A NEW NON-ARITHMETIC LATTICE IN $PU(3,1)$

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Abstract. We study the arithmeticity of the Couwenberg-Heckman-Looijenga lattices in $PU(n,1)$, and show that they contain a non-arithmetic lattice in $PU(3,1)$ which is not commensurable to the non-arithmetic Deligne-Mostow lattice in $PU(3,1)$.

1. Introduction

Lattices in the isometry groups of most symmetric spaces of non-compact type are arithmetic, due to celebrated superrigidity results by Margulis [23] (symmetric spaces of higher rank), Corlette [7], and Gromov-Schoen [20] (quaternionic hyperbolic spaces, and the octonionic hyperbolic plane).

For small values of n, it is fairly easy to construct non-arithmetic lattices in $SO(n,1)$ by using Coxeter polyhedra (a criterion due to Vinberg gives a simple computational way to determine the arithmeticity of these groups). For n large enough, there are no Coxeter polytopes in $H^n_{\mathbb{R}}$, but there are non-arithmetic lattices in $SO(n,1)$ for arbitrary n by a beautiful construction due to Gromov and Piatetski-Shapiro [19]. Their construction produces infinitely many commensurability classes of non-arithmetic lattices in any dimension. Note that the general structure of lattices in $SO(n,1)$ remains mysterious.

The situation is even more mysterious for lattices in $PU(n,1)$, $n \geq 2$, which is (up to finite index) the isometry group of complex hyperbolic space $H^n_{\mathbb{C}}$. Here there is currently no analogue of the Gromov-Piatetski-Shapiro construction (there exist no real totally geodesic hypersurfaces in $H^n_{\mathbb{C}}$, so there is no reasonable gluing interface to construct hybrids). In fact, only finitely many commensurability classes of non-arithmetic lattices in $PU(n,1)$ are known, only for very low values of n.

The first examples in $PU(2,1)$ were due to Mostow [24], and his construction was soon generalized to produce several more examples in $PU(2,1)$, and a single one in $PU(3,1)$ see [10]. For some decades, the Deligne-Mostow examples were the only known examples, even though some alternative constructions were given, see [29] for instance. To this day, it is still unknown whether there exist non-arithmetic lattices in $PU(n,1)$ for any $n > 3$.

A slightly different construction was given by Hirzebruch (see [2]), based on the equality case in the Miyaoka-Yau inequality, i.e. an orbifold version of the fact that a compact complex surface X of general type with $c_1^2(X) = 3c_2(X)$ is covered by the ball. Given such an X, the existence of a lattice Γ in $PU(2,1)$ such that $X = \Gamma \setminus \mathbb{B}^2$ is guaranteed, but it is not obvious how to describe the lattice explicitly (the existence of a Kähler-Einstein metric is obtained by showing existence of a solution to a Monge-Ampère equation).

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In fact, the arithmetic structure of the Hirzebruch examples seems not to have been worked out anywhere in the literature, apart from a small number of examples where coincidences with some arithmetic groups were found (see the work of Holzapfel [21], [22], and also the more recent [12]).

The Deligne-Mostow construction and the Barthel-Hirzebruch-Höfer construction were given a common generalization by Couwenberg, Heckman and Looijenga [8], but their work barely brushes the discussion of arithmeticity (they mention that the examples derived from real Coxeter groups are arithmetic, without any details). It was recently observed [12] that some of the non-arithmetic lattices in $PU(2,1)$ produced by the author, Parker and Paupert [16] were in fact conjugate to some specific Couwenberg-Heckman-Looijenga lattices.

The main goal of the present paper is to give a systematic study of the arithmeticity of the Couwenberg-Heckman-Looijenga lattices. We write $\mathcal{C}(G, p_1, \ldots, p_k)$ for the CHL lattice derived from the Shephard-Todd group G, generated by complex reflections of angle $2\pi/p_j$. As mentioned in [8], when G is the Weyl group of type A_n or B_n, the lattices of the form $\mathcal{C}(G, p)$ are all commensurable to Deligne-Mostow lattices (the Deligne-Mostow construction gives lattices in $PU(n,1)$ only for $n \leqslant 9$). Note also that the Shephard-Todd group G_{32} is obtained from the Couwenberg-Heckman-Looijenga construction starting with the group $W(A_4)$ so, just as in [8], we omit G_{32} from our lists (it would again produce lattices commensurable with Deligne-Mostow lattices). The imprimitive Shephard-Todd groups $G(m,p,n)$ can also be obtained from classical groups by the CHL construction, so the corresponding ball quotients are also commensurable to Deligne-Mostow ball quotients.

We refer to (primitive) Shephard-Todd groups not of type A_n or B_n as *exceptional complex reflection groups*, and we refer to the corresponding mirror arrangements as *exceptional arrangements*. Via the CHL construction, the exceptional arrangements produce lattices in $PU(n,1)$ only for $n \leqslant 7$.

The CHL lattices in $PU(2,1)$ were already mentioned in [12] and [13]. It turns out that, for $n \geq 3$, all the (non Deligne-Mostow) CHL lattices in $PU(n,1)$ are arithmetic except for one.

Theorem 1.1. *Let Γ be a CHL lattice derived from an exceptional finite complex reflection group G acting irreducibly on $V = \mathbb{C}^{n+1}$, $n \geqslant 3$. Then Γ is arithmetic, unless $\Gamma = \mathcal{C}(G_{29}, 3)$.*

More precisely, we state the following.

Theorem 1.2. *The lattice $\mathcal{C}(G_{29}, 3)$ is a non-arithmetic, non-cocompact lattice, with adjoint trace field $\mathbb{Q}(\sqrt{3})$. It is not commensurable to any Deligne-Mostow lattice.*

Recall that the Deligne-Mostow list of lattices contains only one non-arithmetic lattice in $PU(n,1)$ with $n \geq 3$, namely the lattice Γ_μ for $\mu = (3,3,3,3,5,7)/12$; so the main additional content of Theorem 1.2 is the claim that $\mathcal{C}(G_{29}, 3)$ is not commensurable to that specific Γ_μ.

Given the commensurability analysis in [15], putting together all known non-arithmetic lattices in $PU(n,1)$, we see that there are currently 22 known commensurability classes in $PU(2,1)$ and 2 commensurability classes in $PU(3,1)$.

The basic tool for proving these results is the knowledge of explicit presentations of the braid groups associated to the Shephard-Todd groups (see the conjectural statements in [5], later proved in [4]). Using braid relations between the generators, we study the irreducible representations of the corresponding braid groups that send the generators to complex reflections of the appropriate angle (the values of the angle for the discrete holonomy groups in Couwenberg-Heckman-Looijenga have been tabulated, see section 8 of [8]).

It turns out there are finitely many such representations, and the finite number is usually very small. Basic geometric considerations (using cocompactness or discreteness arugments) allow us to single out (a group conjugate to) the Couwenberg-Heckman-Looijenga holonomy group. Along the way, we find explicit matrices for generators for the holonomy groups, which may be of independent interest (but these were not given in [8]).

Each holonomy group preserves an explicit Hermitian form (by irreducibility, such an invariant Hermitian form is unique up to scaling). The strategy for determining arithmeticity is then to

(1) Find coordinates such that the Hermitian form has entries in a number field;
(2) Check that the above number field is as small as possible;
(3) Find coordinates where the matrices of the generators are actually algebraic integers.

It is known that (1) can always be achieved, because of Calabi-Weil local rigidity of lattices, see chapter VI of [25]. In general, it is not easy to make that result effective, but it turns out to be fairly easy in the cases we consider in the paper.

Step (2) follows from the determination of the adjoint trace field, i.e. the field generated by traces in the adjoint representation, which is a well known commensurability invariant for lattices (in fact for Zariski dense groups).

It is not known whether step (3) can always be achieved, even though it is strongly believed to be the case for every lattice in $PU(n,1)$ (for cocompact lattices, it follows from very recent work of Esnault and Groechenig [17]). Recall that there are so-called quasi-arithmetic lattices in $SO(n,1)$ for every n, i.e. lattices where arithmeticity fails only by failure of integrality (see [3] and [28]).

We will go through steps (1) through (3) by explicit case by case computation. In fact, we follow a suggestion of the referee and combine steps (2) and (3).

