Generalized Sum-Free Sets of Integers

Louis Funar

Department of Mathematics, University of Bucharest, Academiei 14
Bucharest, R-70109, Romania

Let $A=(a_1,\ldots,a_k)$ be a vector with positive integer components. The set $M\subset Z_+$ is called A-sum-free if for any choice of $x_1,\cdots,x_k\in M$, $a_1x_1+\cdots+a_kx_k\notin M$ holds. For $A_i\in Z_+^{k_i}$, $1\leq i\leq r$, denote by $f(n;A_1,\cdots,A_r)$ the greatest integer h such that for some partition of the set $\{n,\cdots,h\}$ of consecutive integers into $S_1,\cdots,S_r$ the sets S_i are A_i-sum-free for all i. In this paper a lower bound for $f(n;A,B)$ is given and some special cases are treated which support the conjecture that $f(n;A,B)$ always equals this lower bound.

Let $A=(a_1,\cdots,a_k)$ be a vector with positive integer components. The set $M\subset Z_+$ is called A-sum-free if for any choice of $x_1,\cdots,x_k\in M$, $a_1x_1+\cdots+a_kx_k\notin M$ holds. In the case when $a_1=\cdots=a_k=1$ the classical definition of k-sum-free sets of integers is obtained. For $A_i\in Z_+^{k_i}, i\in\{1,\cdots,r\}$, let $f(n;A_1,\cdots A_r)$ be the greatest number h such that there exists a partition of the set $\{n,\cdots,h\}$ of consecutive integers into $S_1,\cdots,S_r$ with the property that all S_i are A_i-sum-free for all $i\in\{1,\cdots,r\}$.

If $k_1=\cdots=k_r=1$ then $f(n;A_1,\cdots,A_r)$ is not a finite number because we can get decompositions of Z_+ into $S_1,\cdots,S_r$ such that $x\in S_i$ implies $A_ix\notin S_i$. We describe below two cases: 1) the A_i are distinct prime numbers, and 2) all A_i are equal. In the first case consider the decomposition of an arbitrary integer $n=A_1^{w_1}\cdots A_r^{w_r}M$ where $(M,A_i)=1$ for every i, and set $\overline{w}_i$ for the residue modulo 2 of w_i considered as an element of $\{0,1\}$. Therefore

$$S_i=\{n;\overline{w}_1+\cdots+\overline{w}_r=i(\mathrm{mod}\,r)\},\ i\in\{1,\cdots,r\}$$

is a partition of Z_+ with $A_iS_i\subset S_{i+1}$ so that the S_i are A_i-sum-free. In the second case take $n\in Z_+$ and write $n=A^pm$, $m\in Z_+$, $p\in Z_+$ maximal. Then set

$$S_i=\{n;p=i(\mathrm{mod}\,r)\},\ i\in\{1,\cdots,r\}.$$

Similar but much more involved decompositions of Z_+ can be given for general $A_i\in Z_+$. This suggests that we must consider $k_i\geq 2$. Moreover we don't know what happens when some k_i equal 1, others being greater or equal than 2. Now if $A_i=(a_{ij}), j\in\{1,\cdots,k_i\}$, $k_i\geq 2$, and $A_i'=(a_{i1}, a_{i2}+\cdots+a_{ik_i})\in Z_+^2$, then obviously

$$f(n; A_1, \cdots, A_r) \leqslant f(n; A_1', \cdots, A_r') \tag{1}$$

so that the finiteness question may be reduced to the case $k_i = 2$. Under this assumption, for $r=2$, a result of RADO [7] assures an upper bound for $f(n; A, A)$, namely:

$$f(1; A, A) \leqslant \max\{(bmc^2 - 1)(c-1) + bmc,\ bmc^2(c-1)/a\} \tag{2}$$

where

$$A = (a,b),\ m = a/(a,b),\ c = \max(x_0, y_0, z_0)$$

(x_0, y_0, z_0) being a minimal solution for the diophantine equation $ax + by = z$. In general, for $r \geqslant 3$ and A_i arbitrary it is not known whether $f(n; A_1, \cdots, A_r)$ is finite, but there are a lot of particular results, especially for $A_i = (1, \cdots, 1)$ (we shall denote $(1, \cdots, 1) \in Z^k$ by $k\langle 1\rangle$). For example, KASA [6] proved that:

