due to Murakami [8]. This is related to Jones polynomial at 4th roots of unity. Specifically let $I$ denote the link invariant defined by

$$I(\text{unknot}) = 1$$

$$I(\text{unknot} \bigcup K) = \sqrt{2} I(K), \text{ for any link } K,$$

and the skein relation

$$I(L_+) + I(L_-) = \sqrt{2} I(L_0),$$

where $L_+, L_-$ are the left- and right-hand twists and $L_0$ is the 2-parallel string diagram, and the rest of the diagrams are the same (see Fig. 6).

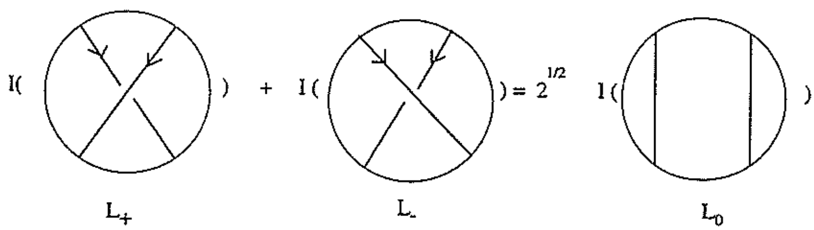


Fig. 6. The skein relation.

Then for a link $L$ we have

$$I(L) = \begin{cases} (-1)^{\varepsilon}\sqrt{2^{n_L - 1}} & \text{if } Arf(L) = \varepsilon \\ 0 & \text{if } L \text{ is not proper} \end{cases}$$

Remember that the link $L$ is proper if, for each sub-link $K$ the linking number $lk(K, L-K)$ is even.

### 4.2. *Jones polynomial at 4th roots of unity*

Consider now $J_L = J_{L,\mathbf{2}}$, where $\mathbf{2}$ is the coloring of all components of $L$ by the module $V^2$. Then $J_L$ is a variant of Jones polynomial according to [7]. In fact we

have:

$$J_L = t^{3L\cdot L}\sqrt{2}I(L)\,,$$

where $t = e^{\frac{2\pi\sqrt{-1}}{16}}$.

### 4.3. *The cabling formula*

If $L$ is a framed link and $\mathbf{k}$ a coloring of $L$ then the following formula permits to compute $J_{L,\mathbf{k}}$ in terms only of Jones polynomial of cablings of $L$. Specifically we have

$$J_{L,\mathbf{k}} = \sum_{\mathbf{j}\,=\,0}^{\mathbf{n}/2} (-1)^{\mathbf{j}} C^{\mathbf{j}}_{\mathbf{n}-\mathbf{j}} J_{L^{\mathbf{n}-2\mathbf{j}}} \tag{7}$$

where $\mathbf{n} = \mathbf{k} - \mathbf{1}$, and we set $f(\mathbf{n}) = \prod_i f(n_i)$, $\mathbf{m} < \mathbf{n}$ if $m_i < n_i$ for all $i$'s etc. Also $L^{\mathbf{c}}$ is the c-cabling of $L$ which consists in replacing the $i^{th}$ component of $L$ by $c_i$ parallel copies.

In case when $r = 4$ and the link $L$ is a twisted Whitehead's link (see Fig. 8) we choose the framings of the two components $K$ and $H$ (both unknotted in $\mathbf{R}^3$) such that $H \cdot H = 0$ and $K \cdot K = 0$.

Then the possible values for $J_{L,\mathbf{k}}$ are

$$\begin{aligned} J_{L,(1,1)} &= 1\,,\\ J_{L,(1,2)} &= J_H = \sqrt{2}\,,\\ J_{L,(2,1)} &= J_K = \sqrt{2}\,, \end{aligned}$$

and using the 1-colored components removing lemma (see [7], p. 511),

$$\begin{aligned} J_{L,(1,3)} &= J_{H,3} = J_{H^2} - 1 = 1\,,\\ J_{L,(3,1)} &= J_{K,3} = J_{K^2} - 1 = 1\,,\\ J_{L,(2,2)} &= J_L\,,\\ J_{L,(3,2)} &= J_{K^2H} - J_H = J_{K^2H} - \sqrt{2}\,,\\ J_{L,(2,3)} &= J_{KH^2} - J_K = J_{KH^2} - \sqrt{2}\,,\\ J_{L,(3,3)} &= J_{K^2H^2} - J_{K^2} - J_{H^2} + 1 = J_{K^2H^2} - 3\,. \end{aligned}$$

Notice that the symmetry principle (see [7], p. 513) implies that the only non-trivial entry is $J_L$ because

$$J_{L,(3,2)} = J_{L,(2,3)} = J_{L,(1,2)} = J_{L,(2,1)} = \sqrt{2}\,.$$

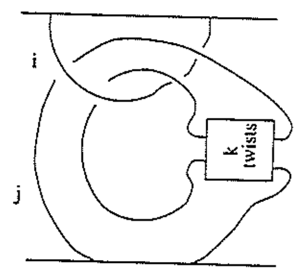


Fig. 7. The special graph $\Gamma$.

