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2 + 1-D Topological Quantum Field Theory and 2-D Conformal Field Theory

Louis Funar

Institute of Mathematics, Romania, Univ. Paris-Sud and (permanent address) Institut Fourier, BP 74, Université Grenoble, Mathématiques, F-38402 Saint-Martin-d'Hères Cedex, France

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Abstract: We describe the relation between three dimensional topological quantum field theory and two dimensional conformal field theory. Some applications to quantum knot invariants leading to the equivalence of Chern–Simons–Witten and Kohno's approaches are outlined.

Contents

1. Introduction

After Witten [Wit89] introduced his invariants for 3-manifolds much work has been done on understanding them from the mathematical point of view. A counterpart to the Feynmann path integral formalism in the Chern–Simons theory has been given via quantum groups by Reshetikhin and Turaev [RT91]. The $SU(2)$-theory has been extensively studied in [RT91, BHMV92, KM91, Koh92]. Recently the quantum group construction of invariants has been extended to the simple Lie groups in the series A, B, C, D by Turaev and Wenzl [TW93]. Several generalizations were given by Crane [Cra91] and Degiovanni [Deg92] which started from Rational Conformal Field Theories (abbrev. RCFT) in dimension 2 and derived Topological Quantum Field Theories (abbrev. TQFT) in dimension $2 + 1$. Also Kohno [Koh92] computed the mapping class group representation arising in the SU(2)–WZW model and show how we can construct topological invariants from this data, by pointing out that these ideas work more generally for any RCFT.

The case of $\mathbf{Z}/k\mathbf{Z}$-fusion rules which turns out to be the same as the abelian Witten's theory, has been discussed in [Koh92], and from a different point of view in [Deg90, Fun91, Fun93b, Goc92, MOO92]. In fact the Dijgraaf–Witten's approach ([DW90]) in the case of abelian groups provides the same system of homotopy invariants. The TQFT based on a finite group was completely described by Freed and Quinn [FQ93].

In this paper we wish to give an axiomatic treatment of the topological invariants. Our main result can be stated as follows (see for more precise statements 2.5, 2.9, 4.3):

Main Theorem 1.1. *There is an equivalence between*:

(i) *multiplicative invariants for closed oriented* 3*-manifolds* (*rational and unitary*).

(ii) *TQFT in dimension* 3 (*rational and unitary*).

(iii) *RCFT in dimension* 2 (*unitary*).

Some of the implications were previously known:

(i) $\Rightarrow$ (ii) is greatly inspired from [BHMV92] where the SU(2) case is treated.

(iii) $\Rightarrow$ (ii) was sketched first by Kontsevich [Kon88] and by Crane [Cra91], and detailed proofs in the case of the SU(2)-model were given by Walker [Wal92] and Kohno [Koh92], and in whole generality by Degiovanni [Deg92] using surgery presentations of 3-manifolds. Moreover in Witten's approach his Chern–Simons invariants are actually based on the WZW-model and he guesses that there is in fact an equivalence between (ii) and (iii). A proof, more physical than mathematically rigorous and using another definitions than us, was sketched in [LY90]. The definition we used for the RCFT is the combinatorial one due to Moore and Seiberg [MS89], but its equivalence may be proved with the analytic formulation proposed by Segal [Seg88] (from a mathematical viewpoint). The equivalence between TQFTs and modular categories (weaker than our assumptions) was proved by Lyubashenko [Mal94] and by Quinn ([Qui92]), and also by Turaev [Tur94]. Remark also that a result of Ocneanu [Ocn92] describes the RCFTs (and the TQFTs arising from state sums over triangulations) in terms of systems of bimodules over II_1-factors.

The strategy of our proof goes as follows: in Sect. 2 we introduce the tensor representations of mapping class groups. It turns out that all multiplicative invariants of closed 3-manifolds come from such representations and furthermore extend canonically to TQFTs. Thereafter if we restrict to the finite dimensional case we outline in Sect. 3 a splitting procedure which permits to decompose the target spaces of tensor representations according to the sewing rules of conformal blocks. Next we extend these representations to the duality groupoid in Sect. 4 and explain why the representations split in some pieces of data corresponding to the RCFTs as axiomatized by Moore and Seiberg. We shall use here the completeness theorem ([MS89], Appendix A) and their reduction of the Frenkel–Shenker flat bundles to a system of matrices. In Sect. 5 we get back the TQFT from the RCFT in case of cobordisms along the ideas outlined by Crane [Cra91].

To a colored framed link $K \hookrightarrow S^3$ we can furthermore associate a topological invariant in two ways:

(i) looking at the manifold with boundary $S^3 - T(K)$, where $T(K)$ is a tubular neighborhood of K and compute the TQFT associated, and

(ii) writing K as Artin's closure of some braid and solving inductively the crossing singularities arising in its plane picture by means of a state sum based on the braiding matrices of the RCFT (as is done for example in [Deg92]).

Our next theorem states that the two ways give the same invariants for links (see for a more precise statement Theorem 6.3). We derive:

Corollary 1.2. *A TQFT is uniquely determined by the matrices S, T (corresponding to the monodromy in genus* 1) *and the braiding matrices B (corresponding to the monodromy on the punctured sphere).*

We give an immediate application on Witten's theory in the SU(2)-case. We shall make a distinction between the WZW-invariant I_{WZW} constructed in a rigorous manner by Kohno [Koh92] (using mapping class group representations and Heegaard splittings) and the Chern–Simons–Witten invariants whose definition we outline briefly below in an algebro-geometric context, following Witten [Wit89]. He associates vector spaces $W(\Sigma_g,k)$ to every Riemann surface of genus g obtained from the quantization of $M(\Sigma_g)$ the space of representations of $\pi_1(\Sigma_g)$ into the gauge group G (modulo conjugation). If $G = SU(n)$ then a theorem of Narasimhan–Seshadri identifies $M(\Sigma_g)$ with the moduli space of rank n semi-stable holomorphic vector bundles of degree 0 over Σ_g. The Picard group $Pic(M(\Sigma_g)) \cong \mathbf{Z}$ is generated by an ample line bundle L and $W(\Sigma_g,k) = H^0(M(\Sigma_g), L^{\otimes k})$ becomes the fibers of a projectively flat hermitian vector bundle over the Teichmuller space, by using the HADW-connection defined in [Hit90, ADW91]. It follows that the monodromy representation $\rho\colon \mathscr{M}_g \to PU(W(\Sigma_g,k))$ of the mapping class group determines the TQFT. Actually Witten defined the invariants by means of the path integral, but the axiomatic behind it implies that whenever the functional integration exists it must be equal to

$$I_{CSW}(M^3) = c^{-g}\langle \rho(\varphi)w_g, \overline{w_g}\rangle \,,$$

where $M^3 = T_g \cup_\varphi \overline{T_g}$ is a Heegaard splitting of the 3-manifold into two handlebodies glued together via the homeomorphism φ (whose class in $\mathscr{M}_g$ we denoted also by φ), $w_g \in W(\Sigma_g,k)$ is some weight vector $\rho(\mathscr{M}_g^+)$-invariant and c is a normalization constant. If framings are taken into account then a correction term must be added since the representation ρ is a projective one. Therefore, away from its original definition, the CSW-invariant is described by the above stated formula which is still valid for any solvable model.

The identification of $W(\Sigma_g,k)$ with the space of conformal blocks in the Wess–Zumino Witten model of RCFT (see [Koh92, MS89]) was recently obtained (see [BL93, Ber92, Fal94, KNR94]). To our knowledge nobody has computed the monodromy representation ρ and verified that it agrees with the natural representation coming from the WZW-model (based on the same gauge group). This has been done in the simplest setting for the abelian case $G = U(1)$ in [Fun91, Fun93, Goc92] in the case when no punctures occur on the surface Σ_g or else in a simplified model for a general gauge group G, which we called semi-abelian, in [Fun93d, Fun93b]. We state:

Theorem 1.3. *The CSW-invariant is well-defined and coincides with the WZW-invariant (as defined by Kohno starting from the WZW-model) in the case when the gauge group is SU*(2).

Proof. We use the degeneration theorem of Daskalopoulos and Wentworth [DW93] for a family of Riemann surfaces. Their statement may be reformulated as a splitting formula:

$$W(\Sigma_g, k) = \bigoplus_{\lambda} W(\Sigma_h, k)_\lambda \otimes W(\Sigma_{g-h}, k)_\lambda \,.$$

This formula allows us to identify the vector w_g as the image of $\otimes W(\Sigma_0, k)_0$, corresponding to the degeneration of a Riemann surface into a union of projective lines. It suffices now to compare the matrices S, T, B. The monodromy of 1-loop functions was computed by Jeffrey via invariant theta functions (see [Jef92]) and the matrices S and T are the same in the CSW and WZW approaches. Next the braiding matrices were computed by Kohno in the WZW-model (see [Koh92]) and by Tsuchyia and Kanie [TK88] for the monodromy of the Kniznik–Zamolodchkikov connection (which is the analog of the HADW connection) and they coincide. Using the corollary our claim follows. □

Remark that in the SU(2) case Piunikhin [Piu93] proved that the WZW-invariant and the invariant defined by Reshetikhin and Turaev [RT91] starting from the quantum SU(2) coincide also. This way we may speak about the (quantum) SU(2) invariant which does not depend on the various ways we used for its construction. It seems that this assertion is valid for all gauge groups G (see [TW93, AC92] for a definition in the case of quasi-quantum groups associated to Lie groups in the series A, B, C, D).

Observe that our main theorem expresses in fact a certain homogeneity for the representations of mapping class groups which are yielding to topological invariants.

This completes the axiomatic approach of Wenzl [Wen93] which expressed an arbitrary multiplicative invariant of a 3-manifold obtained by Dehn surgery on a link, as the thermodynamic limit of the associated invariant for cablings of the link. The masterpieces of his construction are the Markov traces on ribbon links. In some sense our corollary is equivalent to his statement.

In the appendix we discuss the simplest examples of RCFTs which we called abelian. There the monodromy representations factor through the symplectic group.

This is part of author's Ph.D. thesis at Univ. Paris-Sud and some of the results have been announced in [Fun94]. I am indebted to my thesis advisor V. Poenaru, to P. Vogel, V. Turaev, G. Masbaum, L. Guillou, C. Lescop, V. Sergiescu, F. Constantinescu and the referee for their careful reading of the different versions of this paper, for their suggestions and corrections which considerably improved its accuracy.

2. Multiplicative Invariants for Closed 3-Manifolds

We shall consider in this paper only the case of orientable 3-manifolds. We choose an oriented Heegaard splitting of the closed 3-manifold $M = H_g \cup_\varphi \overline{H}_g$ into two genus g handlebodies, where $\varphi \in Homeo(\Sigma_g)$ states for the gluing homeomorphism and Σ_g is the surface of genus g. The Reidemester–Singer stabilization theorem ([Sie80]) states that the homeomorphism type of M is uniquely determined by the Heegaard splitting modulo the following (elementary) operations:

1. replacing an Heegaard splitting by an isomorphic one,
2. taking the connected sum with the standard Heegaard splitting of the sphere S^3 into two genus one handlebodies.

So two 3-manifolds are homeomorphic iff any two Heegaard splittings of them are stably isomorphic. But the Heegaard splitting consists in the data (g, φ), where $\varphi \in \mathcal{M}_g$ is the class of φ in the mapping class group of Σ_g. We wish first to translate the Reidemester–Singer criterion into a purely algebraic one.

Remark firstly that φ is not uniquely defined. In fact different identifications of ∂H_g with the genus g surface Σ_g may give distinct classes in $\mathcal{M}_g$. On the other hand $\mathcal{M}_g$ itself is $Out^+(\pi_1\Sigma_g)$ and there are as many self-identifications as generator systems for $\pi_1\Sigma_g$. All these choices correspond to the first type operation: two Heegaard splittings determined by the pairs (g, φ) and (g, φ') are isomorphic if and only if $\varphi' = c\varphi d$, where $c, d \in \mathcal{M}_g^+$, where $\mathcal{M}_g^+$ is the subgroup of $\mathcal{M}_g$ of the classes of homeomorphisms $\psi\colon \Sigma_g \to \Sigma_g$ which extend to homeomorphisms of the handlebody H_g. A system of generators for $\mathcal{M}_g^+$ was given by Suzuki in [Suz77].

We wish now to obtain the algebraic counterpart of the connected sum of Heegaard splittings. The interesting feature of the tower of groups $\mathcal{M}_g$ is that no group homeomorphisms $\mathcal{M}_g \times \mathcal{M}_h \to \mathcal{M}_{g+h}$ actually exist away from the trivial one (as was pointed out to me by F. Laudenbach). Nevertheless we dispose of a multivalued mapping

$$\sigma\colon \mathcal{M}_g \times \mathcal{M}_h \to \mathcal{M}_{g+h}\,,$$

which makes the following diagram commutative:

$$\begin{array}{ccccc}
\mathcal{M}_{g,1} & \times & \mathcal{M}_{h,1} & & \\
 & & & \otimes & \\
\pi_g \downarrow & & \pi_h \downarrow & \searrow & \\
\mathcal{M}_g & \times & \mathcal{M}_h & \xrightarrow{\sigma} & \mathcal{M}_{g+h}\,,
\end{array}$$

where we denoted by $\mathcal{M}_{g,1}$ the mapping class group of the genus g surface with a disk removed, π_g is the usual projection and $\otimes$ is the group morphism induced by composition of homeomorphisms. Specifically

$$\sigma(x, y) = \{a \otimes b; a \in \pi_g^{-1}(x), b \in \pi_h^{-1}(y)\} \subset \mathcal{M}_{g+h}\,.$$

We can identify $\mathcal{M}_1$ with $SL(2, \mathbf{Z})$. Then the standard Heegaard decomposition of the sphere S^3 has the gluing morphism $\tau = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, up to right and left multiplication by an element from

$$\mathcal{M}_1^+ = SL^+(2, \mathbf{Z}) = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}, n \in \mathbf{Z} \right\} \subset SL(2, \mathbf{Z})\,.$$

Set now $M(\varphi) = H_g \cup_\varphi \overline{H}_g$ for $\varphi \in \mathcal{M}_g$. We can rephrase the Reidemester–Singer criterion as follows:

Proposition 2.1. *The manifolds $M(\varphi)$ and $M(\varphi')$ are homeomorphic if and only if $\varphi, \varphi' \in \mathcal{M}_\infty = \cup_{g>0}\mathcal{M}_g$ are equivalent under the equivalence relation generated by the following elementary moves*:

1. $\varphi \sim c\varphi d$, *for* $\varphi \in \mathcal{M}_g, c, d \in \mathcal{M}_g^+$,
2. $\varphi \sim \psi$ *for any* $\varphi \in \mathcal{M}_g, \psi \in \sigma(\varphi, \tau) \subset \mathcal{M}_{g+1}$.

It should be interesting if we can replace σ by an univalent mapping such that the conclusion of the proposition remains valid. If we should look only at the homological information carried by φ (i.e. we consider its image in the symplectic group $Sp(2g, \mathbf{Z})$), then a similar question would have an affirmative answer (see [Fun91, Fun93c, Fun93b, Fun93d]).

Consider K a field and $\mathscr{I}_K$ the K-algebra of (3-manifold) invariants, i.e. the set of graded functions $f_* : \mathscr{M}_* \longrightarrow K$, which fulfill

$$f_g(cxd) = f_g(x), \text{ for all } x \in \mathscr{M}_g, c, d \in \mathscr{M}_g^+, g \in N,$$

$$f_{g+1}(x') = f_g(x), \text{ for all } x \in \mathscr{M}_g, x' \in \sigma(x, \tau).$$

We shall say that a set of invariants $R \subset \mathscr{I}_K$ is complete if the following condition:

$$\varphi_1 \cong \varphi_2 \text{ if and only if } f_*(\varphi_1) = f_*(\varphi_2) \quad \text{for all } f_* \in R$$

holds.

Proposition 2.2. *If K is infinite then the whole K-algebra of invariants $\mathscr{I}_K$ is complete.*

Proof. Consider the (1) map $x \rightarrow \sigma(x, \tau)$ which enables us to identify $\mathscr{M}_g$ with a subset of $\mathscr{M}_{g+1}$. This map induces another map between cosets

$$\mathscr{M}_g^+ \backslash \mathscr{M}_g / \mathscr{M}_g^+ \rightarrow \mathscr{M}_{g+1}^+ \backslash \mathscr{M}_{g+1} / \mathscr{M}_{g+1}^+ .$$

Then the set of (closed oriented) 3-manifolds (modulo a homeomorphism) may be identified with the direct limit of the system

$$Manif = \lim_{\rightarrow} \mathscr{M}_g^+ \backslash \mathscr{M}_g / \mathscr{M}_g^+ .$$

But $\mathscr{M}_g^+ \backslash \mathscr{M}_g / \mathscr{M}_g^+$ has cardinal at most $\aleph_0$ hence $Manif$ is countable. Thus the direct limit will admit an injective map into K, from which we can recover an universal invariant, and we are done. □

This says that theoretically there is an universal invariant which classifies 3-manifolds, but we don't know if it is algorithmically computable.

We call $f_* \in \mathscr{I}_K$ a multiplicative invariant if the following condition is fulfilled

$$f_{g+h}(z) = f_g(x) f_h(y) \quad \text{for all } x \in \mathscr{M}_g, \ y \in \mathscr{M}_h, \ z \in \sigma(x, y).$$

The 3-manifold invariant associated to f_* is obviously given by

$$f(M(\varphi)) = f_g(\varphi), \text{ if } \varphi \in \mathscr{M}_g .$$

Then f_* is a multiplicative invariant iff

$$f(M \sharp N) = f(M) f(N)$$

holds for all closed manifolds M, N, where $\sharp$ denotes the connected sum of the manifolds. We denote by $\mathscr{M}\mathscr{I}_K$ the set of multiplicative invariants.

Proposition 2.3. *For an infinite K the set $\mathscr{M}\mathscr{I}_K$ is a complete set of invariants.*

Proof. Set $P_g = \{\varphi \in \mathscr{M}_g$ such that $M(\varphi)$ is prime$\}$. The set P_g has not a subgroup structure. Let $x \in \sigma(P_g, \mathscr{M}_1)$, so $x \in \sigma(y, \lambda)$, and $M(x) = M(y) \sharp M(\lambda)$. Suppose $x \in P_{g+1}$. If we agree that S^3 will be not prime then $\lambda \cong \tau$. Therefore if λ is not equivalent to τ then the map $\sigma(*, \lambda): P_g \to \mathcal{M}_{g+1}$ has image disjoint from P_{g+1}. Now the direct limit

$$PManif = \lim_{\to} \mathcal{M}_g^+ \backslash P_g / \mathcal{M}_g^+ ,$$

may be injectively mapped into K. This gives us a collection of maps $f_g: P_g \to K$ fulfilling the conditions stated in Proposition 2.1, and which classifies prime 3-manifolds. Using the multiplicativity one may extend it to all 3-manifolds. Now a well-known theorem of Milnor (see [Hem76]) asserts the uniqueness of the decomposition of 3-manifolds into prime manifolds (modulo connected sums with S^3), and we are done. □

From now on we shall consider $K = \mathbf{C}$, and that the multiplicative invariants are sensitive to the change of orientation, i.e.

$$f(\overline{M}) = \overline{f(M)} .$$

$\overline{M}$ being the manifold M with opposite orientation, and the bar on the right hand being the complex conjugation. We may restrict again, without loss of generality to the study of these multiplicative invariants.

We define next the hermitian tensor representations (abbrev. h.t.r.) of $\mathcal{M}_*$: consider an indexed family of complex vector spaces W_g endowed with non-degenerate hermitian forms $\langle \cdot \rangle$. Set

$$U(W_g) = \{A \in GL(W_g) : \langle Ax, Ay \rangle = \langle x, y \rangle \text{ for all } x, y \in W_g\} .$$

We assume that W_* has a tensor structure, i.e. a multiplication map

$$\otimes : W_g \times W_h \to W_{g+h}$$

which is compatible with the hermitian structures, hence

$$\langle x \otimes x', y \otimes y' \rangle = \langle x, y \rangle \langle x', y' \rangle, \quad \text{if } x, x' \in W_g, \ y, \ y' \in W_h .$$

We have a family of ("unitary") group representations

$$\rho_g : \mathcal{M}_g \to U(W_g)$$

such that

$$\langle \rho_{g+h}(c)(x \otimes y), x' \otimes y' \rangle = \langle \rho_g(a) x, x' \rangle \langle \rho_h(b) y, y' \rangle ,$$

for all $x, x' \in W_g$, y, $y' \in W_h$, and $a \in \mathcal{M}_g$, $b \in \mathcal{M}_h$, and $c \in \sigma(a,b) \subset \mathcal{M}_{g+h}$.