$$f(1; 2\langle 1\rangle, k\langle 1\rangle) = \begin{cases} 3k-3, \text{ if } k \text{ is odd} \\ 3k-2, \text{ if } k \text{ is even} \end{cases} \tag{3}$$

and for n even, $k \geqslant 2$,

$$f(n; 2\langle 1\rangle, k\langle 1\rangle) = (2k+1)n - 1. \tag{4}$$

SERESS [9] extended (2) to all $n > 1$ and also gave

$$f(n; m\langle 1\rangle, k\langle 1\rangle) = (mk + m - 1)n - 1 \tag{5}$$

for $3 \leqslant m \leqslant k$. Earlier SCHUR [8] has proved that

$$f(1; \underbrace{2\langle 1\rangle, 2\langle 1\rangle, \cdots, 2\langle 1\rangle}_{k}) < k!e. \tag{6}$$

Sharp estimates for Schur numbers may be found in [5]. Related questions about sum-free sets are contained in a lot of papers from which we mention [1,2,3,4]. Our paper focusses on the case $r=2$.

For $A = (a_1, \cdots, a_k) \in Z_+^k$ we put $s(A) = a_1 + \cdots + a_k$ and $t(A) = \min_{1 \leqslant i \leqslant k} a_i$. If $B = (b_1, \cdots, b_m) \in Z_+^m$ then set

$$h(A) = t(A)ns(A)s(B) + n(s(A) - t(A)) - 1 \tag{7}$$

$$h(B) = t(B)ns(A)s(B) + n(s(B) - t(B)) - 1$$

$$h = \begin{cases} h(A), \text{ if } s(A) < s(B) \\ h(B), \text{ if } s(A) > s(B) \\ \max(h(A), h(B)), \text{ if } s(A) = s(B). \end{cases}$$

Suppose that $s(A) < s(B)$ and consider the sets

$$S_1 = \{n, n+1, \cdots, ns(A)-1\} \cup \{ns(A)s(B), \cdots, h\}$$
$$S_2 = \{ns(A), \cdots, ns(A)s(B)-1\}.$$

If $x_1, \cdots, x_m \in S_2$ then $b_1x_1 + \cdots + b_mx_m \geqslant ns(A)s(B)$ so that S_2 is B-sum-free. For $x_1, \cdots, x_k \in S_1$ there are two possibilities: we have $x_1 \leqslant ns(A)-1$, for all i when the following inequalities hold

$$ns(A) \leqslant a_1x_1 + \cdots + a_kx_k \leqslant s(A)(ns(A)-1) < ns(A)s(B)$$

else some x_i lies in $\{ns(A)s(B), \cdots, h\}$ so that

$$a_1x_1 + \cdots + a_kx_k \geqslant h+1.$$

Reasoning in a similar way in the remaining cases it follows that

$$f(n;A,B) \geqslant h. \tag{8}$$

We now state the following

Conjecture: $f(n;A,B) = h$

which we shall be able to prove in several cases. As before, $k\langle d\rangle$ denotes the vector $(d, \cdots, d)$ with k components.

THEOREM.

$f(n;A,B) = h$ if *one of the following conditions is fulfilled*:

1. $t(A)=t(B)=1, s(A)=s(B)$ and $k, m \geqslant 3$
2. $A=B=k\langle d\rangle$, d even and $k \geqslant d$
3. $A=B=k\langle d\rangle$, d odd, $k \geqslant 3(d+1)/2$ and k odd
4. $A=B=k\langle d\rangle$, d off, $k \geqslant d$ and k even
5. $A=B=k\langle d\rangle$, d arbitrary and $k=2,3,4,5,6$

PROOF 1. Set $s=s(A)$. According to (4) and (5) we must prove that $\{n, \cdots, n(s^2+s-1)\}$ cannot be partitioned into two sets S_1, S_2 such that S_1 is A-sum-free and S_2 is B-sum-free. We suppose the contrary. Let $n \in S_1$. If $x_1 = \cdots = x_k = n \in S_1$ then $ns = a_1x_1 + \cdots + a_kx_k \notin S_1$ so $ns \in S_2$. Put $x_1 = \cdots = x_n = ns \in S_2$. We obtain $ns^2 \in S_1$. Now suppose that $a_1 = b_1 = 1$; if $x_1 = ns^2, x_2 = \cdots = x_k = n \in S_1$ then because S_1 is A-sum-free it follows that $n(s^2+s-1) \in S_2$.