### 4.4. *A surgical description of the cobordism $X_k$*

We come back now to the cobordism $X_k$ from (2.4). We have a simple surgical description for $X_k$ since both tori $T_0$ and $T_1$ are unknotted in $S^3$. We can choose for example the special graph $\Gamma$ from Fig. 7. Taking into account that the intermediary link of $\Gamma$ is trivial this time we see that

$$Z_j^i(\Gamma) = c^{-3} J_{\Gamma,(i,j)}, \quad \text{for} \quad i,j \in \{1,2,3\}\,.$$

When properly interpretated $J_{\Gamma,(i,j)}$ is $J_{L,(i,j)}$ where $L$ is the Whitehead link (see Fig. 8). Notice that the basis of the two genus 1 spaces associated to $S^1 \times S^1$ corresponds to the framings we choosed for $L$. Then we may choose for our computations arbitrary framings for the two components, in particular those from above (this of course does not apply to the intermediary components for a general surgery). In fact if we change the framing of a component the result changes by multiplication by a diagonal (unitary) matrix $T$. Since both components of $L$ are unknotted we may apply the previous formulas. We compute first by recurrence on $k$

$$I(L) = (-1)^k\sqrt{2},\ I(K^2H) = I(KH^2) = 2,\ I(K^2H^2) = 2\sqrt{2}\,.$$

Therefore up to a root of unity the morphism $Z(X_k)$ is given by the matrix

$$\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & (-1)^k 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}.$$

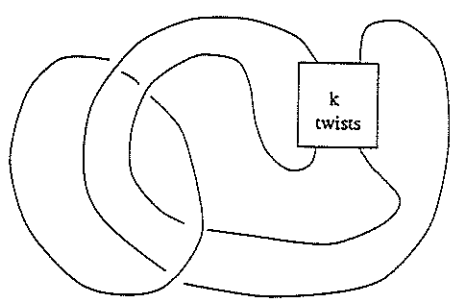


Fig. 8. The twisted Whitehead link.

**Remark.** *For even $k$ this matrix has rank 1. For odd $k$ it is simple to check that $Z(X)^n$ has rank 2 for all $n \neq 0$. Therefore the inductive limit $\lim_{\to}(\mathbf{C}, Z(X))$ of iterates of the map $Z(X)$ is isomorphic to $\mathbf{C}$ for even $k$. It is not clear yet that for odd $k$ the limit is $\mathbf{C}^2$, since the two basis, source and target, in which we computed the matrix of $Z(X_k)$ are different. So taking a power of $Z(X_k)$ we are not sure to compute the right rank.*

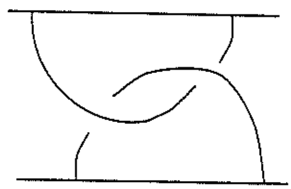

Fig. 9. The Hopf link.

To come back to the initial basis we need to multiply by the matrix associated to the special link from Fig. 9. We should start also directly with the surgery presentation obtained by gluing (composing) the two special presentation from Figs. 7 and 9. Indeed a basis corresponds to an isotopy class of a curve on the torus. By taking the framing of the knot we fixed this curve. Remark that this curve transported on the outer boundary of the cobordism is no more a longitude but a meridian. Therefore we need to change this meridian into a longitude. On the torus level this amounts to consider the map $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in SL_2(\mathbf{Z})$. The action of this element on the space associated to the torus is given by the so-called S-matrix of the theory, which it turns to be just the matrix associated to the Hopf link above. For the $sl_2(\mathbf{C})$-theory one may compute easily that the S-matrix is given (for arbitrary $r$) by the formula

$$S_{ij} = \sin \frac{ij\pi}{r}, i, j = 1, 2, ..., r-1 \,.$$

We derive the matrix of $Z(X_k)$ in the new (longitude-longitude) basis is

(1) for even $k$:

$$\begin{pmatrix} 1 & \sqrt{2} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(2) for odd $k$:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

Now for odd $k$ we may compute $Z(X_k)^n$ using this matrix. We derive that the square of this map has rank 1, so that

$$Z_\infty(Wh(k)) \cong \mathbf{C},$$

for all $k$.

