DEFORMING THE $\mathbb{R}$-FUCHSIAN (4,4,4)-TRIANGLE GROUP INTO A LATTICE

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Abstract. We prove that the last discrete deformation of the $\mathbb{R}$-Fuchsian (4,4,4)-triangle group in $PU(2,1)$ is a cocompact arithmetic lattice. We also describe an experimental method for finding the combinatorics of a Dirichlet fundamental domain, and apply it to the lattice in question.

1. Introduction

A lot of interest for complex hyperbolic geometry has been generated by Mostow's work in the late 1970's, exhibiting the first examples of nonarithmetic lattices in $PU(n,1)$ (in fact the current list of examples is only slightly larger, and all known examples in dimension four or higher are arithmetic).

The major difficulty to construct such groups is to find efficient methods for proving directly that a given group, for instance given by a number of generators, acts discretely on complex hyperbolic space. The construction of fundamental domains is much more complicated than in spaces with constant sectional curvature, since there are no totally geodesic real hypersurfaces. In particular there is no canonical choice for faces of a polyhedron, and the bare hands proofs of discreteness which have appeared to this day rely on using various kinds of hypersurfaces, adapted to the situation at hand (bisectors in [11], $\mathbb{C}$-spheres in [6], hybrid cones in [19], cones over totally geodesic subspaces in [5]).

In this paper, we shall focus on Dirichlet domains, where the faces of the fundamental polyhedra are on bisectors, i.e. hypersurfaces equidistant between two points in complex hyperbolic space. Given a group $\Gamma \subset PU(2,1)$, the Dirichlet domain centered at p_0 is the set

$$F_\Gamma = \{x \in H^2_{\mathbb{C}} : d(x,p_0) \le d(x,\gamma p_0), \forall \gamma \in \Gamma\} \tag{1.1}$$

The group is discrete if and only if F_Γ has nonempty interior, and in that case F_Γ is a fundamental domain for Γ (modulo the action of the stabilizer of p_0 in Γ).

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As discussed in [3], one needs to be extremely cautious when using experimental methods to determine the combinatorics of Dirichlet domains, and in fact several errors can be found in the founding paper [11]. A slightly weaker version of the determination of the combinatorics of F_Γ consists in finding the smallest subset $W \subset \Gamma$ such that $F_W = F_\Gamma$, where we define the partial Dirichlet domain F_W as $\{x \in H^2_{\mathbb{C}} : d(x, p_0) \leq d(x, \gamma p_0), \forall \gamma \in W\}$.

The key point is to be able to determine whether or not, for a given set W, F_W has side pairings, in the sense of the Poincaré polyhedron theorem. We describe a method for doing this efficiently in section 7. The idea is to use somewhat rough necessary conditions to have side pairings, namely W should be Giraud-closed, see Proposition 7.1. If W is not Giraud-closed, then we give an explicit set of group elements that must be added to W (and repeat this procedure until W is Giraud-closed). If it is Giraud-closed, then one needs to work a little to check whether or not F_W has side pairings.

Unfortunately, we are still quite far from an actual algorithm for deciding whether a group is discrete or not. Indeed, not only does our procedure depend on having a computer with infinite precision, but there is no reason why it should stop after a finite number of steps. However, our methods are very helpful in order to guess whether a given group is discrete and, if so, to guess what a fundamental polyhedron should look like. Giving an actual proof of these guesses would then involve a somewhat prohibitive amount of numerical analysis.

One place where we do get away with only approximate constructions is the following. Suppose a group is known to be discrete (which can sometimes be checked by arithmetic means) and, by using experimental methods, we get the impression that it could be cocompact. Then it is reasonably easy to confirm rigorously that this impression is correct (see the techniques of section 6).

We illustrate our method by studying a specific group, which is a deformation of a certain $\mathbb{R}$-Fuchsian triangle group in $PU(2,1)$ and show that, at some point in the deformation, the group becomes a cocompact lattice (the corresponding representation of the triangle group is not faithful).

To put this in perspective, we mention that R. Schwartz has studied many deformations of triangle groups, and found ingenious ways to prove discreteness, but since his analysis is done by finding fundamental domains on the boundary $\partial H^2_{\mathbb{C}}$, these methods do not allow us to handle lattices in any straightforward way. In fact the group studied here was already alluded to in [18], and it was already known to be discrete.

The group $G(4,4,4;5)$ is generated by three complex reflections of order two, I_1, I_2 and I_3, whose mirrors make an angle $\pi/4$. There is one additional parameter that expresses how far we are from the $\mathbb{R}$-Fuchsian situation. More precisely, denoting by e_1, e_2 and e_3 some unit vectors polar to the mirrors of these three reflections, the angle condition between the mirrors translates into $|\langle e_i, e_j\rangle| = 1/\sqrt{2}$. The additional parameter can be chosen to be the triple Hermitian inner product $\langle e_1, e_2, e_3\rangle = \langle e_1, e_2\rangle\langle e_2, e_3\rangle\langle e_3, e_1\rangle$.

Choosing the parameter so that $I_1I_2I_3I_2$ is elliptic of order n gives a family of groups $G(4,4,4;n)$; the case $n = 7$ corresponds to the group studied in [19], which is not a lattice (it has infinite covolume).

Our main theorem is the following:

Theorem 1.1. *The group $G(4,4,4;5)$ is a cocompact lattice in $PU(2,1)$.*

The main original aspect of our work is the method for proving that the group is cocompact, even though some parts of the general study of Dirichlet domains in complex hyperbolic geometry seem not to be well known (see however [9], [10], [15]). We expect that our topological methods for proving that the 2-faces of F_W are bounded (see section 6) can be refined to give efficient ways to determine the combinatorics of 2-faces rigorously.

In section 7, we give a conjectural picture of the Dirichlet domain for our group, but it is conceivable that a more refined computer analysis would reveal that we have missed some of its faces (in a situation parallel to the one described in [3] for Mostow's groups). The results of that section give a good illustration of the possible complexity of Dirichlet domains (see Figures 8-11).

Finally we mention that the result of Theorem 1.1 seems to hold only for very few deformations of $\mathbb{R}$-Fuchsian triangle groups. The author was informed by J. Parker that the group $G(5,5,5;5)$ has the same property; in fact, that group can be checked to be a subgroup of index 60 in Mostow's lattice $\Gamma(5,7/10)$, by using an explicit presentation (see [13] and [14]). No such simple description is known for $G(4,4,4;5)$. It is the author's impression that these two deformed triangle groups are the only lattices among all the $G(n,n,n;p)$.

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2. Describing the group

For background on complex hyperbolic geometry, we refer the reader to [9]. We consider the deformation space of the $\mathbb{R}$-Fuchsian $(4,4,4)$-triangle group, generated by three complex reflections I_1, I_2 and I_3 that preserve a totally real plane in $H^2_{\mathbb{C}}$. As an abstract group, it is given by

$$\langle \iota_1, \iota_2, \iota_3 | \iota_j^2 = (\iota_j \iota_k)^4 = 1 \rangle$$

The order four relation imposes that we maintain the angle between the mirrors of the reflections at $\pi/4$ throughout the deformation.

We exclude the $\mathbb{C}$-Fuchsian representation, since it cannot be deformed (see [21]). Except for the latter representation, the mirrors are not orthogonal to a common complex geodesic, hence their orthogonal complements are linearly independent, and we may choose unit vectors e_1, e_2 and e_3 as basis vectors for $\mathbb{C}^3$, in such a way that the mirror of I_j is $e_j^{\perp}$. We must have

$$|\langle e_i, e_j \rangle| = r = \cos(\pi/4) = 1/\sqrt{2}$$

and we are free to choose the argument of this complex number. By rescaling the vectors e_i, we may assume, without loss of generality, that

$$\langle e_1, e_2 \rangle = \langle e_2, e_3 \rangle = \langle e_3, e_1 \rangle = -r\varphi \tag{2.1}$$

The minus sign is included to stick with the notation in [11] and [3]. We summarize the above discussion in the following:

Lemma 2.1. *The deformation space of the $\mathbb{R}$-Fuchsian $(4,4,4)$ triangle group is one-dimensional, parameterized by the argument of the triple Hermitian product $\langle e_1, e_2, e_3 \rangle = \langle e_1, e_2 \rangle \langle e_2, e_3 \rangle \langle e_3, e_1 \rangle$.*

In the basis adapted to the mirrors of the generators, the Hermitian form is given by $\langle v, w \rangle = w^* H v$ where

$$H = \begin{bmatrix} 1 & -r\overline{\varphi} & -r\varphi \\ -r\varphi & 1 & -r\overline{\varphi} \\ -r\overline{\varphi} & -r\varphi & 1 \end{bmatrix} \tag{2.2}$$

This is a real matrix when $\varphi = 1$, which corresponds to the $\mathbb{R}$-Fuchsian case. The matrix has determinant

$$1 - 3r^2 - r^3(\varphi^3 + \overline{\varphi}^3) = -\frac{1}{2} - \frac{1}{\sqrt{2}} \cos t$$

where we denote by t the argument of φ^3, i.e. $\varphi^3 = e^{it}$. The form has signature $(2,1)$ if and only if its determinant is negative, which is equivalent to

$$|t| < \frac{3\pi}{4} \tag{2.3}$$

This gives an explicit interval parameterizing the deformation space of the $\mathbb{R}$-Fuchsian $(4,4,4)$-triangle group. Similar descriptions are of course easily obtained for other triangle groups. For completeness we write down the matrix of one of the generating reflections:

$$I_1 = \begin{bmatrix} 1 & -2r\overline{\varphi} & -2r\varphi \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{2.4}$$

The other two generators are obtained by symmetry, $I_2 = JI_1J^{-1}$, $I_3 = JI_2J^{-1}$ where J is the isometry corresponding to the natural 3-cycle on the standard basis vectors, $Je_i = e_{i+1}$ (indices modulo 3).

For convenience, we shall use shortcut notation for words in the generators, since they are described by giving a sequence of integers. We shall write I_{1232} or sometimes even simply 1232 for $I_1I_2I_3I_2$, etc.

From the extensive experiments conducted by Schwartz (see the survey paper [18]), one expects that discreteness conditions can be expressed in terms of certain explicit short words in the generator. Specifically, one needs to verify that the group elements I_{123} and I_{1232} are not elliptic of infinite order. In the terminology of [18], the $(4,4,4)$-triangle groups are of type A, i.e. the behavior of the deformed groups should be controlled by the element I_{1232}.

It is completely elementary to determine the values of t for which I_{1232} is elliptic; one computes the trace of $I_1I_2I_3I_2$, either by computing its matrix explicitly, or by using the formula from [16] (here we use a slightly different convention for the parameter, due to the minus sign in (2.1); our parameter t is related to Pratoussevitch's parameter α by $t = \pi - \alpha$).

One checks that

$$\mathrm{T}r(I_1I_2I_3I_2) = 5 + 4\sqrt{2}\cos t \tag{2.5}$$

hence the element I_{1232} is elliptic for $\cos t < -\frac{1}{2\sqrt{2}}$, i.e. $|t| > t_0$, where $t_0 = 1.93216345\ldots$ (we still take $|t| < 3\pi/4$, see (2.3)).

Conjecture 2.1 (Schwartz)**.** *The deformed representation is discrete and faithful if and only if I_{1232} is not elliptic, i.e. if and only if $|t| \leq t_0$. In the range $t_0 < |t| < 3\pi/4$, there is a countable collection of values of t where the representation is discrete but not faithful.*

If I_{1232} is elliptic, then in order to get a discrete representation, one needs to impose that I_{1232} have finite order. This leads to define the following parameter values:

Definition 2.2. $G(4,4,4;n)$ is the image of the representation ρ_n of the $(4,4,4)$-triangle group such that

$$\mathrm{T}r(I_1I_2I_3I_2) = 1 + 2\cos\frac{2\pi}{n} \tag{2.6}$$

This is equivalent to

$$\cos t = \frac{\cos\frac{2\pi}{n} - 2}{2\sqrt{2}} \tag{2.7}$$

and one checks that this satisfies (2.3) only for $n \geq 5$.

The group $G(4,4,4;7)$ is studied in great detail in [19]. It has infinite covolume, and Schwartz shows that its manifold at infinity has a real hyperbolic structure. The main technique there is to construct a fundamental domain on the boundary.

The groups corresponding to $n = 5$ and $n = 6$ are mentioned in [18] as being difficult to analyze, even though they are known to be discrete. For $n = 5$, the difficulty is explained by Theorem 3.1, which implies that the limit set is the whole boundary $\partial H^2_{\mathbb{C}}$ (though this does not seem to hold for $n = 6$).