(i) if $n(s+1) \in S_1$ we take $x_1 = n$, $x_2 = \cdots = x_k = n(s+1)$, and from $ns^2 \in S_1$ it yields that S_1 is not A-sum-free, which is false.

(ii) if $n(s+1) \in S_2$ we take $x_1 = ns$, $x_2 = \cdots = x_m = n(s+1)$ and $n(s^2+s-1) \notin S_2$ is obtained because S_2 is B-sum-free, which is a contradiction.

2. It is clear that we may treat only the case when $n=1$, which yields $h = h(k\langle d\rangle, k\langle d\rangle) = k^2d^2 + kd - d - 1$. Suppose it is possible to partition $\{1, 2, \cdots, k^2d^3 + kd - d\}$ into two sets S_1 and S_2, neither of which contains a solution of

$$d(x_1 + \cdots + x_k) = x_{k+1}. \tag{9}$$

Then, if $1 \in S_1$, we must have $dk \in S_2$ and $d^2k^2 \in S_1$. Furthermore $1 \in S_1$ and $d^2k^2 \in S_1$ implies $k^2d^3 + kd - d \in S_2$, since otherwise $x_1 = d^2k^2$, $x_2 = x_3 = \cdots = x_k = 1$, $x_{k+1} = k^2d^3 + kd - d$ is a solution of (9) in S_1. We claim that the following hold:

$$(d^2 - ld)k^2 + l(d+1)k - l \in S_1 \tag{10}$$

$$(d^3 - ld^2)k^2 + (ld + l + 1)kd - (l+1)d \in S_2 \tag{11}$$

for all l such that the expressions involved are positive integers.

This is proved by induction on l. It has been shown for $l = 0$. Suppose we have shown it for $l = 0, \cdots, m-1$. If $(d^2 - md)k^2 + m(d+1)k - m \in S_2$ then $x_1 = (d^2 - md)k^2 + m(d+1)k - m$, $x_2 = \cdots = x_k = dk$, $x_{k+1} = (d^3 - (m-1)d + m)dk - md$ is a solution of (9) in S_2. Thus (10) holds for $l = m$. Also if $(d^3 - md^2)\ k^2 + (md + m + 1)dk - (m+1)d \in S_1$ then $x_1 = (d^2 - md)\ k^2 + m(d+1)k - m$, $x_2 = \cdots = x_k = 1$, $x_{k+1} = (d^3 - md^2)$ $k^2 + (md + m + 1)dk - (m+1)d$ is a solution of (9) in S_1. Thus (11) holds for $l = m$. This establishes the claim. Now take $l = d$ in (10). This shows that $d(d+1)k - d \in S_1$. Take $l = d - 1$ in (11). This shows that $d^2k^2 + d^3k - d^2 \in S_2$. Now if $2dk - 1 \in S_2$, then $x_1 = \cdots = x_d = 2dk - 1$, $x_{d+1} = \cdots = x_k = dk$, $x_{k+1} = d^2k^2 + d^3k - d^2$ is a solution of (9) in S_2. Thus $2dk - 1 \in S_1$. Since $(d+1, 2) = 1$ we can find non negative integers s and t such that $(d+1)s + 2t = k$. Then $x_1 = \cdots = x_s = d(d+1)xk - d$, $x_{s+1} = \cdots = x_{s+t} = 2dk - 1$, $x_{s+t+1} = \cdots = x_{k+1} = d^2k^2$ is a solution of (9) in S_1, a contradiction.

3. As in case 2. we have $2dk - 1 \in S_1$, $d^2k^2 + d^3k - d^2 \in S_2$. If $1/2d(dk-1) + dk \in S_2$ we find that $x_1 = x_2 = 1/2d(dk-1) + dk$, $x_3 = x_4 = \cdots = dk$, $x_{k+1} = d^2k^2 + d^3k - d^2$ is a solution of (9) in S_2. Thus $1/2d(dk-1) + dk \in S_1$. Since $(d+2, 4) = 1$, we may find integers $s, t \geqslant 0$ such that $(d+2)s + 4t = 2k$. (Every integer $n \geqslant 3(d+1)$ has such a representation.) Then $x_1 = \cdots = x_s = 1/2d(dk-1) + dk$, $x_{s+1} = \cdots = x_{s+t} = 2dk - 1$, $x_{s+t+1} = \cdots = x_k = 1$, $x_{k+1} = d^2k^2$ is seen to be a solution of (9) in S_1, a contradiction.