Some parts of the paper require delicate arguments. One is the proof that $\mathcal{C}(G_{29},3)$ is not commensurable to the Deligne-Mostow non-arithmetic lattice in $PU(3,1)$. Indeed, the two groups have the same rough commensurability invariants (cocompactness, adjoint trace field and non-arithmeticity index, as defined in section 6.2 of [15]). We work out an explicit description of the cusps of these two lattices, and show that the cusps themselves are not commensurable. Another delicate part is the determination of the reflection representations for the braid group associated to the Shephard-Todd group G_{31}. This group is not well-generated, in the sense that it is not generated by the right number of reflections for the ambient dimension.

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2. Basic facts about complex reflections.

We start with a complex vector space V equipped with a non-degenerate Hermitian inner product, which we denote by $\langle \cdot, \cdot \rangle$ (we take this to be linear on the first factor, and antilinear on the second factor). A *complex reflection* is a linear transformation of the form $R_{v,z}$ where

$$R_{v,z}(x) = x + (z-1)\frac{\langle x, v\rangle}{\langle v, v\rangle} v, \tag{2.1}$$

for some vector $v \in V$ with $\langle v, v\rangle \neq 0$, and some $z \in \mathbb{C}$ with $|z| = 1$. It is easy to see that such a transformation preserves the Hermitian inner product.

Note that scaling the vector v does not change the above transformation, so v is not uniquely determined by the transformation. The reflection fixes pointwise the complex-linear subspace $v^\perp = \{w \in \mathbb{C}^{n,1} : \langle w, v\rangle = 0\}$, called its *mirror*, and it acts on $\mathbb{C}v$ by multiplication by z. We will call such a vector v the *polar vector* to the mirror (this is only well-defined up to scaling). The complex number z is called the *multiplier*, and its argument is called the *angle* of the complex reflection. In this paper, we will only consider reflections of finite order, i.e. z will actually be a root of unity.

When the Hermitian inner product is positive definite, the condition $\langle v, v\rangle \neq 0$ is of course equivalent to $v \neq 0$, and the definition of a complex reflection agrees with the one in [26]. We will also use the same definition for hyperbolic Hermitian inner products, i.e. those of signature $(n, 1)$, which are related to complex hyperbolic space $H^n_{\mathbb{C}}$ (for basic information about complex hyperbolic space, see [18] for instance). As a set, $H^n_{\mathbb{C}}$ is the set of complex lines spanned by vectors v with $\langle v, v\rangle < 0$, and the metric is built in such a way that the linear isometries of the Hermitian inner product induce isometries of $H^n_{\mathbb{C}} \subset \mathbb{P}(V)$ (in fact, the corresponding group $PU(n, 1)$ has index two in the full isometry group, the latter being obtained by adjoining any antiholomorphic isometry).

In the hyperbolic case, we require moreover that v in equation (2.1) be a positive vector, i.e. $\langle v, v\rangle > 0$. In that case, the restriction of the Hermitian form to $v^\perp$ has signature $(n-1, 1)$, and the set of negative vectors in $v^\perp$ projects down to a totally geodesic copy of $H^{n-1}_{\mathbb{C}}$. Moreover since we are free to scale v, we can (and will often) assume $\langle v, v\rangle = 1$.

For $k \in \mathbb{N}^*$, two group elements a and b are said to satisfy a braid relation of length k if

$$(ab)^{k/2} = (ba)^{k/2}.$$

In that case, we write $\mathrm{br}_k(a, b)$. When k is odd, the notation $(ab)^{k/2}$ stands for an alternating product $a \cdot b \cdot a \cdots b \cdot a$ with k factors. Note that when $\mathrm{br}_k(a, b)$ holds, $\mathrm{br}_{nk}(a, b)$ also holds for every $n \geq 1$. The smallest k such that $\mathrm{br}_k(a, b)$ holds is called the braid length of a and b, which we denote by $\mathrm{br}(a, b)$.

It is often convenient to describe reflection groups by a (complex) Coxeter diagram. The diagram is attached to a generating set of reflections (with pairwise distinct mirrors). It has one vertex for each generating complex reflection, and vertices are represented by a circled

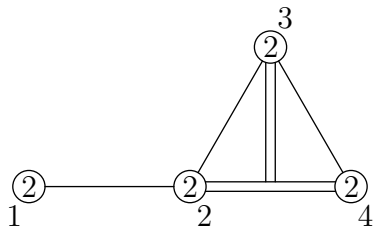

FIGURE 1. Coxeter diagram for G_{29}

integer, to indicate the order of the corresponding complex reflection (more precisely, a node with a circled p stands for a complex reflection with multiplier $e^{2\pi i/p}$).

A pair of vertices is joined by an edge labelled k if the corresponding reflections satisfy a braid relation of length $k \geq 3$. Braid relations of length 3 are called standard braid relations, and the corresponding edge in the graph is drawn, but the label 3 is usually omitted. Braid relations of length 4 are often drawn by doubling the corresponding edge in the graph (and then omitting the label 4).

In order to get abstract presentations for Shephard-Todd groups, we often need to include extra relations, which are indicated with extra decorations of the diagram, see the Appendix 2 in [6]. For instance, in the diagram (d) in Figure 2, the vertical double bar indicates a braid relation $\mathrm{br}_4(R_3, R_2R_4)$.

The Coxeter diagrams for the Shephard-Todd groups that we will need in this paper can be deduced from the graphs given in Table 2 by replacing the nodes by circled 2's. For example, the Coxeter diagram for G_{29} is the one in Figure 1. We have numbered the nodes, and write R_j for the reflection corresponding to the node labelled j ($j = 1, 2, 3, 4$). The corresponding presentation is the one given in equation (5.7) (p. 11), with extra relations $R_j^2 = Id$ (in fact only one of these four relations suffices, since $\mathrm{br}_3(R_j, R_{j+1})$ implies that R_j and R_{j+1} are conjugate).

We only list exceptional Shephard-Todd groups in $U(n+1)$ ($n \geqslant 3$) with a generating set consisting of reflections of order 2, since the ones with higher order generators do not produce any more lattices (see [8]).

We briefly sketch the general strategy for writing matrices for the Hermitian forms and complex reflections generating CHL lattices. For well-generated groups, i.e. reflection groups in $U(n+1)$ generated by $n+1$ reflections, we will use coordinates given by the basis obtained by choosing polar vectors for these $n+1$ reflections; in other words, the polar vectors are simply given by the standard basis vectors $e_1, \ldots, e_{n+1}$ of $\mathbb{C}^{n+1}$. The reflections are uniquely determined by the matrix of the Hermitian form in that basis, given by $H_{j,k} = \langle e_k, e_j \rangle$.

We may assume that $\langle e_j, e_j \rangle = 1$, so we will take H to have ones on the diagonal. For $j \neq k$, the braid length $\mathrm{br}(R_j, R_k)$ determines $|\langle e_j, e_k \rangle|$ (see section 2.2 of [24]). We can choose the argument of $\langle e_j, e_k \rangle$ freely, by replacing e_k by λe_k for some $|\lambda| = 1$, and the goal will be to get the corresponding reflections to have algebraic integer entries (in the

discussion below, we will call a choice of argument *reasonable* if the corresponding matrices have algebraic integer entries).

For triples j, k, l of pairwise distinct indices, we can freely choose the arguments of $\langle e_j, e_k\rangle$ and $\langle e_k, e_l\rangle$, but then the third one $\langle e_k, e_l\rangle$, cannot usually be chosen arbitrarly, unless one of the inner product is zero (for more details on this, see section 2.3 of [24]).

An important special case is the one where the Coxeter diagram is a tree (see (a), (b), (c), (e), (i), (j), (k) in Figure 2). In that case, the arguments of the inner products $\langle e_j, e_k\rangle$ can be chosen freely (for instance we may assume all inner products are real, but this is not always a "reasonable" choice, in the above sense).

In other well-generated cases (see (d), (g), (h) in Figure 2), we consider triangles in the diagram. If a triangle corresponds to polar vectors e_j, e_k, e_l, we fix a reasonable choice of $\langle e_j, e_k\rangle$ and $\langle e_k, e_l\rangle$, and use the presentation of the braid group to find admissible values of $\langle e_j, e_l\rangle$ (in well-generated cases, there are only finitely many admissible values).

