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On the cohomology of weighted complete intersections

By

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The weighted projective space $P(a_0, a_1, \ldots, a_n)$ is defined as the quotient of CP^n by the following action of $G = \mathbb{Z}/a_0\mathbb{Z} \oplus \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_n\mathbb{Z}$:

$$(k_0, k_1, \ldots, k_n)(z_0, z_1, \ldots, z_n) = (\zeta_{a_0}^{k_0} z_0, \zeta_{a_1}^{k_1} z_1, \ldots, \zeta_{a_n}^{k_n} z_n),$$

where $\zeta_l = \exp(2\pi i/l)$.

It is known that the integral homology groups of $P(a_0, a_1, \ldots, a_n)$ are torsion free (see [6, 3]) so they are isomorphic to the homology groups of CP^n. An entirely elementary computation was carried out for $n = 2$ in [4].

Let now $(V, 0)$ be an isolated singularity of complete intersection in C^{n+k+1} defined by the weighted homogeneous polynomials $f = (f_1, f_2, \ldots, f_k)$. We suppose that f_i has degree d_i with respect to the weights $w(z_j) = a_j$, $j = 0, 1, \ldots, n + k$. There are two spaces naturally associated to the singularity $(V, 0)$, namely the link $K = V \cap S^{2(n+k)+1}$ and the quasi-smooth weighted complete intersection Y_∞ defined by the polynomials f_i in $P(a_0, a_1, \ldots, a_{n+k})$. Notice that K is a smooth compact oriented $(2n+1)$-dimensional manifold which is $(n-1)$-connected (see [5]). The middle Betti numbers of K have been computed in terms of the a_i's and the d_i's by Dimca ([2]). The aim of this note is to give a brief insight into the cohomology of Y_∞. All the cohomology groups considered below have integer coefficients. We say that $(a_0, a_1, \ldots, a_{n+k+1})$ is m-prime if the greatest common divisor of any m of the a_i's equals one.

Proposition 1. *Suppose that $(a_0, a_1, \ldots, a_{n+k+1})$ is m-prime. Then the relative cohomology groups vanish:*

$$H^i(P(a_0, a_1, \ldots, a_{n+k}), Y_\infty) = 0 \quad \textit{for } i \leqq n - m + 1.$$

Proof. Consider $F_i(z) = f_i(z_0^{a_0}, z_1^{a_1}, \ldots, z_{n+k}^{a_{n+k}})$ and set Z_∞ for the complete intersection defined by the polynomials F_i in CP^{n+k}. Remark that the G-action on CP^{n+k} leaves Z_∞ invariant and we have $Z_\infty/G = Y_\infty$. Let now $P = \mathbb{Z}/p^{\alpha_0}\mathbb{Z} \oplus \mathbb{Z}/p^{\alpha_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_{n+k}}\mathbb{Z} \subset G$ be a p-subgroup of G. Therefore p^{α_i} divides a_i for all $i = 0, 1, \ldots, n + k$. Then the P-invariant subsets are

$$(CP^{n+k})^P = \{\alpha_i z_i = 0\},$$

and

$$(Z_\infty)^P = Z_\infty \cap \{\alpha_i z_i = 0\}.$$

Since Z_∞ is a complete intersection $(Z_\infty)^P$ is also a complete intersection, eventually using only part of the original equations F_i. Next the number of non-zero α_i's cannot exceed $(m-1)$ because $(a_0, a_1, \ldots, a_{n+k})$ is m-prime. Then Lefschetz's theorem for complete intersections implies

$$\pi_j((CP^{n+k})^P, (Z_\infty)^P) = 0 \quad \text{for } j \leqq n - m + 1.$$

But this holds for all primes p and all maximal p-subgroups P so from [1] we derive our claim. □

Corollary 2. *For a prime number p write $a_i = p^{r_i} c_i$ with r_i maximal. Choose a permutation σ of $\{0, 1, 2, \ldots, n+k\}$ such that*

$$r_{\sigma(1)} \geqq r_{\sigma(2)} \geqq \ldots \geqq r_{\sigma(n+k)} \geqq r_{\sigma(0)} = 0,$$

and set:

$$b_i(p) = \prod_{0 \leqq j \leqq i} p^{r_{\sigma(i)}} \quad \textit{and} \quad b_i = \prod_p b_i(p).$$

If $(a_0, a_1, \ldots, a_{n+k})$ is m-prime then the set of numbers

$$R_{ij} = b_i b_j / b_{i+j} \quad \textit{with} \quad 0 \leqq i, j, i+j \leqq (n-m+1)/2$$

is a topological invariant of the isolated singularity $(V, 0)$.