4. We may argue as in case 2. that $2dk - 1 \in S_1$. Let $k = 2p$. Then $x_1 = \cdots = x_p = 2dk - 1$, $x_{p+1} = \cdots = x_k = 1$, $x_{k+1} = d^2k^2$ is a solution of (9) in S_1, which is a contradiction.

5.
(a) $k = 2$. Let $1 \in S_1$. Then $2d \in S_2$ so $4d^2 \in S_1$. If $x_1 = 4d^2$, $x_2 = 1$ then it follows that $4d^3 + d \in S_2$. Take $l = 2d - 1$ in (11) and we obtain $4d^2 \in S_2$, a contradiction.
(b) $k = 3$. We have $1 \in S_1$ and $3d \in S_2$. If d odd we get on setting $l = d$ in (10) that $3d^2 + 2d \in S_1$ and on setting $l = 1/2(3d-1)$ in (11) that $6d^2 + d \in S_2$. It now follows that $1/2(3d+1) \in S_2$, since otherwise $x_1 = x_2 = 1/2(3d+1)$, $x_3 = 1$ $x_4 = 3d^2 + 2d$ is a solution of (9) in S_1. But then $x_1 = x_2 = 1/2(3d+1)$, $x_3 = 3d$, $x_4 = 6d^2 + d$ is a solution of (9) in S_2, a contradiction. If d is even, we find that on setting $l = d$ in (10) $3d \in S_1$,

another contradiction.

(c) $k=4$. We have $1 \in S_1$, $4d \in S_2$. If $d=0 \pmod 3$ put $l=4d/3$ in (10). This shows that $4d \in S_1$, a contradiction. If $d=1 \pmod 3$ put $l=(4d-1)/3$ in (10). This shows that $8d-1 \in S_1$. Setting $l=0$ in (10) shows $16d^2 \in S_1$. But then $x_1=x_2=8d-1$, $x_3=x_4=1$, $x_5=16d^2$ is a solution of (9) in S_1, a contradiction. If $d=1 \pmod 3$ put $l=(4d-1)/3$ in (10). This shows that $8d-1 \in S_1$. Setting $l=0$ in (10) shows $16d^2 \in S_1$. But then $x_1=x_2=8d-1$, $x_3=x_4=1$, $x_5=16d^2$ is a solution of (9) in S_1, a contradiction. If $d=2 \pmod 3$ put $l=(4d-2)/3$ in (10). This shows that $12d-2 \in S_1$. We must then have $24d^2-3d \in S_2$, since otherwise $x_1=x_2=12d-2$, $x_3=x_4=1$, $x_5=24d^2-2d$ is a solution of (9) in S_1. It now follows that $8d-1 \in S_1$ since otherwise $x_1=x_2=8d-1$, $x_3=x_4=4d$, $x_5=24d^2-2d$ is a solution of (9) in S_2. But now $x_1=x_2=8d-1$, $x_3=x_4=1$, $x_5=16d^2$ is a solution of (9) in S_1, a contradiction.