The h.t.r. is a weight h.t.r. (or a h.t.r. of $(\mathcal{M}_*, \mathcal{M}_*^+)$) if we have a weight vector $w_g \in W_g$ in every level g satisfying

$$w_{g+h} = w_g \otimes w_h .$$

$$\rho_g(c)(w_g) = w_g, \text{ for all } c \in \mathcal{M}_g^+ ,$$

and

$$d = \langle w_1, \rho_1(\tau)(w_1) \rangle \neq 0 .$$

Denote by WHTR the set of weight h.t.r. of $\mathcal{M}_*$. We associate to every element $(\rho_*, W_*) \in$ WHTR a function $f_* = f(\rho_*, W_*)$ by the formula:

$$f_g(x) = d^{-g} \langle \rho_g(x) w_g, \overline{w}_g \rangle \text{ if } x \in \mathcal{M}_g .$$

Proposition 2.4. *The functions $f(\rho_*, W_*)$ define a multiplicative* **C**-*invariant.*

Proof. Let $a, b \in \mathscr{M}_g^+$. Then

$$f_g(axb) = d^{-g}\langle \rho_g(axb)w_g, \overline{w}_g \rangle = d^{-g}\langle \rho_g(ax)w_g, \overline{w}_g \rangle \, .$$

since $\rho_g(b)w_g = w_g$. Also

$$\rho_g(a)(\mathbf{C}\langle w_g \rangle^{\perp}) = \mathbf{C}\langle w_g \rangle^{\perp}$$

holds from the unitarity condition. Therefore $f_g(axb) = f_g(x)$.

Further we have

$$f_{g+h}(z) = d^{-g-h}\langle \rho_{g+h}(z)w_{g+h}, \overline{w}_{g+h} \rangle = d^{-g-h}\langle \rho_{g+h}(z)w_g \otimes w_h, \overline{w}_g \otimes \overline{w}_h \rangle$$
$$= f_g(x) f_h(y)$$

for all $z \in \sigma(x, y)$. But $f_1(\tau) = 1$, hence $f_{g+1}(x \otimes \tau) = f_g(x)$. The unitarity implies now that f_* is sensitive to the orientation, which ends the proof. □

We obtained a map $f\colon \mathrm{WHTR} \to \mathscr{M}\mathscr{I}_{\mathbf{C}}$. We can state now the main result of this section:

Theorem 2.5. *The map f is surjective hence any multiplicative* **C**-*invariant* (*always sensitive to the orientation*) *arise from a weight hermitian tensor representation.*

Proof. Consider the set of (compact orientable) 3-manifolds with marked boundary:

$MB_g = \{(M, \varphi)\colon M$ is a 3-manifold with boundary and $\varphi : \partial M \to \Sigma_g$ is an orientation preserving homeomorphism$\}$/modulo homeomorphisms compatible with the markings φ's on the boundary.

Here Σ_g denotes the standard genus g surface. More precisely Σ_g is a tubular neighborhood of the graph shown in Fig. 1, hence it inherits a natural cut system $c_{0,*}$, it bounds the standard handlebody H_g and there are two disks δ_g^+, δ_g^- embedded in Σ_g (see Fig. 2). There is a canonical way to fix the marking φ in terms of a combinatorial data on ∂M. We choose a cut system c_* on ∂M having the dual graph Γ isomorphic to the dual graph of $c_{0,*}$. We can see the graph Γ as the spine of the surface ∂M. A framing of the graph Γ will be an embedding into ∂M. We suppose that the intersection of the framing with each trinion is the neighborhood of a vertex in the graph (a star configuration). The surface with the cut system and the framing of the dual graph satisfying the above written condition will be called a rigid surface. The reason is very simple: once we have two rigid surfaces with an identification of the dual graphs there is an unique homeomorphism ψ (up to an isotopy) between the rigid surfaces extending the combinatorial isomorphism at the graph level. In fact if a trinion is cut along the framing we obtain a disk (whose homeomorphisms are all isotopic) which implies our claim. So instead of marking the boundary we can add a rigid structure on the boundary. This will be useful in the further sections.

Consider now F_* a non-trivial multiplicative invariant. There is a induced map

$$B_F^{(1)}\colon \mathbf{C}\langle MB_g \rangle \times \mathbf{C}\langle MB_g \rangle \to \mathbf{C}$$

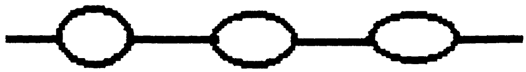

Fig. 1. The spine of the standard H_g.

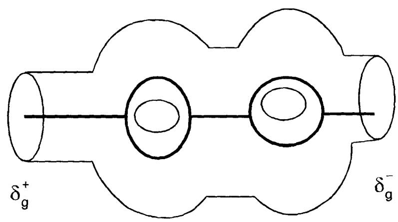

Fig. 2. The standard Σ_g.

defined on generators by

$$B_F^{(1)}((M,\varphi),(N,\psi)) = F(M \cup_{\varphi\psi^{-1}} N),$$

where the manifold on the right is obtained by gluing the boundaries according to the prescribed homeomorphism. Then $B_F^{(2)}(x,y) = B_F^{(1)}(x,\bar{y})$, where the bar denotes the complex conjugation of the coordinates (in the canonical basis) is a hermitian bilinear form on the huge space $\mathbf{C}\langle MB_g\rangle$. Set

$$W_g^1 = \mathbf{C}\langle MB_g\rangle/\ker B_F^{(2)}.$$

Now we may assume that $\overline{(M,\varphi)} = (\overline{M},\varphi)$, where the first bar is the complex conjugation on coordinates (the complex structure), and the second denotes the reversal of the orientation. This may be achieved by passing to a quotient which we shall call W_g. Next $B_F^{(2)}$ induce a hermitian form

$$B_F : W_g \times W_g \to \mathbf{C}.$$

We have a mapping $\rho_g : \mathscr{M}_g \to GL(W_g)$ given by

$$\rho_g(x)[M,\varphi] = [M,\tilde{x}\varphi]$$

where $[,]$ denotes the class of the corresponding element in W_g and $\tilde{x} \in Homeo(\Sigma_g, \delta_g^+,\delta_g^-)$ is an arbitrary lift of $x \in \mathscr{M}_g$.

Lemma 2.6. *The mapping ρ_g is a well-defined group representation.*

Proof. It suffices to prove that whenever $h \in Homeo(\Sigma_g,\delta_g^+,\delta_g^-)$ is isotopic to identity on Σ_g (by an isotopy not necessarily trivial on the two disks) the following identity $[M,\varphi] = [M,h\varphi]$ holds in W_g. Let h_t be an isotopy with $h_0 = h$ and $h_1 = id$. Consider the pseudo-isotopy

$$H : \partial M \times [0,1] \to \partial M \times [0,1]$$

given by $H(x,t) = (\varphi^{-1}h_t^{-1}\varphi(x),t)$.

We identify $\partial M \times [0,1]$ with a collar V of ∂M in M. Define further $\phi: M \to M$ by

$$\phi(x) = \begin{cases} H(x) & \text{if } x \in V \\ x & \text{elsewhere} \end{cases} .$$

Then ϕ is a homeomorphism of M and the following diagram is commutative:

$$\begin{array}{ccccc} M & \supset & \partial M & & \\ \varphi \downarrow & & \varphi^{-1}h^{-1}\varphi \downarrow & \overset{\varphi}{\searrow} & \\ M & \supset & \partial M & \overset{h\varphi}{\rightarrow} & (\Sigma_g, \delta^+ g, \delta_g^-), \end{array}$$

which implies that $[M, \varphi] = [M, h\varphi]$ getting our claim. Finally every $x \in \mathcal{M}_g$ has a lift $\hat{x} \in Homeo(\Sigma_g)$, which may be isotoped on Σ_g to $\tilde{x} \in Homeo(\Sigma_g, \delta^+ g, \delta_g^-)$. Since the connected component of the identity, $Homeo^0(\Sigma_g)$, acts trivially on W_g it follows that ρ_g is a representation of $\mathcal{M}_g$. $\square$

It is clear that $\rho_g(x)$ is an isometry with respect to the bilinear form B_F. We shall define now the tensor structure on W_*. Let $[M, \varphi] \in W_g$ and $[N, \psi] \in W_h$. Consider the tubular neighborhoods (see Fig. 3) in M and N which satisfy:

$$\begin{array}{ccccccc} V_0 & \subset & V_1 & \subset & V & \subset M \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow & \\ \varphi^{-1}(\delta_g^+) \times \left[0, \frac{1}{2}\right] & \subset & \partial M \times \left[0, \frac{1}{2}\right] & \subset & \partial M \times [0,1] & \end{array}$$

and respectively

$$\begin{array}{ccccccc} T_0 & \subset & T_1 & \subset & T & \subset N \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow & . \\ \psi^{-1}(\delta_h^-) \times \left[0, \frac{1}{2}\right] & \subset & \partial N \times \left[0, \frac{1}{2}\right] & \subset & \partial N \times [0,1] & \end{array}$$

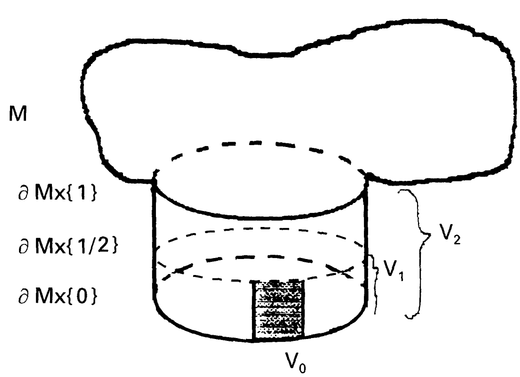

Fig. 3. The tubular neighborhoods V_0, V_1, V_2.

We construct the 3-manifold

$$X = M - int(V_0) \bigcup_{\partial V_0 \cong \partial T_0}^{id} N - int(T_0)\,,$$

which has the boundary

$$\partial X = \partial M - int(\varphi^{-1}(\delta_g^+)) \bigcup_{\partial\varphi^{-1}(\delta_g^+)) \cong \partial\psi^{-1}(\delta_h^-)} \partial N - int(\psi^{-1}(\delta_h^-))\,.$$

Now we have an homeomorphism

$$\partial X \xrightarrow{\varphi\sharp\psi} \Sigma_g - int(\delta_g^+)) \bigcup_{\partial\delta_g^+ \simeq \partial\delta_h^-} \Sigma_h - int(\delta_h^-) = \Sigma_{g+h}$$

of ∂X in the standard surface of genus $g+h$ by simply taking the connected sum of the homeomorphism φ and ψ (on the respective subsets). The uniqueness of the tubular neighborhood implies that $(X, \varphi\sharp\psi) \in MB_g$ does not depend on the various choices we made but only on (M, φ) and (N, ψ). We put therefore

$$[M, \varphi] \otimes [N, \psi] = [X, \varphi\sharp\psi]\,,$$

which may be extended to a tensor structure on W_* by linearity. Suppose now that for another two elements $(M', \varphi') \in MB_g, (N', \psi') \in MB_h$ the same construction yields the marked manifold $(X', \varphi'\sharp\psi') \in MB_{g+h}$. Then we have an homeomorphism between the closed 3-manifolds $X \cup_{\varphi\sharp\psi\circ(\varphi'\sharp\psi')^{-1}} X'$ and $M \cup_{\varphi\psi'^{-1}} M'\sharp N \cup_{\psi\,\psi'^{-1}} N'$. Since F is a multiplicative invariant we derive the compatibility of the tensor structure on W_* and the bilinear forms B_F.

Set now $w_g = [H_g, id] \in W_g$. We show that w_g is the weight vector. Obviously $w_{g+h} = w_g \otimes w_h$. Let $a \in Homeo(\Sigma_g, \delta_g^+, \delta_g^-)$ representing a class in $\mathcal{M}_g^+$. Then

$$\rho_g(a)([H_g, id]) = [H_g, a]\,,$$

and

$$B_F([H_g, id], [M, \psi]) = F(H_g \cup_{\psi^{-1}} M)\,,$$

$$B_F([H_g, a], [M, \psi]) = F(H_g \cup_{a\psi^{-1}} M)\,.$$

But a extends to a homeomorphism $A: H_g \longrightarrow H_g$. Therefore we have an homeomorphism between $H_g \cup_{\psi^{-1}} M$ and $H_g \cup_{a\psi^{-1}} M$ obtained by gluing A and id_M and taking a quotient. This gives $F(H_g \cup_{\psi^{-1}} M) = F(H_g \cup_{a\psi^{-1}} M)$. Since B_F is non-degenerate we derive $[H_g, 1] = [H_g, a]$, hence w_g is $\rho_g(\mathcal{M}_g^+)$-invariant. Now

$$F(M(\varphi)) = B_F([H_g, \varphi], [\overline{H_g}, 1]) = \langle \rho_g(\varphi) w_g, \overline{w}_g \rangle\,.$$

Since f is non-trivial and multiplicative $F(S^3) = 1$ so $d = 1$. This ends the proof of our theorem. □

Consider now the set of cobordisms M with boundary $\partial M = \partial_1 M \cup \partial_2 M$, where we suppose for simplicity that $\partial_j M$ are connected. Denote

$$MB_{g_1,g_2} = \{(M, \varphi_1, \varphi_2), \varphi_j: \partial_j \to \Sigma_{g_j}\}/\text{modulo}$$
$$\text{homeomorphisms compatible with the markings } \varphi_j\,.$$

We have a multilinear mapping induced by the invariant F

$$u_F^1 : \mathbf{C}\langle MB_g\rangle \times \mathbf{C}\langle MB_{g,h}\rangle \times \mathbf{C}\langle MB_h\rangle \longrightarrow \mathbf{C}$$

defined on the generators by

$$u_F^1((M,\varphi),(N,\varphi_1,\varphi_2),(P,\psi)) = F(M \cup_{\varphi\,\varphi_1^{-1}} N \cup_{\varphi_2\psi_1^{-1}} P)\,.$$

Suppose that $[M_1,\varphi]=[M_2,\psi]$. Then

$$F(M_1 \cup_{\varphi\theta^{-1}} Q) = F(M_2 \cup_{\psi\theta^{-1}} Q)$$

for any $(Q,\theta)\in MB_g$. So u_F^1 induces a map

$$u_F : \mathbf{C}\langle MB_{g,h}\rangle \to W_g \otimes W_h\,.$$

We can identify W_h and its dual W_h^* by means of the form B_F, so we think of u_F as having an image in $Hom(W_g,W_h)$. We have also a twist composition

$$\mathscr{M}_h \times \mathbf{C}\langle MB_{g,h}\rangle \times \mathbf{C}\langle MB_{h,k}\rangle \longrightarrow \mathbf{C}\langle MB_{g,k}\rangle$$

extending linearly the composition

$$(\varphi,(M,\psi_1,\psi_2),(N,\mu_1,\mu_2)) \to (M \cup_{\psi_2\varphi\mu_1^{-1}} N,\psi_1,\mu_2)\,.$$

We denote $\xi_1=(M,\psi_1,\psi_2)$, $\xi_2=(N,\mu_1,\mu_2)$ and their twist composition by $\xi_1\times_\varphi\xi_2$. Observe that u_F has a simple expression as element of $Hom(W_g,W_h)$, given by

$$u_F(\xi_1)([Q,\theta]) = \left[Q\cup_{\theta\psi_1^{-1}} M,\psi_2\right]\,.$$

We have also a twisted version of the composition of morphisms:

$$\mathscr{M}_h \times Hom(W_g,W_h)\times Hom(W_h,W_k) \longrightarrow Hom(W_g,W_k)\,,$$

given by

$$(\varphi,a,b)\longrightarrow b\times_\varphi a = b\circ\rho_h(\varphi)\circ a\,.$$

Proposition 2.7. *We have* $u_F(\xi_1\times_\varphi\xi_2)=u_F(\xi_1)\times_\varphi u_F(\xi_2)$.

Proof. Consider (Q,θ) arbitrary. Then

$$u_F(\xi_1\times_\varphi\xi_2)([Q,\theta]) = \left[Q\cup_{\theta\psi_1^{-1}} M\cup_{\psi_2\varphi\mu_1^{-1}} N,\mu_2\right]$$

and

$$\begin{aligned} u_F(\xi_1)\times_\varphi u_F(\xi_2)([Q,\theta]) &= u_F(\xi_2)\circ\rho_h(\varphi)\left(\left[Q\cup_{\theta\psi_1^{-1}} M,\psi_2\right]\right)\\ &= u_F(\xi_2)\circ\left(\left[Q\cup_{\theta\cup_1^{-1}} M,w_2\varphi\right]\right)\\ &= \left[Q\cup_{\theta\psi_1^{-1}} M\cup_{\psi_2\varphi\mu_1^{-1}} N,\mu_2\right]\end{aligned}$$

and we are done. □

Remark 2.8. We observe that when we write the morphism u_F the dependence on F is not explicit. In fact from W_g and ρ_g we recover all the information on F. In the same manner we can get a functor from the category of all cobordisms (so with not necessarily connected boundaries) into the category of hermitian vector spaces. This is usually called a topological quantum field theory TQFT (in dimension $2+1$) ([Ati89, Wit89]).

Corollary 2.9. i) *Any multiplicative invariant extends canonically to a TQFT.*
ii) *The invariants coming from TQFTs form a complete set of invariants.*

Remark that the computation of the TQFT extending an invariant is not always obvious. An example is given in [Fun93c]. The general case will be discussed in Sect. 5, once we obtain the structure of a WTHR, following the same pattern.

We can make further some easy simplifications. First we consider the orbit of the weight vector $O_g = \rho_g(\mathcal{M}_g(w_g))$ and set $\widetilde{W_g} = Span(O_g) \subset W_g$.

Lemma 2.10. ρ_* *restricts to a tensor representation on* $\widetilde{W_*}$.

Proof. It suffices to prove that $\widetilde{W_*}$ has a tensor structure, so

$$\widetilde{W_g} \otimes \widetilde{W_h} \subset \widetilde{W_{g+h}}\,.$$

Let us consider

$$x = \sum_i a_i \rho_g(g_i) w_g \in \widetilde{W_g} \quad \text{and} \quad y = \sum_i b_i \rho_g(g_i') w_g \in \widetilde{W_h}, a_i, b_i \in \mathbf{C}\,.$$

Then we can write

$$x \otimes y = \sum_{i,j} a_i b_j \rho_g(g_i) w_g \otimes \rho_h(g_j') wh\,.$$

Now

$$\rho_g(x) w_g = [H_g, \tilde{x}] \quad \text{and} \quad \rho_h(y) w_h = [H_h, \tilde{y}]$$

for two lifts $\tilde{x}, \tilde{y}$ in the appropriated homeomorphisms groups. The construction of the tensor structure enables us to obtain

$$\rho_g(x) w_g \otimes \rho_h(y) w_h = [H_{g+h}, \tilde{x} \# \tilde{y}] = \rho_{g+h}(\tilde{x} \# \tilde{y}) w_{g+h}\,,$$

from which we derive our claim. □

Thus we may restrict ourselves to the case when W_g is spanned by O_g since $f(\rho_*, W_*) = f(\rho_*, \widetilde{W_*})$. In this case the h.t.r. will be called a cyclic h.t.r.

We shall make some remarks concerning the irreducibility of h.t.r. The tensor subspace $H_* \subset W_*$ is an invariant tensor subspace if H_g is an invariant subspace of W_g for all g and $H_g \otimes H_h \subset H_{g+h}$ (it is a tensor vector subspace). If equality holds above we say that H_* is fully invariant. An h.t.r. is (weakly) irreducible if it does not contain proper fully invariant tensor subspaces. Set

$$H_g^{\perp} = \{z \in W_g \colon \langle x, V \rangle = 0 \text{ for all } v \in H_g\}\,.$$

Suppose we have a cyclic but not irreducible h.t.r. and H_* is an invariant tensor subspace. Set

$$\pi_1 \colon H_g \longrightarrow Z_g = H_g / H_g \cap H_g^{\perp}\,,$$

$$\pi_2 \colon H_g^{\perp} \longrightarrow V_g = H_g^{\perp} / H_g \cap H_g^{\perp}$$

for the cononical projections. We shall decompose

$$w_g = z'_g + v'_g \quad \text{with } z'_g \in H_g, v'_g \in H_g^{\perp}.$$

This decomposition is not necessarily unique. We have induced hermitian forms $\langle \cdot \rangle$ on Z_g and V_g. Since ρ_* is unitary we find that $H_g^{\perp}$ is an invariant subspace, henceforth $H_g \cap H_g^{\perp}$ is also invariant. Thus we have two induced representations of $\mathcal{M}_g$ into $U(Z_g)$ and $U(V_g)$ respectively. Set $z_g = \pi_1(z'_g)$ and $v_g = \pi_2(v'_g)$. Since the h.t.r. is cyclic the vectors z_g, v_g are nonzero.