From this point on, we focus on analyzing the group $G(4,4,4;5)$. In that case, using the fact that $\cos(2\pi/5) = (\sqrt{5}-1)/4$, we have

$$t = \arccos\left(-\frac{9-\sqrt{5}}{8\sqrt{2}}\right) \approx 2.211616098 \tag{2.8}$$

and

$$\varphi^3 = -\frac{9-\sqrt{5}}{8\sqrt{2}} + i\sqrt{3}\frac{3+\sqrt{5}}{8\sqrt{2}} \tag{2.9}$$

One can get an explicit expression for φ itself, by rewriting

$$\varphi^3 = -\overline{\omega}\frac{\sqrt{5}+3i\sqrt{3}}{4\sqrt{2}} \tag{2.10}$$

where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Note that

$$-\frac{\sqrt{5}+3i\sqrt{3}}{4\sqrt{2}} = \left(\frac{\sqrt{5}-i\sqrt{3}}{2\sqrt{2}}\right)^3 \tag{2.11}$$

Accordingly, we write

$$\varphi = -\overline{\omega}\frac{\sqrt{5}-i\sqrt{3}}{2\sqrt{2}}e^{\pi i/9} = \frac{\sqrt{5}-i\sqrt{3}}{2\sqrt{2}}e^{4\pi i/9} \tag{2.12}$$

One could of course multiply this value by ω without changing the group, which is really determined by φ^3 (the above value is chosen so that its argument is closest to 0).

Recall that the integers in $\mathbb{Q}(\sqrt{5})$ are given by $\mathbb{Z}+\mathbb{Z}\frac{1+\sqrt{5}}{2}$, so that φ^3/r^3 is an algebraic integer. We shall see that more can be said along those lines (see Proposition 2.5).

Note that the phase shift is not rational (in the terminology of [11]), i.e. φ is not a root of unity or, in other words, t is not a rational multiple of π. Perhaps surprisingly, a well chosen reflection subgroup does have rational phase shift, as we now justify (see Proposition 2.3(2)).

We first compute

$$I_1I_2I_3 = \begin{bmatrix} -3-2\sqrt{2}\overline{\varphi}^3 & -2\varphi^2-\sqrt{2}\overline{\varphi} & \sqrt{2}\varphi+2\overline{\varphi}^2 \\ -\sqrt{2}\varphi-2\overline{\varphi}^2 & -1 & \sqrt{2}\overline{\varphi} \\ -\sqrt{2}\overline{\varphi} & -\sqrt{2}\varphi & 1 \end{bmatrix} \tag{2.13}$$

Its characteristic polynomial is

$$\begin{aligned} &\lambda^3 - \frac{3+\sqrt{5}}{2}\omega\lambda^2 + \frac{3+\sqrt{5}}{2}\overline{\omega}\lambda - 1 \\ &= \overline{\omega}(\lambda-\omega)[(\overline{\omega}\lambda)^2 - \frac{1+\sqrt{5}}{2}(\overline{\omega}\lambda)+1] \end{aligned}$$

Noting that $\frac{1+\sqrt{5}}{2} = 2\cos\frac{\pi}{5}$, we find that the eigenvalues of I_{123} are

$$\omega, \omega e^{\pi i/5}, \omega e^{-\pi i/5} \tag{2.14}$$

In particular, $(I_{123})^5$ is a complex reflection, and it turns out its mirror contains one of the 2-faces of the Dirichlet domain for our group (see Section 8). The mirror in question is given by $v_{123}^{\perp}$, where

$$v_{123} = [e^{-\pi i/9}, 1, e^{\pi i/9}]^T \tag{2.15}$$

is an eigenvector for I_{123} with eigenvalue ω. Its other eigenvectors can be written as

$$u_{123} = [e^{-2\pi i/45}, 1, e^{2\pi i/45}]^T, \quad w_{123} = [e^{-38\pi i/45}, 1, e^{38\pi i/45}]^T \tag{2.16}$$

The negative vector among the three is u_{123}, giving the isolated fixed point of I_{123} in hyperbolic space (it has eigenvalue $e^{\pi i/5}\omega$ for I_{123}). This point turns out to be one of the vertices of our fundamental domain.

Proposition 2.3.

(1) $\dfrac{|\langle v_{123}, v_{231}\rangle|}{\sqrt{\langle v_{123}, v_{123}\rangle\langle v_{231}, v_{231}\rangle}} = \dfrac{1}{2\sin\frac{\pi}{10}}$

(2) $\arg(\langle v_{123}, v_{231}\rangle\langle v_{231}, v_{312}\rangle\langle v_{312}, v_{123}\rangle) = -\dfrac{2\pi}{3}$

Proof: The result follows from a tedious calculation.

$$\begin{aligned}
&\langle v_{123}, v_{123}\rangle = 3 - 2\mathfrak{Re}\{2r\varphi e^{-\pi i/9} + r\overline{\varphi}e^{-2\pi i/9}\}\\
&= 3 - 2\mathfrak{Re}\{2r\overline{\varphi}e^{\pi i/9}(2-\omega)\}\\
&= 3 + 2\mathfrak{Re}\{\frac{\sqrt{5}+i\sqrt{3}}{2}(2\omega - \overline{\omega})\}\\
&= \frac{3-\sqrt{5}}{4} = 2\sin^2\frac{\pi}{10}
\end{aligned}$$

Now we compute

$$\begin{aligned}
&\langle v_{123}, v_{231}\rangle = 2e^{\pi i/9} + e^{-2\pi i/9} - 3r\varphi - r\overline{\varphi}(2e^{-\pi i/9} + e^{2\pi i/9})\\
&= e^{\pi i/9}(2-\omega) + 3\overline{\omega}\frac{\sqrt{5}-i\sqrt{3}}{4}e^{\pi i/9} + \omega\frac{\sqrt{5}+i\sqrt{3}}{4}e^{-2\pi i/9}(2-\overline{\omega})\\
&= e^{\pi i/9}\{2-\omega+3\overline{\omega}\frac{\sqrt{5}-1}{4} + 3\overline{\omega}(-\frac{\omega}{2}) + (2+3\omega)\frac{\sqrt{5}-1}{4} - (2+3\omega)\frac{\overline{\omega}}{2}\}\\
&= e^{\pi i/9}\{2-\omega-\frac{3}{2}-\overline{\omega}-\frac{3}{2}+\frac{\sqrt{5}-1}{4}(2+3\omega+3\overline{\omega})\}\\
&= e^{\pi i/9}\{-1-\omega+1+\omega+\frac{\sqrt{5}-1}{4}(-1)\}\\
&= \frac{1-\sqrt{5}}{4}e^{\pi i/9}
\end{aligned}$$

This means that $|\langle v_{123}, v_{231}\rangle| = \dfrac{\sqrt{5}-1}{4} = \sin\dfrac{\pi}{10}$, from which the results of the proposition easily follow. □

Remark 2.4. The triple Hermitian inner product is

$$(-\frac{\sqrt{5}-1}{4}e^{\pi i/9})^3 = (\frac{\sqrt{5}-1}{4})^3 e^{-2\pi i/3}$$

The angle between the two complex geodesics $v_{123}^\perp$ and $v_{231}^\perp$ implies that reflections of order 10 along those mirrors would braid. We do not have such reflections in our group, but I_{123} is regular elliptic and its fifth power is a reflection of order 2. In other words, our group and Mostow's group $\Gamma(10, 2/3)$ share a subgroup generated by three complex reflections of order two. Mostow's group is commensurable to a hypergeometric monodromy group Γ_μ, where $\mu = (1, 6, 6, 6, 11)/15$ (see [2] or [20]). This group Γ_μ is not discrete, see [12] (in fact, it is easy to find a non-discrete triangle subgroup). In particular the common subgroup has infinite index in one of the two groups.

We do not know if our group is a subgroup of finite index in any of the Mostow lattices (such a description is known explicitly for the group $G(5,5,5;5)$, see [13]).

In fact more can be said about coordinates given by the v_{ijk}:

Proposition 2.5. *The group $G(4,4,4;5)$ can be conjugated to consist of matrices with entries in the ring of integers in $\mathbb{Q}(\sqrt{5},\omega)$.*

Proof: This follows from a direct calculation, based on Proposition 2.3. We shall write $\tau = (\sqrt{5}-1)/2$. In the basis $\sqrt{\frac{2}{\tau}}\{v_{312}, e^{\pi i/9}v_{123}, e^{2\pi i/9}v_{231}\}$, one verifies that I_1, I_2 and I_3 are given respectively by the matrices

$$(2.17)\qquad \begin{bmatrix} -1 & 0 & 0 \\ \tau\overline{\omega} & 0 & -1 \\ -\tau\overline{\omega} & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \tau\omega & \overline{\omega} \\ 0 & -1 & 0 \\ \omega & \tau\overline{\omega} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & -\tau\omega \\ -1 & 0 & \tau\omega \\ 0 & 0 & -1 \end{bmatrix}$$

and the Hermitian form is given by the matrix

$$(2.18)\qquad \begin{bmatrix} \tau & -1 & \overline{\omega} \\ -1 & \tau & -1 \\ \omega & -1 & \tau \end{bmatrix}$$

Observe that the entries of the above four matrices are algebraic integers. Indeed the integers in $\mathbb{Q}(\sqrt{5})$ are given by $\mathbb{Z}[\tau]$, and the integers in $\mathbb{Q}(\omega)$ by $\mathbb{Z}[\omega]$. □

Corollary 2.6. *The group $\Gamma = G(4,4,4;5)$ is discrete.*

Proof: Observe that $\tau^2 = 1-\tau$, and compute the determinant of the Hermitian matrix (2.18), which is

$$(2.19)\qquad \tau^3 - 1 - 3\tau = -2-\tau = \frac{-5-\sqrt{5}}{2} < 0$$

Note in particular that the form H has signature $(2,1)$, since $\det(H) < 0$ and there exist vectors that are positive for H (the diagonal entries are equal to $\tau > 0$).

The determinant remains negative when applying $\sqrt{5} \mapsto -\sqrt{5}$, so the conjugate form H^σ has signature either $(0,3)$ or $(2,1)$. To distinguish between those two cases, we consider the span of the first two basis vectors (let us call these v_1 and v_2).

$$(2.20)\qquad ||v_2 - \frac{\langle v_2, v_1\rangle}{\langle v_1, v_1\rangle}v_1||^2 = \frac{\langle v_1, v_1\rangle\langle v_2, v_2\rangle - |\langle v_1, v_2\rangle|^2}{\langle e_1, e_1\rangle}$$

For either Hermitian form H or H^σ, the latter expression is equal to $(\tau^2-1)/\tau = ((\tau^\sigma)^2-1)/\tau^\sigma = -1$. For H^σ, this exhibits two orthogonal negative vectors, hence H^σ is negative definite.

The integers of $\mathbb{Q}(\sqrt{5},\omega)$, embedded in $\mathbb{C}\times\mathbb{C}$ by the product of two non complex embeddings $\mathbb{Q}(\sqrt{5},\omega)$, is a discrete set, hence $\Gamma\times\Gamma^\sigma$ is discrete in $U(H)\times U(H^\sigma)$. The group $U(H^\sigma)$ is compact, hence the projection onto $U(H)$ maps discrete subsets to discrete subsets. $\square$

Remark 2.7.

(1) The above shows that Γ, if a lattice, is in fact an arithmetic subgroup of $U(H)\approx U(2,1)$. The fact that there is an arithmetic reason for its discreteness is already mentioned in [18].
(2) The matrices (2.17) make it easy to test equality between words in the generators simply by performing algebra in $\mathbb{Z}[\tau,\omega]$, which reduces to repeated use of the relations $\tau^2=1-\tau$, $\omega^2=-1-\omega$. We shall not make much use of this observation in the present paper.

3. Proof of cocompactness

Theorem 3.1. *The group $\Gamma=G(4,4,4;5)$ is a cocompact arithmetic lattice.*

We construct a bounded polyhedron that contains a fundamental domain for the action of the group. Since we know that the group is discrete, the Dirichlet construction is guaranteed to provide a fundamental domain.

We shall use the coordinates given by Hermitian form (2.2), where generators have matrices of the form (2.4). We take for the center p_0 of the domain the natural point fixed by the symmetry of order three given by J, i.e. $p_0=[1,1,1]^T$, and define

$$F=\{x\in H^2_{\mathbb{C}} : d(x,p_0)\leq d(x,\gamma p_0)\ \forall\gamma\in\Gamma\} \tag{3.1}$$

It is quite difficult to find the combinatorics of the polyhedron F, and we shall only state the results without proof in section 7. Observe however that F is of course contained in any "partial" Dirichlet domain

$$F_W=\{x\in H^2_{\mathbb{C}} : d(x,p_0)\leq d(x,\gamma p_0)\ \forall\gamma\in W\} \tag{3.2}$$

where W is any finite set of group elements.