If the corresponding matrices have algebraic integer entries in the correct number field, we are done. Otherwise, we use ad-hoc changes of coordinates that produce algebraic integer entries (see section 5.1 for the group G_{28}).

The only group we need to consider that is not well generated is $G_{31} \subset U(4)$. In fact that group is generated by $5 = 4 + 1$ reflections. In that case, we will use the same parametrization as above using only 4 of the 5 reflections, and then use braid relations in the Bessis-Michel presentation to determine the polar vector to the mirror of the fifth reflection. For more details on this, see section 5.4).

3. The Couwenberg-Heckman-Looijenga lattices

In [8], Couwenberg-Heckman-Looijenga give a general construction of affine structures on the complement of hyperplane arrangements in projective space, parametrized by angles related to the holonomy around the hyperplanes in the arrangement. They also give necessary and sufficient conditions for the completion of that structure to be an orbifold (i.e. the holonomy is discrete, and the completion is a quotient of the appropriate complex space form).

It is unclear how often these conditions are satisfied, but there is a somewhat large list of examples associated to finite unitary groups generated by complex reflections (these were classified by Shephard and Todd [26]). That list contains a lot of the previously known examples of lattices in $PU(n, 1)$ generated by complex reflections, namely the Deligne-Mostow lattices [10], as well as the ones constructed by Barthel, Hirzebruch and Höfer [2]. Note that some examples in [15] are still not covered by the Couwenberg-Heckman-Looijenga construction, see [13].

The Couwenberg-Heckman-Looijenga lattices are decribed by giving:

- an irreducible Shephard-Todd group G;
- a positive integer $p_j \geqslant 2$, $j = 1, \ldots, k$ for each of the k orbits of mirrors of complex reflections in G.

We denote by $\mathcal{C}(G, p_1, \ldots, p_k)$ the corresponding group. In this paper, we only consider the exceptional Shephard-Todd groups, since the other ones are covered by Deligne-Mostow theory.

It turns out (exceptional) Shephard-Todd groups have at most two orbits of mirrors, so we only take $k \leq 2$. In fact, there is a single orbit of mirrors (i.e. $k = 1$) for all but one group, namely $G = G_{28}$, which is isomorphic to the Coxeter group F_4.

The CHL structures are obtained as structures on the complement of the union of the mirrors of reflections in $G \subset U(n+1)$; we follow the notation in [8] and write $V = \mathbb{C}^{n+1}$, $\mathcal{H}$ for the union of mirrors of reflections in G, and $V^0 = V \setminus \mathcal{H}$. In particular, by construction, the holonomy group is a quotient of $\pi_1(\mathbb{P}(V^0/G))$, which is often called a *braid group*.

It is reasonably easy (especially using modern computer technology, and more so in low dimensions) to write down explicit group presentations in terms of generators and relations for the Shephard-Todd groups. This was done by Coxeter, see [9] and also Appendix 2 in [6] for a convenient list.

This gives presentations for some quotients of the braid group $\pi_1(\mathbb{P}(V^0/G))$, namely the orbifold fundamental group of the quotient $\mathbb{P}(V/G)$, but it is not completely obvious how to deduce a presentation for $\pi_1(\mathbb{P}(V^0/G))$. Roughly speaking, one would like to cancel the relations expressing the order of reflections, and keep the braid relations, but this is of course not well-defined. Presentations for $\pi_1(\mathbb{P}(V^0/G))$ were proposed by Bessis and Michel in [5], and their conjectural statements have later been proved in [4].

Note that the Bessis-Michel presentations are given in such a way that the generators correspond to suitably chosen simple loops around hyperplanes in the arrangement. It follows that the Couwenberg-Heckman-Looijenga holonomy groups $\mathcal{C}(G, p)$ (resp. $\mathcal{C}(G, p_1, p_2)$) are homomorphic images of the braid group $\pi_1(\mathbb{P}(V^0/G))$, such that the corresponding homomorphism maps the Bessis-Michel generators to complex reflections of angle $2\pi/p$ (resp. $2\pi/p_1$ and $2\pi/p_2$). One can in fact obtain an explicit presentation for the lattices in terms of these generators (see Theorem 7.1 in [8] or section 4 in [14]).

For arrangements of type A_n or B_n, the corresponding lattices are commensurable to Deligne-Mostow lattices, and the list is a bit too long to be reproduced here, see p.157-159 of [8]. The other CHL lattices (in $PU(n,1)$ with $n \geqslant 3$) are listed in Tables 3 and 4 in appendix B.

4. Arithmeticity

We will use the following arithmeticity criterion, which is proved in [24] (see also [10]). We refer to it as the Vinberg/Mostow arithmeticity criterion. In what follows, $\mathrm{tr}Ad\Gamma = \mathbb{Q}(\{\mathrm{tr}Ad\gamma : \gamma \in \Gamma\})$ is the field generated by traces of elements of Γ in the adjoint representation.

Theorem 4.1. *Let H be a Hermitian form of signature $(n,1)$, defined over a CM field $\mathbb{L} \supset \mathbb{K}$. Let Γ be a lattice in $SU(H, \mathcal{O}_{\mathbb{L}})$, such that $\mathrm{tr}Ad\Gamma = \mathbb{K}$. Then Γ is arithmetic if and only if H^σ is definite for every $\sigma \in Gal(\mathbb{L})$ acting non-trivially on $\mathbb{K}$.*

Recall that a CM field is a purely imaginary quadratic extension of a totally real number field, we denote by $\mathbb{K}$ the totally real field and by $\mathbb{L}$ the imaginary quadratic extension. As

usual, $\mathcal{O}_{\mathbb{L}}$ denotes the ring of algebraic integers. Note that not every lattice is commensurable to a lattice as in the above statement, which are sometimes called lattices *of simplest type* (for the general case, one needs to consider division algebras over a CM field).

Because of the fact that the adjoint representation of a unitary representation ρ is isomorphic to the tensor product $\rho \otimes \overline{\rho}$, we have

$$\mathrm{tr} Ad\gamma = |\mathrm{tr}\gamma|^2$$

for all $\gamma \in \Gamma$, which we will repeatedly use in the sequel.

5. Explicit generators and arithmeticity

The goal of this section is to give explicit matrix generators for the CHL lattices, as well as explicit Hermitian forms, and to use these to apply the arithmeticity criterion stated in section 4. We only work on lattices derived from exceptional complex reflection groups acting on $\mathbb{C}^{n+1}$ with $n \geq 3$, which give an action on $\mathbb{P}^n$ with $n \geq 3$. Indeed, non-exceptional ones yield Deligne-Mostow groups (explicit matrices can easily be deduced from [10], see also [27]); 2-dimensional examples turn out to be commensurable to groups that have been studied elsewhere (see [15], for instance).

We go through a somewhat painful case by case analysis in sections 5.1 through 5.9. The groups in sections 5.1 through 5.4 give lattices in dimension 3, the next ones in slightly higher dimension (the list of groups and respective dimensions is given in appendix B).

Note that, just as in [8], we do not include the group G_{32} in the list, since it can be seen as a group derived from the A_4 arrangement, hence the corresponding lattices already appear in the Deligne-Mostow list (see [14] for more details).

5.1. **Lattices derived from G_{28}.** Recall that G_{28} has two orbits of mirrors of reflections (see [8] for instance), hence the corresponding CHL lattices depend on two integer parameters. We denote the corresponding groups by $\mathcal{C}(G_{28}, p, q)$.

We call $r_1, \ldots, r_4$ generators of G_{28}, numbered according to the numbering of the nodes in Figure 2(c) (page 28). The orbits of mirrors can be checked to be represented by the mirrors of r_1 and r_4.

Couwenberg, Heckman and Looijenga show that there exist representations of G_{28} into $PU(3,1)$, with lattice image, mapping r_1, r_2 to complex reflections with multiplier $e^{2\pi i/p}$ and r_3, r_4 to complex reflections of multiplier $e^{2\pi i/q}$, for (p,q) given by $(2,q)$, $q = 4,5,6,8,12$, $(3,q)$ for $q = 3,4,6,12$, $(4,4)$ and $(6,6)$.