Proposition 2.11. *Suppose that H_* is a fully invariant tensor subspace. Then the induced representations $\rho_{g,Z}$ and $\rho_{g,V}$ are in WHTR, with weight vectors z_g and v_g respectively. The associated invariants satisfy*

$$f(\rho_*, W_*) = f(\rho_{*,Z}, Z_*) + f(\rho_{*,V}, V_*).$$

Proof. Take $a \in \mathcal{M}_g^+$. Then $\rho_g(a)w_g = w_g$, hence

$$\rho_g(a)z'_g - z'g = \rho_g(v'_g) - v'g \in H_g \cap H_g^{\perp}.$$

Therefore $\rho_g(a)z_g = z_g, \rho_g(a)v_g = v_g$. Now we claim that $H_*^{\perp}$ is a tensor vector space with the induced structure. In fact $x \in H_g^{\perp}, y \in H_h^{\perp}$ implies $\langle x \otimes y, z \rangle = 0$ for all $z \in H_g \otimes H_h = H_{g+h}$. Thus $x \otimes y \in H_{g+h}^{\perp}$. This implies that $H_* \cap H_*^{\perp}$ is a tensor (vector) subspace, hence V_g and Z_g will be tensor vector spaces. The compatibility between the hermitian and the tensor structures is immediate. Finally

$$\begin{aligned}\langle \rho_g(x)(w_g), \overline{w}_g \rangle &= \langle \rho_g(x)(z'_g), \overline{z'}_g \rangle + \langle \rho_g(x)(v'_g), \overline{v'}_g \rangle \\ &= \langle \rho_g(x)(z_g), \overline{z}_g \rangle + \langle \rho_g(x)(v_g), \overline{v}_g \rangle,\end{aligned}$$

and this ends the proof of the proposition. □

Remark 2.12. In the case when the hermitian form $\langle \cdot \rangle$ is positive we may complete W_g to a tensor structure of Hilbert spaces. This will be called the geometric (or unitary) situation. Then $H_g \cap H_g^{\perp} = \phi$ and we find that we have an induced h.t.r. on H_* for any invariant (not necessary fully invariant) tensor subspace H_*.

3. Geometric Representations of the Mapping Class Group

We shall restrict now to the geometric situations, and also, we assume that the representations ρ_g are finite dimensional. The invariants which will be derived are called rational.

Let us consider $c_* = \{c_1, c_2, \ldots, c_{3g-3}\}$ be a cut system (see [HT82]) on Σ_g. The Dehn twists around the curves in the cut system generate an abelian subgroup $\mathbf{Z}^{3g-3}$ of $\mathcal{M}_g$. Now we know that a finite family of pairwise commuting unitary operators on W_g could be simultaneously diagonalized. We shall carry out this diagonalization procedure in all genera by taking into account the tensor structure of W_*. Then the decomposition of W_g into the sum of eigenspaces of a fixed operator will be iterated and we shall obtain the sewing rules of conformal blocks in a RCFT.

We wish to derive firstly a comparison result for the blocks W_g in different genera. Consider some curve c lying in the cut system c_* on Σ_{g+h}.

We suppose that c is a separating curve so $\Sigma_{g+h} - c = \Sigma_{g,1} \cup \Sigma_{h,1}$. Set $\mathcal{M}_{g+h}(c)$ for the subgroup of $\mathcal{M}_{g+h}$ generated by the homeomorphisms φ having the property that $\varphi(c)$ is isotopic to c. We put then

$$W_{g+h|1,0} = \mathrm{Span}\langle \rho_{g+h}(x) w_{g+h}; x \in \mathcal{M}_{g+h}(c)\rangle \ .$$

Let d_c denote the Dehn twist around c and $t_c = \rho_{g+h}(d_c)$. We consider the eigenspaces of t_c, namely

$$W_{g+h|\lambda} = \langle x \in W_{g+h}; t_c x = \lambda x\rangle \ .$$

Remark that all these subspaces are $\mathbf{C}[\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}]$-modules. In fact $\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1} \subset \mathcal{M}_{g+h}$ is a subgroup contained in the centralizer of d_c so

$$t_c \rho_{g+h}(u) w_{g+h} = \rho_{g+h}(u d_c) w_{g+h} = \lambda \rho_{g+h}(u) w_{g+h}$$

for all $u \in \mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}$ having the property that $\rho_{g+h}(u) w_{g+h} \in W_{g+h|\lambda}$. The algebra $\mathbf{C}[\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}]$ is an integral domain hence $W_{g+h|\lambda}$, for $\lambda \neq 1$ splits into simple modules:

$$W_{g+h|\lambda} = \bigoplus_i W_{g+h|\lambda,i} \ ,$$

where $W_{g+h|\lambda,i}$ are simple cyclic $\mathbf{C}[\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}]$-modules. When $\lambda = 1$ we observe that $W_{g+h|1,0} \subset W_{g+h|1}$ and the above decomposition takes the form

$$W_{g+h|1} = W_{g+h|1,0} \bigoplus_i W_{g+h|1,i} \ .$$

Here $W_{g+h|1,0}$ is not necessary simple but all the rest are simple cyclic $\mathbf{C}[\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}]$-modules. Consider now

$$W_{g|1} = \mathrm{Span}\langle \rho_{g+h}(z) w_{g+h}; z \in \sigma(\mathcal{M}_g, 1) = \mathcal{M}_{g,1} \otimes 1 \subset \mathcal{M}_{g+h}\rangle \ ,$$

$$W_{h|1} = \mathrm{Span}\langle \rho_{g+h}(z) w_{g+h}; z \in \sigma(1, \mathcal{M}_h) = 1 \otimes \mathcal{M}_{g,1} \subset \mathcal{M}_{g+h}\rangle \ .$$

We have natural isomorphisms $W_g \simeq W_{g|1}$ and $W_h \simeq W_{h|1}$ given respectively by

$$x \to x \otimes w_h \quad \text{and} \quad x \to w_g \otimes x \ .$$

Denote for instance by $\tilde{\otimes}$ the tensor structure on W_* which a priori has nothing to do with the natural tensor product of vector spaces.

Lemma 3.1. *The natural map*

$$\theta\colon W_{g|1} \otimes W_{h|1} \simeq W_g \otimes W_h \overset{\tilde{\otimes}}{\to} W_{g+h|1,0}$$

is an isomorphism.

Proof. Since $W_{g+h|1,0}$ is a cyclic $\mathbf{C}[\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}]$-module, $w_g \tilde{\otimes} w_h = w_{g+h}$, and $\theta(W_{g|1} \otimes W_{h|1})$ is also a $\mathbf{C}[\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}]$-module it follows that θ is onto. It remains to prove that θ is injective. Consider

$$z = \sum_{i,j} a_{ij} z_i \otimes t_j \in \ker(\theta) \ ,$$

where $z_i = \rho_{g+h}(\sigma(x_i, 1))w_{g+h}$ and $t_j = \rho_{g+h}(\sigma(1, y_j))w_{g+h}$, with $x_i \in \mathcal{M}_g, y_j \in \mathcal{M}_h$. We can compute now

$$\theta(z) = \sum_{i,j} \rho_{g+h}(\sigma(x_i, y_j))w_{g+h} = 0 .$$

Therefore

$$\langle \theta(z), u \tilde{\otimes} v \rangle = 0$$

holds for all $u \in W_g$ and $v \in W_h$. This implies that

$$\sum_{i,j} a_{ij} \langle \rho_g(x_i)w_g, u \rangle \langle \rho_h(y_j)w_h, v \rangle = 0$$

for all u and v. Since the hermitian product $\langle , \rangle$ is non-degenerate we derive $a_{ij} = 0$ hence $z = 0$ and our claim follows. □

As a consequence we derive that the map $\tilde{\otimes} : W_g \otimes W_h \to W_{g+h}$ is injective hence

$$\dim(W_{g+h}) \geq \dim(W_g)\dim(W_h) .$$

Suppose now that $(\lambda, i) \neq (1, 0)$. We consider the generators $w_{g+h}(c; \lambda, i)$ for the $\mathbf{C}[\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}]$-modules $W_{g+h|\lambda,i}$. We set

$$Z_{g|\lambda,i} = \mathrm{Span}\langle \rho_{g+h}(z)w_{g+h}(c; \lambda, i); z \in \sigma(\mathcal{M}_g, 1)\rangle .$$

$$Z_{h|\lambda,i} = \mathrm{Span}\langle \rho_{g+h}(z)w_{g+h}(c; \lambda, i); z \in \sigma(1, \mathcal{M}_g)\rangle .$$

In an obvious manner $Z_{g|\lambda,i}$ is a $\mathbf{C}[\mathcal{M}_{g,1}]$-module which decompose further into simple (and cyclic) $\mathbf{C}[\mathcal{M}_{g,1}]$-modules:

$$Z_{g|\lambda,i} = \sum_{j=1}^{s^+(i)} W_{g|\lambda,i,j} ,$$

and in a similar manner

$$Z_{h|\lambda,i} = \sum_{j=1}^{s^-(i)} W_{h|\lambda,i,j} ,$$

We wish to construct a natural mapping

$$\theta_{i,j,k}: W_{g|\lambda,i,j} \otimes W_{h|\lambda,i,k} \to W_{g+h|\lambda,i}$$

similar to θ. We choose the generators $w^+_{g+h}(c; \lambda, i, j)$ for the cyclic $\mathbf{C}[\mathcal{M}_{g,1}]$-modules $W_{g|\lambda,i,j}$ and the generators $w^-_{g+h}(c; \lambda, i, k)$ for the $\mathbf{C}[\mathcal{M}_{h,1}]$-modules $W_{h|\lambda,i,k}$. Observe that $w^+_{g+h}(c; \lambda, i, j), w^-_{g+h}(c; \lambda, i, k) \in W_{g+h|\lambda,i}$. Consider $z = x \otimes 1 \in \mathcal{M}_{g,1} \otimes 1 \subset \mathcal{M}_{g+h}$ and $t = 1 \otimes y \in 1 \otimes \mathcal{M}_{g,1} \subset \mathcal{M}_{g+h}$. We set

$$\theta_{i,j,k}(\rho_{g+h}(z)w^+_{g+h}(c; \lambda, i, j) \otimes \rho_{g+h}(t)w^-_{g+h}(c; \lambda; i, k)) = \rho_{g+h}(x \otimes y)w_{g+h}(c; \lambda, i) ,$$

which extends by linearity to $W_{g|\lambda,i,j} \otimes W_{h|\lambda,i,k}$. This map is well-defined. Indeed suppose that

$$v_0 = \sum_u a_u \rho_{g+h}(z_u)w^+_{g+h}(c; \lambda, i, j) = 0 .$$

Since $w^+_{g+h}(c; \lambda, i, j) \in W_{g+h,\lambda,i}$ we find that

$$\rho_{g+h}(s)v_0 = 0$$

for all $s \in \mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}$. But such s and z commute with each other. On the other hand the module L defined by

$$0 \subset L = \mathrm{Span}\langle \rho_{g+h}(\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}) w^{+}_{g+h}(c; \lambda, i, j)\rangle \subset W_{g+h|\lambda,i} ,$$

is a nontrivial $\mathbf{C}[\mathcal{M}_{g,1} \otimes \mathcal{M}_{h,1}]$-module so we derive

$$L = W_{g+h|\lambda,i} .$$

Thus $w_{g+h}(c; \lambda, i) \in L$ so

$$\sum_u a_u \rho_{g+h}(z_u) w_{g+h}(c; \lambda, i) = 0 ,$$

which implies $\theta_{i,j,k}(v_0 \otimes w) = 0$ for all w so $\theta_{i,j,k}$ is well-defined. The same argument based on simplicity implies that $\theta_{i,j,k}$ is onto.

Lemma 3.2. *The map $\theta_{i,j,k}$ is injective.*

Proof. Consider $s_0 = \sum_{u,v} a_{uv} X_u \otimes Y_v \in \ker(\theta_{i,j,k})$, where

$$X_u = \rho_{g+h}(z_u) w^{+}_{g+h}(c; \lambda, i, j) ,$$

$$Y_v = \rho_{g+h}(t_v) w^{-}_{g+h}(c; \lambda, i, k)$$

are chosen so that $\{X_u; u\}$ and $\{Y_v; v\}$ are bases of $W_{g|\lambda,i,j}$ and $W_{h|\lambda,i,k}$ respectively. We suppose that a_{uv} are not all zero and let

$$L = \mathrm{Span}\langle X_u; u \text{ is such that } a_{uv} \neq 0 \text{ for some } v\rangle .$$

Therefore $\rho_{g+h}(t) s_0 = 0$ for all $t \in \mathcal{M}_{g,1} \otimes 1$ hence $0 \subset L \subset W_{g|\lambda,i,j}$ is a nontrivial $\mathbf{C}[\mathcal{M}_{g,1}]$-module. The simplicity hypothesis implies that $L = W_{g|\lambda,i,j}$. Therefore we have some unitary matrices L_u acting on $W_{g|\lambda,i,j}$ such that:

i) For any $X \in W_{g|\lambda,i,j}$ the elements $\{L_u(X); u\}$ form a basis of $W_{g|\lambda,i,j}$,
ii) We have $\sum_{u,v} a_{uv} L_u(X) \otimes Y_v = 0$ for all X.

A similar reasoning on the Y_v's yields the existence of the unitary matrices S_v satisfying the analog of condition (i) and

$$\sum_{u,v} a_{uv} L_u \otimes S_v(X \otimes Y) = 0$$

for all X, Y. But the matrices $\{L_u \otimes S_v; u, v\}$ are linearly independent in $\mathrm{End}(W_{g|\lambda,i,j} \otimes W_{h|\lambda,i,k})$ so $a_{uv} = 0$. Thus our claim follows. □

As an immediate consequence the spaces $W_{g|\lambda,i,j}$ for arbitrary j are all isomorphic. Let us denote by $W_{g|\lambda,i}$ this isomorphism class if $(\lambda, i) \neq (1,0)$ and $W_{g|1,0} = W_g$ elsewhere. The above two lemmas permit to conclude

Proposition 3.3. *To a separating curve c in the cut system there is associated the following splitting of the target space of an unitary weight h.t.r.:*

$$W_{g+h} \simeq \bigoplus_{(\lambda,i)} W_{g|\lambda,i} \otimes W_{h|\lambda,i} .$$

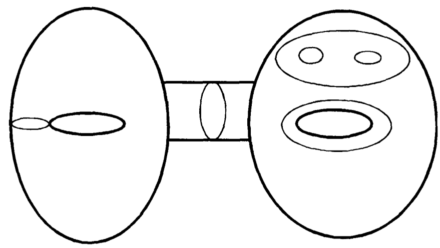

Fig. 4. An extended cut system.

It is clear that for a non-separating curve c the space W_g splits into the eigenspaces of t_c which are also $\mathbf{C}[\mathcal{M}_{g,2}]$-modules. We consider

$$W_{g+1|1;1} = \langle x \in W_{g+1}; t_e x = x \rangle ,$$

where e is the edge associated to the non-separating curve c. Denote also by

$$S_{g+1} = \operatorname{Span}\langle \rho_{g+1}(\mathcal{M}_{g,2} \otimes 1) w_{g+1} \rangle \subset W_{g+1|1;1} .$$

Both S_{g+1} and $W_{g+1|1;1}$ are $\mathbf{C}[\mathcal{M}_{g,2}]$-modules. Now the tensor product with w_1 establishes an isomorphism between W_g and S_{g+1} which will be useful further.

So we obtained upon now some natural inclusions

$$W_g \otimes W_h \rightarrow W_{g+h|1,0} \hookrightarrow W_{g+h}$$

and

$$W_g \rightarrow S_{g+1} \hookrightarrow W_{g+1}$$

depending on the choice of some curve in the cut system. On the other hand we have the splitting of the block W_g according to Proposition 3.3. There is an obvious one in the non-separating case. We wish to iterate this procedure until all the curves of the cut system are cut off.

A cut system c_* defines a dual 3-valent graph Γ of genus g which is usually called by physicists a ϕ^3-diagram. Its vertices are in one-to-one correspondence with the connected components of $\Sigma_g - c_1 \cup c_2 \cup \ldots \cup c_{3g-3}$, which are all isomorphic to a sphere with 3 holes ($g > 1$). Two vertices are adjacent if the boundaries of the closures of the corresponding components contain the same curve c_j. It is convenient to enlarge the notion of cut system such that the case $g = 1$ fits also in this description. An extended cut system $c_* = \{c_1, c_2, \ldots, c_{3g-3+2h}\}$ (on Σ_g) is given by a collection $\{c_{3g-2+h}, c_{3g-1+h}, \ldots, c_{3g-3+h}\}$ of h disjoint embedded circles in Σ_g which bound the 2-disks $\delta_1, \delta_2, \ldots, \delta_h \subset \Sigma_g$ together with the cut system on the h-holed surface $\Sigma_{g,h} = \Sigma_g - \cup_{i=1}^{h} \delta_i$. The associated graph $\Gamma = \Gamma(c_*)$ has $2g - 2 + h$ vertices of valence 3 and h vertices of valence 1 which we call leaves. Let $V(\Gamma)$ denote the set of 3-valent vertices of Γ, $\partial\Gamma$ be the set of leaves, $E(\Gamma)$ be the set of edges and $F(\Gamma)$ be the subset of edges incident to the leaves. The graph Γ is planar. Once we have chosen an orientation of the plane, say the clockwise one, we have a cyclic order on the set of edges incident to a vertex. If $v \in V(\Gamma)$ let $\{e_1(v), e_2(v), e_3(v)\}$ be the set of the edges incident to v which are clockwise ordered. We shall write e also for the curve of the cut system associated to the edge e when no confusion arises.

Define

$$Z(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3)) = \langle x; t_{e_i(v)}x = \lambda_i x\rangle \subset W_g \,.$$

Further the choice of some $v \in V(\Gamma)$ determines an embedding $\Sigma_{0,3}(v) \subset \Sigma_g$, hence a morphism $\mathscr{M}_{0,3} \simeq \mathscr{M}_{0,3} \otimes 1 \to \mathscr{M}_g$, (which is an injection if the vertex is 3-valent) corresponding to take the connected sum with the identity outside $\Sigma_{0,3}(v)$. This induces on $Z(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3))$ a structure of a $\mathbf{C}[\mathscr{M}_{0,3}]$-module since any $\varphi \in \mathscr{M}_{0,3} \otimes 1$ commutes with $d_{e_i(v)}$.