We shall choose a certain small set of words W, making the analysis of the combinatorics more tractable. The following result clearly implies 3.1, since $F\subset F_W$.

Theorem 3.2. *Let W consist of all group elements obtained from words of length three in the generators I_1, I_2 and I_3. Then the polyhedron F_W is bounded.*

One advantage of using the set W is that it is much smaller than the actual set of elements needed for the Dirichlet fundamental domain (see section 7). More importantly, it turns out that all intersections of bisectors defining F_W are "normal", in the sense that the codimension k-faces are on precisely k bounding bisectors. This simplifies matters greatly when determining the combinatorics.

Recall the distance formula

$$\cosh \frac{1}{2} d(x, y) = \frac{|\langle x, y\rangle|}{\sqrt{\langle x, x\rangle\langle y, y\rangle}}$$

where we normalize the metric to be pinched between -1 and $-1/4$, and we abuse notation by using the same symbol for the points of $H^2_{\mathbb{C}}$ and vector representatives in $\mathbb{C}^3$.

The bisector $B(a, b)$ equidistant from a and b is defined by

$$B(a, b) = \{x \in H^2_{\mathbb{C}} : d(x, a) = d(x, b)\} \tag{3.3}$$

We shall always normalize the vectors a and b so that $\langle a, a\rangle = \langle b, b\rangle$, in which case the equation defining the bisector $B(a, b)$ takes the simple form

$$|\langle x, a\rangle| = |\langle x, b\rangle|.$$

When dealing with a Dirichlet domain with center p_0, we use the notation

$$\widehat{\gamma} = B(p_0, \gamma p_0). \tag{3.4}$$

Every such bisector is diffeomorphic to a 3-ball. Its trace on the boundary of $H^2_{\mathbb{C}}$ is a smooth 2-sphere (called a spinal sphere, see [9]).

The intersection of two bisectors can be quite complicated (see [9]), and in general it is neither connected nor transverse. The basic fact, which is also due to Goldman, is that these pathologies do not occur in the case of coequidistant bisectors (in particular for all the pairs of bisectors bounding a Dirichlet domain). The intersection of two coequidistant bisectors is a smooth 2-disk, whose trace on the boundary of $H^2_{\mathbb{C}}$ is a smooth circle. We shall come back to this in section 4.

Theorem 3.2 can be reformulated as follows.

Proposition 3.3. *Each 3-face of F_W has at least one 2-face, and all its 2-faces are bounded.*

We first prove that the theorem is indeed a consequence of the proposition.

Proof: We write Z for the trace of F_W on the boundary $\partial H^2_{\mathbb{C}}$, i.e. the projectivization of the set of null vectors x satisfying $|\langle x, p_0\rangle| \leq |\langle x, wp_0\rangle|$ for all $w \in W$. F_W is bounded if and only if Z is empty.

It is enough to show that for each $w \in W$ the 3-face $\widehat{w} \cap F_W$ is bounded. Indeed, suppose that this is the case. Then $Z \subset \partial H^2_{\mathbb{C}}$ is closed by definition, and we show that it is open as well. If $w \in W$, then every point in the spinal sphere on the boundary of $\widehat{w}$ is outside Z, hence Z can be defined as the projectivization of the set of null vectors x satisfying the strict inequality $|\langle x, p_0\rangle| < |\langle x, wp_0\rangle|$ for all $w \in W$. Now Z is open since it is the intersection of finitely many open sets, and it is clearly not the whole boundary $\partial H^2_{\mathbb{C}}$ (W is non empty). This implies that Z is empty, since $\partial H^2_{\mathbb{C}}$ is connected.

In order to show that each 3-face is bounded, we apply the same connectedness argument as above, but one dimension lower. Each 3-face $\widehat{w} \cap F_W$ is a polyhedron in the bisector $\widehat{w}$ (which is a 3-ball and whose trace on the boundary is a 2-sphere). Since all 2-faces are bounded we see as above, using the connectedness of S^2, that the trace on the boundary of $\widehat{w} \cap F_W$ is either empty or the whole spinal sphere bounding $\widehat{w}$. The latter is excluded because every 3-face of F_W has at least one 2-face. □

Remark 3.4. Note that the argument shows that it suffices to prove that all the 1-faces are bounded. This in turn can be done by analyzing all possible intersections $\widehat{\alpha} \cap \widehat{\beta} \cap \widehat{\gamma}$, hence one is reduced to checking that a finite number of intervals are compact. We shall explain a similar approach in section 6.3.

We defer the proof of Proposition 3.3 to section 5, where we shall use the computer to analyze the 2-faces of F_W individually, each of which is contained in the intersection of two bisectors $\widehat{\gamma_1} \cap \widehat{\gamma_2}$. Since there are twelve group elements in W and two types of 3-faces, we need to check the claim only for twenty-two such intersections.

Moreover, there is a natural symmetry between the generators, given by antiholomorphic involutions σ_{ij} that conjugates I_i into I_j (and I_k into itself). For instance, the map

$$\sigma_{12} : [x_1, x_2, x_3]^T \mapsto [\overline{x}_2, \overline{x}_1, \overline{x}_3]^T \tag{3.5}$$

clearly induces an isometry in view of the Hermitian form (2.2). Note that the product of any two such involutions is given by the 3-cycle J or its inverse.

The finite set W is preserved by the symmetry, hence the polyhedron F_W has at most two isometry type of faces, contained in bisectors for I_{iji} and I_{ijk} respectively (here and in what follows we mean the indices i, j and k to be distinct integers between 1 and 3). It will become clear in Section 5 that the two faces on bisectors for I_{iji} and I_{ijk} are not isometric to each other.

4. Intersections of bisectors

In order to analyze the 2-faces of Dirichlet polyhedra, we describe a convenient set of coordinates for bisector intersections, deduced from the slice decomposition.

Given two distinct points p_0 and p_1 in $H^2_{\mathbb{C}}$ (represented by vectors with the same square norm, see section 3), the bisector $B(p_0, p_1)$ is the projectivization of the set of negative vectors x with

$$|\langle x, p_0\rangle| = |\langle x, p_1\rangle| \tag{4.1}$$

In particular, the (non necessarily negative) vectors in $\mathbb{C}^3$ that satisfy (4.1) are in the orthogonal complement of one and only one vector of the form $p_0 - \alpha p_1$ where $|\alpha| = 1$. This gives a foliation of B by complex geodesics, given by the linear hyperplanes $(p_0 - \alpha p_1)^\perp$, where α ranges over the arc of the unit circle where the square norm of $p_0 - \alpha p_1$ is positive. This is the so-called **slice decomposition** of bisectors, described in [11] or [9]. The circle of vectors $p_0 - \alpha p_1$ intersects the ball in a geodesic that is contained in B, called the **real spine** of B. The complex geodesic that contains it is the **complex spine** of B.

Under some genericity assumption to be discussed shortly, we can parameterize the intersection of two bisectors by their respective spinal coordinates. For simplicity of the exposition, we shall only consider the intersection of coequidistant bisectors, i.e. bisectors equidistant from a common point (more general intersections are considered in [9]).

Let D be the equidistant locus between p_0, p_1 and p_2, which is the intersection of the two bisectors $B_j = B(p_0, p_j)$, $j = 1, 2$. In what follows we assume that D is non empty. Each $x \in D$ is in a unique slice of B_1 and in a unique slice of B_2. In other words, we have $x \in (p_0 - u_1 p_1)^\perp \cap (p_0 - u_2 p_2)^\perp$ for a unique pair $(u_1, u_2) \in S^1 \times S^1$. We denote by $s : D \to S^1 \times S^1$ the map defined by

$$s(x) = (u_1, u_2) \tag{4.2}$$

It is readily seen that

$$u_j = \frac{\langle x, p_j\rangle}{\langle x, p_0\rangle} \tag{4.3}$$

Definition 4.1. The bisectors B_1 and B_2 are called **cospinal** if they have the same complex spine (this is equivalent to saying that the vectors p_0, p_1 and p_2 are linearly dependent in $\mathbb{C}^3$).

Lemma 4.2. *The spinal coordinate map s is injective if and only if B_1 and B_2 are not cospinal.*

Proof: Suppose the complex spines are distinct, i.e. that the vectors p_j, $j = 0, 1, 2$ are independent. Then $p_0 - u_1 p_1$ and $p_0 - u_2 p_2$ are independent for all $u_j \neq 0$, hence the corresponding polar complex lines are distinct, and they intersect in at most one point. In particular x is uniquely determined by u_1 and u_2, since it is simply given by $(p_0 - u_1 p_1)^\perp \cap (p_0 - u_2 p_2)^\perp$. This shows that s is injective.

If the complex spines coincide, it follows from the slice decomposition that the bisectors intersect in a slice (which goes through the point of intersection of the real spines of the two bisectors). In that case the map s is constant. □

If B_1 and B_2 are not cospinal, the image of the map is a disk, as was first proved by Goldman (see Proposition 4.3). For x not necessarily in the intersection of the two bisectors, (u_1, u_2) with $u_j = \langle x, p_j\rangle / \langle x, p_0\rangle$ gives a system of inhomogeneous coordinates on the complex hyperbolic plane (but in those coordinates, the complex hyperbolic plane is usually not given by the unit ball).

It is convenient to choose lifts to homogeneous coordinates for a point (u_1, u_2). Writing P for the matrix that passes from the standard basis to $\{p_0, p_1, p_2\}$, we may take

$$x = (P^* H)^{-1} U \tag{4.4}$$

where $U^T = [1, u_1, u_2]$. This corresponds to normalizing the homogeneous coordinates so that $\langle x, p_0\rangle = 1$. We shall sometimes write the above change of coordinates as

$$x = \mathbf{a_0} + \mathbf{a_1} u_1 + \mathbf{a_2} u_2 \tag{4.5}$$

so that the vectors $\mathbf{a_j}$ are the columns of $A = (P^* H)^{-1}$.

The equations defining the complex hyperbolic plane in the coordinates (u_1, u_2) are then simply

$$\langle \mathbf{a_0} + \mathbf{a_1} u_1 + \mathbf{a_2} u_2, \mathbf{a_0} + \mathbf{a_1} u_1 + \mathbf{a_2} u_2\rangle < 0$$

4.1. **Cospinal bisectors.** If the three points p_0, p_1 and p_2 are in a common geodesic, then the bisectors $B(p_0, p_j)$, $j = 1, 2$ are cospinal. In that case it is easy to see that the bisectors intersect if and only if their real spines intersect, in which case the intersection consists of one complex geodesic, orthogonal to their common complex spine, through the intersection of the real spines.

Since this situation occurs in the context of Dirichlet domains, it is important to be able to describe the intersection of a complex geodesic C with another bisector (or with the corresponding half spaces). This is done in detail in [9], and we only review some of the results that we need.

We take an orthonormal basis $\{v_1, v_2\}$ for the complex 2-plane that projects to the geodesic, so that $\langle v_1, v_2\rangle = 0$, $\langle v_1, v_1\rangle = -1$ and $\langle v_2, v_2\rangle = 1$. This identifies the complex geodesic with the unit disk $|z| < 1$ by taking vectors to be of the form $v_1 + zv_2$.

It is then easy to describe the intersection $B(p_0, p_3) \cap C$ of any bisector with C. Indeed in projective space one simply has the following equation

$$|\langle v_1, p_0\rangle + z\langle v_2, p_0\rangle| = |\langle v_1, p_3\rangle + z\langle v_2, p_3\rangle| \tag{4.6}$$

which yields a circle (in degenerate cases, it could be empty, or a line, or the whole complex plane). In general this circle is not a geodesic, but it is a hypercycle, i.e. it is at a constant distance from a certain geodesic (see [9]). One may check that it is a geodesic if and only if C intersects the real spine of the bisector, possibly in projective space (see [5]).

The 2-faces of a Dirichlet domain F_W that lie on complex geodesics are obtained by intersecting a number of regions delimited by arcs of circles in the unit disk (note that in general such a region is not convex in the hyperbolic metric, contrary to the claim in Lemma 3.3.1 of [11]).

4.2. **Generic intersections.** If the three points p_0, p_1 and p_2 are not in a common complex geodesic, we shall say that the two bisectors have **generic intersection**. We write $D = B_1 \cap B_2$ and T for its extension to projective space, i.e. the set of solutions of equation (4.1) where x is no longer required to have negative square norm. In the terminology of [9], T is a Clifford torus, which is the intersection of the two extors that extend the bisectors B_j.

Proposition 4.3. *D is a smooth disk (in particular it is connected).*

This result is proved by Goldman in [9], by analyzing in detail the possible tangencies between bisectors.

Definition 4.4. We call T a **Giraud torus** and its intersection with complex hyperbolic space $D = T \cap H^2_{\mathbb{C}}$ a **Giraud disk**.