For a generic value of p, q, we set up the Hermitian form as

$$\begin{pmatrix} 1 & \alpha & 0 & 0 \\ \bar{\alpha} & 1 & \beta & 0 \\ 0 & \bar{\beta} & 1 & \gamma \\ 0 & 0 & \bar{\gamma} & 1 \end{pmatrix}, \tag{5.1}$$

and the generators are given by $R_{e_1,z}, R_{e_2,z}, R_{e_3,w}, R_{e_4,w}$, where $z = e^{2\pi i/p}$, $w = e^{2\pi i/q}$ and the e_j, $j = 1,2,3,4$ are the standard basis vectors of $\mathbb{C}^4$. The corresponding matrices are

given in equation (5.2).

$$R_1 = \begin{pmatrix} z & \alpha(z-1) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \bar{\alpha}(z-1) & z & \beta(z-1) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.2}$$

$$R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \bar{\beta}(w-1) & w & \gamma(w-1) \\ 0 & 0 & 0 & 1 \end{pmatrix}, R_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \bar{\gamma}(w-1) & w \end{pmatrix}. \tag{5.3}$$

The braid relation $\mathrm{br}_3(R_1, R_2)$ is equivalent to $|\alpha| = \frac{1}{|z-1|} = \frac{1}{2\sin\frac{\pi}{p}}$ (see [24] for instance). Similarly $\mathrm{br}_3(R_3, R_4)$ is equivalent to $|\gamma| = \frac{1}{|w-1|}$.

One checks that $\mathrm{br}_4(R_2, R_3)$ is equivalent to $\beta = 0$ or

$$\beta^2 = \frac{z+w}{z+w-1-zw} = \frac{\cos(\frac{\pi}{p} - \frac{\pi}{q})}{2\sin\frac{\pi}{p}\sin\frac{\pi}{q}}.$$

We first rule out the case $\beta = 0$.

Proposition 5.1. *In the CHL lattice $\mathcal{C}(G_{28}, p, q)$, R_3 and R_4 do not commute.*

Proof: Let S_2 and S_3 denote reflections in G_{28} acting on $P^3_{\mathbb{C}}$ that correspond to R_2 and R_3 in $\mathcal{C}(G_{28}, p, q)$. By "corresponding", we mean that R_j and S_j are images of the same element r_j in the braid group $\pi_1(P(V^0/G_{28}))$.

Then S_2 and S_3 generate a group of order 8, isomorphic to the imprimitive Shephard-Todd group $G(4,4,2)$, and the arrangement has 4 planes intersecting along the mirror intersection $L_2 \cap L_3$, namely the mirrors of S_2, S_3, $S_2S_3S_2$, $S_3S_2S_3$.

The branch locus of the quotient map $\mathbb{C}^2 \to \mathbb{C}^2/G(4,4,2)$ has local analytic structure $z_1^4 = z_2^2$, which gives two tangent components (this can be seen by computing the invariant polynomial ring, see [1] for instance). This gives the structure of the quotient $P^3_{\mathbb{C}}/G_{28}$, near a generic point of the intersection of the mirrors of S_2 and S_3, the branch locus is given locally analytically by the same equation $z_1^4 = z_2^2$ (but there is a third variable, say z_3).

If R_2 and R_3 were to commute, their mirrors would have to be orthogonal (they cannot coincide, otherwise the monodromy group would not act irreducibly on $\mathbb{C}^4$), and the branch locus in the quotient would have two smooth transverse components, which is a contradiction. □

We take $\alpha = \frac{1}{z-1}$ and $\gamma = \frac{1}{w-1}$ (this is a natural choice, given the shape of the matrices for R_1 and R_3 in equation (5.2)), $\beta = \sqrt{\frac{z+w}{z+w-1-zw}}$, and we take as the basis for $\mathbb{C}^4$ the vectors $e_1, e_2, Je_2, J^{-1}e_2$ where $J = R_2R_3R_4$ (where the e_j are simply the standard basis vectors of $\mathbb{C}^4$). These vectors do indeed form a basis as long as $w + z \neq 0$, which will be the case for all relevant pairs (p, q).

We write Q for the corresponding matrix

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -w & \bar{z} \\ 0 & 0 & \frac{z+w}{\beta(1-z)} & \frac{z+w}{\beta zw(z-1)} \\ 0 & 0 & 0 & \frac{z+w}{\beta zw(z-1)} \end{pmatrix},$$

and get the matrices $\widetilde{R}_j = Q^{-1}R_jQ$ to have entries in $\mathbb{Z}[z,w]$, namely

$$\widetilde{R}_1 = \begin{pmatrix} z & 1 & -w & \bar{z} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{R}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -z & z & -z(w+1) & 1+\bar{w} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.4}$$

$$\widetilde{R}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+w & -w & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{R}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1+w & -\bar{z}\bar{w} \\ 0 & 0 & zw^2 & 0 \end{pmatrix}, \tag{5.5}$$

which preserve a Hermitian form defined over $\mathbb{Q}(z,w)$, in fact

$$Q^*HQ = \begin{pmatrix} 1 & \frac{1}{z-1} & \frac{w}{1-z} & \frac{1}{z(z-1)} \\ \frac{1}{\bar{z}-1} & 1 & \frac{z(w+1)}{1-z} & \frac{1+\bar{w}}{z-1} \\ \frac{\bar{w}}{1-\bar{z}} & \frac{\bar{z}(\bar{w}+1)}{1-\bar{z}} & 1 & \frac{\bar{w}-\bar{w}^2}{1-z} \\ \frac{1}{\bar{z}(\bar{z}-1)} & \frac{1+w}{\bar{z}-1} & \frac{w-w^2}{1-\bar{z}} & 1 \end{pmatrix} \tag{5.6}$$

Proposition 5.2. *The adjoint trace field of $\mathcal{C}(G_{28},p,q)$ is given by $\mathbb{Q}(\cos\frac{2\pi}{l})$, where l is the least common multiple of p and q.*

Proof: Denote by $\mathbb{K}$ the adjoint trace field $\mathrm{tr}Ad\Gamma$, where $\Gamma = \mathcal{C}(G_{28},p,q)$. From the construction of the matrices $\widetilde{R}_j$, we get traces in $\mathbb{Q}(z,w) = \mathbb{Q}(\zeta_l)$, where $\zeta_d = e^{2\pi i/l}$. This implies $\mathbb{K} \subset \mathbb{Q}(\cos\frac{2\pi}{l})$.

Since $\mathrm{tr}(R_1) = 3+z$ and $\mathrm{tr}(R_3) = 3+w$, we get $\cos(\frac{2\pi}{p}), \cos(\frac{2\pi}{q}) \in \mathbb{K}$, which implies $\cos(\frac{2\pi}{l}) \in \mathbb{K}$. □

Proposition 5.3. *The CHL lattices derived from the group G_{28} are all arithmetic.*

Proof: We apply Theorem 4.1 to the group generated by the matrices $\widetilde{R}_j$ given in equation (5.4). Note that the hypotheses of that theorem are satisfied, since the entries of R_j are algebraic integers in the CM field $\mathbb{Q}(\zeta_l)$, and the adjoint trace field is equal to the maximal totally real subfield $\mathbb{Q}(\cos\frac{2\pi}{l})$ (as in Proposition 5.2, l denotes the lcm of p and q).

For the list of pairs (p,q) and the corresponding adjoint trace fields, see Table 4. The only cases that require work are those where the trace field is not $\mathbb{Q}$. In each case, we need to compute the signature of non-trivial Galois conjugates of the Hermitian matrix of equation (5.6), and check that they are all definite (see Theorem 4.1).

For $(p,q) = (2,5)$, up to complex conjugation, there is only one non-trivial Galois automorphism, given by $\zeta_{10} \mapsto \zeta_{10}^3$, which changes $\sqrt{5}$ to $-\sqrt{5}$. The Hermitian form (5.6) has signature $(3,1)$ for $z = -1$, $w = \zeta_{10}^2 = \zeta_5$, but it is positive definite for $z = -1$, $w = \zeta_{10}^6 = \zeta_5^3$.

For $(p,q) = (2,8)$, we need to consider the automorphism defined by $\zeta_8 \mapsto \zeta_8^3$, which changes $\sqrt{2}$ to $-\sqrt{2}$.

