PRODUCTS OF BIJECTIONS

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1. Introduction and statement of results

The aim of this paper is to give some necessary or sufficient conditions for a map from a group $(B,*)$ in itself being the product of two bijections. The product of the mappings $f,g:B\to B$ is defined by $(f*g)(x) = f(x)*g(x)$. The case when the group is $(R, +)$ is assigned to M. Rădulescu and S. Rădulescu (see [1]) and treated by J.Ceder ([1]) ; also appeared formulated by L.Funar [2] as an open problem.

The main results of Ceder are concerned in :

THEOREM 1 Every map $f:R\to R$ can be written as a sum of three bijections of R in itself.

THEOREM 2 If the map $f: R\to R$ is constant or has an uncountable range, then f can be written as the sum of two bijections.

It has been not settled the case when f is not constant and its range is at most countable. Miller [4] extends theorem 2, by proving the following:

THEOREM 3 For every map $f :R\to R$ there exist a finite set $A\subset R$ and the bijections $g, h: R\to R$ such that $f(x)=g(x)+h(x), \forall x\in R\setminus A$.

Closely related are the problems when the bijections are constrained to be isomorphisms for some additional structures, by example homeomorphisms of R .We mention here the result of Ceder [1] and Ho [3]:

THEOREM 4 The continuous mapping $f: R\to R$ is the sum of two homeomorphisms of R iff f has finite total variation on each finite interval.

Our main result is an improvement of theorems 1,2,3 above stated. Below $(B,*)$ is a fixed group.

THEOREM 5. (i) If B is infinite and $f : B \to B$ has infinite range then f is the product of two bijections.

(ii) If B is not finitely generated, $f(B) = \{a_1,\dots,a_k\}$ and $f^{-1}(a_j)$ are infinite $\forall j \in \{1,\dots,k\}$ then f is the product of bijections

(iii) If B is a torsion free abelian group, $f(B) = \{a_1,a_2\}$ and f is the product of two bijections, then $f^{-1}(a_i)$ are infinite $\forall i \in \{1,2\}$.

(iv) If B is an abelian group $f(B) = \{a_0,a_1,\dots,a_k\}$ card $f^{-1}(a_i) = y_i$ for $i \in \{1,\dots,k\}$, then f is the product of two bijections iff $a_1^{y_1} a_2^{y_2} \dots a_k^{y_k} = a_0^{y_1+\dots+y_k}$

As a corollary we shall obtain :

<u>THEOREM 6</u> If B is infinite, $f : B \to B$, then f is the sum of three bijections.

We denote below card A or $|A|$ the cardinal of A, rg f the range and dom f the definition set of the map f. Our proof follows the lines developed in [4].

<u>2. Preliminary lemmas</u>

For the proof of the above theorems we need some prerequisites.

<u>Lemma 1</u> If G is a subgroup of B, $f : B \to B$ is a map such that rg $f \subseteq G$, card rg f = card G, and for every $x \in$ rg f, $f^{-1}(x) \cap G \neq \emptyset$ then there exist two bijections $h,g : G \to G$ such that $f|_G = h * g$.

Proof. Let k = card rg f, γ be the least ordinal of cardinal k and consider a well ordering of $G = \{y_\alpha,\ \alpha < \gamma\}$. Set Γ_φ for the graph of the map φ. Then we have an order relation defined by $f_0 \subseteq f_1$ iff $\Gamma_{f_0} \subseteq \Gamma_{f_1}$. For $X \subseteq G$ set $\varphi(X)$ for the least ordinal α with $y_\alpha \notin X$.

Denote by $\mathfrak{F}$ the family of maps (h,g) having the following properties :

1. $\operatorname{dom} h = \operatorname{dom} g \subseteq G$, $\operatorname{rg} f, \operatorname{rg} g \subseteq G$
2. $f|\operatorname{dom} h = h * g$
3. $|\Gamma_h| = |\Gamma_g| \leqslant |\varphi(\operatorname{dom} h)|$
4. $\varphi(\operatorname{dom} h) \leqslant \varphi(\operatorname{rg} h), \varphi(\operatorname{rg} g)$
5. h, g are injective