The terminology comes from the fact that these disks appear in the theorem of Giraud, see Theorem 4.6, which turns out to be crucial in analyzing Dirichlet domains.

Concretely, it is useful to write explicit inequalities defining the Giraud disk in the Clifford torus. These can be obtained by solving $\langle x, x\rangle \leq 0$ with $x = \mathbf{a_0} + \mathbf{a_1}u_1 + \mathbf{a_2}u_2$ and $|u_1| = |u_2| = 1$ (see (4.5)). We shall write

$$f(u_1, u_2) = \langle \mathbf{a_0} + \mathbf{a_1}u_1 + \mathbf{a_2}u_2, \mathbf{a_0} + \mathbf{a_1}u_1 + \mathbf{a_2}u_2\rangle \tag{4.7}$$

It is easy to see that, for each u_1, there is a (possibly empty) interval of values of u_2 that satisfy $f(u_1,u_2)\leq 0$.

One way to check this is to use the following elementary fact, that will be used several times throughout the paper.

Lemma 4.5. *Let $\beta\in\mathbb{R}$. The equation*

$$2\mathfrak{Re}(\alpha z)=\beta \tag{4.8}$$

has a solution on the unit circle $|z|=1$ if and only if

$$|\beta|\leq 2|\alpha| \tag{4.9}$$

There is exactly one solution when equality holds, and the solution is arbitrary if and only if $\alpha=\beta=0$. Otherwise there are two values of z, corresponding to intersecting two circles in the plane.

The solutions can be obtained for instance by rewriting equation (4.8) as the following quadratic equation

$$\alpha z^2-\beta z+\overline{\alpha}=0 \tag{4.10}$$

whose solutions are on the unit circle if and only if condition (4.9) holds. One gets

$$z=\frac{\beta\pm i\sqrt{4|\alpha|^2-\beta^2}}{2\alpha} \tag{4.11}$$

For a given value of u_1, the values of u_2 for which $f(u_1,u_2)=||\mathbf{a_0}+\mathbf{a_1}u_1+\mathbf{a_2}u_2||^2=0$ are obtained by solving an equation of the form (4.8), where

$$\begin{aligned}\alpha&=\alpha(u_1)=-\langle\mathbf{a_2},\mathbf{a_0}+\mathbf{a_1}u_1\rangle\\ \beta&=\beta(u_1)=-||\mathbf{a_0}+\mathbf{a_1}u_1||^2-||\mathbf{a_2}||^2\end{aligned} \tag{4.12}$$

Hence, for each u_1, there is an interval of values of u_2 for which $f\leq 0$. It can be checked that, when nonempty, this interval yields a hypercycle in the corresponding complex slice of B_1 (see [9]).

Determining the Giraud disk amounts to finding the values of u_1 such that the interval is non empty. Using (4.9), this translates into an inequality of the form

$$|\mu_0+\mu_1u_1+\overline{\mu}_1\overline{u}_1|\leq 2|\nu_0+\nu_1u_1| \tag{4.13}$$

where

$$\begin{aligned}\mu_0&=\langle\mathbf{a_0},\mathbf{a_0}\rangle+\langle\mathbf{a_1},\mathbf{a_1}\rangle+\langle\mathbf{a_2},\mathbf{a_2}\rangle\\ \mu_1&=\langle\mathbf{a_1},\mathbf{a_0}\rangle\\ \nu_0&=\langle\mathbf{a_0},\mathbf{a_2}\rangle\\ \nu_1&=\langle\mathbf{a_1},\mathbf{a_2}\rangle\end{aligned} \tag{4.14}$$

After squaring both sides, this can be rewritten as

$$\tau_0 + \tau_1 u_1 + \overline{\tau}_1 \overline{u}_1 + \tau_2 u_1^2 + \overline{\tau}_2 \overline{u}_1^2 \leq 0 \tag{4.15}$$

For completeness, we give a formula for the coefficients that appear:

$$\begin{aligned} \tau_0 &= \mu_0^2 + 2|\mu_1|^2 - 4|\nu_0|^2 - 4|\nu_1|^2 \\ \tau_1 &= 2\mu_0\mu_1 - 4\overline{\nu}_0\nu_1 \\ \tau_2 &= \mu_1^2 \end{aligned} \tag{4.16}$$

The left hand side of (4.15) is a degree 2 expression in the real and imaginary parts of u_1, so equality holds along a conic. It is quite clear from the geometry of our situation that the conic will not be the unit circle, so that it intersects $|u_1| = 1$ in at most four points.

In particular, there are at most two intervals for u_1 where $f \leq 0$ has solutions, and each interval corresponds to a connected component of the intersection of the bisectors (indeed, the intersection fibers over each interval, with intervals as fibers).

The above analysis makes it clear that D has at most two connected components; the number of components can be found by plotting the graph of $\Delta(u_1) = \beta(u_1)^2 - 4|\alpha(u_1)|^2$. The content of Proposition 4.3 is that this function has at most two zeros on the unit circle.

For concreteness, we now describe one specific example, for $p_1 = I_{121}p_0$ and $p_2 = I_{131}p_0$. We write $u_1 = e^{2\pi i t_1}$, $0 \leq t_1 \leq 1$ and plot the graph of the discriminant as a function of t_1 in Figure 1. The minimum

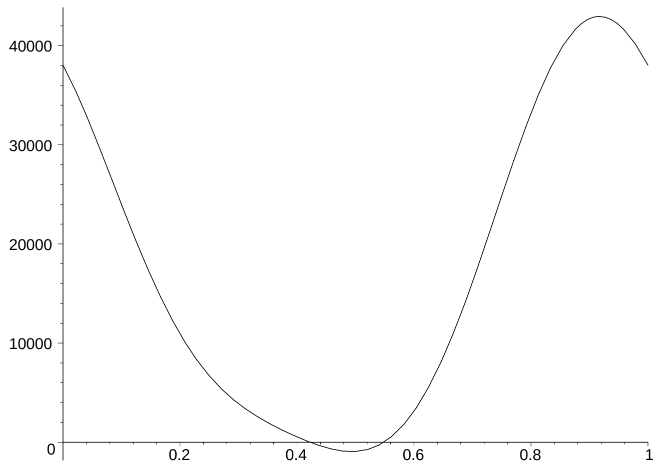


FIGURE 1. The graph of $\beta^2 - 4|\alpha|^2$, as a function of $\frac{1}{2\pi} arg(u_1)$.

and maximum values of t_1 on the boundary of the ball, up to five decimal places, are given by $t_1^{min} = 0.42219\ldots$ and $t_1^{max} = 0.54845\ldots$.

The trace of the boundary of the ball on the torus T can be plotted by using formula (4.11), and plotting the two branches $u_2 = \phi_\partial^\pm(u_1)$. For convenience replace u_j by $t_j = \arg(u_j)/(2\pi)$, with a little care needed in order to choose the argument in a continuous fashion (in the case at hand it is easier to choose the argument between 0 and 2π).

One obtains the graph of $\widehat{121} \cap \widehat{131} \cap \partial H^2_{\mathbb{C}}$, given in Figure 2. Note

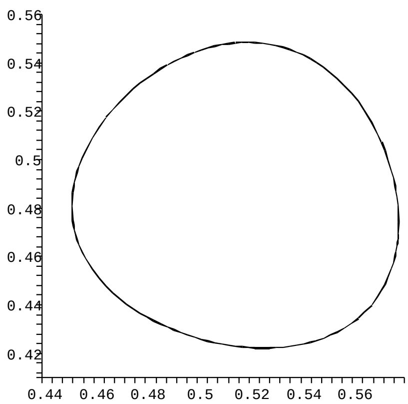


FIGURE 2. The trace on the torus $u_1 = e^{2\pi i t_1}$, $u_2 = e^{2\pi i t_2}$ of the boundary of the ball.

that the maximum value of ϕ^+ and the minimum value of ϕ^- can be obtained by repeating the above process, now writing u_1 as a function of u_2. One finds that $t_2^{min} = 0.45155\dots$ and $t_2^{max} = 57781\dots$.

4.3. **Intersection of three bisectors.** Given a third bisector $B_3 = B(p_0, p_3)$, it is clear that the intersection $T \cap B_3$ has an equation of the form

$$|\delta_0 + \delta_1 u_1 + \delta_2 u_2| = 1 \tag{4.17}$$

where $\delta_j \in \mathbb{C}$ are simply obtained by expressing $p_3 = \sum_{j=0}^2 \overline{\delta}_j p_j$. The corresponding region closer to p_0 than p_3 is given by the inequality $|\delta_0 + \delta_1 u_1 + \delta_2 u_2| \geq 1$.

The solution set of equation (4.17) in the torus $|u_1| = |u_2| = 1$ is a (possibly singular) curve, unless p_3 is equal to one of the three points p_0, p_1 and p_2 (i.e. only one of the δ_j is nonzero). This is one way to formulate Giraud's theorem (see [9]):

Theorem 4.6. *Suppose p_0, p_1 and p_2 are not contained in a common geodesic. Then the disk $D = B_1 \cap B_2$ is not totally geodesic, and it is contained in precisely three bisectors, namely $B_1 = B(p_0, p_1)$, $B_2 = B(p_0, p_2)$ and $B(p_1, p_2)$.*

Note that the curves (4.17) are often reducible. We shall describe in detail how to plot them in section 4.4.

The curves $u_1 = c$ are by definition the intersection of the torus T with the complex slices of B_1. Of course the roles of u_1 and u_2 are symmetric, so $u_2 = c$ gives the slices of B_2. The geometric interpretation also suggests that u_1, u_2 behave similarly as $\overline{u}_1 u_2 = u_2/u_1$. Indeed, the curves $u_2 = cu_1$, $|c| = 1$, are in the complex slices of the third bisector $B(p_1, p_2)$.

Definition 4.7. We shall refer to intersections of T with the complex slices of any of the three bisectors intersecting in T (see Proposition 4.3) as the complex slices of T. More specifically, we shall refer to the curves $u_1 = c$, $u_2 = c$ and $u_2 = cu_1$ with $|c| = 1$ as vertical, horizontal and diagonal complex slices of T.

In order to determine the 2-faces of the Dirichlet domain F_W, we look at the pairs w_1, w_2 of group elements in W such that the corresponding bisectors are not cospinal (the cospinal case is much easier, and was discussed in the previous section). The 2-face is contained in the disk $\widehat{w}_1 \cap \widehat{w}_2$, and it is delimited by the various $\widehat{w}$, $w \in W, w \neq w_1, w_2$. Note that the 2-dimensional region delimited on our disk is piecewise smooth, but not necessarily connected.

The corresponding 2-face of F_W are obtained by solving a system of inequalities, written as

$$|\delta_0^{(w)} + \delta_1^{(w)} u_1 + \delta_2^{(w)} u_2| \geq 1 \tag{4.18}$$

for each $w \in W$, $w \neq w_1, w_2$.

This can be done by plotting the various curves $|\delta_0^{(w)} + \delta_1^{(w)} u_1 + \delta_2^{(w)} u_2| = 1$ on the torus, and analyzing the various connected components cut out on the corresponding Giraud disk. We shall describe a number of examples in section 5.

4.4. **Parameterizing the 1-faces.** As discussed in the previous section, the intersection of three coequidistant bisectors two of which are not cospinal can be written in the natural torus coordinates $|u_1| = |u_2| = 1$ in the form

$$|\delta_0 + \delta_1 u_1 + \delta_2 u_2| = 1 \tag{4.19}$$

We now explain in detail how plotting the solutions of (4.19) on the torus amounts to solving a family of quadratic equations.

4.4.1. *Degenerate case.* We first consider the special case where one of the δ_j is equal to zero, which occurs precisely when one pair of bisectors among the three is cospinal (see equation 4.17).

Say for instance that $\delta_2 = 0$ (we assume also that none of the other two δ_j is zero, otherwise two of the bisectors are equal). Geometrically this means that first and third bisectors are cospinal, hence their intersection is either a complex geodesic or empty, depending on whether their real spines intersect or not.

On the level of projective space, it is clear that $|\delta_0 + \delta_1 u_1| = 1$ yields 0, 1 or 2 values $u_1 = u_1^{(i)}$ such that u_2 is arbitrary, obtained by intersecting the circle $\delta_0 + \delta_1 u_1$ with the unit circle. At most one of the circles $u_1 = u_1^{(i)}$ can intersect the ball, because of the above geometric interpretation.