We deduce a splitting

$$Z(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3)) = \bigoplus_j W(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3))(j)$$

into simple and cyclic $\mathbf{C}[\mathscr{M}_{0,3}]$-modules, each of them generated by some $w(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3))(j) \in W_g$. This means that

$$W(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3))(j) = \mathrm{Span}\langle \rho_g(\mathscr{M}_{0,3}) w(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3))(j)\rangle \,.$$

On the other hand suppose that a labeling $l: E(\Gamma) \to \mathbf{C}$ is chosen. It will be always supposed that $l(F(\Gamma)) = 1$. We set

$$W_g(l) = \langle x; t_e x = l(e)x; e \in E(\Gamma)\rangle \subset W_g$$

for the eigenspace corresponding to l. It follows that $W_g(l)$ is a $\mathbf{C}[\mathscr{M}_{0,3} \otimes \mathscr{M}_{0,3} \otimes \cdots \otimes \mathscr{M}_{0,3}]$-module. This structure is induced from the map

$$\mathscr{M}_{0,3} \otimes \mathscr{M}_{0,3} \otimes \cdots \otimes \mathscr{M}_{0,3} \to \mathscr{M}_g$$

which represents the connected sum of homeomorphisms defined on the various components $\Sigma_{0,3}$ using the graph Γ. Therefore $W_g(l)$ splits into simple and cyclic submodules

$$W_g(l) = \bigoplus_j W_g(l)(j)$$

which are respectively generated by $w(l, j)$. Set also

$$W(\Gamma, l) = \bigotimes_{v \in V(\Gamma)} W(\Gamma, v, (l(e_1(v)), l(e_2(v)), l(e_3(v))))(j_v) \,.$$

We claim that we have an isomorphism of $\mathbf{C}[\mathscr{M}_{0,3} \otimes \mathscr{M}_{0,3} \otimes \cdots \otimes \mathscr{M}_{0,3}]$-modules given by

$$\bigotimes_v \sum_i a_{iv} \rho_g(x_{iv} w(\Gamma, v, (l(e_1(v)), l(e_2(v)), l(e_3(v)))(j_v))$$

$$\to \sum_{i_1,\dots,i_r} \left(\prod_{s \in V(\Gamma)} a_{i_s s} \right) \rho_g(x_{i_1 1} \otimes x_{i_2 2} \otimes \cdots \otimes x_{i_r r}) w(l, j) \,,$$

where r is the cardinal of $V(\Gamma)$. The fact that this application is well-defined follows as in Lemma 3.2. Also as a morphism between simple modules it is an isomorphism. We derive that $W(\Gamma, v(\lambda_1, \lambda_2, \lambda_3))(j)$ are isomorphic for all j. $W_g(l)(j)$ are also isomorphic for all j, and we denote by $W(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3))$ and respectively by $W(\Gamma, l)$ these isomorphism classes.

Set

$$L = \{\lambda \in \mathbf{C}^*; \text{ such that } \lambda \text{ or } \lambda^{-1} \text{ is an eigenvalue for some } t_e, e \in E(\Gamma)\} \,.$$

Then we may restrict ourselves to the set of labelings $\mathscr{L}$ taking values in L.

We obtained the following splitting

$$W_g = \bigoplus_{l \in \mathscr{L}} \bigoplus_{j=1}^{s(l)} \bigotimes_{v \in V(\Gamma)} W(\Gamma, v, (l(e_1(v)), l(e_2(v)), l(e_3(v))))$$

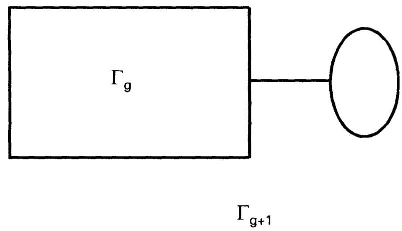

Fig. 5. The inclusion $\Gamma_g \subset \Gamma_{g+1}$.

into the primary blocks $W(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3))$. A priori these primary blocks may depend upon the extended cut system c_*, the choice of $v \in \Gamma$ and of the ordered set $e_1(v), e_2(v), e_3(v)$.

Extension Lemma 3.4. *The primary blocks do not depend upon the extension* $\tilde{c}_*$ *of the cut system* c_*.

Proof. This is clear since c_i are bounding for $i > 3g-3$ so $t_{c_i} = 1$. □

First Stabilization Lemma 3.5. *Assume that there is only one vector* $w_1 \in W_1$ *which is* $SL^+(2, \mathbf{Z})$*-invariant. Consider* $c_{*,g} \subset \Sigma_g$ *and* $c_{*,g+1} \subset \Sigma_{g+1}$ *having the properties*:

1. *if we identify* Σ_{g+1} *as* $\Sigma_g \# S^1 \times S^1$ *then* $c_{*,g+1|\Sigma_g} = c_{*,g}$.
2. *If* Γ_g *and* Γ_{g+1} *are the dual graphs then these are positioned as in Fig. 5. Let* $v \in \Gamma_g \subset \Gamma_{g+1}$.

Therefore we have an isomorphism

$$W(\Gamma_g, v, (\lambda_1, \lambda_2, \lambda_3)) \simeq W(\Gamma_{g+1}, v, (\lambda_1, \lambda_2, \lambda_3)) .$$

Proof. We choose the labels of the additional edges to be 1. These outer labels are irrelevant in the definition of $W(\Gamma_g, v, (\lambda_1, \lambda_2, \lambda_3))$. We claim that

$$W_g(l) \simeq W_{g+1}(l') ,$$

where l' is the extension of the labeling l by 1. Consider that e is the new separating edge (see Fig. 5). Then we have

$$\begin{aligned} W_{g+1}(l') &= \langle x \in W_{g+1|1}; t_f x = l'(f)x \text{ for all } f \neq e \rangle \\ &= \langle x \in W_{g+1|1,0}; t_f x = l'(f)x \rangle \bigoplus_{j>0} \langle x \in W_{g+1|1,j}; t_f(x) = l'(f)x \rangle . \end{aligned}$$

Further we know from Lemma 3.1. that the first space decompose as a tensor product

$$\langle x \in W_{g+1|1,0}; t_f x = l'(f)x \rangle = \langle x \in W_g; t_f x = l(f)x \rangle \otimes W_1^{SL^+(2,\mathbf{Z})} .$$

On the other hand each space from the second term decompose also in a tensor product according to Proposition 3.3.

$$\langle x \in W_{g+1|1,j}; t_f x = l'(f)x \rangle = \langle x \in W_{g|1,j}; t_f(x) = l(f)x \rangle \otimes \langle x \in W_{1,j}; t_a x = x \rangle ,$$

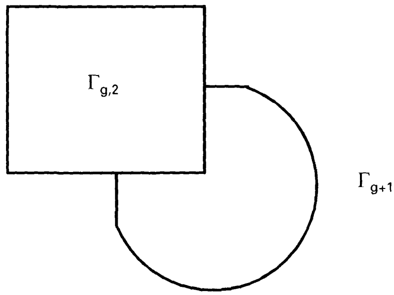

Fig. 6. The inclusion $\Gamma_{g,2} \subset \Gamma_{g+1}$.

where a is the meridian of the torus. We know that $W_{1|1} = \oplus_{j \geqq 0} W_{1|1,j}$ and the assumption of lemma implies

$$\langle x \in W_{1|1,j}; t_a x = x \rangle = 0 \text{ if } j > 0 .$$

This will establish our claim. But now we find that

$$Z(\Gamma_g, v, (\lambda_1, \lambda_2, \lambda_3)) \simeq Z(\Gamma_{g+1}, v, (\lambda_1, \lambda_2, \lambda_3))$$

as $\mathbf{C}[\mathcal{M}_{0,3}]$-modules so the lemma follows. □

Second Stabilization Lemma 3.6. *Assume that there is only one vector $w_1 \in W_1$ which is $SL^+(2, \mathbf{Z})$-invariant. Consider $c_{*,g,2} \subset \Sigma_{g,2}$ and $c_{*,g+1} \subset \Sigma_{g+1}$ having the properties:*

1. *if we identify Σ_{g+1} as $\Sigma_g \# S^1 \times [0,1]$ then $c_{*,g+1|\Sigma_{g,2}} = c_{*,g,2}$.*
2. *if $\Gamma_{g,2}$ and Γ_{g+1} are the dual graphs then these are positioned as in Fig. 6. Let $v \in \Gamma_{g,2} \subset \Gamma_{g+1}$ and suppose the leaves of $\Gamma_{g,2}$ are labeled by 1. Consider a simple path p in $\Gamma_{g,2}$ between the endpoints of the new attached edge e and a vertex v not incident to the path p. Therefore we have an isomorphism $W(\Gamma_{g,2}, v, (\lambda_1, \lambda_2, \lambda_3)) \simeq W(\Gamma_{g+1}, v, (\lambda_1, \lambda_2, \lambda_3))$.*

Proof. We use the same method as above but we look this time at the non-separating curve corresponding to the edge e. We shall use now only the labelings l which take the value 1 on the edges of the path p, and denote by l' the extension by 1 on e. We claim that

$$W_g(l) \simeq W_{g+1}(l')$$

holds. Remark that

$$W_{g+1}(l') = W_{g+1|1:1} \cap \langle x \in W_{g+1}; t_f x = l(f)x \rangle .$$

and

$$W_{g+1|1:1} \supset S_{g+1} .$$

Then we have an isomorphism

$$S_{g+1} \cap \langle x \in W_{g+1}; t_f x = l(f)x \rangle \simeq W_g(l) .$$

coming from the identification of S_{g+1} and W_g. Consider the circuit $p \cup e$ which from the geometric viewpoint represents a great (holed) torus which is attached to a surface of genus h with s holes for obtaining the surface of genus $g+1$. Remark that this torus is attached in s places depending on the combinatorics of the path p (see Fig. 7).

Now a decomposition principle holds also in the non-separating case as

$$W_{g+1|1:1} = S_{g+1} \oplus_r W_{g,r} ,$$

where $W_{g;r}$ are isomorphic simple $\mathbf{C}[\mathcal{M}_{g,2}]$-submodules of W_g. The great torus has the attaching edges $f_1, f_2, \ldots, f_s$ all labeled by 1. We wish now to change the splitting procedure as follows: we cut first all the edges $f_1, f_2, \ldots, f_s$ and in final the edge e. This does not matter for the primary blocks we considered. The first $s-1$ edges now are non-separating and the last one is separating. A recurrence on s permits to obtain

$$\langle x \in W_{g+1}; t_{f_i}x = x; i = 1, s-1\rangle \simeq W_{g-s+1} \oplus_r W_{g-s+1;r} ,$$

where $W_{g-s+1,r}$ are simple cyclic $\mathbf{C}[\mathcal{M}_{g-s+1,s}]$-submodules of W_{g-s}. But the last move will separate the genus $g+1$-surface into a genus $g-s$ surface with s holes and the great torus. Following the Extension Lemma the space associated to this torus does not depend upon the number of leaves, being in fact isomorphic to W_1. We have according to the splitting principle

$$\langle x \in W_{g+1}; t_{f_i}x = x; i = 1, s\rangle \simeq W_{g-s} \otimes W_1 \oplus_r W_{g-s|r} \otimes W_{1|r} ,$$

hence

$$\langle x \in W_{g+1}; t_{f_i}x = x; i = 1, s \text{ and } t_e x = x\rangle$$

$$\simeq W_{g-s} \otimes W_1^{SL^+(2,\mathbf{Z})} \oplus W_{g-s|1,j} \otimes W_{1|1,j}^{SL^+(2,\mathbf{Z})} .$$

As in the previous lemma we conclude that

$$\langle x \in W_{g+1}; t_{f_i}x = x; i = 1, s \text{ and } t_e x = x\rangle \simeq W_{g-s} .$$

This implies our claim and we are done. □

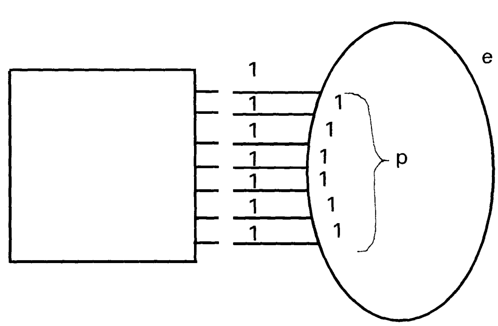

Fig. 7. The attached great torus.

Homogeneity Lemma 3.7. *Let $\sigma \in Aut(\Gamma)$ be a combinatorial isomorphism preserving the cyclic order on edges incident to a vertex. Then*

$$W(\Gamma, v, (\lambda_1, \lambda_2, \lambda_3)) \simeq W(\Gamma, \sigma(v), (\lambda_1, \lambda_2, \lambda_3))$$

holds.

Proof. Any such σ admits a lift $\varphi \in Homeo(\Sigma_g, c_*)$. Therefore $\rho_g(\varphi)$ induces the wanted isomorphism. □

Lemma 3.8. *The primary blocks do not depend upon the choice of the vertex $v \in \Gamma$.*

Proof. We claim that for every pair of vertices $v_1, v_2 \in \Gamma$ we may use extensions and stabilizations of $\Gamma \subset \Gamma'$ such that the images of v_1 and v_2 become equivalent under $Aut(\Gamma')$. Then the homogeneity lemma will conclude.

Also it suffices to check our claim for pairs of adjacent vertices by using a recurrence on the length of the shortest path between them (Γ is arcwise connected).

We may enlarge the stabilization procedure to include also the transformation from Fig. 8. The conclusion of the stabilization lemmas remains valid for this type of transformations on the cut system level because we may use a recurrence. Here A and B stands for 3-valent graphs eventually with leaves.

Now the general situation of v_1 and v_2 in Γ is depicted in Fig. 9, where some of the graphs $A, B, \ldots, H$ may be void and B, D, G, E may be disconnected. We stabilize this graph using the pattern from Fig. 10.

Now v_1 and v_2 are equivalent under the rotation of angle π of the plane. □

So we can drop the index v from the indices of a primary block.

Lemma 3.9. *The cyclic permutations on the labels don't change the isomorphism class of primary blocks.*

Proof. We use the same method as above. The general position of a 3-valent vertex in Γ is described in Fig. 11. We stabilize Γ as in Fig. 12. Then we may perform the cyclic permutations of the edges e_1, e_2, e_3 using the automorphism of the stabilized graph. The homogeneity lemma proves our claim. □

Lemma 3.10. *The label set L and the spaces $W(\Gamma, (\lambda_1, \lambda_2, \lambda_3))$ do not depend on the cut system.*

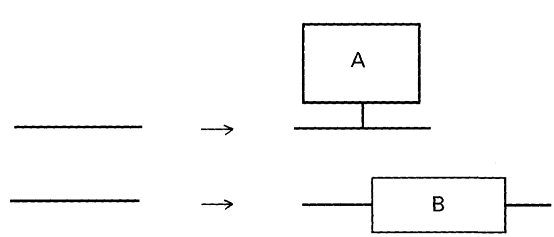

Fig. 8. The stabilization procedure.

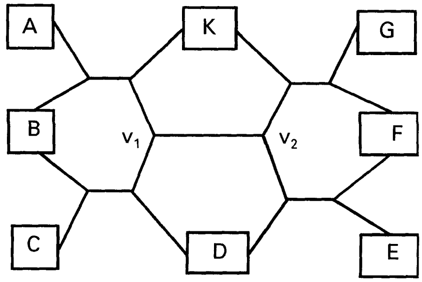

Fig. 9. The relative position of vertices.

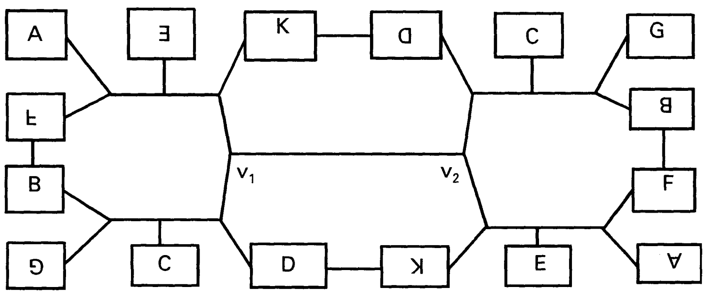

Fig. 10. The stabilized graph.

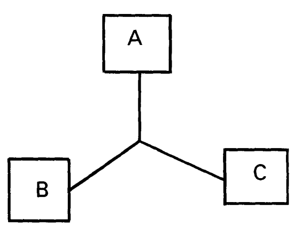

Fig. 11. The position of a vertex.

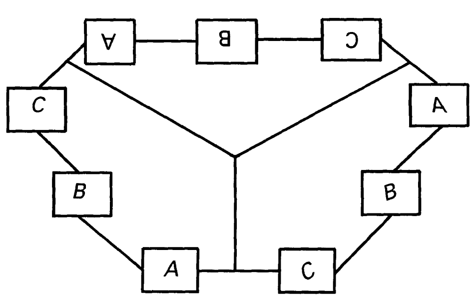

Fig. 12. The stabilized graph.

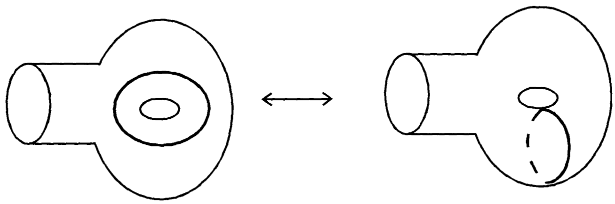

Fig. 13. The C operation.

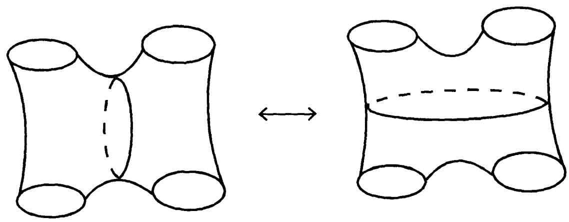

Fig. 14. The F operation.

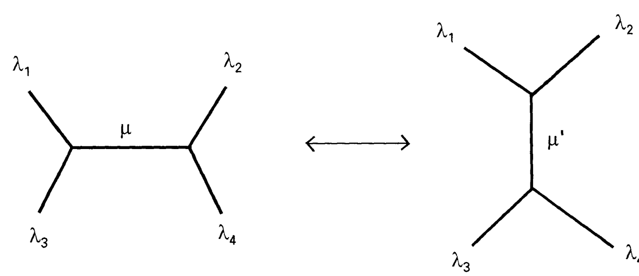

Fig. 15. The move F on the graph level.

Proof. A theorem of Hatcher and Thurston ([HT82]) states that two cut systems $c_{*,0}$ and $c_{*,1}$ on a surface are obtained one from the other by a sequence of operations C and F and their inverses. The operations C and F are described in Figs. 13 and 14.

The move C does not affect the graph Γ and replace α by β. Now the following relation

$$\alpha\beta\alpha = \beta\alpha\beta$$

holds in $\mathcal{M}_{1,1}$. Therefore $\beta = \alpha\beta\alpha(\alpha\beta)^{-1}$ is conjugate to α so the eigenvalues of t_α and t_β coincide. Further the map

$$\rho_g(t_\alpha t_\beta \otimes 1) : W(\Gamma, (\lambda_1, \lambda_2, \lambda_3)) \to W(C\Gamma, (\lambda_1, \lambda_2, \lambda_3))$$

is an isomorphism if the vertex v considered is incident to α in Γ.

The move F changes the graph according to picture 15.

Consider now ω_i the class of the homeomorphism which interchanges e_i and e_{i+1} in the mapping class group $\mathcal{M}_{0,4}$. It is well-known that $\omega_i, i = 1, 2, 3$ and

$t_{e_i}, i = 1,2,3,4$ generate $\mathscr{M}_{0,4}$. We have further

$$t_{c_1} = t_{e_2}^{-1} t_{e_3}^{-1} t_{\omega_2}^2 ,$$

$$t_{c_2} = t_{e_1}^{-1} t_{e_2}^{-1} t_{\omega_1}^2 ,$$

so $t_{c_2} = t_{e_1}^{-1} t_{e_3} \omega_1 \omega_2 t_{c_1} (\omega_1 \omega_2)^{-1}$. We wish to realize the primary block $W(\Gamma,(\lambda_1, \lambda_3, \mu))$ in both labeled graphs Γ and $F\Gamma$. So in the two labeled graphs from picture 16 we must find a pair of vertices having the same circular labels. From the previous two lemmas it suffices to check only one case, namely $\mu = \lambda_1, \mu' = \lambda_3$. Thus $\mu' = \lambda_1^{-1}\lambda_3\mu$. So from the relation $t_{c_1}x = \mu x$ we shall derive $t_{c_2}\rho_g(\omega_1\omega_2 \otimes 1)x = \mu' \rho_g(\omega_1\omega_2 \otimes 1)x$. Hence the map $\rho_g(\omega_1\omega_2 \otimes 1)$ gives an isomorphism between the primary blocks $W(\Gamma,(\lambda_1,\lambda_2,\lambda_3))$ and $W(F\Gamma,(\lambda_1,\lambda_2,\lambda_3))$ corresponding to the fixed vertices. This proves also that the label set is invariant. □

As an immediate consequence of these lemmas we derive

Theorem 3.11. *Assume that the cyclic vector generating the h.t.r. is the unique vector $SL^+(2,\mathbf{Z})$-invariant in genus $g = 1$. Then the target spaces of a cyclic geometric h.t.r. of $\mathscr{M}_*$ have the following decomposition*:

$$W_g \overset{i_\Gamma}{\simeq} \bigoplus_{l\in\mathscr{L}} \bigoplus_{j=1}^{s(l)} \bigoplus_{v\in V(\Gamma)} W(l(e_1), l(e_2), l(e_3))$$

into primary blocks $W(i,j,k)$.