### 4.5. *Proof of Theorem 1*

Using Corollary 2.3 we are able to prove that two manifolds $Wh(\mathbf{r})$ and $Wh(\mathbf{r}')$ are not homeomorphic if the sequences $\mathbf{r}$ and $\mathbf{r}'$ are not cofinally equal. In fact we know that there exists an isotopy which transforms one prime decomposition of a manifold into another. Since the unwrapping number is 2 for each pair of succesive tori $T_1 \hookrightarrow T_0$ we derive that these are prime unknotted pairs. The uniqueness of prime decompositions and the fact that the sequences are not cofinally equal imply that there are the unitary matrices $A$ and $B$ so that

$$Z_{sl_2(\mathbf{C}),2}(X_0) = AZ_{sl_2(\mathbf{C}),2}(X_1)B,$$

where $X_0$ and $X_1$ are the two cobordisms for $k = 2$ and $k = 1$. This cannot be true since the two matrices computed above have different ranks. This proves Theorem 1.

## 5. Proof of Theorem 2

### 5.1. *Level 3 computations*

We consider as above the link $L$ for $k = 1$, this time with different colors corresponding to $r = 5$. We remark that the symmetry principle [7], p. 513 considerably reduce the number of colored link invariants we need to compute, because:

$$\begin{aligned} J_{L,(1,1)} &= J_{L,(4,4)} = J_{L,(1,4)} = J_{L,(4,1)} = 1\,, \\ J_{L,(2,1)} &= J_{L,(1,2)} = J_{L,(4,3)} = J_{L,(3,4)} = J_K\,, \\ J_{L,(2,2)} &= J_{L,(3,2)} = J_{L,(2,3)} = J_{L,(3,3)} = J_L\,. \end{aligned}$$

Since $K, H$ are unknotted with trivial framings, $J_H = J_K = [2]$, and it remains to compute $J_L$. The skein relation (similar to that satisfied by Jones polynomial [7], p. 509) enables us to derive

$$J_L = \bar{q}[2](\bar{q}[2] + u[2]^2 + u^2[2](q - \bar{q}) - u^3)\,,$$

where

$$q = \exp\left(\frac{2\pi\sqrt{-1}}{r}\right), \quad u = \exp\left(\frac{\pi\sqrt{-1}}{r}\right) - \exp\left(\frac{-\pi\sqrt{-1}}{r}\right).$$

For $r = 5$ this reduces to

$$J_L = \bar{q}[2](\frac{1}{2} - (3\sin\frac{\pi}{5} + 3\sin\frac{2\pi}{5})\sqrt{-1})\,.$$

Finally the S-matrix (up to a constant) reads:

$$S = \begin{pmatrix} 1 & [2] & [2] & 1 \\ [2] & 1 & -1 & [2] \\ [2] & -1 & -1 & [2] \\ 1 & -[2] & [2] & -1 \end{pmatrix}.$$

Therefore the linear map $Z(X)$ (in the longitude-longitude basis) is given by the formula

$$Z_{sl_2(\mathbf{C}),3}(X_1) = SJ = \begin{pmatrix} 2+2[2]^2 & 2(J_L+[2]) & 2(J_L+[2]) & 2+2[2]^2 \\ 0 & 0 & 0 & 0 \\ 0 & [2]^2 - J_L & [2]^2 - J_L & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We may check that $J_L \neq [2]^2$, and so the rank of this matrix is 2. Furthermore one may compute also the iterates of this linear map, and we find that (for $n \neq 0$)

$$Z_{sl_2(\mathbf{C}),3}(X_1)^n = \begin{pmatrix} (2+2[2]^2)^n & * & * & (2+2[2]^2)^n \\ 0 & 0 & 0 & 0 \\ 0 & ([2]^2 - J_L)^n & ([2]^2 - J_L)^n & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so that the rank is 2 for all $n$. Therefore

$$(Z_{sl_2(\mathbf{C}),3})_\infty(Wh) \cong \mathbf{C}^2 .$$

This proves Theorem 2 for $k = 1$.

### 5.2. *The general case*

For arbitrary positive $k$ one may use the same method. We set $J_k$ for $J_L$ when $L$ has $k$ twists. The skein relation from [7] gives the recurrence

$$J_k = \bar{q}^2 J_{k-1} + J_1 - \bar{q}^2[2]^2.$$

We thus obtain

$$J_k = \frac{\bar{q}^{2k+2}([2]^2 - J_1) + J_1 - \bar{q}^2[2]^2}{1 - \bar{q}^2}.$$

Furthermore $J_k = [2]^2$ if and only if $k = -1 (\mathrm{mod}\ 5)$. If $k \neq -1 (\mathrm{mod}\ 5)$ we get as above:

$$(Z_{sl_2(\mathbf{C}),3})_\infty(Wh(k)) \cong \mathbf{C}^2,$$

which ends the proof.

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