4.4.2. *General case.* Suppose now that $\delta_j \neq 0$ for all j. We concentrate on plotting u_2 as a function of u_1. For each u_1, we need to find the u_2's that satisfy

$$2\mathfrak{Re}((\overline{\delta}_0 + \overline{\delta}_1 \overline{u}_1)\delta_2\ u_2) = 1 - |\delta_0 + \delta_1 u_1|^2 - |\delta_2|^2 \tag{4.20}$$

Solving the corresponding quadratic equation as in 4.10 yields

$$u_2 = \frac{\beta \pm i\sqrt{4|\alpha|^2 - \beta^2}}{2\alpha} \tag{4.21}$$

where

$$\begin{aligned} \alpha &= (\overline{\delta}_0 + \overline{\delta}_1 \overline{u}_1)\delta_2 \\ \beta &= 1 - |\delta_0 + \delta_1 u_1|^2 - |\delta_2|^2 \end{aligned} \tag{4.22}$$

The solution is on the unit circle if and only if the expression under the squareroot is positive.

Formula (4.21) makes sense only as long as $\delta_0 + \delta_1 u_1 \neq 0$, which can occur for at most one value of u_1, namely $u_1 = -\delta_0/\delta_1$. This is on the torus if and only if $|\delta_0| = |\delta_1|$, and one then gets a "vertical line" on the graph (i.e. u_2 is arbitrary for that value of u_1) if and only if $|\delta_2| = 1$. The latter case can be given a geometric interpretation, considering the definition of the coefficients δ_j. Indeed, one has

$$\delta_0 p_0 + \delta_1 p_1 = p_3 - \delta_2 p_2$$

which means that the two real spines of $B(p_0, p_1)$ and $B(p_2, p_3)$ intersect. Since it comes up somewhat frequently in Dirichlet fundamental domains, we push the analysis a little further. Note however that for the partial Dirichlet domain used defined in the previous section, using only words of length three, all intersections are completely generic (in the sense that they are not cospinal, and their real spines do not intersect) so we do not need to handle this difficulty in order to get Theorem 3.1.

Lemma 4.8. *Suppose* $|\delta_0| = |\delta_1|$ *and* $|\delta_2|$=*1. Then the circle* $u_1 = -\delta_0/\delta_1$ *is contained in the curve (4.19). Moreover,*

$$\lim_{u_1\to(-\delta_0/\delta_1)^+} u_2^+ = \lim_{u_1\to(-\delta_0/\delta_1)^-} u_2^- = -\frac{\delta_0}{|\delta_0|}\frac{1}{\delta_2}$$

$$\lim_{u_1\to(-\delta_0/\delta_1)^-} u_2^+ = \lim_{u_1\to(-\delta_0/\delta_1)^+} u_2^- = +\frac{\delta_0}{|\delta_0|}\frac{1}{\delta_2}$$

Note that it is clear from equation (4.21) that there are indeed two determinations of u_2 near the value $u_1 = -\delta_0/\delta_1$, which we denote by $u_2^\pm$. The meaning of the symbol $\lim_{u_1\to u^\pm}$, where $|u_1| = 1$, stands for a one-sided limit with decreasing or increasing argument, respectively.
Proof: With the hypotheses of the lemma, equation (4.21) becomes

$$\begin{aligned} u_2^\pm &= \frac{-|\delta_0+\delta_1 u_1|^2 \pm i\sqrt{4|\delta_0+\delta_1 u_1|^2 - |\delta_0+\delta_1 u_1|^4}}{2(\overline{\delta}_0+\overline{\delta}_1 u_1)\delta_2} \\ &= \frac{\delta_0+\delta_1 u_1}{|\delta_0+\delta_1 u_1|}\cdot\frac{-|\delta_0+\delta_1 u_1| \pm i\sqrt{4-|\delta_0+\delta_1 u_1|^2}}{2\delta_2} \end{aligned}$$

The second fraction clearly tends to $\pm i/\delta_2$, and the second one is analyzed as follows.

Let us write $\delta_1 = \delta_0 e^{i\mu}$ and $u_1 = e^{i\theta}$, and compute

$$\begin{aligned} \frac{\delta_0+\delta_1 u_1}{|\delta_0+\delta_1 u_1|} &= \frac{\delta_0}{|\delta_0|}\cdot\frac{1+e^{i(\mu+\theta)}}{|1+e^{i(\mu+\theta)|}} \\ &= \frac{\delta_0}{|\delta_0|}\cdot\frac{1+\cos(\mu+\theta)+i\sin(\mu+\theta)}{\sqrt{2}\sqrt{1+\cos(\mu+\theta)}} \end{aligned}$$

The result follows at once from the fact that

$$\begin{aligned} \lim_{x\to\pi^\pm}\frac{1+\cos x}{\sqrt{1+\cos x}} &= 0 \\ \lim_{x\to\pi^\pm}\frac{\sin x}{\sqrt{1+\cos x}} &= \mp\sqrt{2} \end{aligned}$$

□

We end this section with a useful observation, that implies that none of the graphs we plot can have any complicated oscillations. Denote by Λ the set $|\delta_0+\delta_1 u_1+\delta_2 u_2| = 1$ on the torus $|u_1| = |u_2| = 1$.

Lemma 4.9. *Suppose that* Λ *does not contain any complex slice of the torus* T*. Then on any complex slice of* T*, there are at most two points of* Λ*.*

Proof: The fact that vertical complex slices contain at most two points follows at once from the fact that, for a fixed u_1, one can solve for u_2 by solving an equation of degree at most two.

For horizontal slices one solves for u_1 in terms of u_2, and for diagonal ones, one solves for either u_1 or u_2 in terms of $\overline{u}_1 u_2$. $\square$

5. 2-faces of F_W

We now describe the combinatorics of the various 2-faces of F_W, where W consists of all groups elements written as length three words in the generators. The verification for each 2-face relies on computer use quite heavily, and it is highly recommended for the reader to test our claims on their own machine. Computer code that does this efficiently is available on the author's webpage, see [4].

It is enough to describe the faces corresponding to $I_1I_2I_1$ and $I_1I_2I_3$, since the other ones are isometric to one of them by applying the natural symmetry between the generators.

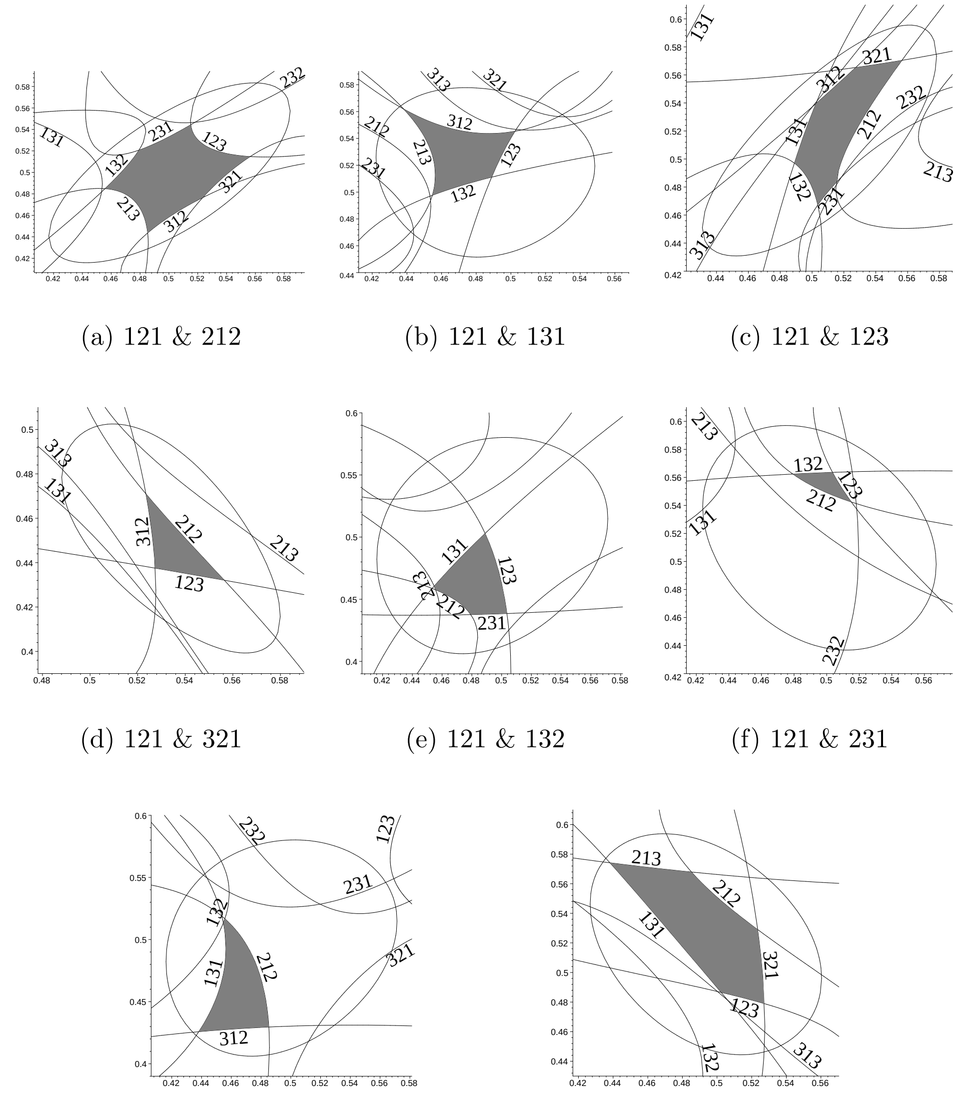
It turns out the face $\widehat{121} \cap F_W$ has eight 2-faces, contained in the intersection with the bisector corresponding to all words in W except 232, 313 and 323 (see Figure 3). The bisectors $\widehat{323}$ does not intersect $\widehat{121}$ (although their analytic continuation to projective space do intersect). The other two intersections $\widehat{121} \cap \widehat{232}$ and $\widehat{121} \cap \widehat{313}$ are both disks in complex hyperbolic space, but their intersection with F_W is empty.

The face $\widehat{123} \cap F_W$ has ten 2-faces, corresponding to all words except 213. All the 2-faces given by the intersection with a bisector corresponding to a word of the form iji is isometric to some 2-face of the face $\widehat{121} \cap F_W$. For instance the isometry σ_{12}, cf. equation (3.5), clearly maps $\widehat{123} \cap \widehat{212}$ to $\widehat{213} \cap \widehat{121}$. We list all the isometries in the table below.

$$
\begin{array}{ccc}
123 \cap 121 & \overset{Id}{\leftrightarrow} & 121 \cap 123 \\
123 \cap 212 & \overset{\sigma_{12}}{\leftrightarrow} & 121 \cap 213 \\
123 \cap 232 & \overset{J^{-1}}{\leftrightarrow} & 121 \cap 312 \\
123 \cap 323 & \overset{\sigma_{13}}{\leftrightarrow} & 121 \cap 321 \\
123 \cap 131 & \overset{\sigma_{23}}{\leftrightarrow} & 121 \cap 132 \\
123 \cap 313 & \overset{J}{\leftrightarrow} & 121 \cap 231
\end{array}
$$

Note also that two 2-faces of $\widehat{123} \cap F_W$ are isometric, since J sends $\widehat{123} \cap \widehat{312} \cap F_W$ to $\widehat{231} \cap \widehat{123} \cap F_W$.

Figure 4 gives a picture of the remaining three 2-faces of $\widehat{123} \cap F_W$.



(a) 121 & 212 (b) 121 & 131 (c) 121 & 123

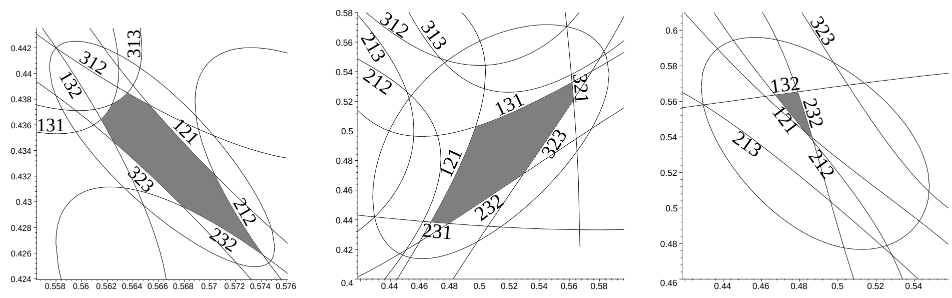
(d) 121 & 321 (e) 121 & 132 (f) 121 & 231

(g) 121 & 213 (h) 121 & 312

FIGURE 3. The 2-faces of $\widehat{121} \cap F_W$, drawn in spinal coordinates.

6. Issues of precision

The results from the preceding section are satisfactory only as long as we can verify that our pictures of the 2-faces of F_W, produced using computer calculations, are precise enough not to be misleading. We list a couple of natural approaches, the most efficient one being presented in section 6.3.



(a) 123 & 321 (b) 123 & 132 (c) 123 & 231

FIGURE 4. The 2-faces of $\widehat{123} \cap F_W$, drawn in spinal coordinates.

