Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 15 (1996), No. 2, 275–278

# A Note on the Bonnet-Myers Theorem

**V. Boju and L. Funar**

**Abstract.** The aim of this note is to derive a compactness result for complete manifolds whose Ricci curvature is bounded from below. The classical result, usually stated as Bonnet-Myers theorem, provides an estimation of the diameter of a manifold whose Ricci curvature is greater than a strictly positive constant. Weaker assumptions that the Ricci curvature function tends slowly to zero (when the distance from a fixed point goes to infinity) were already considered in [2, 3]. We shall improve here their results.

**Keywords:** *Ricci curvature, Jacobi equation*

**AMS subject classification:** 53 C 20, 53 C 21

We will be concerned with the following analytic

**Problem.** Given a function $a : [r_0, +\infty) \to (0, +\infty)$ we consider positive solutions $y = y(r)$ of the differential equation

$$y'' + ay = 0$$

satisfying $y(r_0) = 0$. Obviously, $y$ has to be concave. We have to determine the functions $a = a(r)$ for which $y$ has a further zero $r_1 > r_0$ which may be bounded from above.

It is clear that there is such a bound in case $a$ is a positive constant, but this bound tends to infinity as $a(r) \to 0$. The above problem seems to be interesting for functions $a$ satisfying $\lim_{r\to+\infty} a(r) = 0$. It turns out that the right asymptotic is $a(r) \sim cr^{-2}$, with critical value $c = \frac{1}{4}$. In fact, for $c = \frac{1}{4} + v^2$ one gets the solution $y(r) = r^{\frac{1}{2}} \sin v\big(\log \frac{r}{r_0}\big)$, and hence there is a second zero. In this paper we show that in fact the constant $v^2$ may be replaced by a function which tends as weakly as an iterated logarithm to zero, which enters in our definition of some function $A_{k,v} = A_{k,v}(r)$.

Let us first make some notations. For each natural number $k$ we set

$$Log_0(r) = r$$
$$L_k(r) = \underbrace{\log \ldots \log}_{k} r$$

V. Boju: Univ. of Craiova, Dept. Math., Str. Al. I. Cuza 13, Craiova 1100, Romania
L. Funar: Université de Grenoble I, Institut Fourier, B.P.74, 38402 Saint-Martin-d'Hères Cedex, France. e-mail: funar@fourier.ujf-grenoble.fr

ISSN 0232-2064 / $ 2.50 © Heldermann Verlag

whenever is defined, and

$$A_{k,v}(r) = \frac{1}{4r^2}\left(1 + \frac{1}{L_1(r)^2} + \ldots + \frac{1}{L_1(r)^2 L_2(r)^2 \cdots L_{k-1}(r)^2} + \frac{1+4v^2}{L_1(r)^2 L_2(r)^2 \cdots L_k(r)^2}\right).$$

For a Riemannian manifold $M$, we denote by $Ric_x(Y)$ the Ricci curvature in the direction $Y \in T_x(M)$, for a point $x \in M$ and $T_x(M)$ being the tangent space to $M$ in $x$. The space $M$ is said to have an *almost positive asymptotic Ricci curvature* (abbreviated to be an AP-Riemannian space) if there exist $k, v, r_0 > 0$ and $p \in M$ such that

$$Ric_x(Y) \geq (n-1)\, A_{k,v}(r)|Y|^2 \tag{1}$$

holds for all $x \in M$ whose distance from a fixed point $p$ is $r = dist(p,x) \geq r_0$ and for all vectors $Y \in T_x(M)$. Also $|\cdot|$ stands for the norm in the tangent space induced by the metric, and $n$ is the dimension of $M$.

Our result can be stated now as follows.

**Theorem 1.** *A complete AP-Riemannian manifold is compact, and its diameter* $d(M)$ *is bounded by*

$$d(M) \leq e_{k-1}\left(L_{k-1}\left(\exp\frac{\pi}{v}\max\{r_0, e_k(0)\}\right)\right)$$

*where* $e_0(x) = x$ *and* $e_{m+1}(x) = \exp e_m(x)$ *for* $m > 0$.

Notice that the case $k = 0$ is discussed in [2] and the case $k = 1$ is covered by [3]. Also, Dekster and Kupka [3] proved that the constant $\frac{1}{4}$ is sharp, i.e. for any function $A = A(r)$ using in the place of $A_{k,r}$ so that Theorem 1 holds we must have

$$\lim_{r\to+\infty} A(r)r^2 \geq \frac{1}{4} \qquad \text{and} \qquad \lim_{r\to+\infty}\left(A(r)r^2 - \frac{1}{4}\right)(\log r)^2 \geq 1.$$

So our result identifies the higher order terms which might be added in spite to preserve the boundedness of the manifold. We think that the function $A_{k,v}$ is sharp.

