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GENERALIZED HADWIGER NUMBERS FOR SYMMETRIC OVALS

VALENTIN BOJU AND LOUIS FUNAR

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ABSTRACT. Some estimations for the "juxtaposition function" h_F and an asymptotic formula for the function h_F/h_G, where F, G are central symmetric convex bodies, are given. Hadwiger and Grünbaum gave for $h_F(1)$ the bounds $n^2+n \leq h_F(1) \leq 3^n-1$. Grünbaum conjectured (and proved for $n=2$ in Pacific J. Math. **11** (1961), 215–219) that for every even r between these bounds there exists in E^n an oval F such that $h_F(1)=r$. Lower bounds for h_F could be derived in the same way as in Theorems 1 and 2 from a good estimate of packing numbers on a Minkowski sphere, that is, from solutions to a Tammes-type problem in a Banch space.

For a topological disk $F \subseteq \mathbb{E}^n$ we shall denote by $h_F\colon (0, 1] \to \mathbb{N}$ the "juxtaposition function" introduced by the first author [2, 3] as follows. Let $A_{F,\lambda}$ denote the family of all sets, homothetic to F in the ratio λ, which have only boundary points in common with F. Then $h_F(\lambda)$ is the greatest integer k such that $A_{F,\lambda}$ contains k sets with pairwise disjoint interiors. In particular, $h_F(1)$ is just the Hadwiger number of F.

In case of convex F, Hadwiger [11] and Grünbaum [8] gave for $h_F(1)$ the bounds $n^2+n \leq h_F(1) \leq 3^n-1$. Grünbaum [8] conjectured (and proved for $n=2$; see also Boltjanski and Gohberg [4]) that for every even r between these bounds there exists in $\mathbb{E}^n$ an oval F such that $h_F(1)=r$.

Unless explicitly stated otherwise, throughout this paper F, G will denote symmetric plane ovals. Any such F determines a norm $\| \ \|_F$ by $\|x-y\|_F = \|x-y\|/\|o-z\|$, where $\| \ \|$ is the Euclidean norm, o is the center of F, and z is a point on the boundary ∂F of F such that oz and xy are parallel. With this norm $\mathbb{E}^2$ becomes a Banach space, with unit disk isometric to F. Set $p(F)$ for the perimeter of ∂F in its inner norm.

Theorem 1. *For a symmetric oval F in the plane*

$$p(F) = 2\lim_{\lambda\to 0} \lambda h_F(\lambda). \tag{1}$$

Proof. Let x, y be points of ∂F, and let points x', y' be given by $ox' = (1+\lambda)ox$ and $oy' = (1+\lambda)oy$. Denote by F_x, F_y those sets in $A_{F,\lambda}$ which have centers at x' and y', respectively. If $F_x \cap F_y \neq \varnothing$, it follows from the

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symmetry and convexity of F that $x'y' \subset F_x \cap F_y$. We put $x'y' \cap \partial F_x = \{a, b\}$, $x'y' \cap \partial F_y = \{c, d\}$, and $z \in \partial F$ such that oz is parallel to $x'y'$. Then

$$\|x' - y'\| \leq \|x' - b\| + \|c - y'\| = 2\|x' - b\| = 2\lambda\|o - z\|,$$

hence

$$\|x - y\|_F = \|x' - y'\|/(1+\lambda)\|o - z\| \leq 2\lambda/(1+\lambda).$$

Reversing the reasoning we obtain

$$\text{(2)} \qquad \operatorname{int} F_x \cap \operatorname{int} F_y = \varnothing \quad \text{if and only if} \quad \|x - y\|_F \leq 2\lambda/(1+\lambda).$$

Now consider a maximal collection $\{F_i : i = 1, \dots, k\} \subset A_{F,\lambda}$ of sets with disjoint interiors and the points $x_i \in \partial F$, $i = 1, \dots, k$, for which $F_i = F_{x_i}$. From (2) it follows that $\|x_i - x_{i+1}\|_F \leq 2\lambda/(1+\lambda)$ and thus

$$\sum_{1 \leq i \leq k} \|x_i - x_{i+1}\|_F \leq 2k\lambda(1+\lambda);$$

however,

$$p(k, F) = \sup\left\{\sum_{1 \leq i \leq k} \|x_i - x_{i+1}\|_F,\ x_i \in \partial F\right\} \leq p(F).$$

These inequalities yield

$$\text{(3)} \qquad h_F(\lambda) = k \leq (1+\lambda)p(k, F)/2\lambda < (1+\lambda)p(F)/2\lambda.$$

Conversely, let P_λ be an inscribed polygon with $2k$ vertices $u_1, \dots, u_{2k}$ such that P_λ is symmetric about o and

$$\|u_1 - u_2\|_F = \|u_2 - u_3\|_F = \cdots = \|u_{k-2} - u_{k-1}\|_F = 2\lambda/(1+\lambda),$$
$$2\lambda/(1+\lambda) \leq \|u_{k-1} - u_k\|_F < 4\lambda/(1+\lambda).$$

Then the sets F_{u_i} have disjoint interiors and

$$4(k+1)\lambda/(1+\lambda) > \sum_{1 \leq i \leq 2k} \|u_i - u_{i+1}\|_F \geq 4k\lambda/(1+\lambda).$$

Since $h_F(\lambda) \geq 2k$, it follows that

$$\text{(4)} \qquad 2 + h_F(\lambda) \geq (1+\lambda)\left(\sum_{1 \leq i \leq 2k} \|u_i - u_{i+1}\|_F\right)/2\lambda.$$

If $p(\lambda)_F$ denotes the perimeter of P_λ in the $\| \ \|_F$ norm, then (see [1, 11])

$$\text{(5)} \qquad \lim_{\lambda \to 0} p(\lambda)_F = p(F).$$

For symmetric ovals F, G relations (3)–(5) imply

$$\text{(6)} \qquad \begin{aligned} \lim_{\lambda \to 0} h_F(\lambda)/h_G(\lambda) &\geq \lim_{\lambda \to 0}((-2 + (1+\lambda)p(\lambda)_F)/2\lambda)/((1+\lambda)p(G))/2\lambda \\ &= p(F)/p(G), \end{aligned}$$

and similarly the reverse inequality. Therefore, taking for G a square we obtain the claim which was to be proved.