6.1. **Interval calculus.** One possible way is to use interval arithmetic throughout the calculations, and replace each plotted point by an interval, large enough that we are sure that the actual graph is contained in it.

There are mainly two difficulties with this approach. Note that we are after a precise enough description of the connected components of the complement of a number of curves in the plane. The interval arithmetic would have the effect of “fattening” each curve (in the vertical direction), and it is not completely clear how this affects the connected components of the complement. In particular, it is not at all clear even that the number of complementary regions would be preserved.

The other difficulty is that we need to obtain some control over possible jumps or oscillations in the graphs, in order for the finite number of values sample points on the horizontal axis to be representative of the behavior of the entire graph (note however that complicated oscillation behaviors are prevented by Lemma 4.9).

We do not discuss the details, but simply remark that using so called “double precision” would suffice by far to plot the above graphs, and in fact it is the level of precision used in [4].

6.2. **Large precision software.** Another option is to use mathematical software like Maple, for instance, that allows us to perform the calculations with arbitrary precision. This is how the graphs in this paper were produced. The price to pay consists of longer running times for the computer programs (note that the graphs in the paper were all produced in a matter of minutes, on a conventional workstation).

There are still a couple of issues with this approach. The first one is that Maple gives a plot based on a finite number of sample points, and technically one would need to justify that these are representative of the general behavior of the graph.

The other issue is that one needs to be careful when dividing by numbers very close to zero (here the relevant result is Lemma 4.8, see section 4.4). We do not comment on this issue here because it does not happen for any of the graphs relevant to F_W, hence has no bearing on our main theorem. One would have to analyze this carefully in order to justify our conjectural picture of the complete fundamental domain for the group.

6.3. **Using topology and basic calculus.** A better approach is the following. We have an explicit parameterization of the intersection with the boundary of the ball of a pair of coequidistant bisectors (or rather of the corresponding extors). Recall from section 4 that this is a smooth circle, which we want to prove is completely disjoint from F_W.

It is enough to cover this circle by a finite number of intervals, each of which is entirely outside of some bisector $\widehat{w}$, $w \in W$, where "outside" means closer to wp_0 than to p_0. Such intervals were of course implicitly defined by Figures 3 and 4, but the advantage of the present formulation is that it now relies on finitely many numerical verifications.

We shall verify all the details for the 2-face corresponding to 121 and 131, drawn in Figure 3(b). The other 2-faces are of course entirely similar. We saw in section 4 how to parameterize the intersection of the two extors extending $\widehat{121}$ and $\widehat{131}$ as $|u_1| = |u_2| = 1$, as well as the curves on the torus corresponding to the boundary of the ball and the intersection with other bisectors.

Recall that for each $w \in W$, the intersection of $\widehat{w}$ with the torus can be written in the form (4.11), hence it has a parameterization $u_2 = \phi_w^{\pm}(u_1)$, valid on at most two intervals of values of u_1. In principle there could be values of u_1 where the denominator vanishes, but we disregard this issue here as it does not come up in any of the 2-faces of F_W.

Recall that the above expression parameterizes not only the curves $\widehat{121} \cap \widehat{131} \cap \widehat{w}$, but also their extension to projective space. In particular, one can find with arbitrarily large precision all the intersections of these curves with the boundary of the ball. Concretely, one solves the equations

$$||\mathbf{a_0} + \mathbf{a_1} u_1 + \mathbf{a_2} \phi_w^{\pm}(u_1)||^2 = 0 \tag{6.1}$$

for u_1, with arbitrarily large precision. In turn, this breaks the topological circle $\widehat{121} \cap \widehat{131} \cap \partial H^2_{\mathbb{C}}$ into a number of intervals that are either closer to or further from p_0 than to wp_0.

A schematic picture of those intervals is drawn in Figure 5. Table 1 gives all the intersections with the boundary for $w = 123$, 132, 213 and 312. Observe that finding the solutions amounts to finding the

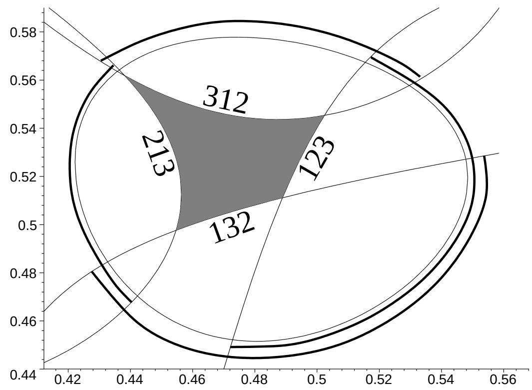


FIGURE 5. Each thick curve corresponds to an interval in the topological circle $\widehat{121} \cap \widehat{131} \cap \partial H^2_{\mathbb{C}}$ that is entirely outside of some bisector $\widehat{w}$ for some $w \in W$.

w	(t_1, t_2)	ϕ^+_∂ or ϕ^-_∂
123	(0.47286, 0.45218)	$-$
	(0.51546, 0.56734)	$+$
132	(0.43267, 0.48454)	$-$
	(0.54783, 0.52714)	$+$
213	(0.44224, 0.47035)	$-$
	(0.43641, 0.56427)	$+$
312	(0.43578, 0.56359)	$-$
	(0.52965, 0.55776)	$+$

TABLE 1. The coordinates of the endpoints on the boundary of some curves $\widehat{121} \cap \widehat{131} \cap \widehat{w}$, with 5 decimal places. The functions $\phi^\pm_\partial$ correspond to the top and bottom halves of the boundary circle, respectively.

roots of a polynomial (even though this is not completely apparent from equation (6.1)), which reduces the problem to a basic calculus problem.

We close this section with a remark about empty 2-faces. As mentioned in section 5, the intersection $\widehat{121} \cap \widehat{323}$ (or more generally $\widehat{iji} \cap \widehat{kjk}$) is empty. Recall from section 4 that the connected components of their intersections correspond to intervals on the unit circle $|u_1| = 1$

where an inequality of the form

$$\tau_0 + \tau_1 u_1 + \overline{\tau}_1 \overline{u}_1 + \tau_2 u_1^2 + \overline{\tau}_2 \overline{u}_1^2 \leq 0$$

holds. The fact that the two bisectors are in fact disjoint can be checked by verifying that the expression (4.15) is strictly positive for any u_1. A Maple plot of its graph is given in Figure 6.

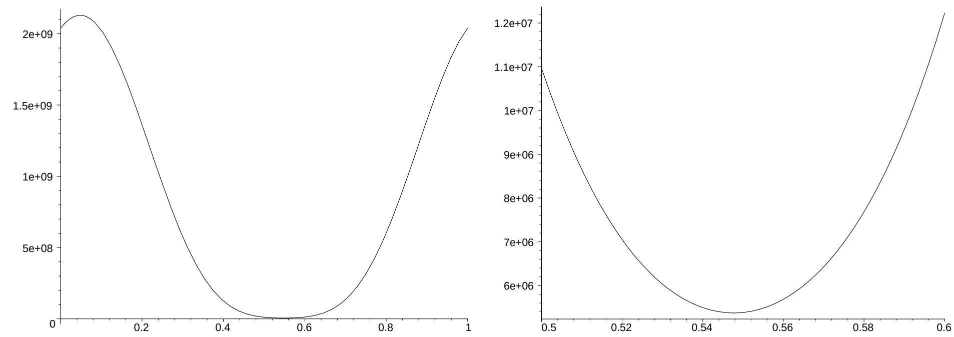


FIGURE 6. The graph of $\tau_0 + 2\mathfrak{Re}\{\tau_1 e^{2\pi i t} + \tau_2 e^{4\pi i t}\}$ for $\widehat{121} \cap \widehat{323}$. The right part of the figure zooms around the minimum value (and makes it apparent that the function stays above 5,000,000).

The other empty 2-faces (like $\widehat{121} \cap \widehat{232}$ for instance, see section 5) are of a different nature, since the corresponding bisectors do intersect (in a Giraud disk). Notice that for our main theorem, what matters is that the 2-face does not approach the boundary, and that can be checked just like in the beginning of this section. Figure 7 illustrates why the 2-face is actually empty (the arrow on each curve points to the side where one gets closer to p_0).

7. Description of a fundamental domain

There is an algorithm to decide whether a given set of matrices in $PU(1,1)$ generate a discrete group, provided the generators are explicitly arithmetically defined (see [17], and also [8]).

The algorithm can be summarized as follows; fix a generating set S for Γ, and let W_n be the set of group elements that can be expressed as words of length at most n in S. For simplicity we assume that no non trivial element of Γ fixes p_0. Start with $n = 1$ and do the following:

(1) Check if the polyhedron F_{W_n} is a fundamental domain for Γ (using the Poincaré polyhedron theorem). If it is, we know Γ is discrete and stop the procedure. Else go to step (2).

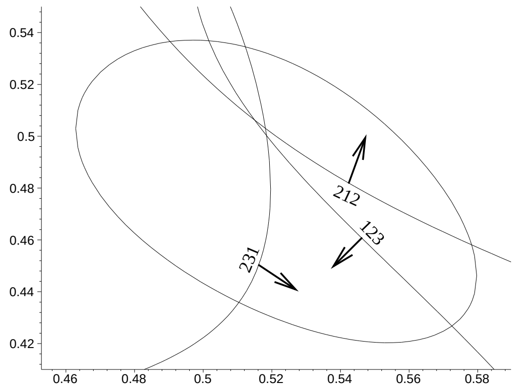


FIGURE 7. $\widehat{121} \cap \widehat{232}$ is a disk, but its intersection with F_W is empty.

(2) If F_{W_n} has side pairings, check if the cycle transformations have finite order (if not, the group is not discrete and we stop the procedure). If F_{W_n} does not have side pairings, but there is no obvious reason why the group is not discrete, replace n by $n+1$ and repeat step (1).

In step (2), the obvious reasons why the group is not discrete correspond to pairs of elements of W_n for which Jørgensen's inequality is violated, or whose product is elliptic of infinite order. In $PU(1,1)$, the procedure is known to stop in finite time (and for 2-generator groups, one can get a universal estimate on this finite time, see [8]).

Some of Nielsen's algorithm described in [17] does carry over to higher dimensions. As mentioned in Remark 2, Proposition 2.5 gives a way to solve the word problem for our group, and we may think of the generating matrices as being explicitly arithmetically defined.

Several technical points of step (1) are much more difficult in $H^n_{\mathbb{C}}$, $n \geq 2$. In fact, the exact determination of the combinatorics of the domain F_{W_n} is already quite problematic. The algebra involved in finding coordinates for the vertices of a Dirichlet polyhedron is much more difficult in the non constant curvature setting; note that, even in homogeneous coordinates, the equations defining the faces of a Dirichlet domain cannot be made simultaneously linear (they are quadratic in the real and imaginary parts of the coordinates). As observed by Mostow, coordinates for the vertices can be obtained by solving polynomial equations of degree six in one real variable, with coefficients in the field generated by the entries of the matrices (see section 8 of [11]). In particular, in order to determine the exact combinatorics of a given domain by algebraic means, one is led to taking several possibly complicated extensions of the field where the matrix entries lie.

Another related difficulty is that the half spaces bounded by bisectors are not convex (see [5] for instance), and very few faces of the skeleton are totally geodesic. In other words, in order to show that a bisector is disjoint from a certain polyhedron P, it is not enough to show that the vertices of P are all in the same half space determined by the bisector. A more topological approach to the determination of precise combinatorics would be to try and adapt the methods of section 6.3, but this is far from straightforward; specifically, many vertices of the fundamental domains turn out to be on more than four bisectors bounding the polyhedron (or at least they seem to be according to numerical experimentation). It is then difficult to prove rigorously that such coincidences actually occur, since we need exact calculations to show that an explicit point lies on a bisector (even though it can be checked with arbitrary precision).

More importantly, it is well known that, in higher-dimensional situations, Nielsen's procedure cannot stop in finite time in general, as Dirichlet domains often have infinitely many faces (see [1], and [10] for the complex hyperbolic analogue). Moreover, given a generating set of matrices, there is no way to know a priori if the group is geometrically finite (examples of geometrically infinite groups in the context of complex hyperbolic geometry can be found in [13], for instance).

We now go back to some of the details of the complex hyperbolic version of Nielsen's algorithm, assuming have an efficient procedure for determining the combinatorics of F_W for a given W (in other words we assume we have a computer with infinite precision). We start with step (1), and discuss possible ways to determine whether the polyhedron F_W is a fundamental domain for Γ.

The key is to use the Poincaré polyhedron theorem (or rather a non constant curvature version of it, see [11] for instance). The starting point is to check whether each isometry $w \in W$ gives a well-defined side pairing of F_W, i.e. it induces a bijection between $\widehat{w^{-1}} \cap F_W$ and $\widehat{w} \cap F_W$.