For $(p,q) = (2,12)$, $(3,4)$ or $(3,12)$, we need to consider $\zeta_{12} \mapsto \zeta_{12}^5$, which changes $\sqrt{3}$ to $-\sqrt{3}$. □

5.2. **Lattices derived from G_{29}.** It follows from the results by Broué, Malle, Rouquier [6] and Bessis and Michel [4], [5] that the corresponding braid group is given by

$$\langle\, r_1, r_2, r_3, r_4 \,|\, \mathrm{br}_2(r_1,r_3), \mathrm{br}_2(r_1,r_4), \mathrm{br}_3(r_1,r_2), \mathrm{br}_3(r_2,r_3), \mathrm{br}_3(r_3,r_4), \mathrm{br}_4(r_2,r_4), \mathrm{br}_4(r_3,r_2r_4)\,\rangle. \tag{5.7}$$

Couwenberg, Heckman and Looijenga show that there are representations into $PU(3,1)$ with lattice image, that map every r_j to a complex reflection R_j of angle $2\pi/p$, where p is either 3 or 4. We denote the corresponding groups by $\mathcal{C}(G_{29}, p)$.

As before, we denote by v_j a polar vector for the mirror of R_j (this simply means that the mirror is the orthogonal complement of v_j with respect the Hermitian inner product). Note that these four vectors must be linearly independent, because the group generated by the R_j must act irreducibly on $\mathbb{C}^4$.

We take the vectors v_j as the basis for $\mathbb{C}^4$, and because of the braid relations $\mathrm{br}_3(R_j, R_{j+1})$, we can normalize them so that the Hermitian form has the shape

$$\begin{pmatrix} 1 & \alpha & 0 & 0 \\ \bar{\alpha} & 1 & \alpha & \beta \\ 0 & \bar{\alpha} & 1 & \alpha \\ 0 & \bar{\beta} & \bar{\alpha} & 1 \end{pmatrix} \tag{5.8}$$

where $z = e^{2\pi i/p}$, $\alpha = 1/(z-1)$ and β is a complex number to be determined.

By computing the matrices for the reflections R_j and comparing the (2,2)-entries of $(R_2R_4)^2$ and $(R_4R_2)^2$, it is easy to see that the braid relation $\mathrm{br}_4(R_2, R_4)$ implies $|\beta|^2 = 0$ or

$$|\beta|^2 = \frac{2}{|z-1|^2}. \tag{5.9}$$

The case $\beta = 0$ is ruled out exactly as in Proposition 5.1.

By computing the (2,2)-entry of $(R_3(R_2R_4))^2$ and $((R_2R_4)R_3)^2$, we get the equation

$$|\beta|^2 z(z-1) - \bar{\beta} + \beta z^3 = 0,$$

which, together with equation (5.9), implies

$$\mathfrak{Re}(\beta z(z-1)) = 1,$$

hence

$$\beta = \frac{\mu}{z(z-1)},$$

where $\mu = 1 \pm i$.

The corresponding matrices R_j are given by

$$R_1 = \begin{pmatrix} z & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -z & z & 1 & \mu\bar{z} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -z & z & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\bar{\mu}z^2 & -z & z \end{pmatrix}, \tag{5.10}$$

which preserve the Hermitian form

$$H = \begin{pmatrix} 1 & \frac{1}{z-1} & 0 & 0 \\ \frac{1}{\bar{z}-1} & 1 & \frac{1}{z-1} & \frac{\mu\bar{z}}{z-1} \\ 0 & \frac{1}{\bar{z}-1} & 1 & \frac{1}{z-1} \\ 0 & \frac{\bar{\mu}z}{\bar{z}-1} & \frac{1}{\bar{z}-1} & 1 \end{pmatrix}. \tag{5.11}$$

One verifies that the Hermitian forms (5.11) corresponding to the two choices $\mu = 1 \pm i$ both have signature $(3, 1)$, for both values $p = 3$ and $p = 4$. We now identify which choice corresponds to the lattice $\mathcal{C}(G_{29}, p)$.

The first remark is that in the case $p = 3$, the two matrices (5.11) are Galois conjugate. Indeed the automorphism $\mathbb{Q}(\zeta_{12})$ that maps ζ_{12} to ζ_{12}^7 fixes ζ_3 while changing ζ_4 to $\bar{\zeta}_4$.

Proposition 5.4. *The group $\mathcal{C}(G_{29}, p)$ for $p = 3, 4$ corresponds to choosing $\mu = 1 + i$ in formulas* (5.10) *and* (5.11).

Proof: Let us first assume $p = 3$. We denote by H^+ (resp. H^-) the matrix H of equation (5.11) for $\mu = 1 + i$ (resp. $\mu = 1 - i$), and $z = e^{2\pi i/3}$. We also write R_j^+ (resp. R_j^-) for the reflections preserving H^+ (resp. H^-), and finally we write Γ^+ (resp. Γ^-) for the group generated by the R_j^+ (resp. R_j^-).

The lower right 3×3 submatrix of H^- gives a positive definite Hermitian form, so the subgroup Γ_{234}^- of Γ^- generated by R_2^-, R_3^-, R_4^- has a fixed point inside the ball.

One easily checks that $R_2^- R_3^- R_4^-$ is elliptic (the 1-eigenvector is negative for H_-), but has infinite order. The easiest way to check this is to consider the Galois conjugate group, where the corresponding matrix $R_2^+ R_3^+ R_4^+$ is loxodromic.

This implies that the group corresponding to $\mu = 1 - i$ is not discrete.

Let us now assume $p = 4$. The argument uses a bit of CHL-theory, see [8] or [14]. We consider the line $L = m_2 \cap m_3 \cap m_4$ which is the intersection of three mirrors of the reflections S_2, S_3 and S_4 in G_{29} (as previously, S_j and R_j correspond to the same braid element). Its fixed point stabilizer in G_{29} is isomorphic to the imprimitive Shephard-Todd group $G(4, 4, 3)$, which has order 96, and has 12 mirrors of reflections. This implies that L is contained in 12 mirrors of the arrangement for G_{29}.

For the group $\mathcal{C}(G_{29}, 4)$, the parameter κ_L is given by $\kappa_L = \frac{12}{\mathrm{codim} L}(1 - \frac{2}{p}) = 4(1 - \frac{2}{4}) = 2$ (see p. 88 of [8]). The fact that $\kappa_L > 1$ implies that the subgroup of $\mathcal{C}(G_{29}, 4)$ generated by R_2, R_3 and R_4 must preserve a totally geodesic copy of $H^2_{\mathbb{C}}$, hence the restriction of the Hermitian form to the span of the last three standard basis vectors e_2, e_3, e_4 must have signature $(2, 1)$.

One easily checks that, among the matrices H^+ and H^- in equation (5.11) by taking ($z = i$ and) $\mu = 1 \pm i$, only H_+ has a lower-right 3×3 block of signature $(2, 1)$. □

Proposition 5.5. *The adjoint trace field of $\mathcal{C}(G_{29},p)$ is $\mathbb{Q}(\sqrt{3})$ for $p=3$, and $\mathbb{Q}$ for $p=4$.*

Proof: Given the above matrices and the fact that $\mu = 1 \pm i$, we clearly have matrices with entries in $\mathbb{Q}(i,z)$.

For $p=4$, $\mathbb{Q}(i,z)=\mathbb{Q}(i)$, so the adjoint trace field is $\mathbb{Q}$. For $p=3$, $\mathbb{Q}(i,z)=\mathbb{Q}(\zeta_{12})$, since z is a primitive cube root of unity. This shows that the adjoint trace field is contained in $\mathbb{Q}(\sqrt{3})$.

To show the other inclusion, we compute $\mathrm{tr}(R_4R_3R_2) = 1+iz$, which for $z=\frac{-1+i\sqrt{3}}{2}$ gives $|\mathrm{tr}(R_4R_3R_2)|^2 = 2+\sqrt{3}$. □

Proposition 5.6. *The lattice $\mathcal{C}(G_{29},p)$ is non-arithmetic for $p=3$, and arithmetic for $p=4$.*

Proof: Note that all the entries of the matrices in equation (5.10) are algebraic integers, and their entries are in a CM field with maximal totally real subfield equal to the adjoint trace field, so we can apply Theorem 4.1.

The arithmeticity in the case $p=4$ is obvious, since the trace field is $\mathbb{Q}$ and there is no nontrivial Galois conjugate to consider.