Remark now that the tensor structure $W_g \otimes W_h \rightarrow W_{g+h}$ is given by the usual tensor product of vector spaces according to Lemma 3.1. Now once we have chosen an embedding of graphs $\Gamma_g \cup \Gamma_h \hookrightarrow \Gamma_{g+h}$ we have a corresponding multiplication rule for labelings $\mathscr{L}_g \times \mathscr{L}_h \hookrightarrow \mathscr{L}_{g+h}$ by extending the product labeling by 1 on the new edge and preserving the labels of an edge after we introduced a new vertex on it (so defining two adjacent edges). This induces the tensor structure on the decomposed blocks in an obvious manner.

Remark 3.12. In the infinite dimensional unitary context the h.t.r. of $\mathscr{M}_*$ into $U(W_*)$ has a Hilbert completion to a h.t.r. into $U(\overline{W_*})$. Then the set of labels may be infinite and the direct sum replaced by an integral but the same decomposition principle holds for the completed blocks. The proof is essentially the same.

Observe finally that we have chosen an orientation of each circle of the cut system, without any restriction because we must distinguish between t_e and t_e^{-1}. The change of the orientation of a curve corresponds to change the eigenvalue λ into $\frac{1}{\lambda}$. But we may restrict to some almost canonical choices. We look at the standard surface of genus g without the two disks bounded by δ_g^+, δ_g^- as being an oriented cobordism between the two circles. Each trinion lying will be therefore an oriented cobordism between its positive boundary and its negative boundary. Suppose we have 2 circles labeled j and k in the positive boundary and one circle labeled by i as the negative boundary. Therefore we specify in the primary block associated to the vertex-trinion $W(i,j,k)$ by putting the indices differently as W_{jk}^i. So we shall encounter 4 types of (oriented) primary blocks $W^{ijk}, W_{j^*k^*}^i, W_{k^*}^{ij}, W_{i^*j^*k^*}$ which are all isomorphic. But when we write the decomposition of the block W_g this notational convention specifies the orientation of all circles in the cut system.

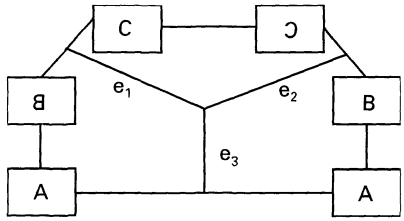

Fig. 16. Another stabilization.

Lemma 3.13. *We have the symmetries*

$$W_{jk}^{-i} \simeq W_{kj}^{i} ,$$

$$W_{jk}^{i} \simeq W_{ji^{-1}}^{k^{-1}} .$$

Proof. The proof is similar to that of the invariance of the primary blocks to cyclic permutations. Specifically we stabilize the graph from Fig. 11 to arrive at the graph depicted in Fig. 16. In the first case, when the edges e_1 and e_2 correspond to oriented circles lying on the positive boundary of the trinion. We can interchange e_1 and e_2 using a homeomorphism $\varphi \in Homeo(\Sigma_g, c_*)$ preserving the orientation. In the second case the homeomorphism φ interchanges e_1 and e_2^{-1}, hence the change of the labeling. □

This permits to start with a cut system and to obtain the decomposition specifying the orientation of each circle.

4. The Structure of Rational Geometric Invariants

Our aim now is to get a similar decomposition for the representation ρ_* which follows the decomposition of target spaces.

We shall consider a groupoid which is closely related to the mapping class group having a tensor structure itself, and which is called the Teichmuller groupoid in [Dri91] or the duality groupoid in physical literature [MS89]. If T_g denotes the Teichmuller space [Abi77, Gro84] then $\mathscr{M}_g$ acts properly discontinuous on T_g and the quotient $\mathscr{M}_g = T_g/\mathscr{M}_g$ is the moduli space of genus g non-singular algebraic curves. Due to the presence of curves with automorphisms M_g is not smooth but a V-manifold (see [Sat75, Wol83]) or a Q-manifold [Mat72, Mum74]. The set of its non-singular points M_g^{ns} is an open manifold, and we shall consider its (fundamental) path groupoid $\Pi_1(M_g^{ns})$. This is the duality groupoid D_g. It will become clear that it has a tensor structure when we derive another description of D_g.

We remember that an alternative description of M_g is as the moduli space of hyperbolic structures on Σ_g (or conformal structures). For $c \in c_*$ we set $l(c)$ for the hyperbolic length of the geodesic lying in the isotopy class of c, for an hyperbolic structure on Σ_g. But now the hyperbolic trinions up to conformal or anticonformal equivalence are determined by the lengths of boundary circles (which we suppose to be geodesic). Consider now the geodesic connecting two boundary circles and which are orthogonal to them. Fix the order of the loops in the cut system. There are two orthogonal geodesics which intersect a boundary circle c. Set Δl for the oriented-distance between their endpoints and consider the angles $\theta(c) = \mathrm{Arcsin}(\Delta l / l(c)) \in [0, 2\pi)$. Now the $(3g-3)$ pairs $(l(c), \theta(c))$ give a function $f_{c*}: T_g \to \mathbf{R}^{3g-3} \times (S^1)^{3g-3}$. It is a result of Bers which says that f_{c_*} is a $\mathbf{Z}^{3g-3}$-covering and the Galois group is the subgroup of $\mathcal{M}_g$, generated by the Dehn twists around the cut circles. These are the so-called Fenchel–Nielsen coordinates on Teichmuller space; notice they are real analytic coordinates (see [Abi77]).

To an unitary representation $\rho_g: \mathcal{M}_g \to U(W_g)$ there is associated an holomorphic flat hermitian and $\mathcal{M}_g$-invariant vector bundle over T_g, such that the monodromy of the mapping class group is precisely ρ_g. Further this bundle descends to a flat holomorphic V-bundle E_g on M_g. Equivalently the pull-back of E_g on a smooth finite covering of M_g is a flat holomorphic bundle. Such a smooth covering is well-known to be the moduli space of algebraic curves with a level l structure.

Now there is a canonical identification of W_g with the space of flat sections of the V-bundle $E_g \mid_{M_g}$. The set $f_{c*}^{-1}((0, \varepsilon)^{3g-3} \times (0, \pi)^{3g-3}))$ is a disjoint union of contractible open sets in T_g (for little ε) on which $\mathbf{Z}^{3g-3}$ acts freely. The flat and $\mathcal{M}_g$-invariant sections over one such contractible set U_{c*} may be analytically continued at all of T_g (modulo the path groupoid action). The monodromy representation we get this way is nothing but the initial ρ_g from the beginning. Taking another cut system c'_* or another coordinate chart, (i.e. we consider $f_{c*}^{-1}(\prod_{j=1,3g-3}(l_j, l_j + \varepsilon) \times \prod_{j=1,3g-3}(v_j, v_j + \pi)))$. We shall get a matrix which relates the two basis of flat sections $\mathcal{M}_g$-invariant obtained by analytic continuation. Therefore we have a representation of the groupoid G_g acting on the set of cut systems, so in particular on labeled 3-valent graphs (with leaves). In our case the particular labelings are the Fenchel–Nielsen coordinates and some extra marking from the identification of $\mathbf{Z}^{3g-3}$ as a subgroup of $\mathcal{M}_g$. We can get a covering for T_g by taking a sufficiently large family of points (l_j, v_j). Now we project on M_g and we find that we can extract an open covering with contractible sets of $M_g^{ns} - \{$a neighborhood of the variety of singular points$\}$. Since the path groupoid is a homotopy invariant and the singular locus is triangulable we derive $G_g \simeq D_g$. Hence we may describe D_g by looking only at its action on labeled 3-valent graphs. It is a result of Moore and Seiberg [MS89] (which in particular settles a question raised by Grothendieck) which asserts that D_g is generated by finitely many moves and relations among them. Specifically the five duality moves can be described geometrically as in Pictures 17–21. There is another operation called braiding which can be described as the composition of F and Ω moves (see Picture 22) or alternatively, by a change in the pants decomposition (the cut system is changed but the dual graph remains the same), as in Picture 23.

The fact that these five moves suffice to generate D_g is easy to prove: in fact S and F act transitively on the set of 3-valent graphs (with a fixed number of leaves) with fixed labels. T ensures the $\mathbf{Z}^{3g-3}$ marking, and Ω, Θ acts transitively on the set of Fenchel–Nielsen labels.

Another way to look at these moves is the following: observe that S and T are classes in $\mathcal{M}_{1,1}$, Ω and Θ are lying in $\mathcal{M}_{0,3}$ and $F \in \mathcal{M}_{0,4}$. The original statement of the Grothendieck conjecture states that $\mathcal{M}_{1,1}, \mathcal{M}_{0,3}, \mathcal{M}_{0,4}$ generate the whole tower of groups $\mathcal{M}_{*,*}$. What it means to generate is clear: to every decomposition of a h-holed surface $\Sigma_{g,h}$ into pieces homeomorphic to a 1-holed torus, a trinion or a 4-holed sphere we get a subgroup of $\mathcal{M}_{g,h}$ by gluing the homeomorphisms defined

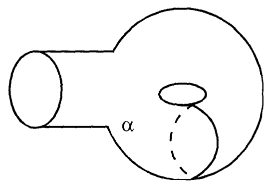

Fig. 17. T = Dehn twist around α.

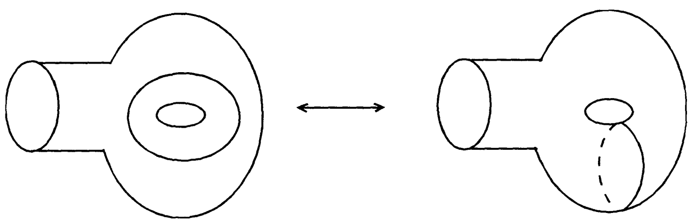

Fig. 18. S corresponding to the C-move on cut systems.

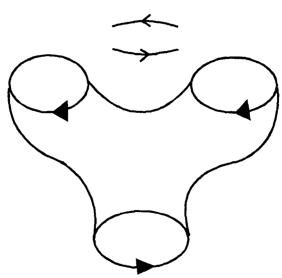

Fig. 19. Ω interchanges two boundary circles.

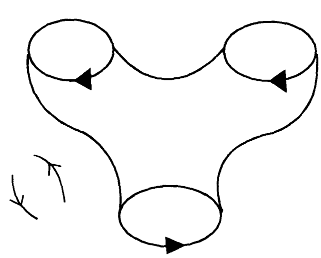

Fig. 20. Θ interchanges two boundary circles differently oriented.

on each piece. When we carry out this procedure for all possible decompositions we obtain a family of subgroups which together generate $\mathscr{M}_{g,h}$.

Proposition 4.1. *The representation ρ_* extends naturally to a h.t.r. of the whole duality groupoid D_*.*

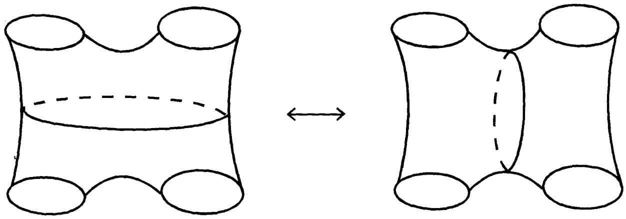

Fig. 21. F coming from the F move on cut systems.

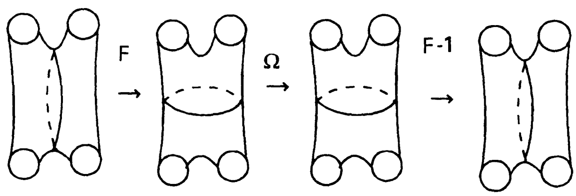

Fig. 22. The braiding move B.

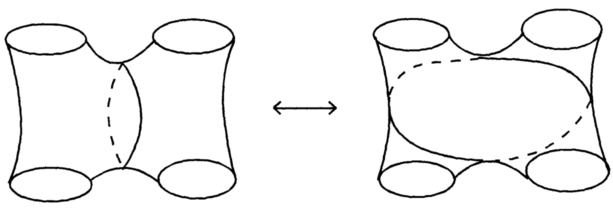

Fig. 23. Changing the cut system for braiding.

We have seen that ρ_g extends naturally to a representation of D_g. Consider $\Sigma_{1,1}$ a 1-holed torus embedded in Σ_g. Then there exists a cut system c_* on Σ_g containing the boundary of $\Sigma_{1,1}$. Actually when looking at W_g as being identified to $i_{\Gamma(c_*)}(W_g)$ we see that $\mathcal{M}_{1,1} \otimes 1 \subset \mathcal{M}_g$ acts only on the primary blocks corresponding to the vertex associated to $\Sigma_{1,1}$. So we have a family of transformations

$$S(j)\colon \bigoplus_i W^i_{ji} \to \bigoplus_i W^i_{ji},\ j \in L\,,$$

$$T(j)\colon \bigoplus_i W^i_{ji} \to \bigoplus_i W^i_{ji},\ j \in L\,,$$

which together give a representation of $\mathcal{M}_{1,1}$ for each j. But the map $T(j)$ acts by multiplication by i on W^i_{ji}, hence $T(j) = T$ is a diagonal matrix which does not depend upon the external index j. A priori all these representations depend upon the choice of the particular embedding of the 1-holed torus in Σ_g. Fortunately this is not the case due to

Lemma 4.2. *The primary blocks $W(i, j, k)$ are $\mathbf{C}[\mathcal{M}_{0,3}]$-modules, not only vector spaces, which depend only on the labels not on the particular choices we made in the previous section.*

Proof. All the isomorphism we get in Lemmas 3.1–3.9 are module isomorphisms. □

Since $\mathcal{M}_{1,1} \hookrightarrow \mathcal{M}_{0,3}$ we derive that $S(j)$ and T are independent on the particular embedding chosen.

For the moves Θ and Ω we obtain in the same manner the family of isomorphisms

$$\Omega^i_{jk}(-): W^i_{jk} \to W^i_{jk},\ \Omega^i_{jk}(+) = \Omega^i_{jk}(-)^*,$$

$$\Theta^i_{jk}(-): W^i_{jk} \to W^{k-1}_{ji-1},\ \Theta^i_{jk}(+) = \Theta^i_{jk}(-)^*.$$

Geometrically these arise as follows: we identify the trinion with a domain in the plane $D - D_1 \cup D_2$, where $D_i \subset D$ are equal 2-disks. Consider another disk $D_0 \subset D$ containing D_i and an homeomorphism of D which is identity outside D_0, and the rotation by π which interchanges the disks D_1 and D_2 on a smaller disk contained in D_0.

This time it is not a representation of $\mathcal{M}_{0,3}$ which is obtained but of an object related to it. Let $\varepsilon : \{1, 2, 3\} \to \mathbf{Z}/2\mathbf{Z}$ be the signature of the boundary where the circle numbered $j = 1, 2, 3$ lies on. Here we adopt the previous convention by looking at the 3-holed sphere as to an oriented cobordism. A homeomorphism h of $\Sigma_{0,3}$ which preserves globally the boundary (but not necessarily pointwise) induces a permutation of the boundary circles leading to another marking $h^*(\varepsilon) \in (\mathbf{Z}/2\mathbf{Z})^3$. We consider the triples (ε, h(*modulo isotopy*), $h^*(\varepsilon)$). Their set is the mapping class groupoid $\mathcal{M}_{0,3}(2)$ of the 2-colored (or oriented) 3-holed sphere. In the same manner the mapping class groupoid of c-colored h-holed surface of genus g could be defined. So actually the mappings Θ and Ω (together with S and T) define a representation of this groupoid $\mathcal{M}_{0,3}(2)$. Again this structure is uniquely defined from the previous considerations and Lemma 3.13.

Finally the move F (called also the fusion move) define the isomorphisms

$$F\begin{bmatrix} i & j \\ k & l \end{bmatrix} : \bigoplus_{r\in L} W^k_{ir} \otimes W^r_{jl} \to \bigoplus_{s\in L} W^k_{sl} \otimes W^s_{ij}.$$

Its action is induced from that of $t_{\omega_1} t_{\omega_2}$. But ω_i are both lying in a $\mathcal{M}_{0,3}$-factor (for two different cut systems). So each of them is canonically defined henceforth the mappings F do not depend on the particular 4-holed sphere used. Otherwise it is simple to check that the spaces on which F acts are $\mathcal{M}_{0,4}$-modules intrinsically defined.

On the other hand these isomorphisms must define a representation of the mapping class group. Using the identities from [MS89] we derive that the following conditions must be verified:

$$F(\Omega(\varepsilon) \otimes 1)F = (1 \otimes \Omega(\varepsilon))F(1 \otimes \Omega(\varepsilon)), \tag{1}$$

$$F_{23}F_{12}F_{23} = P_{23}F_{13}F_{12}, \tag{2}$$

$$S^2(j) = \bigoplus_{i\in L} \Theta^i_{ji}(-), \tag{3}$$

$$S(j)TS(j) = T^{-1}S(j)T^{-1}, \tag{4}$$

$$(S \otimes 1)(F(1 \otimes \Theta(-)\Theta(+)F^{-1})(S^{-1} \otimes 1) = FPF^{-1}(1 \otimes \Omega(-)) \tag{5}$$

with the usual convention: F_{ij} acts on the i^{th} and j^{th} factors of a tensor product, and P_{ij} interchanges these factors.

But once these conditions are satisfied we know from [MS89] that the five moves define a representation of whole duality groupoid D_g which respect to the tensor structure. □

Several comments are necessary now. We know that the h.t.r. admits also a weight vector w_g, which is uniquely defined by the weight condition at level 1. We say that the vacuum is irreducible if this condition is fulfilled in each genus. We have the splitting

$$W_g \simeq \bigoplus_{l\in\mathscr{L}} \bigoplus_{j=1}^{s(l)} W(\Gamma_g, l),$$

where we denoted

$$W(\Gamma_g, l) = \bigotimes_{v\in V(\Gamma)} W(\Gamma_g, v, l(e_1), l(e_2), l(e_3)).$$

Since w_1 is uniquely determined we derive

$$W_{11}^1 \simeq \mathbf{C} w_0$$

and $w_1 = w_0 \otimes w_0$. In particular

$$w_g = w_0^{\otimes 2g} \in W_{11}^1 \otimes \cdots \otimes W_{11}^1 = W(\Gamma_g, \mathbf{1}),$$

where we used for Γ_g the simplest 3-valent graph of genus g with 2 leaves. Above $\mathbf{1}$ stands for the labeling identical 1. In particular if the vacuum is irreducible it follows that $s(\mathbf{1}) = 1$. Because the theory is a cyclic one generated by w_g and the representation of D_g is defined on the primary blocks directly (and not on sums of primary blocks) we obtain $s(l) = 1$ for all labelings l. So the splitting principles has the canonical form

$$W_g \simeq \bigoplus_{l\in\mathscr{L}} \bigotimes_{v\in\Gamma_g} W(l(e_1), l(e_2), l(e_3)).$$

We use now this expression to compute W_1 in the case of two graphs which may be seen in Fig. 24. Suppose all the representations ρ_g are finite dimensional and denote by $n_g = \dim_{\mathbf{C}} W_g$ and $n_{jk}^i = \dim_{\mathbf{C}} V_{jk}^i$. It follows $n_1 = \sum_i n_{1i}^i = \sum_{i,j} (n_{1j}^i)^2$. Therefore

$$W_{1j}^i \simeq \delta_{ij} \mathbf{C},$$

where δ_{ij} states for the Kronecker symbol.

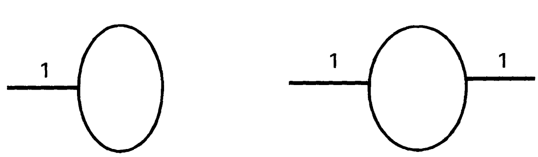

Fig. 24. Two graphs of genus 1.

Remark also that $w_g = w_0^{2g}$ is in fact a weight vector for our representation. For the group of Dehn twists around curves which bound in the handlebody this is already clear. But from the description given by Suzuki (see also [Cra91, Koh92]) we derive that w_g is in fact $\mathcal{M}_g^+$-invariant.

As a notational convenience we denote by $\exp(2\pi\sqrt{-1}\Delta_j)$ the eigenvalue corresponding to j, this time j being a natural number. This is possible since all the matrices are unitary.