This amounts to showing that the combinatorics of $\widehat{w} \cap F_W$ and $\widehat{w^{-1}} \cap F_W$ are the same, and then to go through their skeletons and verify that the k-faces of their respective skeletons, $k = 0, 1, 2$, are identified by w (this is the approach taken in [11]). Clearly this involves painful bookkeeping, but it is not particularly difficult (provided once again that we can determine the exact combinatorics of the polyhedron F_W).

If F_W has side pairings, then it is quite easy to check whether it is a fundamental domain by using the Poincaré polyhedron theorem. The cycle conditions for generic 2-faces are trivially satisfied because of Giraud's theorem (see [3], for instance). If e is a complex geodesic 2-face of F_W, let us write R_e for the corresponding cycle transformation. Then R_e is a complex reflection, and in order to check that the images of F_W under R_e tile a neighborhood of e, it is enough to compare the rotation angle of R_e with the angle between the two faces of F_W that intersect in e (cospinal bisectors intersect with constant angle, unlike general pairs of bisectors).

If F_W does not have side pairings (but the group has not been found to be indiscrete up to this point, i.e. no cycle transformation has infinite order), we need to change the set W. The standard technique (see step (2)) is to take W to be the set of all group elements that can be written as words of length $\leq n$ in a given generating set, for increasing values of n.

Here we give a different way to enlarge the relevant sets of group elements, which in many cases seems to be more efficient (this is somewhat important, since it is quite time consuming to find the combinatorics of the partial Dirichlet domain at each stage in the procedure). The idea is to get simple necessary conditions for a given F_W to have side pairings, and to include new elements in W only when we know we have to, namely when the necessary conditions are violated.

Proposition 7.1. *Suppose that no nontrivial group element in W fixes p_0. If the Dirichlet polyhedron F_W has side pairings (i.e. each w maps $\widehat{w^{-1}} \cap F_W$ isometrically onto $\widehat{w} \cap F_W$), then*

(1) W is symmetric, $w^{-1} \in W$ whenever $w \in W$.

(2) W is Giraud-closed, i.e. for any non totally geodesic 2-face of F_W corresponding to a pair of elements $v, w \in W$, the word $v^{-1}w$ is in W.

Proof: It follows from the definition of bisectors that

$$v : \widehat{v^{-1}} \to \widehat{v} \tag{7.1}$$

and also that

$$v^{-1} : \widehat{v} \cap \widehat{w} \to \widehat{v^{-1}} \cap \widehat{v^{-1}w} \tag{7.2}$$

If F_W has side pairings, then the latter intersection must be contained in some intersection $\widehat{v^{-1}} \cap \widehat{u}$ for some $u \in W, u \neq v^{-1}$. By Giraud's theorem, see Theorem 4.6, for any such u, up_0 must be one of p_0, $v^{-1}p_0$ or $v^{-1}wp_0$. Since we assumed no nontrivial element of W fixes p_0, this implies $u = v^{-1}w \in W$. $\square$

For instance, the set W of words of length 3 which we have used in section 5 is symmetric but not Giraud-closed, hence the corresponding polyhedron F_W does not have side pairings.

Remark 7.2. There is a heuristic reason not to worry about complex faces, namely given any complex geodesic 2-face, all of its neighbors must be generic (two complex geodesics that share a curve are equal).

The analog of the necessary condition (2) for complex 2-faces can be formulated as follows. If F_W has side pairings and it has a complex 2-face corresponding to $v, w \in W$ (i.e. the bisectors $\widehat{v}$ and $\widehat{w}$ are cospinal, and they share a slice S that contains a 2-face of F_W), then there must be an element $u \in W$ such that $\widehat{u}$ and $\widehat{v^{-1}}$ are cospinal, and their complex geodesic intersection contains a face of F_W. Unlike the conclusion of Proposition 7.1, in general, u might be different from $v^{-1}w$. It is quite clear, however, that these two elements must differ by a complex reflection (provided once again that no nontrivial group element fixes p_0).

The conditions of the proposition were only shown to be necessary for F_W to have side pairings. It would be very interesting to know whether or not they are also sufficient (they seem to be sufficient for all the Dirichlet domains in [3], see Remark 7.4).

Proposition 7.1 gives a way to obtain natural candidates for sets W of group elements to define a Dirichlet domain with side pairings, by successively refining them in order to force the necessary conditions to be satisfied. We sometimes refer to the following as the Giraud procedure, since it is based mainly on using Giraud's theorem.

Procedure:

(1) Start with a symmetric set of generators W_0^+ and determine the combinatorics of $F_{W_0^+}$. Let $W_0 \subset W_0^+$ be the set of elements $w \in W_0^+$ such that $\widehat{w} \cap F_{W_0^+}$ has dimension three.

(2) Take W_1^+ to be the union of W_0 with the set of group elements of the form $v^{-1}w$, where $\widehat{v} \cap \widehat{w} \cap F_{W_0}$ is a non totally geodesic 2-face of F_{W_0}. Find the combinatorics of $F_{W_1^+}$, and let W_1 be the set of elements $w \in W_1^+$ such that $\widehat{w} \cap F_{W_1^+}$ has dimension three.

(3) If W_i is not Giraud-closed (see Prop. 7.1), repeat the above process to get $W_{i+1}^+ \supset W_i$ and then $W_{i+1} \subset W_{i+1}^+$. If W_i is Giraud-closed, check whether F_{W_i} has side pairings (see the discussion after Remark 7.2).

Remark 7.3. (1) Along this process, we find some relations in the group. Indeed, at each stage one needs to verify whether certain products $v^{-1}w$ are already in the previous set of group elements. For our particular group of interest $G(4,4,4;5)$, this can be done algorithmically, using (2.17) and algebra in $\mathbb{Z}[\tau,\omega]$.

Note that these relations correspond to cycles of generic 2-faces of the polyhedron (in the terminology of the Poincaré polyhedron theorem, see [3]), and these cycles have length three because of Giraud's theorem.

(2) For the particular group we analyze in this paper, no nontrivial group element seems to fix p_0. This is of course a consequence of the conjectural assertion that $F_W = F_{W_4}$ is indeed a fundamental domain for the group.

In general, as long as the group is discrete, the isotropy group of a point is a finite group. If p_0 has non trivial stabilizer, one might simply change the center of the domain, but this might not be desirable, since it is convenient to have a high degree of symmetry in the Dirichlet domains (note also that the combinatorics of a Dirichlet domain depend on its center in a significant way, see [10]).

The above procedure is much more tedious to state when p_0 has non trivial stabilizer, and we only sketch some of the difficulties. The basic observation is that the Dirichlet domain F_W is of course no longer a fundamental domain, but it can be split as a union of finitely many isometric copies of a fundamental domain. One then needs to find an appropriate fundamental domain for the finite group fixing p_0, which can be done for instance by considering its action on $P^1_{\mathbb{C}}$ (see [7]).

In order to use the procedure in that case, one needs to keep track not only of W_n but also of the list of elements $v,w\in W_n$ such that $v^{-1}w$ fixes p_0. We denote by H_n the set of elements of $\mathrm{Stab}(p_0)$ of the form $v^{-1}w$, with $v,w\in W_n$. For such an element, we have $\widehat{v}=\widehat{w}$, and we might need to split $\widehat{v}\cap F_W$ into two different faces, with respective side pairings given by v^{-1} and w^{-1} (only one of these two faces would appear in a fundamental domain for the group).

In the notation of Proposition 7.1, if $\widehat{v}\cap\widehat{w}\cap F_W$ is a generic 2-face, then we must have an element in W of the form $v^{-1}ws$, where $s\in\mathrm{Stab}_\Gamma(p_0)$. For the next step of the procedure we include all the words of the form $v^{-1}wh$, $h\in H_n$; we include them either in W^+_{n+1} or in H_{n+1} depending on whether or not they fix p_0. There is then an obvious extension of the notion of being

Giraud-closed, namely W_n is Giraud-closed if $W_{n+1} = W_n$. If W_n is Giraud- closed, we find a fundamental domain for H_n, and check whether its intersection with F_{W_n} is a fundamental domain for Γ (if it is, then H_n is the full stabilizer of p_0 in Γ).

Although the procedure might not stop in finite time (and even if it does, the resulting domain might not have side pairings), it gives a useful experimental method to search for relevant sets of group elements W.

Remark 7.4. (1) For the Mostow reflection groups $\Gamma(p,t)$, the Giraud procedure stops after very few steps, and the corresponding polyhedra can be checked to have side pairings (for all $p = 3, 4, 5$ and $t \in \mathbb{R}$ such that $|t| < 3(\frac{1}{2} - \frac{1}{p})$, see [11]).

One may then check that the conditions of the Poincaré polyhedron theorem hold if and only if $(\frac{1}{4} - \frac{1}{2p} \pm \frac{t}{2})^{-1} \in \mathbb{Z} \cup \{\pm\infty\}$, these conditions coming from the requirement that the cycle transformations corresponding to complex geodesic 2-faces rotate by an angle which is a integer fraction of π.

Note that the corrections to Mostow's fundamental domains given in [3] were discovered simply by applying the Giraud procedure.

(2) For the deformed triangle groups $G(4,4,4;7)$, which is studied in [19], the Giraud procedure seems not to stop (even though after some steps, the new words become predictable, and one could hope to describe the combinatorics of a locally finite Dirichlet domain with infinitely many faces).

We now present the result of the above procedure for the deformed triangle group $G(4,4,4;5)$, starting with $W_0 = \{I_1, I_2, I_3\}$ and when p_0 is the center of mass of the mirrors of the three generating reflections. Assuming that our computer calculations are correct, we obtain the list of sets of words given in Table 2 (see the notation in the description of the procedure). The set W_4 is then Giraud-closed, i.e. $W_5^+ = W_5 = W_4$ (at least conjecturally, since it depends on the correctness of the announced combinatorics). Along this process, we gather the following observations:

(1) Since the generators have order two, there are two 3-faces contained in the same bisector $\widehat{I_i}$, intersecting along the mirror of I_i. Note that these two faces meet at an angle π, and the cycle transformation for their intersection is trivial (in particular the conditions of the Poincaré polyhedron theorem hold for those

step	$\|W_n\|$	words kept
0	3	i
1	9	ij
2	33	$iji, ijk, ijik, kiji$
3	81	$ijikji, ijkiji, ijikij, jikiji, ijikiji, ijikjij, ijkij, ijkjijk$
4	99	$(ijkj)^2i, (ijik)^2i, (ijik)^2ij$

step	$\|W_n^+\|$	words removed
0	3	
1	9	
2	42	$ijki, ijij$
3	93	$ijkijk, ijkjkji$
4	135	$(ijik)^2ijki, ijki(jiki)^2, ijikjijkjikj, ikjijikijkik,$ $ijikjijkjikjk, (ijik)^2ijkik, ijikj(ijkj)^2$

TABLE 2. The group elements added or removed during the course of the Giraud procedure. The set W_4 is has 99 elements and is (conjecturally) Giraud-closed.

2-faces). A similar description holds for all involutive side pairings, which are the conjugates of I_i given by I_{iji}, $I_{ijikiji}$ and $I_{k(ijik)^2}$.

(2) There are other geodesic 2-faces, contained in the intersection of the bisectors corresponding to the pairs of words of the form $ijkij$ and $kjikj$. These two bisectors are cospinal, and they meet at (constant) angle $\pi/3$. Their intersection is the mirror of the complex reflection I_{ijk}^5 (the fact that I_{ijk}^5 is indeed a complex reflection was already discussed in section 2, see (2.14)). The latter mirror can also be described as $v_{ijk}^{\perp}$ (see equation (2.15)).

There are three such 2-faces, which get identified by the side pairings, and it is readily verified that the cycle transformation is precisely

$$I_{ijk}^5 = I_{ijkij} I_{kijki} I_{jkijk}.$$

Note that the symmetry J permutes these three 2-faces, but it is of course not equal to the side pairing (J preserves the fundamental polyhedron F, whereas $F \cap \gamma F = \widehat{\gamma} \cap F$ for all side pairing γ). One can check that J and I_{ijkij} differ by a complex reflection of order 6.

The conditions of the Poincaré theorem hold on these 2-faces: the cycle I_{ijk}^5 is a reflection of order 2, and a neighborhood of this 2-face is tiled by six wedges of angle $\pi/3$.

(3) As stated in Remark 7.3, one finds certain relations in the group during the course of the Giraud procedure. One of them is $ijiji = jij$, which is of course equivalent to saying that I_iI_j has order four. Another non obvious relation is $ik(ijik)^2 = ji(kiji)^2$, which is equivalent to saying that $I_iI_jI_iI_k$ has order 5.