**Proof of Theorem 1.** We write the Jacobi equation associated to the sectional curvature function $A_{k,v}$, namely

$$y'' + A_{k,v}(r)y = 0. \tag{2}$$

We claim that this equation admits the basic solutions

$$\Psi_0 = \Phi_k(r)\cos vL_k(r) \qquad \text{and} \qquad \Psi_1 = \Phi_k(r)\sin vL_k(r)$$

where

$$\Phi_k(r) = r^{\frac{1}{2}} L_1(r)^{\frac{1}{2}} \cdots L_{k-1}(r)^{\frac{1}{2}}.$$

For $k = 1$ it is easy to see that $r^{\frac{1}{2}}\cos(v\log r)$ and $r^{\frac{1}{2}}\sin(v\log r)$ are solutions for equation (2). By recurrence we prove first that the following relations are fulfilled (for $k = 1$ they are simply to check):

$$\Phi_k'' + A_{k,0}\Phi_k = 0 \qquad \text{and} \qquad 2\Phi_k' L_k' + \Phi_k L_k'' = 0.$$

In fact we have

$$\Phi_{k+1} = \Phi_k L_k^{\frac{1}{2}} \qquad \text{and} \qquad L_{k+1} = \log L_k,$$

hence

$$2\Phi_{k+1}' L_{k+1}' + \Phi_{k+1} k L_{k+1}'' = \left(2\Phi_k' L_k' + \Phi_k L_k'\right) L_k^{-\frac{1}{2}} = 0.$$

On the other hand

$$\frac{\Phi_{k+1}''}{\Phi_{k+1}} = -A_{k,0} + (L_k')^2 L_k^{-2} = -A_{k+1,0}$$

holds and the two relations stated above are proved.

Furthermore we verify that

$$\Psi_0'' = \Phi_k'' \cos(vL_k) - v\left(2\Phi_k' L_k' + \Phi_k L_k'\right)\sin(vL_k) - v^2 \Phi_k L_k'^2 \cos(vL_k).$$

The two relations stated above and the obvious identity

$$L_k' = L_0^{-1} L_1^{-1} \cdots L_{k-1}^{-1}$$

complete the proof of our claim for $\Psi_0$ (the case of $\Psi_1$ is similar).

Both $\Psi_0$ and $\Psi_1$ are defined on the interval $[e_k(0), +\infty)$. Set $r_1 = \max\{r_0, e_k(0)\}$. Therefore, for each $\lambda \geq e_k(0)$ the linear combination

$$Y_{k,v,\lambda}(r) = -\sin(vL_k(\lambda))\,\Psi_0(r) + \cos(vL_k(\lambda))\,\Psi_1(r) \tag{3}$$

is a solution for equation (2), which satisfies also $Y_{k,v,\lambda}(\lambda) = 0$. Also, we may write

$$Y_{k,v,\lambda}(r) = \sin(v(L_k(r) - L_k(\lambda))\,\Phi_k(r)\,L_k(r)$$

so that $Y_{k,v,\lambda}$ is positive on the interval $(\lambda, \beta(\lambda))$ where $\beta(\lambda) = e_{k-1}\left(L_{k-1}(\lambda)\exp\left(\frac{\pi}{v}\right)\right)$ and vanishes again in $\beta(\lambda)$. This is a consequence of the straightforward formula

$$L_k(\beta(\lambda)) - L_k(\lambda) = \frac{\pi}{v}.$$

A standard argument (see, for instance, [1]) proves that the diameter of the manifold $M$ is less than $\beta(r_1)$. Since $M$ is complete from the Hopf-Rinow theorem it follows that $M$ is in fact compact and this ends the proof of the theorem ∎

**Remark 2.** The form of the function $A_{k,v}$ is in some sense sharp. In fact, for $v = 0$ the analog result is false: We may choose on $M = \mathbb{R}^n - K$, with $K$ being a sufficiently large compact, the metric with radial symmetry $dr + P_k(r)d\theta$ (in polar coordinates) where $d\theta$ is the metric form on the standard sphere $S^{n-1}$ and

$$P_k(r) = r\left(\sum_{i=1}^{k} L_i(r)^{-2}\right)^{-\frac{1}{2}}.$$

Then a straightforward computation shows that $Ric_x(Y) = A_{k,0}(r)|Y|^2$ for all points $x$ outside the compact $K$ and all tangent vectors $Y$.

**Acknowledgements.** We wish to thank the referees for pointing out some errors and missprints in the earlier version of this note. Their suggestions substantially improved the revision.

## References

[1] Cheeger, J. and D. Ebin: *Comparison Theorems in Riemannian Geometry* (North Holland Mathematical Library: Vol. 9). Amsterdam: North Holland Publ. Comp. 1975.

[2] Cheeger, I., Gromov, M. and L. Taylor: *Finite propagation speed, normal estimates for functions of the Laplace operator and the geometry of complete Riemannian manifolds.* J. Diff. Geom. 17 (1982), 15 – 53.

[3] Dekster, B. and I. Kupka: *Asymptotics of curvature in a space of positive curvature.* J. Diff. Geom. 15 (1980), 553 – 568.

[4] Myers, B.: *Riemann manifolds with positive mean curvature.* Duke Math. J. 8 (1941), 401 – 404.

Received 10.06.1995; in revised form 15.01.1996