The non-arithmeticity in the case $p=3$ follows immediately from the above discussion. Indeed, one checks that the matrices obtained in equation (5.11) for $\mu=1+i$ and $\mu=1-i$ both have signature (3,1). □

5.3. **Lattices derived from the group G_{30}.** The computations are similar to those in section 5.1. Because of the braid relations

$$\mathrm{br}_2(R_1,R_3), \mathrm{br}_2(R_1,R_4), \mathrm{br}_2(R_2,R_4), \mathrm{br}_3(R_1,R_2), \mathrm{br}_3(R_2,R_3), \mathrm{br}_5(R_3,R_4)$$

we may take the invariant Hermitian form to be

$$\begin{pmatrix} 1 & \alpha & 0 & 0 \\ \bar{\alpha} & 1 & \alpha & 0 \\ 0 & \bar{\alpha} & 1 & \beta \\ 0 & 0 & \bar{\beta} & 1 \end{pmatrix}, \tag{5.12}$$

where

$$\alpha = \frac{1}{z-1}, \quad \beta = \varphi\frac{1}{z-1},$$

where $\varphi = \frac{1\pm\sqrt{5}}{2}$ (one could use $\frac{-1\pm\sqrt{5}}{2}$, but this would give a conjugate subgroup of $GL(4,\mathbb{C})$).

In order to get a form with signature $(3,1)$ (at least when $p=3$ or 5), we need to take $\varphi = \frac{1+\sqrt{5}}{2}$, which we will do in the sequel.

Proposition 5.7. *The adjoint trace field of $\mathcal{C}(G_{30},p)$ is equal to $\mathbb{Q}(\cos 2\pi p, \varphi)$.*

Proof: We denote by $\mathbb{K}$ the adjoint trace field, and by $\mathbb{L}$ the field $\mathbb{Q}(\cos\frac{2\pi}{p},\varphi)$. Since the Hermitian form (5.12) is defined over $\mathbb{Q}(z,\varphi)$, we have $\mathbb{K}\subset\mathbb{L}$.

On the other hand, we have $\mathrm{tr}(R_1) = 3+z$ and $\mathrm{tr}(R_1R_2R_3R_4) = z(1-\varphi)$. This implies $|\mathrm{tr}(R_1)|^2 = 13+6\cos\frac{2\pi}{p}$ and $|\mathrm{tr}(R_1R_2R_3R_4)|^2 = 2-\varphi$, so $\mathbb{L}\subset\mathbb{K}$. □

From the Hermitian form, one can compute the matrices for R_j, which has mirror polar to e_j (the j-th vector in the standard basis for $\mathbb{C}^4$), and multiplier $z = e^{2\pi i/p}$. We get

$$R_1 = \begin{pmatrix} z & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -z & z & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -z & z & \varphi \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; R_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\varphi z & \omega \end{pmatrix}. \tag{5.13}$$

Note that these matrices have entries in $\mathbb{Z}[z, \varphi]$.

Proposition 5.8. *The group $\mathcal{C}(G_{30}, p)$ is a lattice if $p = 3, 5$, and in both cases it is cocompact. Both groups are arithmetic, with $\mathbb{Q}(\mathrm{tr} Ad\Gamma) = \mathbb{Q}(\sqrt{5})$.*

Proof: The entries of the matrices in equation (5.13) are algebraic integers in the CM field $\mathbb{Q}(z, \varphi)$, which is $\mathbb{Q}(i\sqrt{3}, \sqrt{5})$ for $p = 3$ and $\mathbb{Q}(\zeta_5)$ for $p = 5$. Both these fields have maximal totally real subfield equal to $\mathbb{Q}(\sqrt{5})$, which is equal to the adjoint trace field.

In order to show arithmeticity, by Theorem 4.1, we need to show that non-trivial Galois conjugates of the Hermitian form (5.12) are definite.

For $p = 3$, up to complex conjugation, the only non-trivial Galois automorphism is given by $\sqrt{5} \mapsto -\sqrt{5}$ (and we may assume $i\sqrt{3}$ is left unchanged). The signature of the Hermitian form (5.12) is $(4, 0)$ for $z = \frac{-1+i\sqrt{3}}{2}$ and $\varphi = \frac{1-\sqrt{5}}{2}$.

For $p = 5$, there is only one automorphism to consider (up to complex conjugation), given by $\zeta_5 \mapsto \zeta_5^2$, which again changes $\frac{1+\sqrt{5}}{2}$ into $\frac{1-\sqrt{5}}{2}$. Taking $z = e^{4\pi i/5}$ and $\varphi = \frac{1-\sqrt{5}}{2}$ in the matrix (5.12) gives signature $(4, 0)$. □

5.4. **Lattices derived from the group G_{31}.** There are two groups in the CHL list, corresponding to $p = 3$ and $p = 5$. These are a bit more difficult computationally, but not conceptually.

The initial difficulty is that the corresponding Shephard-Todd is not well-generated, i.e. it requires five generators (and not four as one may expect from the dimension). We first parametrize quadruples of reflections that satisfy the braid relations not involving R_5, and express them in terms of off-diagonal entries of the Hermitian form, see α, β in (5.15). Next, we use restrictions on α and β that come from the the existence of a 5-th reflection that satisfies the appropriate relations with the first 4 reflections (this is expressed in terms of the parameters α, β and the coordinates of a suitably normalized polar vector for the reflection R_5, see the parameter z below).

According to [5], the group is generated by reflections $R_1, \dots R_5$ that satisfy

$$\begin{gathered} \mathrm{br}_3(R_1, R_2), \mathrm{br}_3(R_2, R_5), \mathrm{br}_3(R_5, R_3), \mathrm{br}_3(R_3, R_4), \\ \mathrm{br}_2(R_2, R_4), \mathrm{br}_2(R_1, R_3), \mathrm{br}_2(R_2, R_3), \\ R_5 R_4 R_1 = R_4 R_1 R_5 = R_1 R_5 R_4. \end{gathered} \tag{5.14}$$

We denote by v_j a polar vector to the mirror of R_j. Note that the last relation implies that the polar vectors v_1, v_4, v_5 are linearly dependent.

Since the action of the group generated by all the R_j must be irreducible on $\mathbb{C}^4$, the vectors v_1, v_2, v_3, v_4 must be linearly independent. We write the Hermitian form in the

corresponding basis. The right angles coming from the above commutation relations imply that we may assume the corresponding Hermitian matrix has the form

$$H = \begin{pmatrix} 1 & \alpha & 0 & \beta \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & \alpha \\ \beta & 0 & \alpha & 1 \end{pmatrix}, \tag{5.15}$$

and we can choose $\alpha = 1/(z-1)$ as in the previous sections.

We write v_5 for a polar vector to the mirror of R_5. Because of the linear dependence between v_1, v_4 and v_5, we can write $v_5 = (x_1, 0, 0, x_4)$. If R_1 and R_5 have the same mirror, then they coincide, and this would imply that R_1, R_4 commute, in which case the action cannot be irreducible on $\mathbb{C}^4$.

Hence we must have $x_1 \neq 0$, and we can take $w = (1, 0, 0, \mu)$ for some $\mu \in \mathbb{C}$; by a similar reasoning, we must have $\mu \neq 0$. Writing out matrices for R_1, R_4 and R_5 in terms of the parameters z, β, μ, and comparing the $(1,3)$-entries of $R_5R_4R_1$ and $R_4R_1R_5$, we must have

$$|\mu| = 1.$$

Adjusting v_4 (and v_3 as well, since we want to keep the $\langle v_3, v_4\rangle$ unchanged) by multiplying them by a complex number of modulus one, we may assume further that $\mu = 1$, i.e. we assume

$$v_5 = (1, 0, 0, 1).$$

Comparing the $(1,2)$-entries of $R_5R_4R_1$ and $R_4R_1R_5$, we now get the equation

$$|\beta|^2(1-z) + \beta - \bar{\beta}z = 0. \tag{5.16}$$

It is fairly easy to see that the braid relation $\mathrm{br}_3(R_2, R_5)$ is equivalent to the relation

$$\beta + \bar{\beta} + 1 = 0. \tag{5.17}$$

Combining equations (5.16) and (5.17), we get

$$\beta = \frac{1}{\bar{r} - 1}, \tag{5.18}$$

where $r = \sqrt{z}$ denotes one of the two complex numbers whose square is z.