Using the relations in $\mathcal{M}_{0,3}$ we derive that $\Omega(\varepsilon)^2$ can be expressed in terms of the Dehn twists around the boundary circles as

$$\Omega(-)^2 = t_1^{-1} t_2 t_3 .$$

This implies that

$$\Omega^i_{jk}(-)^2 = \exp(2\pi\sqrt{-1}(\Delta_j + \Delta_k - \Delta_i))\mathbf{1}_{n^i_{jk}} ,$$

$$\Theta^i_{jk}(-)^2 = \exp(2\pi\sqrt{-1}(\Delta_i + \Delta_k - \Delta_j))\mathbf{1}_{n^i_{jk}} ,$$

where $\mathbf{1}_n$ stands for the identity matrix of rank n. From the geometric interpretation we shall have natural identifications of the bases on the spaces W^i_{jk}, W^i_{kj} and $W^{k^*}_{ij^*}$ which we call σ_{12}, σ_{13} and σ_{23} respectively. These will produce a representation of the symmetric group S_3 and we are able to get the following form for the matrices Ω and Θ (in this bases):

$$\Omega^i_{jk}(-) = \exp(\pi\sqrt{-1}(\Delta_j + \Delta_k - \Delta_i))(\delta_{a,\sigma_{12}a}) ,$$

$$\Theta^i_{jk}(-) = \exp(\pi\sqrt{-1}(\Delta_j + \Delta_k - \Delta_i))(\delta_{a,\sigma_{13}a}) ,$$

where the indices a run in a basis for W^i_{jk}.

Now from this data we can recover the representation ρ_* as follows: Suppose we take Γ_g be again the simplest 3-valent graph with 2-leaves (see Fig. 25). Then W_g is identified with

$$\oplus W^1_{i_1 i_1} \otimes W^{i_1}_{i_1 j_1} \otimes W^{j_1}_{i_2 k_2} \otimes \cdots \otimes W^{j_{g-1}}_{i_g i_g} \otimes W^{i_g}_{i_g 1} .$$

We consider as generators of the mapping class group the Dehn twists around the curves $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \delta_2\}$ as in Picture 26 (see [MS89, Bir74]). Then

$$\rho_g(\alpha_1) = T_1^{-1} , \tag{6}$$

$$\rho_g(\alpha_l) = T_{i_{l-1}}^{-1}(B^-_{j_{l-1}}[i_{l-1} \quad i_l \quad k_{l-1} \quad k_l] B^-_{j_{l-1}} \begin{bmatrix} i_l & i_{l-1} \\ k_l k_{l-1} & \end{bmatrix}) T_{i_l}^{-1}$$

$$= F_{j_{l-1}} \begin{bmatrix} i_{l-1} & i_l \\ k_{l-1} & k_l \end{bmatrix} T_{j_{l-1}} F^{-1}_{jl-1} \begin{bmatrix} i_l & i_{l-1} \\ k_l & k_{l-1} \end{bmatrix}$$

for $l > 1$.

$$\rho_g(\beta_l) = T_{k_l} S_{k_l i_l}(j_{l-1}, j_l) T_{k_l} , \tag{7}$$

$$\rho_g(\delta_2) = T_{i_2} . \tag{8}$$

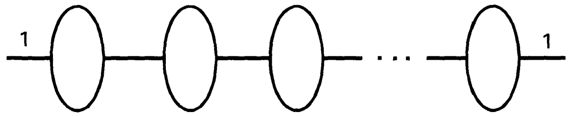

Fig. 25. The graph Γ_g.

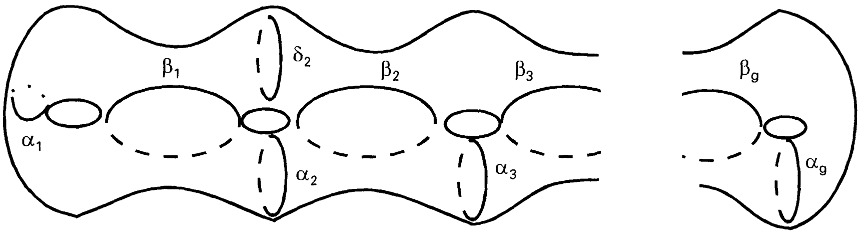

Fig. 26. Generators for $\mathscr{M}_g$.

Above we used the braiding matrix B given by

$$B = F^{-1}(1 \otimes \Omega(-))F\,.$$

Also the indices on the linear transformations tell us on which of the subspaces it acts on. Remark that what we have obtained as data for the h.t.r. is exactly the axiomatic definition of a unitary RCFT (see [MS89, Deg92]) having the central charge $c = 0$(modulo 24). This is due to the fact that we have a representation of the mapping class group not one of a central extension of it.

Theorem 4.3. *A geometric h.t.r. of* $\mathscr{M}_*$ *with irreducible vacuum is equivalent to a RCFT of central charge* 0 (*modulo* 24).

We suppose from now on that we are working with rational unitary invariants given by a h.t.r. with irreducible vacuum so the theorem above applies.

5. Reconstruction of the TQFT for Cobordisms

We obtained in the previous section the combinatorial data of a RCFT having the central charge $c = 0$(modulo 24). This allowed us to reobtain the initial representation ρ_g in terms of $(F, S, T, \Omega, \Theta)$. Our invariant is therefore given by the formula

$$F(M(\varphi)) = d^{-g}\langle \rho_g(\varphi)w_g, \overline{w_g}\rangle\,,$$

where

$$d = S(0)_{11}\,.$$

We wish to obtain a similar description for the TQFT extending the invariant F. We start with an oriented cobordism M^3 having the positive boundary ∂_+M^3 and the negative boundary ∂_-M^3. We have an analog of the Heegaard splitting for cobordisms by using instead of handlebodies the compression bodies (see [Cra91]). A compression body C may be obtained as follows: consider $c_1, c_2, \ldots, c_s \subset \Sigma_g$ disjointly embedded circles (which we suppose to be pairwise non-isotopic) which bound in H_g. Then consider

$$C_1 = \Sigma_g \times [0, 1] \cup \text{ 2-handles attached on } c_1, c_2, \dots, c_s .$$

A general compression body has the form

$$C = C_1 \cup \text{ 3-handles,}$$

permitting thus capping off the S^2 components of the boundary. We assume that $\partial_+ C = \Sigma_g$. Now a Heegaard splitting of M is a decomposition into compression bodies

$$M^3 = C_+ \cup \overline{C_-} ,$$

where the boundaries are identified as

$$\partial_- C_+ \simeq \partial_- M^3 ,$$

$$\partial_- C_- \simeq \partial_+ \overline{C_-} \simeq \partial_+ M^3 .$$

The compression bodies C_+ and C_- are glued together along their boundary components $\partial_+ C_+$ and $\partial_- \overline{C_-} \simeq \partial_+ C_-$ using some homeomorphism of Σ_g whose class in the mapping class group is φ. In order to find $F(M^3)$ it suffices to know the value of invariants on compression bodies (see [Ati89]).

We construct first the functor F on surfaces. Set

$$F(\Sigma_g) = W_g \text{ with its hermitian structure,}$$

$$F(\overline{\Sigma_g}) = W_g^* \text{ if the orientation changes,}$$

$$F(\phi) = \mathbf{C} .$$

Further for a disjoint union of surfaces we have

$$F\left(\bigcup_{i=1}^{r} \Sigma_{g_i}\right) = \bigotimes_{i=1}^{r} F(\Sigma_{g_i}) .$$

Next we have the morphisms

$$F(C_+) : F(\partial_+ C_+) \to F(\partial_- C_+) = F(\partial_- M^3) ,$$

$$F(\overline{C_-}) : F(\partial_- \overline{C_+}) \to F(\partial_+ \overline{C_+}) .$$

The second morphism is the transposed of $F(C_-)$. Using Proposition 2.7 we derive that

$$F(M^3) = F(C_+) \circ \rho_g(\varphi) \circ F(\overline{C_-}) .$$

On the other hand F is defined for cobordisms with marked boundaries, i.e. some fixed homeomorphisms

$$\varphi_+ \colon \partial_+ M^3 \to \bigcup_i \Sigma_{g_i} ,$$

$$\varphi_- : \partial_- M^3 \to \bigcup_i \Sigma_{h_i} ,$$

Suppose we choose once and for all the cut systems $c_*^0 \subset \Sigma_g$. For the compression body C we have $\partial_+ C = \Sigma_g$, $\partial_- C \simeq \cup_i \Sigma_{h_i}$, where $\sum_i h_i = g - s$. Once we have chosen a cut system $\gamma_*^+ \subset \partial_+ C$ we have the natural marking

$$\varphi^0(\gamma_*^+) : \partial_+ C \to \Sigma_g .$$

Let Γ^+ be the dual graph of γ_*. The surface Σ_g could be identified with the boundary of a tubular neighborhood of the 3-valent graph Γ embedded trivially into $\mathbf{R}^3$. Since the graph is actually planar the blackboard framing f provides the surface of a rigid structure. Then $\varphi^0(\gamma_*^+)$ is the homeomorphism of Σ_g respecting the rigid structure. A similar canonical marking may be defined on $\partial_- C$ if a cut system γ_*^- and a framing are chosen. Set Γ^+, Γ^- for the corresponding dual graphs of γ_*^+ and γ_*^- respectively. We may suppose, for simplicity, that $\partial_- C$ is connected so it is a surface of genus $h = g - s$. We start with the (eventually extended) cut system $\gamma_* \subset \Sigma_g$ which contains the attaching circles of the 2-handles, hence $\gamma_i = c_i$ for $i = 1, s$. Each curve c_i has a natural framing given by $c_i \times [-\varepsilon, \varepsilon] \subset \Sigma_g$. For small ε these tubular neighborhoods remain disjoint. Consider

$$X = \Sigma_g - \bigcup_{i=1}^{s} c_i \times [-\varepsilon, \varepsilon] \bigcup_{i=1}^{s} d_{i1} \cup d_{i2} ,$$

where d_{i1} are 2-disks (disjointly embedded in H_g) bounding $c_i \times \{-\varepsilon\}$, and respectively d_{i2} are 2-disks capping off $c_i \times \{\varepsilon\}$. Therefore

$$X = \Sigma_h \bigcup_j S^2 .$$

We shall identify the negative boundary of C with the surface Σ_h which is a boundary component of X. Consider the curves γ_i, for $i > s$, which remain drawn on this surface Σ_h. We add those curves $c_i \times \{+ - \varepsilon\}$ which also lie on Σ_h. Their set represents an extended cut system on $\Sigma_h = \partial_- C$ which we denote by $[C]\gamma_*$ and we call the transport by C of γ_*. The pieces of the framing which remain on Σ_h give the transport of the framing, hence a rigid structure on the negative boundary. Let Γ^- be its dual graph. A labeling l of $[C]_{\gamma*}$ is admissible if

$$l(x) = 1 \text{ if } x \text{ is not in } \{\gamma_i, i > s\}.$$

Any such labeling extends to a labeling l^e of γ_* (or, equivalently Γ^+) by 1. Further we have a canonical isomorphism (by the stabilization lemmas) between $W(\Gamma^-, l)$ and $W(\Gamma^+, l^e)$. We obtain a natural injective mapping

$$W_h \overset{i_{\Gamma^-}}{\simeq} \bigoplus_l W(\Gamma^-, l) \simeq \bigoplus_l W(\Gamma^+, l^e) \subset \bigoplus_l W(\Gamma^+, l) \overset{i_{\Gamma^+}}{\simeq} W_g ,$$

where in the first two direct sum the l's run over all admissible labelings of Γ^-, while the third sum is taken over all labelings of Γ^+.

Now we can get the expression of $F(C)$ for some special markings of the boundary. This is sufficient since $\mathcal{M}_g \times \mathcal{M}_h$ acts transitively on the markings. Namely we choose $\varphi^+ = \varphi^0(\gamma_*)$, and $\varphi^- = \varphi^0([C]\gamma*)$. We can state now

Proposition 5.1. *The morphism*

$$F(C, \varphi^+, \varphi^-) : W_g \to W_h$$

is the projection dual to the above described inclusion mapping.

Proof. Observe first that for a handlebody F has the wanted description because $F(H_g, id) = w_g$. This equality follows from the proof of Theorem 2.5.

On the other hand it suffices to check the result for a particular cut system since $\mathcal{M}_g$ acts transitively on the set of cut systems, and in a compatible manner on $F(C, \varphi^+, \varphi^-)$ as given above. So we consider

$$X_{g,h} = \Sigma_h \times [0, 1] \bigcup_{\Sigma_h \times 1 \supset b_i} H_{g-h},$$

where after we take the union we identify the 2-disks b_i leaving in $\Sigma_h \times 1$ and ∂H_{g-h}. Let us consider some $\varphi \in \mathcal{M}_g$, $\varphi = \varphi_1 \# id$, with $\varphi_1 \in Homeo(\Sigma_h, b_1, \dots, b_r)$. Therefore

$$X_{g,h} \cup_\varphi \overline{H_g} = \Sigma_h \times [0, 1] \cup_{\varphi_1} \overline{H_h} \#_{b_i} H_{g-h} \cup \overline{H_{g-h}}.$$

Next for any $\psi \in \mathcal{M}_h$ we have

$$F(H_h \cup_\psi X_{g,h} \cup_\varphi \overline{H_g}) = F(H_h \cup_{\varphi_1^{-1}\psi} \overline{H_h} \#_{b_i} H_{g-h} \cup \overline{H_{g-h}}),$$

since the two considered manifolds are homeomorphic. We wish to replace the quotient space on the right by an usual connected sum. Choose a null homotopic curve which passes through the centers of the 2-disks b_i in both manifolds. Then Dehn's lemma gives us two embedded disks (in $M(\varphi_1^{-1}\psi)$ and S^3 respectively) D_1 and D_2. The usual connected sum may be carried out by identifying some collars of these two disks. This says that replacing the quotient space with the connected sum has the effect of a connected sum with the S^3. Thus the homeomorphism type does not change. It follows from the multiplicativity of F that

$$F(H_h \cup_\psi X_{g,h} \cup_\varphi \overline{H_g}) = F(M(\varphi_1^{-1}\psi))F(S^3) = F(M(\varphi_1^{-1}\psi)).$$

Let $Z = \text{Span}\ \langle \rho_g(\mathcal{M}_h \otimes 1)w_g \rangle \subset W_g$. The above formula reads

$$F(X_{g,h}, id, id)|_Z = 1.$$

On the other hand $Z \simeq W_h$ which implies that we have a cross section of $F : W_g \to W_h$ given by $x \to x \otimes w_{g-h}$. Then the position of Z in W_g is that arising from the inclusion of graphs $\Gamma^- \subset \Gamma^+$. This establishes our claim. □

Remark 5.2. The value of F on compression bodies is universal because it does not depend upon the particular invariant chosen but only on the primary blocks. As a direct consequence this value (for compression bodies only) is the same in the classical RCFT associated to a compact group and for the quantum RCFT obtained from the associated quantum group (for a parameter value not a root of unity).

In the abelian TQFT (the gauge group $U(1)$) coming from the Chern–Simons–Witten theory the extension to cobordisms was described in [Fun93c].

Remark that

$$F(M \cup_\varphi N) = F(M) \circ \rho_g(\varphi) \circ F(N)$$

from Proposition 1.4, so the twist factor from the middle does not depend upon the choice of the splitting (not necessary a Heegaard splitting).

We shall give an example. If $V \xrightarrow{\pi} S^1$ is a Σ_g-bundle over the circle having the monodromy mapping $\varphi \in \mathcal{M}_g$ we decompose

$$V = \pi^{-1}([0, 1/2]) \cup \pi^{-1}([1/2, 1]).$$

Both components in the right are two cylinders over Σ_g. But the positive boundary of $\pi^{-1}([0, 1/2])$ consists into two copies of Σ_g and the other one is void. The marking may be chosen to be $(1 \otimes 1)$. The negative boundary of $\pi^{-1}([1/2, 1])$ consists also into two copies of Σ_g and we can consider the marking $(1 \otimes \varphi)$. Since

$$F(\Sigma_g \times [0, 1], 1, 1) = 1$$

we derive

$$F(\Sigma_g \times [0, 1], 1 \otimes 1) = \sum_{i=1, k} e_i \otimes e_i^* ,$$

where $\{e_1, e_2, \dots, ek\}$ is a basis for W_g. Thus

$$F(\Sigma_g \times [0, 1], 1 \otimes \varphi) = \sum_{i=1, k} e_i \otimes \rho_g(\varphi)(e_i)^* ,$$

and we can compute

$$F(V) = \sum_{i, j=1, k} \langle e_i \otimes e_i^*, e_j \otimes \rho_g(\varphi)(e_j)^* \rangle = \sum_{i=1, k} \langle e_i, \rho_g(\varphi)(e_i) \rangle$$
$$= \mathrm{tr}(\rho_g(\varphi)) ,$$

which agrees with Atiyah's formula (see [Ati89]).

Corollary 5.3. *Suppose we have a Hilbert h.t.r. yielding unitary invariants for 3-manifolds. Then $\rho_g(\mathcal{M}_g)$ consists in trace class operators on W_g.*

We wish to make a little digression on Hilbert TQFTs. The simplest example is the universal TQFT used in Theorem 2.5 in the case when a complete topological invariant F (which might exist) is chosen. This does not give however any pertinent information on the topology of 3-manifolds. Another way is to consider quasi-rational CFTs, where all spaces of intertwiners are finite dimensional but the label set L is infinite and the right hand of every fusion rule is finite. If we start with the RCFT defined by a quasi-quantum Lie group (as for example the quantum E_8 since no central charge occur) and we consider the level of the theory goes to the infinity we obtain a quasi-rational CFT. No explicit computations were done for the invariants associated to these models on our knowledge.

We outline below another example related to the Casson invariant. If $M(\Sigma_g)$ is the spaces of representations of $\pi_1(\Sigma_g)$ into G we may consider the Lagrangian Chow space $Ch(M(\Sigma_g))$ which is the space generated by all Lagrangian submanifolds (eventually with prescribed singularities) in the middle dimension up to isotopy. We fix w_g as the class of the submanifold $Hom(\pi_1(H_g), G)/G$. As the group of outer automorphisms of $\pi_1(\Sigma_g)$ the mapping class group acts on $Ch(M(\Sigma_g))$. We consider W_g to be the span of the orbit of w_g under the Torelli group. Therefore the lagrangian intersection index provides W_g with a bilinear form and a representation of the Torelli group. This way the Casson invariant for homology spheres is expressed via Theorem 2.5. It seems to be clear that the spaces W_g are infinite dimensional. However the extension of Casson invariant to all 3-manifolds is not multiplicative (as is done by Lescop [Les92]) and the associated TQFT must be a TQFT for a larger category of 3-manifolds with additional structure (see also [Fuk94]). In a sense this theory, for fixed G, is the limit of the CSW theories when the level goes to the infinity and encodes all topological information therein.

6. Colored Link Invariants

Consider $K \subset M^3$ to be a link with k components having the framing f. The framing is equivalent to the choice of k longitudes on the tori bounding the tubular neighborhood $T(K) \subset M^3$. Choose some circle on each torus which bounds a small 2-disk embedded in $S^1 \times S^1$ disjoint from the framing. This gives an extended cut system $c_*(f)$ on ∂M^3. We have further canonical identifications

$$F(\partial(M - T(K)) = W_1^{\otimes k},$$
$$F(M - T(K), f) = F(M - T(K), \varphi^0(c_*(f))) = v \in W_1^{\otimes k}.$$

The second one comes from the choice of the rigid structure on $\partial T(K)$ given by the framing f. Also we know that

$$W_1 = \bigoplus_i W_{1i}^i \quad \text{and} \quad W_{1i}^i \simeq \mathbf{C} e_i$$

with fixed unitary e_i (defined up to a modulus 1 scalar).

Suppose we have a coloring of the components of the link K, say $c : \{1, 2, \ldots, k\} \to L$. We have then a naturally associated invariant for framed colored links given by

$$F(M^3, K, f, c) = \langle v, e_{c(1)} \otimes e_{c(k)} \otimes \cdots \otimes e_{c(k)} \rangle \in \mathbf{C}.$$

Proposition 6.1. *Consider M^3 obtained by Dehn surgery on the framed link $(K, f) \subset S^3$. Then the following formula*

$$F(M) = \sum_{c \text{ coloring}} S(0)_{c(1)1} S(0)_{c(2)1} \cdots S(0)_{c(k)1} F(S^3, K, f, c)$$

holds.