Denote by $[t]$ the integer part of $t \in \mathbb{R}$.

Theorem 2. *For every symmetric oval F in the plane*

$$3+[3/\lambda]\le h_F(\lambda)\le 4(1+\lambda)/\lambda, \tag{7}$$

with equality on the left if and only if $1/\lambda\in\mathbb{N}$, and F is an affine-regular hexagon and equality on the right if and only if $1/\lambda\in\mathbb{N}$ and F is a parallelogram.

Proof. A result of Golab [6] and Reshetnyak [14], generalized by Schäffer [15], asserts that $6\le p(F)\le 8$. Hence we have

$$h_F(\lambda)\le 4(1+\lambda)/\lambda,$$

and, using the existence of an affine-regular hexagon inscribed in F [13], we obtain $h_F(2/(1+k))\ge 6k$. Since $h_F(\lambda)$ is a decreasing function of λ, we are done.

If the dimension of F is greater than two, the situation is essentially different. We shall prove (see also [7])

Theorem 3. *Any symmetric convex body $F\subset\mathbb{E}^n$ satifies the inequality*

$$h_F(\lambda)\le((1+\lambda)^n-1)/\lambda^n, \tag{8}$$

with equality if and only if $1/\lambda\in\mathbb{N}$ and F is a parallelohedral body.

Proof. Let $B_\lambda=\bigcup_{H\in A_{F,\lambda}}H$. We shall prove that

$$B_\lambda\subset(1+2\lambda)F. \tag{9}$$

Indeed, let x be a point on the boundary of F_v, $|ox|\cap\partial F=\{a\}$, $|ov|\cap\partial F_v=\{q\}$, and let vx'' be parallel to qx with $x''\in\partial F$. Then $\measuredangle qv'x=\measuredangle v'xo+\measuredangle v'ox\ge\measuredangle v'ox$, which yields $\measuredangle vox\le\measuredangle vox''=\measuredangle qv'x$. Since F is convex, we can take a point b in the nonempty intersection $|oa|\cap|vx''|$. Then $|vx''|\subset F$, $b\in F$, $b\in|ox|$. Since

$$\|o+a\|/\|o-x\|\ge\|o-b\|/\|o-x\|\ge\|o-v\|/\|o-q\|=1/(1+2\lambda),$$

the point x belongs to $(1+2\lambda)F$, and (9) is proved.

If $\{F_i,\ i=1,\dots,k\}\subset A_{F,\lambda}$ have disjoint interiors, then

$$\bigcup_{1\le i\le k}F_i\subset B_\lambda\subset(1+2\lambda)F;$$

therefore,

$$\mathrm{vol}(F)+\mathrm{vol}(F_1)+\cdots+\mathrm{vol}(F_k)\le(1+2\lambda)^n\,\mathrm{vol}(F)$$

where $\mathrm{vol}(F)$ denotes the volume of F. This gives the desired estimation on $h_F(\lambda)$. The equality case is treated in [7].

Lower bounds for $h_F(\lambda)$ could be derived in the same way as in Theorems 1 and 2 from a good estimate of packing numbers on a Minkowski sphere, that is, from solutions to a Tammes-type problem in a Banach space.

Grünbaum asked what happens to relation (1) in case F is not centrally symmetric. We recall that for an arbitrary oval F and $z\in\operatorname{int}F$ a norm (nonsymmetric, in general) is defined by the Minkowski functional

$$\|x\|_{F,z}=\inf\{\lambda>0:x-z\in\lambda(F-z)\}.$$

Using the (possibly nonsymmetric) distance derived from this norm it is possible to define arc-length for oriented arcs. For an oriented closed curve C let the length of C in the metric derived from $\| \|_{F,z}$ be denoted by $p_{F,z}(C)$. The intrinsic perimeter (self-circumference [6, 10]) of F is $P(F) = \inf\{p_{F,z}(\partial F) : z \in \operatorname{int} F\}$. Then it follows that

$$g(F) = \lim_{\lambda \to 0} \lambda h_F(\lambda)/P(F)$$

is a measure of symmetry (see [8]). By the same method as used above, it is possible to show that $g(F) \leq \frac{1}{2}$, with equality if F is centrally symmetric. If F is a triangle then $g(F) = \frac{1}{3}$, and we conjecture that $g(F) \geq \frac{1}{3}$ for any oval F.

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Department of Mathematics, University of Craiova, A. I. Cuza no. 13, Craiova 1100, Romania

Institute of Mathematics, P.O. BOX 1-700, 70700 Bucharest, Romania

Current address, L. Funar: Université de Paris-Sud, Mathématiques, Bat. 425, 91405, Orsay Cedex, France

E-mail address, L. Funar: funar@matups.matups.fr