(d) $k=5$. We have $1 \in S_1$, $5d \in S_2$. If $d=0 \pmod 4$, put $l=5d/4$ in (10). This gives $5d \in S_1$, a contradiction. If $d=1 \pmod 4$, put $l=(5d-1)/4$ in (10). This shows that $10d-1 \in S_1$. Taking $x_1=10d-1$, $x_2=x_3=x_4=x_5=1$ it follows that $10d^2+3d \in S_2$. It then follows that $(5d+3)/4 \in S_1$ since otherwise $x_1=x_2=x_3=x_4=(5d+3)/4$, $x_5=5d$, $x_6=10d^2+3d$ would be solution of (9) in S_2. Take $l=d$ in (10). This shows $5d^2+4d \in S_1$. But then $x_1=x_2=x_3=x_4=(5d+3)/4$, $x_5=1$, and $x_6=5d^2+4d$ is a solution of (9) in S_1 a contradiction. If $d=2 \pmod 4$, put $l=(5d-2)/4$ in (10). This shows that $15d-2 \in S_1$. Then we must have $45d^2-4d \in S_2$ since otherwise $x_1=x_2=x_3=15d-2$, $x_4=x_5=1$, $x_6=45d^2-4d$ would be a solution of (9) in S_1. It follows from this that $10d-1 \in S_1$ since otherwise $x_1=x_2=x_3=x_4=10d-1, x_5=5d$, $x_6=45d^2-4d$ is a solution of (9) in S_2. But then $x_1=10d-1, x_2=15d-2$, $x_3=x_4=x_5=1$, $x_6=25d^2$ is a solution of (9) in S_1, which is false. If $d=3 \pmod 4$, put $l=(5d-3)/4$ in (10). This shows that $20d-3 \in S_1$. We must have $20d^2+d \in S_2$ since otherwise we could take $x_1=20d-3, x_2=x_3=x_4=x_5=1$, $x_6=20d^2$ as a solution of (9) in S_1. We must have $1/2(5d+1) \in S_1$ since otherwise $x_1=x_2=1/2(5d+1)$, $x_3=x_4=x_5=5d$, $x_6=20d^2+d$ is a solution of (9) in S_2. However we now have $x_1=20-3, x_2=x_3=1/2(5d+1)$, $x_4=x_5=1, x_6=25d^2$ as a solution of (9) in S_1, a contradiction.

(e) $k=6$. We have $1 \in S_1, 6d \in S_2$: If $d=0 \pmod 5$ take $l=6d/5$ in (10). This gives $6d \in S_1$, a contradiction. If $d=1 \pmod 5$ take $l=(6d-1)/5$ in (10). This gives $12d-1 \in S_1$. Then $x_1=x_2=x_3=12d-1$, $x_4=x_5=x_6=1$, $x_7=36d^2$ is a solution of (9) in S_1, a contradiction. If $d=2 \pmod 5$ take $l=(6d-2)/5$ in (10). This shows that $18d-2 \in S_1$. Then $x_1=x_2=18d-2$, $x_3=x_4=x_5=x_6=1$, $x_7=36d^2$ is a solution of (9) in S_1, a contradiction. If $d=3 \pmod 5$ take $l=(6d-3)/5$ in (10). This shows $24d-3 \in S_1$. We cannot have $48d^2-2d \in S_1$ since, if this is so, $x_1=x_2=24d-3$ $x_3=x_4=x_5=x_6=1$, $x_7=48d^2-2d$ is a solution of (9) in S_2. But now we find that $x_1=18d-2, x_2=x_3=x_4=x_5=x_6=6d$, $x_7=48d^2-2d$ verifies (9) so $198d-2 \in S_1$. Also $x_1=x_2=18d-2$, $x_3=x_4=x_5=x_6=1$, $x_7=36d^2$ is a solution of (9) in S_1 contradiction. If

$d=4 \pmod 5$) take $l=(6d-4)/5$ in (10). This shows that $30d-4 \in S_1$. We cannot have $60d^2-4d \in S_1$; $x_1=x_2=30d-4$, $x_3=x_4=x_5=x_6=1$, $x_7=60d^2-4d$ would be a solution of (9) in S_1. Thus $60d^2-4d \in S_2$. We must then have $12d-1 \in S_1$ since otherwise $x_1=x_2=x_3=x_4=12d-1$, $x_5=x_6=6d$, $x_7=60d^2-4d$ is a solution of (9) in S_2. But this shows that $x_1=x_2=x_3=12d-1$, $x_4=x_5=x_6=1$, $x_7=36d^2$ is a solution of (9) in S_1, a contradiction.

REMARK 1. It is easy to see that the conjecture also holds for $d>k$, $d=0(\mathrm{mod}(k-1))$ and possibly for certain specific values of $d(\mathrm{mod}(k-1))$ not necessarily zero, but the above analysis becomes very long for $k \geqslant 7$.

REMARK 2. In all cases covered by the theorem, A and B satisfy $s(A)=s(B)$ and $t(A)=t(B)$. However the conjecture has been verified using a computer in a few cases when A, B do not satisfy these restrictions e.g ., $A=(1,1)$ $B=(1,2)$; $A=(1,3)$, $B=(2,2)$ and $A=(1,4)$, $B=(2,2)$.

ACKNOWLEDGEMENTS. The author wishes to thank R.K. Guy for very helpful discussions about this subject and to the referee for pointing out some errors in the initial version of this paper.

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