Proof. We may decompose $M^3 = S^3 - T(K) \cup_\varphi T(K)$, where $\varphi = \tau \oplus \tau \oplus \cdots \oplus \tau \in SL(2, \mathbf{Z})^k$, under the framing identification. On the other hand $T(K)$ is a union of solid tori (with their canonical markings of their boundaries $\partial H_1 = \Sigma_1$) hence

$$F(T(K)) = w_1^{\otimes k} \in W_1^{\otimes k},$$

if we use the standard marking of the boundary. Therefore

$$\begin{aligned} F(M) &= \langle F(S^3 - T(K)), (\rho_1(\tau) w_1)^{\otimes k} \rangle \\ &= \sum_{c \text{ coloring}} S(0)_{c(1)1} S(0)_{c(2)1} \cdots S(0)_{c(k)1} F(S^3, K, f, c). \qquad \square \end{aligned}$$

This formula permits to recover the invariant for closed 3-manifolds once we know its values for colored links. This way was used in [Deg92, KM91, KT93] to define 3-manifold invariants.

There is another approach to obtain link invariants directly from the data of RCFT. Start with a braid representative for the link K having the strands colored (this coloring is induced from a coloring of the link components) (see Fig. 27).

Define now the spaces $W_{0,n}(c)$ where c is a strand coloration compatible with respect to the Artin's closure. Consider $\Sigma_{0,n}$ to be the sphere with n-holes, having the boundary circles c_i, $i = 1, n$. Extend the set of c_i's to an extended cut system

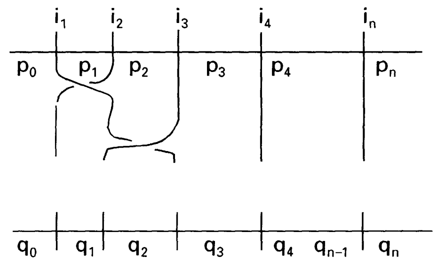

Fig. 27. A colored braid representative.

c_* on $\Sigma_{0,n}$ having the dual graph $\Gamma_{0,n}$. A labeling $l : E(\Gamma_{0,n}) \to L$ is admissible if $l(c_i) = c(i)$, where $c(i)$ is the color of the i^{th} strand. We set further

$$W_{0,n}(c) = \bigoplus_{l \text{ admissible}} W(\Gamma_{0,n}, l).$$

This definition may be done more generally for a h-holed surface $\Sigma_{g,h}$ of genus g having a fixed coloring c of the boundary components. The corresponding spaces are

$$W_{g,h}(c) = \bigoplus_l W(\Gamma_{g,h}, l),$$

the sum being taken over all the labelings extending the boundary one. Remark that whenever $\Sigma_{g,h} \cup \Sigma_{g',h'} = \Sigma_{g+g',h+h'-2s}$ under the identification of s boundary circles we have a splitting

$$W_{g+g',h+h'-2s}(c) = \bigoplus_d W_{g,h}(c_0 d) \otimes W_{g',h'}(c_1 d),$$

where c_0 is the coloring of the $h - s$ circles of $\Sigma_{g,h}$ induced by c and $c_0 d$ is the extension of c_0 by an arbitrary labeling d of the remaining circles (and similar for c_1 and $c_1 d$).

Observe that

$$\begin{aligned} W_{0,n}(i_1, \ldots, i_n) &= \bigoplus_{p_1,\ldots,p_{n-1}} W^0_{i_1 i_1} \otimes W^{i_1}_{i_2 p_1} \otimes \cdots \otimes W^{i_n}_{i_n 0} \\ &\hookrightarrow \bigoplus_{p_1,\ldots,p_{n-1},j} W^0_{i_1 i_1} \otimes W^{i_1}_{i_2 p_1} \otimes \cdots \otimes W^{in}_{inj} \simeq \bigoplus_j W_{0,n+1}(i_1, \ldots, i_n, j). \end{aligned}$$

We have a natural representation of the groupoid of c-colored braids $B_n(c)$ (see [Fun93a]) on $W_{0,n+1}(i_1, i_2, \ldots, i_n, j)$ given by

$$\rho_{0,n,j}(b_s) = 1 \otimes B_{p_s} \begin{bmatrix} i_s & i_{s+1} \\ p_s & p_{s+1} \end{bmatrix} \otimes 1, \quad \text{with } p_n = j.$$

We set

$$\rho_{0,n} = \bigoplus_j \rho_{0,n,j}.$$

We can compute $\rho_{0,n}(x)$ using (a recurrence on) the graphical resolution of crossings from Fig. 28. Finally we obtain an identity as in Fig. 29.

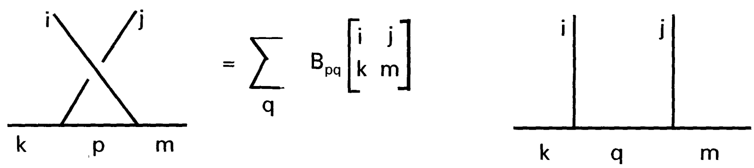

Fig. 28. The resolution of a crossing.

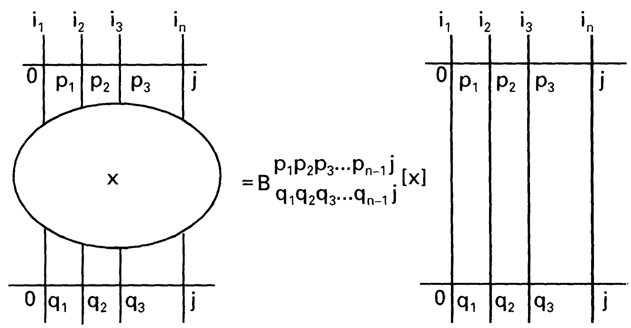

Fig. 29. The expression for $\rho_{0,n}(x)$.

We derive that the trace of the representation

$$\mathrm{tr}(\rho_{0,n}(x)|_{W_{0,n+1}(i_1,\ldots,i_n,j)}) = \sum_{p_1,\ldots,p_n} B^{p_1p_2\cdots p_{n-1}j}_{p_1p_2\cdots p_{n-1}j}(x)\,,$$

where the B^*_* are certain products of braiding matrices, depending on x. Define

$$J(x,c) = \left(\prod_{s=1}^{n} \frac{S(0)_{1i}}{S(0)_{i_s1}}\right) \sum_{j=1}^{n} \frac{S(0)_{j1}}{S(0)_{11}} \mathrm{tr}(\rho_{0,n}(x)|_{W_{0,n+1}(i_1,\ldots,i_n,j)})\,.$$

Proposition 6.2. *Let x be a braid representative for the colored link (K,c). Then $J(K,c) = J(x,c)$ defines an invariant for colored links.*

Proof. It is clear that $J(x,c)$ is constant on conjugation classes. It remains to compute $J(xb_n,c)$ for $x \in B_n(c)$. Since the last two strands belong to the same component of the link (after the closure) the induced color i_{n+1} of the $n+1$-strand is i_n.

We have the graphical identity from Fig. 30. We derive that

$$\mathrm{tr}(\rho_{0,n+1}(xb_n)|_{W_{0,n+2}}(i_1,\ldots,i_n,i_n,j) = \sum_{p_1,\ldots,p_n} B_{p_s}\begin{bmatrix} i_s & i_{s+1} \\ p_s & p_{s+1}\end{bmatrix} B^{p_1\cdots p_{n-1}p_nj}_{p_1\cdots p_{n-1}p_nj}(x)\,.$$

Observe that

$$B^{p_1\cdots p_{n-1}p_nj}_{p_1\cdots p_{n-1}p_nj}(x) = B^{p_1\cdots p_{n-1}p_n}_{p_1\cdots p_{n-1}p_n}(x)$$

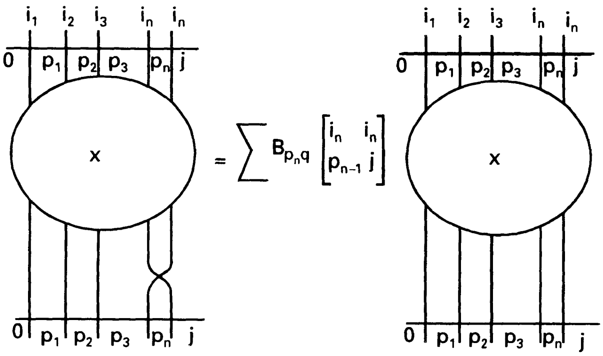

Fig. 30. The resolution for xb_n.

because the last strand is not touched in the resolution process. From the Moore–Seiberg equations we derive the identity:

$$\sum_j \frac{S(0)_{j1}}{S(0)_{11}} B_{pp}\begin{bmatrix} i & i \\ q & j\end{bmatrix}$$

$$= \frac{S(0)_{p1}}{S(0)_{11}} \sum_j B^{-1}_{1j}\begin{bmatrix}p & 1\\ p & 1\end{bmatrix} B^{-1}_{j1}\begin{bmatrix}1 & p\\ p & 1\end{bmatrix} B_{pp}\begin{bmatrix}1 & 1\\ q & j\end{bmatrix}\exp(2\pi\sqrt{-1}(\Delta_j-\Delta_p))$$

$$= \frac{S(0)_{p1}}{S(0)_{11}}\exp(2\pi\sqrt{-1}(\Delta_p-\Delta_q))\sum_j B^{-1}_{1j}\begin{bmatrix}p & 1\\ p & 1\end{bmatrix}B_{pp}\begin{bmatrix}1 & 1\\ q & j\end{bmatrix}B^{-1}_{j1}\begin{bmatrix}1 & p\\ p & 1\end{bmatrix}$$

$$= \frac{S(0)_{p1}}{S(0)_{11}}\exp(2\pi\sqrt{-1}(\Delta_p-\Delta_q)\Omega^q_{p1}(-)\Omega^q_{p1}(-)B^{-1}_{11}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} = \frac{S(0)_{p1}}{S(0)_{11}}\delta_{pq^*}\,.$$

This implies that

$$J(x,c) = J(xb_n,c)\,,$$

proving that J is in fact an invariant for colored links. □

Let f_0 be the blackboard framing of K induced from a braid representative of it. We set

$$J(K,f_0,c) = J(K,c)\,.$$

An arbitrary framing f differs from f_0 by a sequence of integers $r_1, r_2, \dots, r_k$. We define then

$$J(K,f,c) = J(K,f_0,c)\prod_{j=1}^{k}\exp(2\pi\sqrt{-1}\Delta_{c(j)}r_j)\,.$$

Observe that if we alter the framing f_0 by the same sequence of integers in the first definition of the link invariant then $\rho_1(\tau)$ changes to $\rho_1(\tau)T^{r_j}$, hence

$$F(K,f,c) = \prod_{j=1}^{k}\exp(2\pi\sqrt{-1}\Delta_{c(j)}r_j)F(K,f_0,c)\,.$$

This says that the framing dependence is the same in the two approaches. Now we can state the main result of this section:

Theorem 6.3. *The invariants $F(K,f,c)$ and $J(K,f,c)$ coincide.*

Proof. There is perhaps an explicit description which allows us to pass from $x \in B_n$ to its Artin closure $\hat{x}$, to change the Dehn surgery presentation on $\hat{x}$ into a Heegaard splitting and to recover some $x' \in \mathcal{M}_g$ but it seems to be a complicated one. Our strategy is simpler: we show that these invariants extend to invariants of colored framed 3-valent graphs. Further an analog of Dehn surgery could be defined for such framed graphs. The analog Kirby moves may be described and we derive that the formula of Proposition 6.1 gives actually 3-manifold invariants in both cases. Now the corresponding h.t.r. corresponding to the two TQFT are coming from the same RCFT hence the 3-manifold invariants must be the same and our claim will follow.

First step: Let Γ be a connected 3-valent framed graph of genus g embedded in the manifold M^3. A tubular neighborhood $T(\Gamma) \subset M^3$ of Γ bounds a genus g surface $\partial T(\Gamma)$. We have a natural cut system on $\partial T(\Gamma)$ obtained in the following manner: over each edge e of the graph there is a cylinder sitting in $T(\Gamma)$ which is a trivial S^1-bundle over e. We consider the meridian $\gamma(e)$ of this cylinders. Their set give a cut system γ_* on $\partial T(\Gamma)$.

Now a coloring of Γ consists in

i) a coloring of its edges $c\colon E(\Gamma) \to L$,

ii) a labeling c of its vertices: assume we have chosen once and for all the basis B_{ijk} for the primary block W_{ijk}. Then a vertex $v \in V(\Gamma)$ has three incident edges e_i. We consider that $c(v) \in B_{c(e_1)c(e_2)c(e_3)}$.

Consider now the colored graph Γ having r connected components $\Gamma_i, i = 1, r$. Assume that the framing gives a rigid structure on $\partial T(\Gamma)$. Then

$$F(M - T(\Gamma), \varphi^0(\gamma_*, f)) = v \in \bigotimes_{i=1}^{r} F(\partial T(\Gamma_i))$$

and

$$F(\partial T(\Gamma)) = \bigoplus_{l \text{ labeling}} W(\Gamma, l) = \bigoplus_i \bigotimes_{l_i} W(\Gamma_i, l_i)\,.$$

We define

$$F(M, \Gamma, f, c) = \left\langle v, \bigotimes_{v \in V(\Gamma)} c(v) \right\rangle \in \mathbf{C}\,.$$

We wish to define now the Dehn surgery on a framed graph $(\Gamma, f) \subset S^3$. As in the classical case we remove a tubular neighborhood of Γ and glue it back differently

$$D(\Gamma, f) = S^3 - T(\Gamma) \cup_{\varphi(f)} T(\Gamma)\,,$$

where $\varphi(f)$ is a homeomorphism depending on the framing f. We have the cut system γ_* on $\partial T(\Gamma)$. Consider an irreducible cycle z (of length s) in the graph Γ. The part of $\partial T(\Gamma)$ sitting over z is a s-holed torus $T(z)$ (see Fig. 30). The framing of the loop z describes a longitude $f(z)$ of the torus $T(z)$ (avoiding the holes). If we cut the holed torus along $f(z)$ we get a $s+2$-holed torus. We identify again the two new circles but changing the orientation of one of them. We obtain again a s-holed torus (see Fig. 31). This transformation may be described on a fixed (holed) torus by a change in the cut system preserving the dual graph. Each curve $\gamma(e)$ with e an edge in z is sliding over the 1-handle (see Fig. 32). This change on the cut system (see Picture 33), once it was done for all irreducible cycles, define

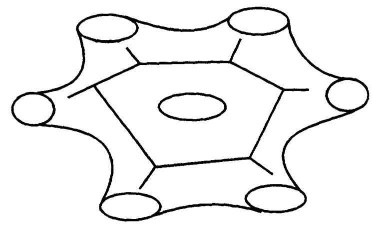

Fig. 31. The s-holed torus.

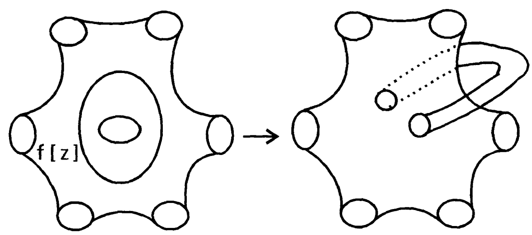

Fig. 32. Reidentification of the s-holed torus.

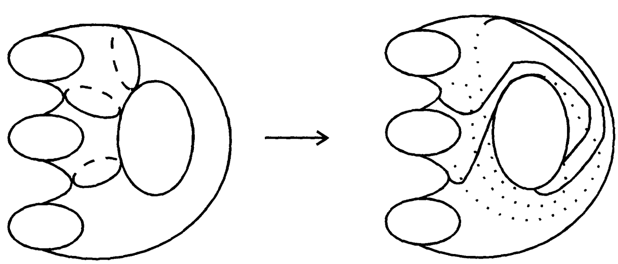

Fig. 33. The change of the cut system.

a homeomorphism $\varphi(f)$ of $\partial T(\Gamma)$. In fact it corresponds to the homeomorphism between the two adjacent rigid surfaces determined by the framings.

We obtain as in 6.1, a decomposition

$$F(D(\Gamma,f)) = \sum_{c \text{ coloring}} [c,\Gamma] F(S^3,\Gamma,f,c),$$

where $[c,\Gamma]$ are certain universal constants. The computation of these constants may be done as follows: At the graph level we perform a transformation $S(z)$ for each irreducible cycle which preserves the dual graph hence we have a mapping

$$1 \otimes S(z) \otimes 1 : W(\Gamma,l) \to \bigoplus_{l(e), e\subset z} W(\Gamma,l).$$

If we have two disjoint cycles the associated transformations commute in an obvious manner. But even if the cycles z_1, z_2 are not disjoint the associated transformations commute. It suffices to look at the images of each curve in the cut system. If e is not a common edge of z_1 and z_2 then only one of the transformations $S(z_i)$ changes

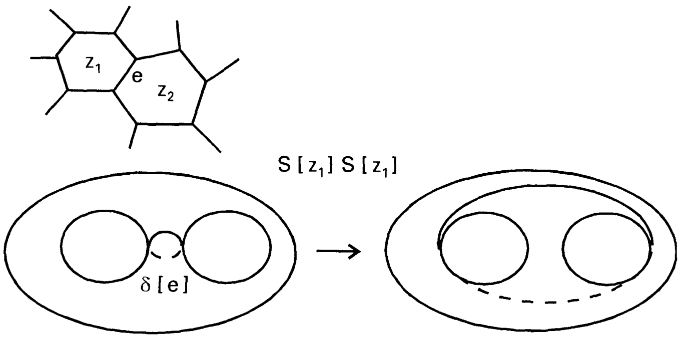

Fig. 34. Commutativity of cycle transformations.

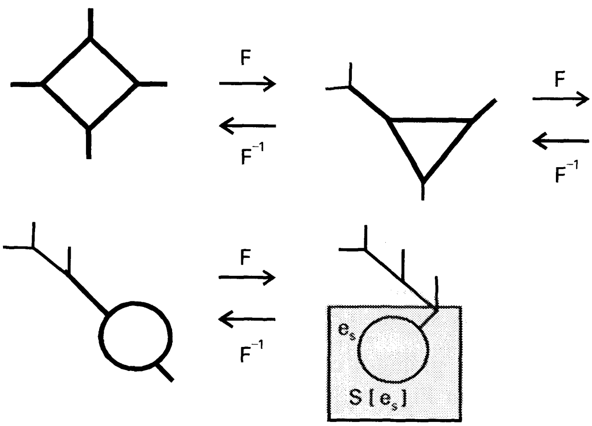

Fig. 35. Getting $S(z)$ from elementary moves.

$\gamma(e)$. If e is a common edge then $S(z_1)S(z_2)\gamma(e)$ is the curve surrounding both 1-handles of the holed genus 2 surface sitting over $z_1 \cup z_2$ (the cycles are irreducible) as can be seen on Fig. 34. Further we restrict to a cycle z and look for the expression of $S(z)$. We may perform s fusion moves to change the initial cut system into a cut system having the dual graph with a length 1 loop as in Fig. 35. Therefore we perform an usual S-move on the 1-loop and we come back using the inverses of the s fusion moves used above. We obtained

$$S(z) = \prod_{i=1}^{s-1} F(e_i) S(e_s) \prod_{i=1}^{s} F^{-1}(e_i),$$

where $F(e_i)$ is the fusion moves which contracts the edge e_i.

However there is not a local formula for

$$[c, \Gamma] = \langle \prod_z S(z) w_g, \bigotimes_{v \in V(\Gamma)} c(v) \rangle$$

because the labeling change at each cycle transformation.

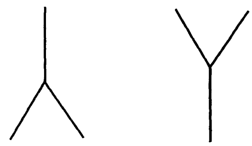

Fig. 36. The vertex elements.

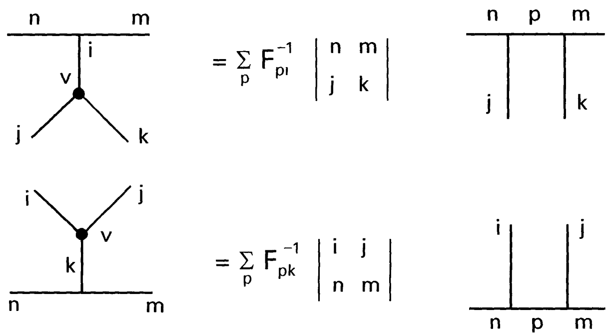

Fig. 37. The resolution of vertices.