One then verifies (most reasonably with a computational software) that all the relations in equation (5.14) are satisfied when we take any of these two values of β.

Proposition 5.9. *The lattices $\mathcal{C}(G_{31}, p)$ for $p = 3, 5$ are obtained by taking $z = e^{2\pi i/p}$, $r = e^{\pi i/p}$ in the Hermitian form H.*

Proof: The fact that we take $z = e^{2\pi i/p}$ is from CHL theory. The two square-roots of z are $\pm e^{\pi i/p}$. For $p = 3$, $r = -e^{\pi i/3}$ gives a form of signature $(4, 0)$.

The case $p = 5$ is a bit more difficult, since both values $r = \pm e^{\pi i/5}$ give a form of signature $(3, 1)$.

In order to rule out $r = -e^{\pi i/5}$, we consider the 3×3 submatrix obtained from H by removing the third row and column from H; for $r = -e^{\pi i/5}$, this submatrix gives a degenerate Hermitian form, which implies that R_1, R_2, R_4 have a common global fixed point

at infinity (explicitly, this fixed point can be obtained by computing $e_1^\perp \cap e_2^\perp \cap e_4^\perp$ using linear algebra). One easily verifies that $R_1R_2R_4$ is a parabolic element (its eigenspace for the eigenvalue 1 is only 1-dimensional, but it is a double root of its characteristic polynomial).

This rules out $r = -e^{\pi i/5}$, since the lattice $\mathcal{C}(G_{31}, 5)$ is cocompact (see p. 160 of [8], where this group appears in boldface). □

We get the following matrices

$$R_1 = \begin{pmatrix} z & 1 & 0 & -r-z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -z & z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.19}$$

$$R_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ r+1 & 0 & -z & z \end{pmatrix}, \quad R_5 = \begin{pmatrix} z+r+1 & 1 & -z & -r-1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ z+r & 1 & -z & -r \end{pmatrix}, \tag{5.20}$$

where $z = e^{2\pi i/p}$ and $r = e^{\pi i/p}$.

We only consider the cases where $p = 3$ or 5, i.e. z is a root of unity of odd order, so the corresponding cyclotomic field $\mathbb{Q}(z)$ contains the square-roots of $z = e^{2\pi i/p}$ (if $p = 2m-1$, the square-roots of z are $\pm z^m$). In particular we get that, in both cases, the adjoint trace field is contained in $\mathbb{Q}(\cos\frac{2\pi}{p})$ and, as before, the fact that $\mathrm{tr}(R_1) = 3 + z$ implies:

Proposition 5.10. *For $p = 3$ or 5, the adjoint trace field of $\mathcal{C}(G_{31}, p)$ is $\mathbb{Q}(\cos\frac{2\pi}{p})$.*

Proposition 5.11. *For $p = 3$ and 5, the lattice $\mathcal{C}(G_{31}, p)$ is arithmetic.*

Proof: The entries of the matrices in equations (5.19), (5.20) are algebraic integers in the cyclotomic field $\mathbb{Q}(z)$ where $z = e^{2\pi i/p}$, and we can apply the arithmeticity criterion of Theorem 4.1.

For $p = 3$, there is nothing to check, since the adjoint trace field is $\mathbb{Q}$ and there is no nontrivial Galois conjugate.

For $p = 5$, the above Hermitian form reads

$$H = \begin{pmatrix} 1 & \frac{1}{z-1} & 0 & -\frac{1}{z^2+1} \\ \frac{1}{\bar{z}-1} & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{z-1} \\ -\frac{1}{\bar{z}^2+1} & 0 & \frac{1}{\bar{z}-1} & 1 \end{pmatrix}, \tag{5.21}$$

where $z = e^{2\pi i/5}$, and this Hermitian form has signature $(3, 1)$.

When changing z to z^2 (which changes $\sqrt{5}$ to $-\sqrt{5}$), the form H becomes definite, so Theorem 4.1 says that $\mathcal{C}(G_{31}, 5)$ is arithmetic. □

5.5. **Lattices derived from the group G_{33}.** Note that the group G_{33} acts on $\mathbb{C}^5$ (hence the corresponding CHL lattice acts on $H^4_{\mathbb{C}}$). More generally, here and in the following few sections, the dimension of the relevant complex hyperbolic space $H^n_{\mathbb{C}}$ can be read off the size $(n+1) \times (n+1)$ of the Hermitian matrix.

The group $\mathcal{C}(G_{33},3)$ is generated by 5 reflections $R_1,\dots,R_5$ of order 3. As before, we write the Hermitian form H in the basis of $\mathbb{C}^5$ given by vectors polar to the mirrors of the reflections R_j. By suitably rescaling these vectors, we may assume H has the shape

$$H=\begin{pmatrix}1 & \alpha & 0 & 0 & 0\\ \bar\alpha & 1 & \alpha & \lambda\alpha & 0\\ 0 & \bar\alpha & 1 & \alpha & 0\\ 0 & \bar\lambda\bar\alpha & \bar\alpha & 1 & \alpha\\ 0 & 0 & 0 & \bar\alpha & 1\end{pmatrix}, \tag{5.22}$$

where $\alpha=1/(z-1)$ as above, and $\lambda\in\mathbb{C}$ is to be determined later.

The braid relation $\mathrm{br}_3(R_2,R_4)$ is equivalent to $|\lambda|=1$. According to Bessis and Michel, we must have

$$(R_2R_4R_3)^2=(R_4R_3R_2)^2=(R_3R_2R_4)^2. \tag{5.23}$$

One can easily write down matrices for the R_j in terms of λ by using equation (2.1) for $\langle v,w\rangle=w^*Hv$. By comparing the (3,1) entries of $(R_2R_4R_3)^2$ and $(R_4R_3R_2)^2$, we see that the relation (5.23) implies $\Re(\lambda z)=\frac{1}{2}$, which gives $\lambda=\frac{1\pm\sqrt{3}}{2}\bar z$.

It is then easy to verify (most conveniently with some computer algebra system) that both values of λ make relation (5.23) hold.

We will only consider the case $p=3$. We write $\omega=e^{2\pi i/3}=\frac{-1+i\sqrt{3}}{2}$, and take $z=\omega$ in the above matrix. The two values of λ are then given by $-\omega$ and $-\bar\omega$.

One verifies that, in that case, the Hermitian form in equation (5.22) is degenerate for $\lambda=-\omega$, whereas is has signature $(4,1)$ for $\lambda=-\bar\omega$. Hence we have the following.

Proposition 5.12. *The group $\mathcal{C}(G_{33},3)$ is isomorphic to the group generated by R_j, $j=1,\dots,5$ as above, preserving the Hermitian form* (5.22) *for $z=\omega$ and $\lambda=-\bar\omega$.*

We write w_j for the j-th row of R_j. Note that R_j-Id has only one non-zero row, so in order to describe R_j, it is enough to list w_j, which is what we do in equation (5.24).

$$\begin{aligned} w_1&=(\omega,1,0,0,0)\\ w_2&=(-\omega,\omega,1,-\omega,0)\\ w_3&=(0,-\omega,\omega,1,0)\\ w_4&=(0,1,-\omega,\omega,1)\\ w_5&=(0,0,0,-\omega,\omega)\end{aligned}\qquad H=\begin{pmatrix}1 & \frac{1}{\omega-1} & 0 & 0 & 0\\ \frac{1}{\bar\omega-1} & 1 & \frac{1}{\omega-1} & \frac{1}{\bar\omega-1} & 0\\ 0 & \frac{1}{\bar\omega-1} & 1 & \frac{1}{\omega-1} & 0\\ 0 & \frac{1}{\omega-1} & \frac{1}{\bar\omega-1} & 1 & \frac{1}{\omega-1}\\ 0 & 0 & 0 & \frac{1}{\bar\omega-1} & 1\end{pmatrix}. \tag{5.24}$$

Equation (5.24) makes it clear that the matrices R_j have algebraic integer entries in $Q(\omega)$, which implies $\mathbb{Q}(\mathrm{tr}Ad\Gamma)=\mathbb{Q}$, so we have the following.