The combinatorics of the faces of the conjectural Dirichlet domain are given in Figures 8-11. Note that we only list one face for each isometry type of faces. The ones that are omitted can all be obtained by applying the natural symmetries σ_{ij}.

We also obtain a presentation for the group (which is conjectural since it depends on the accuracy of our Dirichlet domain):

$$G(4,4,4;5) \simeq \langle \iota_1, \iota_2, \iota_3 | \iota_i^2, (\iota_i\iota_j)^4, (\iota_i\iota_j\iota_k)^{10}, (\iota_i\iota_j\iota_k\iota_j)^5 \rangle \tag{7.3}$$

Out of the geometry of the Dirichlet domain, one easily notices that there are commuting complex reflections in the group, corresponding to $(ijk)^5$ and $jkjijkj$. We leave it as an exercise to the reader to verify that this commuting relation is indeed a consequence of the presentation (7.3).

8. Totally geodesic faces

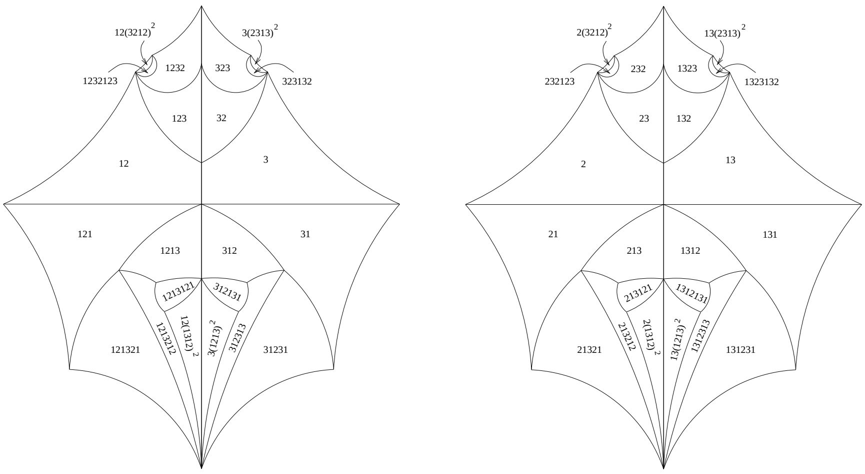
Recall that 3-faces of our polyhedra cannot be totally geodesic, since there are no totally geodesic real hypersurfaces in complex hyperbolic space.

It is an interesting feature of Dirichlet domains that totally geodesic 2-faces can only be in complex geodesics, i.e. they cannot be totally real. Indeed, if $B(p_0,p_1)\cap B(p_0,p_2)$ were to contain a Lagrangian plane L, the involution fixing L would exchange p_0 and p_1, as well as p_0 and p_2, which can only happen if $p_1 = p_2$.

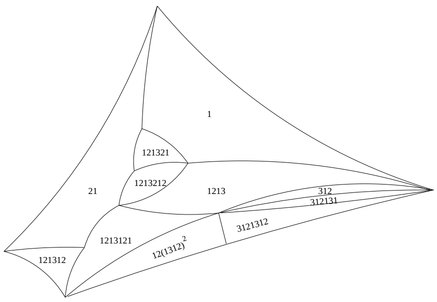
In the fundamental domain for our group $G(4,4,4;5)$, there are five isometry classes of complex 2-faces, on the mirrors of I_i, I_{iji}, $I_{ijikiji}$, $I_{k(ijik)^2}$ and finally on the mirror of I_{ijk}^5. The first four types are what Giraud called "hidden faces", in the sense that the two 3-faces that contain it meet at an angle π, i.e. they are on the same bisector.

Note that the hidden 2-faces are bounded by non-geodesic hypercycles, and the angles between their 1-faces are not all rational multiples of π (such faces appear as the outer polygons in the pictures of Figure 8).

It can be checked that the other three 2-faces, which are on the $v_{ijk}^{\perp}$, are bounded by geodesic quadrilaterals with angles $2\pi/5$, $\pi/2$, $\pi/3$, $\pi/2$ (see the outer polygon in Figure 9(d)).

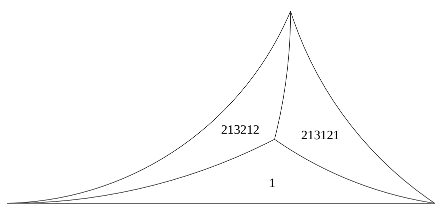


(a) 1



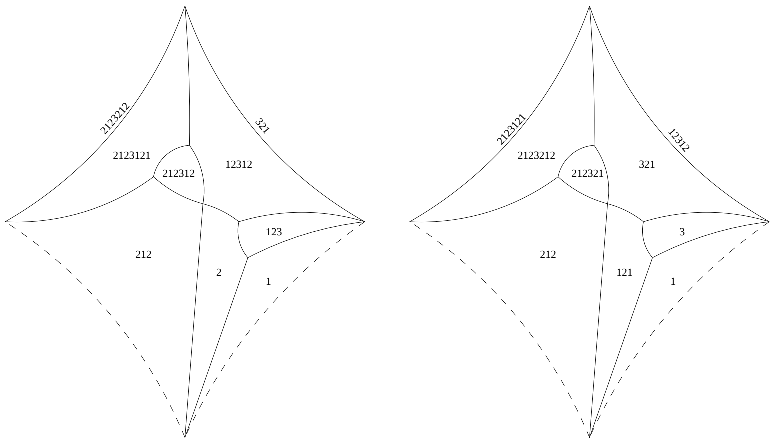
(b) 121

(c) 1213121

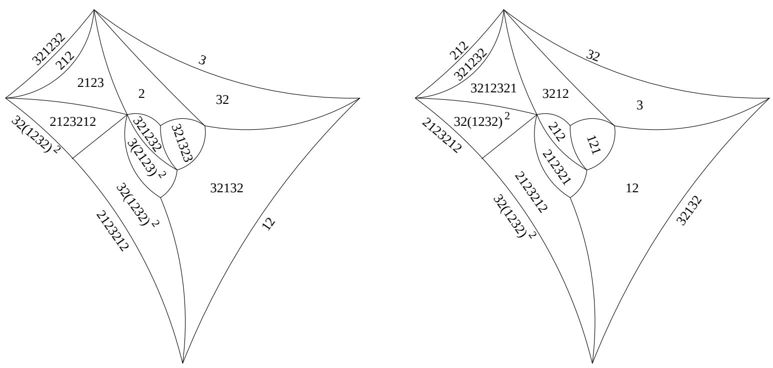


(d) (2131)2(1312)

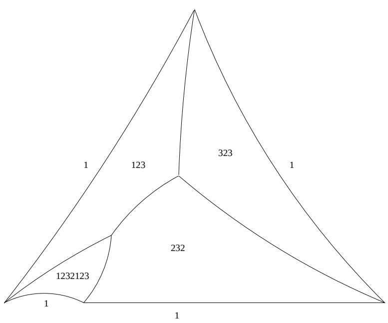
Figure 8. The combinatorics of the 3-faces on $\widehat{w}$ where w is a conjugate of I_j. There are two isometric faces on each such bisector, sharing a complex geodesic 2-face contained in the mirror of w, which is not drawn in the picture (it is bounded by the outer polygon).



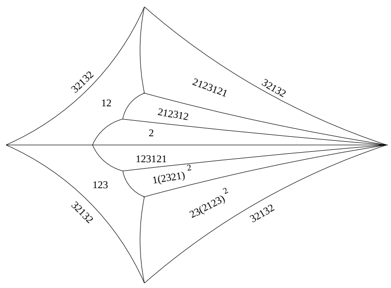
(a) 12



(b) 321

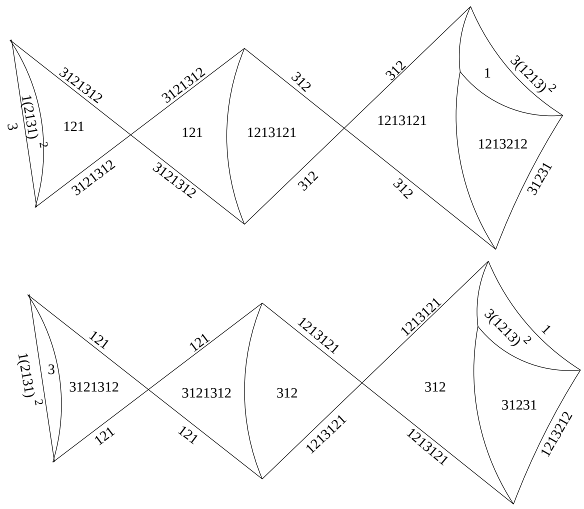


(c) 1232

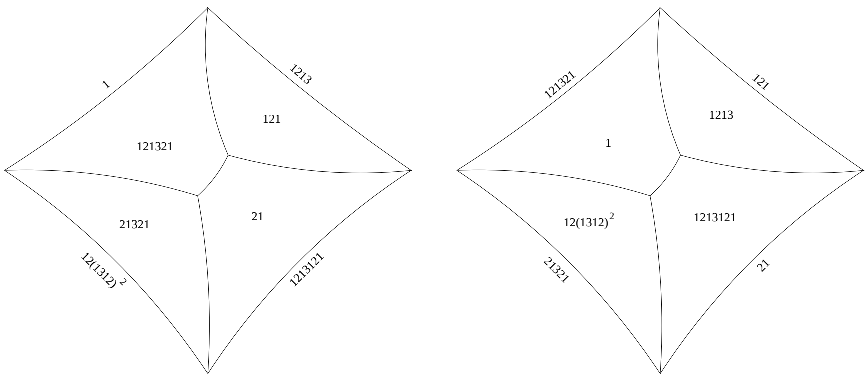


(d) 12312

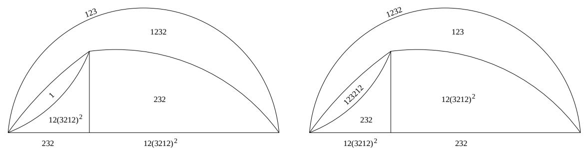
FIGURE 9. The combinatorics of some faces of the Dirichlet domain. Note that dotted curves are not 1-faces of the polyhedron but are simply cuts in 2-faces performed for convenience to draw the picture.



(a) $12(1312)^2$

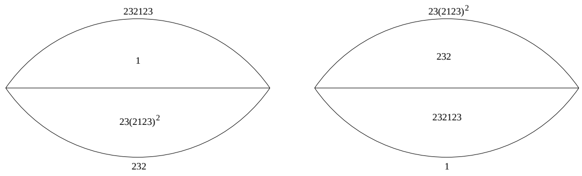


(b) 1213212

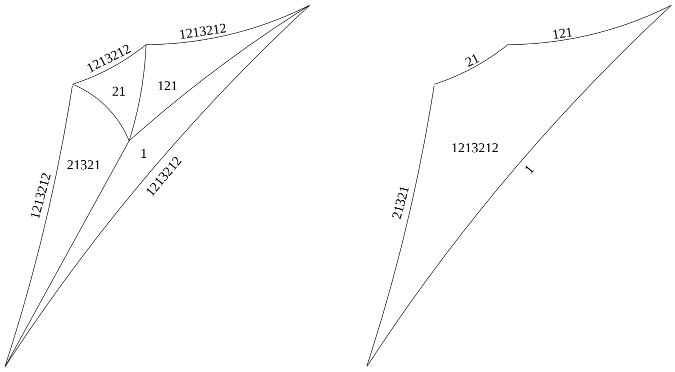


(c) 1232123

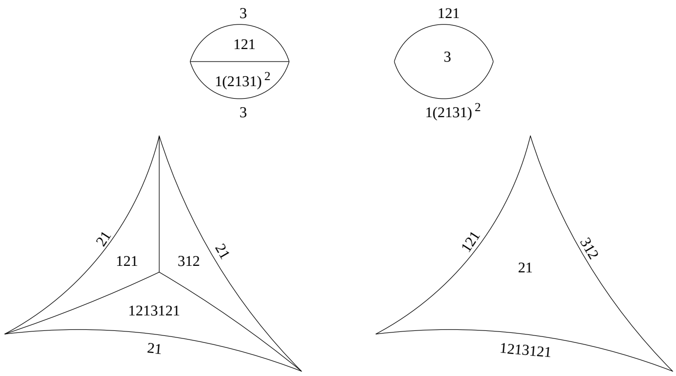
Figure 10. The combinatorics of faces of the Dirichlet domain. Note that $F \cap \widehat{12(1312)^2}$ is not a topological ball, one needs to break it up into three separate faces.



(a) $2(3212)^2$



(b) 121321



(c) 121312

Figure 11. The combinatorics of faces of the Dirichlet domain. Note that $\widehat{121312} \cap F_W$ has two connected components, so there are two different faces on that bisector.

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