We start with $\Gamma_{h_c} = \Gamma_{g_o} = \phi$. The family $\mathfrak{F}$ has an obvious induced ordering. If $K = \{(h_\lambda, g_\lambda)\}$ is a totally ordered sequence inclosed in $\mathfrak{F}$ then $\Gamma_h = \bigcup_\lambda \Gamma_{h_\lambda}$, $\Gamma_g = \bigcup_\lambda \Gamma_{g_\lambda}$ define a majorant for K which lies in $\mathfrak{F}$. By Zorn's lemma $\mathfrak{F}$ has a maximal element say (h,g). Let $\alpha = \varphi(\operatorname{dom} h)$. If $\alpha = \gamma$ then (h,g) will answer our question, else $\alpha < \gamma$. Then there exists $z \in G\setminus \operatorname{rg} h$ such that $z^{-1} * f(y_\alpha) \notin \operatorname{rg} g$. In fact $A = \{z^{-1} * f(y_\alpha) \; ; \; z \in G\setminus \operatorname{rg} h\}$ has all elements distinct, $|A| = |G\setminus \operatorname{rg} h|$ but $|\operatorname{rg} h| \leqslant |\varphi(\operatorname{dom} h)| = |\alpha| < |\gamma|$ hence we cannot have $A \subseteq \operatorname{rg} g$ because $|A| > |\operatorname{rg} g|$. Consider the extensions $\tilde{h}$, $\tilde{g}$ of h, g by $\Gamma_{\tilde{h}} = \Gamma_h \cup (y_\alpha, z)$, $\Gamma_{\tilde{g}} = \Gamma_g \cup (y_\alpha, z^{-1} * f(y_\alpha))$, Now if $y_\alpha \notin \operatorname{rg} \tilde{h}$ we can choose $t \in G\setminus \operatorname{dom} \tilde{h}$ such that $y_\alpha^{-1} * f(t) \notin \operatorname{rg} \tilde{g}$. Indeed set $B = \{y_\alpha^{-1} * f(t) \; ; t \in G\setminus \operatorname{dom} h\}$ then $|B| = |\{f(t) \; ; \; t \in G - \operatorname{dom} h|$. We have $|\{f(t); \; t \in G\}| = |\operatorname{rg} f| = |G|$ because $f^{-1}(x) \cap G \neq \phi \quad \forall x \in \operatorname{rg} f$. It follows that $|B| = |G \setminus \operatorname{dom} h| > |\operatorname{rg} \tilde{g}|$ hence the desired element t could be chosen in $B\setminus \operatorname{rg} \tilde{g}$.

Now put $\Gamma_{h*} = \Gamma_{\tilde{h}} \cup (t, y_\alpha)$, $\Gamma_{g*} = \Gamma_{\tilde{g}} \cup (t, y_\alpha^{-1} * f(t))$. The same arguments hold in the case $y_\alpha \notin \operatorname{rg} g^*$ and a pair (h^{**}, g^{**}) is obtained. Now $(h^{**}, g^{**}) \supsetneq (h,g)$ and lies in $\mathfrak{F}$ contradicting our assumption of maximality of (h,g) in $\mathfrak{F}$.

<u>Lemma 2.</u> Let G be an infinite subgroup of B, $f : B \to B$ with $\operatorname{rg} f \subseteq G$ such that : card $f^{-1}(a) \leqslant$ card G implies $f^{-1}(a) \subseteq G$. Then there exist the bijections $h,g : B\setminus G \to B\setminus G$ for which $f|_{B\setminus G} = h * g$.

Proof. Define the equivalence relation $x \sim y$ iff $x * y^{-1} \in G$ and set $F_o = \{a \in B \mid f^{-1}(a) \notin G\}$, $H_a = f^{-1}(a) \setminus G$ for $a \in F_o$.

If $F_o = \phi$ then $B = G$ and the result is trivial valid. For $F_o \neq \phi$ it follows from hypothese that $\{H_a\}_{a \in F_o}$ is a partition of $B \setminus G$ and card $H_a >$ card G.

Let $\{K_a\}_{a \in F_o}$ a partition of $B \setminus G$ in reunions of cosets with respect to G (i.e. sets $\sim$invariant), and card K_a = card H_a. This is possible because card $H_a >$ card G. Now consider $\pi : B \setminus G \to B \setminus G$ a bijection with the property that $\pi(K_a) = H_a$, $\forall a \in F_o$. Then it is sufficient to prove the lemma for $f_1 = f|_{B \setminus G} \circ \pi$. Observe that $f_1^{-1}(a) = K_a$ so rg $f_1 \subseteq B \setminus G$. Set $h(x) = a * x$ for $x \in K_a$, $g(x) = x^{-1}$. Then $f_1 = h * g$ and h_{K_a} is a bijection onto K_a, so the lemma is proved.