Second step: Also $J(K, f, c)$ extends to 3-valent graphs using the RCFT data. We represent Γ as Artin's closure of a singular braid (as Birman described in the case of 4-valent graphs). A singular braid is the composition of

1) usual braid elements giving a crossing in a generic plane projection,
2) vertex elements as in Fig. 36.

Now the resolution of crossings must take into account the vertices. The two graphical rules from Fig. 37 give the resolution of vertices. One caution is needed. When we pass from the space associated to the upper line (indexed by $p_1, \ldots, p_{n-1}$) to the bottom line, when we encounter a vertex the vector space changes at this level. The change consists of a tensor product with W^i_{jk} (i, j, k are the labels of the three edges incident to the vertex v). We shall identify then the element $x \in W^k_{lm}$ with $x \otimes c(v) \in W^k_{lm} \otimes W^i_{jk}$.

After all singularities are inductively solved we obtain a matrix $\tilde{B}^{p_1 \cdots p_{n-1} j}_{q_1 \cdots q_{n-1} j}(x)$ analog to $B^{p_1 \cdots p_{n-1} j}_{q_1 \cdots q_{n-1} j}(x)$. The formula

$$J(\Gamma, f_0, c) = \left(\prod_{s=1}^{n} \frac{S(0)_{11}}{S(0)_{i_s 1}} \right) \sum_{j=1}^{n} \frac{S(0)_{j1}}{S(0)_{11}} \sum_{p_1, \ldots, p_n} \tilde{B}^{p_1 p_2 \cdots p_{n-1} j}_{p_1 p_2 \cdots p_{n-1} j}(x) .$$

gives a topological invariant for the colored graph Γ (the closure of the singular braid x). This can be derived immediately from the Reidemester's moves for 3-valent graphs. Alternatively the method of Degiovanni ([Deg92] Appendix B.1) gives essentially the same invariant. The change of the framing is the same as in the case of links. In fact it will be clear from below that we may always replace a graph surgery by a link surgery.

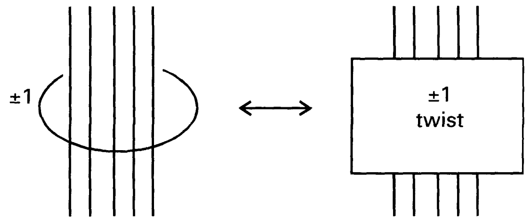

Fig. 38. The K-move.

Set now

$$J(D(\Gamma, f)) = \sum_{c \text{ coloring}} [c, \Gamma] J(\Gamma, f, c) .$$

We claim that J defines a topological invariant for 3-manifolds. We need the analog of Kirby moves for graph surgery. Away from the usual K-move given in Fig. 38 we have another move which permits the reduction of the number of loops in the graph. We choose an irreducible cycle z in Γ having the length $s > 1$. If we have an edge between two distinct vertices we can push one vertex along e in order to get an unknotted edge e in S^3. This may be done for all but one edge of the cycle z. Let e_0 be the edge which remains knotted. Eventually changing the cut system (hence the framing) we perform fusion moves at the graph level which kill the unknotted edges one by one. We arrive at a graph with the cycle z replaced by a single loop e_0. Moreover this loop is disjoint from the rest of the graph.

Consider now (Γ, f) a 3-valent (framed) graph and (K, f_K) a disjoint framed link. Choose two points $x \in \Gamma$, and $y \in K$ and an unknotted arc a between x and y. Then $\Gamma' = \Gamma \cup K \cup a$ is again a 3-valent graph with a natural framing $f' = f \cup f_K$. We claim that

$$D(\Gamma', f') \simeq D(\Gamma \cup K, f \cup f_K) ,$$

where on the right-hand side we have a disjoint union. This is clear from the definition of the graph surgery. So when we try to kill all the loops in the graph Γ we arrive at a g-component link. So the second allowed move is that from Picture 39. If we apply directly the theorem of Kirby [Kir78] it follows that two surgery presentations are equivalent under the equivalence relation given by these two moves, because we may restrict to the link presentations.

Now the same reasoning as in [Deg92] permits to obtain the invariance of J under this generalized Kirby moves.

Third step: We prove that $F(M) = J(M)$ for closed 3-manifolds M. Both are multiplicative invariants which are therefore determined by some h.t.r. ρ_* and $\rho_{J,*}$ respectively.

1) In the definition of $J(M)$ the conformal blocks $W_{J,g} = W(\partial T(\Gamma))$ for a genus g graph may be identified with $\bigoplus_l W(\Gamma, l)$, hence with W_g, or eventually with a quotient if the representation ρ_J splits. Also the label set coincides with L and the primary blocks must be the same.

2) The change of framing is given in both cases by some function on the colors, Γ and the conformal weights. Therefore Δ_j are the same.

3) The fusion matrix corresponds to a change on the cut system of $\partial(M - T(\Gamma))$ as in Fig. 40. This is the same to allow a move on the graph level $\Gamma \to \Gamma'$, where

we assumed that the edge labeled k is unknotted. From the definition of $J(\Gamma, f_0, c)$ we derive that

$$J(\Gamma, f_0, c) = \sum \langle F_{kn} \begin{bmatrix} i & j \\ l & m \end{bmatrix} (c_1 \otimes c_2), c'_1 \otimes c'_2 \rangle J(F\Gamma, f, c') .$$

This proves that the fusion matrices are identically in both approaches.

4) The S-matrix comes from the constants $[c, \Gamma]$, hence it must coincide.

5) Finally the weight vector is unique.

Therefore $F(M) = J(M)$ for closed 3-manifolds. But F extends canonically to manifolds with boundary hence $F(M - T(\Gamma)) = J(M - T(\Gamma))$ and our claim follows. □

Corollary 6.4. *A TQFT is determined by the matrices S,T and the braid matrices B. Equivalently the associated invariants for colored links determine uniquely the TQFT.*

Remark that if the primary blocks W^i_{jk} have dimension 0 or 1 for all labels then we can drop the coloring of vertices. This is the case for example in the quantum (or classical) $SU(2)$-theory. In particular the invariant

$$F_{SU(2)}(S^3 - T(K)) = \sum_{c \text{ coloring}} J_{SU(2)}(K, c) \bigotimes_{j=1}^{k} e_{c(j)} ,$$

where the terms on the right-hand side are the values of Jones polynomial at certain roots of unity for colored links. These are expressed in terms of cablings (see [KM91]) of the link K.

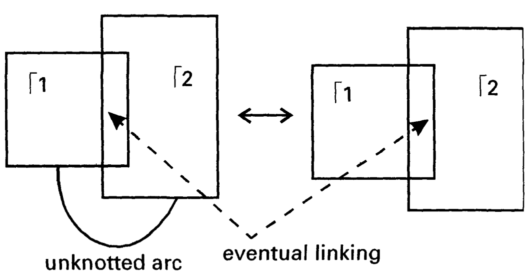

Fig. 39. Stabilization-Destruction move.

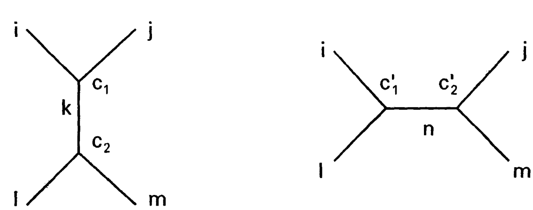

Fig. 40. The fusion move on the surface.

We wish finally to derive a general property fulfilled by the unitary link invariants coming from the RCFT. When all colors to the components are the same $j \in L$ we get an usual Markov trace $t_j : \mathbf{C}[B_\infty] \to \mathbf{C}$. But this Markov trace factors through a filtered quotient P_k of the group algebra of B_k (for each k) which is a finite dimensional matrix algebra. In fact we can take for P_k the endomorphism algebra $End(\bigoplus_{i \in L} W_{0,k+1}(j,j,\ldots,j,i))$. Now this P_k is a $\mathbf{C}^*$-algebra since the representation $\rho_{0,n}$ is unitary. We claim that

$$\mathrm{tr}(xx^*) \geqq 0\,,$$

so the trace is positive. This may be proved directly, but the simplest way is to use the formalism of [Fun93a]. Any link invariant is expressed as

$$t(x) = \langle w_{0,2n}, \tilde{\rho}_{0,2n}(\tilde{x}) w_{0,2n} \rangle$$

for a plat representative $\tilde{x}$ of the Artin closure of x. However the representation $\tilde{\rho}_{0,2n}$ is the same as that described above and the weight vector $w_{0,2n}$ is $e_j^{\otimes 2n}$. It corresponds to the standard semi-link with $2n$ endpoints (see [Fun93a] for details). Now the positivity of the trace is straightforward.

Define generally the definition quotient $D(t)$ of a Markov trace t to be the endomorphism algebra of the smallest nontrivial homogeneous quotient on which t factors.

Proposition 6.5. *Let t be a Markov trace coming from an unitary RCFT. Since t is positive it defines an hermitian product on $D(t)$. Let $\overline{D(t)}$ be the completion of $D(t)$ with respect to the hermitian product. Then $\overline{D(t)}$ is isomorphic to the hyperfinite II_1-factor.*

The proof is straightforward: $\overline{D(t)}$ is a von Neumann algebra by construction which has a Markov trace (unique). Since it is an hyperfinite factor (a quotient of P_∞) it is the hyperfinite II_1-factor. □

We wish to express now the dependence of invariants regardless of the orientation or mirror symmetry. For a link K we denote by $\overline{K}$ its inverse (obtained by reversing all component orientations) and by $K^\sharp$ its mirror image.

Proposition 6.6. *For any RCFT (not necessarily unitary) we have*

$$J(\overline{K}, c) = J(K, c^*)\,.$$

Moreover if the RCFT is unitary then the following identities

$$J(\overline{K}, c) = \overline{J(K,c)}, J(K^\sharp, c) = \overline{J(K,c)}$$

are fulfilled.

Proof. Remark that $F(S^3 - T(K))$ and $F(S^3 - T(\overline{K}))$ are the same vector (the framings are the canonical ones) and both belong to $W_1^{\otimes l}$ (where l is the number of components) but the canonical basis $\{\bigotimes_{i=1,l} e_{c(i)}\}_{c(i) \in L}$ are in fact not the same. Every component of the boundary $\partial T(K)$ is a torus having a canonical basis in homology, say $\{a, b\}$. The corresponding basis for a component of $\partial T(\overline{K})$ will be therefore $\{-a, -b\}$. We can write then

$$e_{c,\overline{K}} = \tilde{\rho}\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right) e_{c,K}\,.$$

We obtain

$$J(\overline{K},c) = \langle F(S^3 - T(K)), e_{c,\overline{K}}\rangle = \langle \rho\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right) F(S^3 - T(K)), e_{c,K}\rangle \,.$$

Write $J(K)$ for the vector $(J(K,c)_{c\in L}) \in \mathbf{C}^n$. We derived that

$$J(\overline{K}) = S(0)^2 J(K)\,.$$

Notice that in general there is not a canonical isomorphism between the lines and columns of the S matrix. Anyway we know that

$$S(0)^2 = \oplus \Theta^i_{0i} = \oplus \sigma_{13}\,,$$

where σ_{13} is the (non-canonical) isomorphism between W^i_{1i} and $W^{i^*}_{1i^*}$. Specifically we have $\sigma_{13}(e_{c,K}) = e_{c^*,K}$ and our first claim follows.

We come back now to the formula for J via traces. We have

$$\rho_{0,n}(x)|W_{0,i_1^* i_2^* \ldots i_n^*, j^*} = B^{i_1^* i_2^* \ldots i_n^*, j^*}_{i_1^* i_2^* \ldots i_n^*, j^*}(x) = \left(B^{i_1 i_2 \ldots i_n, j}_{i_1 i_2 \ldots i_n, j}\right)^*(x)\,,$$

hence

$$\mathrm{tr}(\rho_{0,n}(x)|W_{0,i_1^* i_2^* \ldots i_n^*, j^*}) = \overline{\mathrm{tr}(B^{i_1 i_2 \ldots i_n, j}_{i_1 i_2 \ldots i_n, j}(x)}\,,$$

because $\mathrm{tr}(A^*) = \overline{\mathrm{tr}(A)}$ for an unitary A. Further we know (see [MS89] pg 203) that

$$(S(0)^*)_{ij} = (S(0)^{-1})_{i^*j^*} = (\overline{S(0)}^*)_{i^*j^*} \text{ (again from unitarity)}\,.$$

This means that $S(0)_{ij} = \overline{S(0)}_{i^*j^*}$ and replacing in the formula for J we obtain

$$J(K,c^*) = \overline{J(K,c)}\,,$$

hence our second claim. For the last equality we denote by $\tilde{K}$ the inverse of the mirror image of K. If K is the Artin's closure of the braid x then $\tilde{K}$ is the Artin closure of x^{-1}. But in any unitary representation ρ

$$\mathrm{tr}(\rho(x^{-1})) = \mathrm{tr}(\rho(\overline{x}^*)) = \mathrm{tr}(\rho(x))\,,$$

which implies that

$$J(\tilde{K},c) = J(K,c)\,.$$

Combined with the above identities this gives our last equality. □

Let's comment on the case of the TQFTs coming from a finite gauge group G.

In the untwisted theories (where the cocycle $\alpha \in H^3(G,\mathbf{C}^*)$ is zero) the link invariants could be obtained from a ribbon Hopf algebra by the usual procedure of Reshetikhin and Turaev (see [Fer93]). We put

$$A = \mathbf{C}\langle G \times G\rangle$$

with the composition law

$$(g,h)(g',h') = \delta_{g,h^{-1}g'h}(g,h'h)\,,$$

where δ is the Kronecker delta. The morphism

$$\bar{R}: A\otimes A \to A\otimes A$$

given by

$$\bar{R}((g,h)\otimes(g',h')) = (g,h)\otimes(g',hg^{-1}h^{-1}h')$$

is the right multiplication by the R-matrix

$$R = \sum_{l,\,k\in G}(l,e)\otimes(k,l^{-1}), e \text{ being the identity of } G\,.$$

Put $v = \sum_{k,l\in G}\gamma(k,l^{-1})(l,e)$, where $\gamma(g,h) = (hg^{-1}h^{-1},h^{-1})$. Therefore (A,R,v) is a ribbon Hopf algebra. The elements of L are indexed by the irreducible representations of G and the S-matrix is computed by Freed and Quinn as being

$$\frac{S(0)_{i1}}{S(0)_{11}} = \dim R_i\,,$$

where R_i states for the representation corresponding to i. Viewing R in a specified basis of $A\otimes A$ the formula above written shows that its entries are all real. We derive that $J(K,c)$ is a real number for any coloring c. Furthermore the non-invertibility of links cannot be detected this way.

In the case of the twisted theories the S-matrix has the same expression and all we need is to know whenever the trace of $\rho_{0,n}(x)$ is real or not. If $G = \mathbf{Z}/k\mathbf{Z}$ then $H^3(G,\mathbf{C}^*) = \mathbf{Z}/k\mathbf{Z}$ and it suffices to consider the case when α is a generator. According to ([MS89], p. 251) the fusion matrix F may be considered (after a gauge transformation) to equal 1. Therefore

$$B = \Omega^*\otimes\Omega\,,$$

and $B = B^*$ which implies that $\mathrm{tr}(\rho_{0,n}(x))$ is real for any x. Alternatively we could derive the same invariants using again a ribbon Hopf algebra model (see [MOO92]). It follows that the abelian case does not give any information on the invertibility of links. We don't know however what happens for general G.

There are of course RCFT models having non-symmetric braid matrices as for example the Ising model whose fusion rule algebra is

$$\psi\times\psi = 1, \psi\times\varphi = \varphi, \varphi\times\varphi = 1+\psi\,.$$

We can compute S which is real and

$$B\begin{bmatrix}\varphi & \varphi\\ \varphi & \varphi\end{bmatrix} = \frac{1}{\sqrt{2}}\exp\left(\frac{\pi\sqrt{-1}}{8}\right)\begin{pmatrix}1 & \exp(\frac{3\pi\sqrt{-1}}{2})\\ \exp(\frac{3\pi\sqrt{-1}}{2}) & 1\end{pmatrix}.$$

This model however presents a non-zero central charge and when the correction term is added, the associated invariant is again invariant when passing to the inverse link.

Notice that for the Jones polynomial (or the G-quantum invariants) valued at certain roots of unity the associated RCFT presents a central charge and several normalizations are done. The non-invertibility is not detected but the last formula translates into a $J(K^\sharp,c;q) = J(K,c;\frac{1}{q})$, where q is the deformation parameter, this way permitting us to exhibit examples of non-amphicheiral links.

Therefore in order to have another behaviour of link invariants with respect to the change of orientation we may allow non-unitary RCFT. For example the RCFT derived from a quantum super-group yield indefinite bilinear forms.

A. Appendix

We say that a RCFT is abelian if we have the isomorphisms of vector spaces

$$W_g \simeq W_1^{\otimes g} .$$

Proposition A.1. *An abelian RCFT is determined by a finite abelian group structure on L such that* $W_{hk}^g = \mathbf{C}$ *iff* $g = h + k$, *and otherwise it vanishes. The unity is* 0 *and the involution* * *corresponds to taking the inverse.*

Proof. In genus one we have $n_1 = card(L)$. Further $n_2 = \sum_{i,j,k} n_{ji}^i n_{jk}^k$. This implies $n_{ji}^i = 0$ if $j \neq 0$. This proves also that 0 is a unit. From the expression of n_3 we derive $n_{jk}^i \in \{0, 1\}$ and for fixed j, k there is an unique i with $n_{jk}^i = 1$. We denote it by $j + k$. Since $n_{jk}^i = n_{kj}^i$ this law is commutative. The associativity follows from the fact that F is an isomorphism. Also $n_{jk}^i = n_{ji^*}^{k^*}$ so $k^* = -k$. □

The RCFT determined by a finite group were treated by Dijkgraaf and Witten in [DW90], the abelian theories were classified by Moore and Seiberg in Appendix E of [MS89], and the general case was settled by Freed and Quinn [FQ93].

In particular it follows that the h.t.r. associated factors through the symplectic groups

$$\rho_g : \mathcal{M}_g \to Sp(2g, \mathbf{Z}) \to U(W_g) .$$

The basic data is $(S(0), T)$ since $S(j) = 0$ for $j > 0$ and the fusion matrix is 1.

If we allow projective unitary representations, or unitary invariants for framed 3-manifolds there are some very interesting examples. The general form of these representations (we normalize them to be true representations but allowing that the weight vector be invariant up to a character) is

$$\rho_g \begin{pmatrix} A & 0 \\ 0 & {}^T A^{-1} \end{pmatrix} = (\delta_{T_{A\lambda}, \mu})_{\lambda, \mu \in L^g}, a \in GL(g, \mathbf{Z}) ,$$

$$\rho_g \begin{pmatrix} 1_g & B \\ 0 & 1_g \end{pmatrix} = \mathrm{diag}(\exp(2\pi\sqrt{-1}\Delta_x \langle Bx, x \rangle))_{x \in L^g}$$

for a symmetric matrix B with integer entries, with $\Delta_x = \sum_{i=1}^g \Delta_{x_i}$, the scalar product being the natural one on L^g and $\Delta_j \mathrm{card}(L) \in \mathbf{Z}$ for all j,

$$\rho_g \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} = S^{\otimes g} ,$$

where S and $T = \mathrm{diag}(\exp(2\pi\sqrt{-1}\Delta_x))_{x \in L}$ give a $SL(2, \mathbf{Z})$ representation.

In particular we get the abelian Witten's theory for the gauge group $G = U(1)$ (or equivalently the $\mathbf{Z}/k\mathbf{Z}$-theory) and the family of theories obtained by the semi-abelian quantization in [Fun93a, Fun93c].

The invariants for 3-manifolds we get in the canonical framing are no longer multiplicative invariants. Their modulus corresponds to a multiplicative invariant and therefore is an homotopic invariant determined by the first Betti number and the torsion pairing on $Tors(H_1(M, \mathbf{Z}))$. When also the phase factor is taken into account we obtain in particular Witten's invariants for torus bundles and lens spaces (see [Fun93b, Jef92], hence even in the abelian setting we can obtain non-homotopic invariants.

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