Proof. The $\mathbb{Z}$-cohomology algebra of $P(a_0, a_1, \ldots, a_{n+k})$ is determined in [6]: if g_i is the generator of $H^{2i} P(a_0, a_1, \ldots, a_{n+k})$ then $g_i \cup g_j = R_{ij} g_{i+j}$. But in low rank the cohomology algebra of Y_∞ is induced from that of $P(a_0, a_1, \ldots, a_{n+k})$ (according to Proposition 1) and we are done. □

Set now

$$F = (F_1, F_2, \ldots, F_k),$$
$$\bar{F} = (F_1 - z_{n+k+1}^{d_1}, F_2 - z_{n+k+1}^{d_2}, \ldots, F_k - z_{n+k+1}^{d_k}),$$

and

$$\bar{f} = (f_1 - z_{n+k+1}^{d_1}, f_2 - z_{n+k+1}^{d_2}, \ldots, f_k - z_{n+k+1}^{d_k}).$$

The link of the singularity defined by $\bar{f}$ will be denoted by $\bar{K}$. Let Z be the fibre of F over 1 (the global Milnor fibre) and $\bar{Z}$ its projective closure. Observe that Z_∞ is in fact $\bar{Z} - Z$. In fact $P(a_0, a_1, \ldots, a_{n+k}, 1)$ is the compactification of $\mathbb{C}^{n+k+1}$ whose locus at infinity is precisely $P(a_0, a_1, \ldots, a_{n+k})$. If Y is the global Milnor fibre of f and $\bar{Y}$ is the quasi-smooth weighted intersection in $P(a_0, a_1, \ldots, a_{n+k}, 1)$ associated to $\bar{f}$ then Y may be identified with $\bar{Y} - Y_\infty$. Otherwise we can look at the S^1-action on $(S^{2(n+k)+3}, S^{2(n+k)+1})$ given by

$$\varrho \cdot z = (\varrho^{a_0} z_0, \varrho^{a_1} z_1, \ldots, \varrho^{a_{n+k}} z_{n+k}, \varrho z_{n+k+1}).$$

Then $(\bar{K}, K)/S^1 = (\bar{Y}, Y_\infty)$. Then Y_∞ is called strongly smooth ([2]) if the S^1-action on K is semi-free.

Proposition 3. *Assume that Y_∞ is strongly smooth. Then $H_*(K)$ is torsion free and the Milnor lattice of f is equivalent to the cup product*

$$H^{n+1}(\bar{K},K)\times H^{n+1}(\bar{K},K)\to H^{2n+2}(\bar{K},K)\cong\mathbb{Z}.$$

Moreover if $k=1$ then this may be expressed also as the cup product

$$H^{n+k}(S^{2(n+k)+1},K)\times H^{n+k}(S^{2(n+k)+1},K)\to H^{2(n+k)}(S^{2(n+k)+1},K).$$

Proof. From the Smith-Gysin sequence associated to the S^1-action on K we derive that $H_*(K)$ is torsion free and:

$$H_j(Y_\infty)=H_j(CP^n)\quad\text{for } j\neq n,\ H_n(Y_\infty)=H_n(K)\oplus H_n(CP^n).$$

Now Y_∞ is strongly smooth if and only if $\bar{Y}$ is strongly smooth. The long exact sequence of the pair $(\bar{K},K)$ gives us:

$$H^j(\bar{K},K)=0\quad\text{for } k\neq n+1, n+2, 2n+2, 2n+3,$$
$$H^{2n+2}(\bar{K},K)=H^{2n+3}(\bar{K},K)=Z.$$

But Y has the homotopy type of a bouquet of $(n+1)$-spheres (see [7]) so using the Lefschetz's duality we find:

$$H^j(\bar{Y},Y_\infty)=0\quad\text{for } j\neq n+1.$$

Now from the Smith-Gysin sequence associated to the S^1-action on $(\bar{K},K)$ we obtain:

$$0=H^{2n}(\bar{Y},Y_\infty)\quad=\ker(H^{2n+2}(\bar{Y},Y_\infty)\quad\to H^{2n+2}(\bar{K},K)),$$
$$0=H^{2n+1}(\bar{Y},Y_\infty)=\operatorname{coker}(H^{2n+2}(\bar{Y},Y_\infty)\to H^{2n+2}(\bar{K},K)),$$
$$H^{n+2}(\bar{K},K)\cong H^{n+1}(\bar{Y},Y_\infty),$$
$$H^{n+1}(\bar{Y},Y)\ \cong H^{n+1}(\bar{K},K).$$

Using the functoriality of Lefschetz duality the first part of our claim follows. If k equals one then $\bar{K}-K$ is a non-ramified $\mathbb{Z}/d_1\mathbb{Z}$-covering of $S^{2(n+k)+1}-K$ and the Alexander duality gives the second claim. □

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