<u>LEMMA 3</u> Let $f : B \to B$, F = rg f. There exists an subgroup $G \subseteq B$ which satisfies:

1. $F \subseteq G$ and card $G \leqslant$ card $F + \aleph_0$
2. $\forall x \in F$, $f^{-1}(x) \cap G \neq \phi$
3. $\forall x \in F$ with card $f^{-1}(x) \leqslant$ card G we have $f^{-1}(x) \subseteq G$.

<u>Proof.</u> Let G_o be the group generated by $F \cup \{x_a,\ a \in F\}$ where $x_a \in f^{-1}(a)$. Then set

$$X_o = \bigcup_{\substack{\text{card } f^{-1}(a) \leqslant \text{card } G_o \\ a \in F}} f^{-1}(a)$$

and G be the group generated by $X \cup G_o$. Then card $X \leqslant$ card $G_o + \aleph_0 \leqslant$ card $F + \aleph_0$ so card $G \leqslant$ card $F + \aleph_0$. Also for $a \in F$, $f^{-1}(a) \cap G \ni x_a$ and the group G satisfies the requirements of lemma 3.

<u>3. Proof of theorems and comments</u>

If card $F \geqslant \aleph_0$ and G is the group given by lemma 3 then according to lemmas 1 and 2 f is the product of two bijections. If F is finite, card $B \geqslant \aleph_0$ then f^* defined by $f^*(x) = f(x) * x$ has an infinite range, so f is the product of there bijections, which proves theorem 6 and theorem 5 (i). Let now G be the countable subgroup of B given by lemma 3 and H another subgroup of B such that $G\subseteq H$, card $H/G \geq$ card F .(this is possible since B is not finitely generated). We consider the partition $\{K_a\}_{a\in F}$ of H in G-cosets, and a bijection $\pi : H \to H$ such that $\pi(K_a) = f^{-1}(a) \cap H$. Then set $h(x) = a * x$ for $x\in K_a$, $g(x) = x^{-1}$, which satisfies $f|_H = h * g$. From lemma 2 applied to the subgroup H, theorem 5 (ii) follows. In the casee (iii) we suppose $f = h * g$, $f^{-1}(a_2) = \{z_1, \ldots, z_n\}$, $g(z_i) = t_i^{-1}$. Then $h(z_i) = a_2 * t_i$ and $h(x) = a_1 * g(x)^{-1}$ for $x \notin \{z_1, \ldots, z_n\}$. If we set $T = \{t_1, \ldots, t_n\}$ then it follows that $a_1^{-1} * a_2 * T = T$ so $\prod_{x\in T} x = \prod_{x\in a_1^{-1}*a_2*T} x$. This gives $(a_1^{-1}*a_2)^n = e$, e being the null element of B and since B is torsion free $a_1 = a_2$, a contradiction.

Therefore the assertion of theorem 5 (iii) is valid. In the last case we put $A_i = f^{-1}(a_i)$, $C_i = g(A_i)$. Observe that the family of subsets $\{a_i^{-1} * C_i\}$ $i\in\{0,1,\ldots,k\}$ is a partition of B, hence we can define a map $\tau : B\setminus C_0$ setting $\tau(x) = a_0 * a_j^{-1} * x$ for $x\in C_j$, $1\leq j\leq k$. Then τ is injective, so it is a bijection since C_j are finite for $1\leq j\leq k$. Therefore $\prod_{x\in B\setminus C_0} x = \prod_{x\in B\setminus C_0} \tau(x)$,equivalent to $a_0^{y_1+\ldots+y_k} = a_1^{y_1}\ldots a_k^{y_k}$.Conversely let consider the sets $C_j = \{x \mid x = a_0^{i-1}a_j^{1-i} \prod_{1\leq s\leq k} a_0^{y_s} a_s^{-y_s}$, for $1\leq i\leq y_j\}$, $1\leq j\leq k$ and $C_0 = B\setminus \bigcup_{j=1}^{k} C_j$. Then we can give the bijections $h_i, g_i : A_i \to C_i$ because card A_i = card C_i ,such that $g_i(x) = h_i(x)^{-1} * a_i$ The maps $g(x) = g_i(x)$ for $x\in A_i$,$h(x) = h_i(x)$ for $x\in A_i$ satisfy the requirements of the theorem 5 (iv).

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