Proposition 5.13. *The lattice $\mathcal{C}(G_{33},3)$ is arithmetic with adjoint trace field $\mathbb{Q}$.*

5.6. **The group G_{34}.** The group $\mathcal{C}(G_{34},3)$ is generated by 6 reflections $R_1,\dots,R_6$, the braid group is the same as the previous one, with one extra generator that commutes with the first four, and braids with length 3 with the fifth.

The same computations as in section 5.5 show that we can take H to be

$$H = \begin{pmatrix} 1 & \frac{1}{\omega-1} & 0 & 0 & 0 & 0 \\ \frac{1}{\bar{\omega}-1} & 1 & \frac{1}{\omega-1} & \frac{\lambda}{\omega-1} & 0 & 0 \\ 0 & \frac{1}{\bar{\omega}-1} & 1 & \frac{1}{\omega-1} & 0 & 0 \\ 0 & \frac{\bar{\lambda}}{\bar{\omega}-1} & \frac{1}{\bar{\omega}-1} & 1 & \frac{1}{\omega-1} & 0 \\ 0 & 0 & 0 & \frac{1}{\omega-1} & 1 & \frac{1}{\omega-1} \\ 0 & 0 & 0 & 0 & \frac{1}{\bar{\omega}-1} & 1 \end{pmatrix}, \tag{5.25}$$

where again $\omega = \frac{-1+i\sqrt{3}}{2}$ and λ is either $-\bar{\omega}$ or $-\omega$. One readily checks that both Hermitian forms have signature $(5,1)$ so it is not clear which group corresponds to $\mathcal{C}(G_{34},3)$.

We call H^+ (resp. H^-) the Hermitian form corresponding to $\lambda = \frac{1+i\sqrt{3}}{2}$ (resp. $\lambda = \frac{1-i\sqrt{3}}{2}$). Note that the upper left 4×4 submatrices $K^\pm$ of $H^\pm$ do not have the same signature, namely K^+ has signature $(3,1)$, whereas K^- is degenerate.

Using a bit of CHL theory (see the argument below), the last observation implies the following.

Proposition 5.14. *The group $\mathcal{C}(G_{34},3)$ is conjugate to the group generated by the complex reflections R_j with multiplier ω and polar vectors given by the standard basis vectors e_j of $\mathbb{C}^6$, and Hermitian form H^+.*

Proof: Denote by S_j, $j=1,\dots,6$ the reflections generating G_{34} (with the same numbering as in Figure 2(h) on page 28); moreover, we denote by R_j the corresponding reflections in $\mathcal{C}(G_{34},3)$ (i.e. R_j and S_j are the images of the same element r_j of the relevant braid group, in the Bessis-Michel presentations).

It is quite clear from the Coxeter diagrams in Figure 2 that S_1, S_2, S_3, S_4 and S_5 generate a group isomorphic to G_{33} (this is a group of order 51840). The 1-dimensional intersection of their mirrors in $\mathbb{C}^6$ is contained in 45 mirrors of reflections in G_{34} (this is the number of mirrors of reflections in G_{33}, see p. 302 in [26]).

The weight attached to the stratum $L = L_{12345}$, i.e. the intersection of the mirrors of $S_1,\dots,S_5$ is

$$\kappa_L = \frac{45}{5}(1-\frac{2}{3}) = 3 > 1,$$

where the denominator 5 comes from the codimension of this stratum (see p. 96 of [8]). Since $\kappa_L > 1$, CHL theory predicts that the mirrors of R_1, R_2, R_3, R_4 and R_5 should be orthogonal to a common complex hyperbolic totally geodesic copy of $H^4_{\mathbb{C}}$ in $H^5_{\mathbb{C}}$.

Such a totally geodesic copy it given by the orthogonal complement $v^\perp$ of a vector v with $\langle v,v\rangle > 0$, so the restriction of the Hermitian form to the complex span of $e_1,\dots,e_5$ must have signature $(4,1)$, so we must use H^+ (and not H^-). $\square$

For completeness, as in section 5.5, we describe the (non-obvious rows of) the matrices R_j, $j = 1, \dots, 6$ in equation (5.26) (recall that w_j is the j-th row of R_j).

$$H = \begin{pmatrix} 1 & \frac{1}{\omega-1} & 0 & 0 & 0 & 0 \\ \frac{1}{\bar{\omega}-1} & 1 & \frac{1}{\omega-1} & \frac{1}{\bar{\omega}-1} & 0 & 0 \\ 0 & \frac{1}{\bar{\omega}-1} & 1 & \frac{1}{\omega-1} & 0 & 0 \\ 0 & \frac{1}{\omega-1} & \frac{1}{\bar{\omega}-1} & 1 & \frac{1}{\omega-1} & 0 \\ 0 & 0 & 0 & \frac{1}{\bar{\omega}-1} & 1 & \frac{1}{\omega-1} \\ 0 & 0 & 0 & 0 & \frac{1}{\bar{\omega}-1} & 1 \end{pmatrix}. \quad \begin{aligned} w_1 &= (\omega, 1, 0, 0, 0, 0) \\ w_2 &= (-\omega, \omega, 1, -\omega, 0, 0) \\ w_3 &= (0, -\omega, \omega, 1, 0, 0) \\ w_4 &= (0, 1, -\omega, \omega, 1, 0) \\ w_5 &= (0, 0, 0, -\omega, \omega, 1) \\ w_6 &= (0, 0, 0, 0, -\omega, \omega) \end{aligned} \tag{5.26}$$

The matrices $R_1, \dots, R_6$ have entries in $\mathbb{Z}[\omega]$, which implies once again that $\mathbb{Q}(\mathrm{tr}Ad\Gamma) = \mathbb{Q}$, so we get the following.

Proposition 5.15. *The lattice $\mathcal{C}(G_{34}, 3)$ is arithmetic with adjoint trace field $\mathbb{Q}$.*

5.7. **Lattices derived from the group G_{35}.** In this case, we can write the Hermitian matrix as

$$\begin{pmatrix} 1 & \alpha & 0 & 0 & 0 & 0 \\ \bar{\alpha} & 1 & \alpha & 0 & 0 & 0 \\ 0 & \bar{\alpha} & 1 & \alpha & 0 & \alpha \\ 0 & 0 & \bar{\alpha} & 1 & \alpha & 0 \\ 0 & 0 & 0 & \bar{\alpha} & 1 & 0 \\ 0 & 0 & \bar{\alpha} & 0 & 0 & 1 \end{pmatrix}.$$

where $\alpha = \frac{1}{z-1}$, $z = e^{2\pi i/p}$. The corresponding reflections have entries in $\mathbb{Z}[z]$, hence we have the following.

Proposition 5.16. *The lattices $\mathcal{C}(G_{35}, 3)$ and $\mathcal{C}(G_{35}, 4)$ are both arithmetic with adjoint trace field $\mathbb{Q}$.*

5.8. **The group G_{36}.** Note that G_{36} acts on $\mathbb{C}^7$, the corresponding CHL lattice acts on $H^6_{\mathbb{C}}$. We can write the Hermitian matrix as

$$\begin{pmatrix} 1 & \alpha & 0 & 0 & 0 & 0 & 0 \\ \bar{\alpha} & 1 & \alpha & 0 & 0 & 0 & 0 \\ 0 & \bar{\alpha} & 1 & \alpha & 0 & \alpha & 0 \\ 0 & 0 & \bar{\alpha} & 1 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \bar{\alpha} & 1 & 0 & \alpha \\ 0 & 0 & \bar{\alpha} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \bar{\alpha} & 0 & 1 \end{pmatrix}.$$

where $\alpha = \frac{1}{\omega-1}$. This gives matrices with entries in $\mathbb{Z}[\omega]$, so we have:

Proposition 5.17. *The lattices $\mathcal{C}(G_{36}, 3)$ is arithmetic with adjoint trace field $\mathbb{Q}$.*

5.9. **Lattices derived from the group G_{37}.** We can write the Hermitian matrix as

$$\begin{pmatrix} 1 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{\alpha} & 1 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\alpha} & 1 & \alpha & 0 & \alpha & 0 & 0 \\ 0 & 0 & \bar{\alpha} & 1 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\alpha} & 1 & 0 & \alpha & 0 \\ 0 & 0 & \bar{\alpha} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{\alpha} & 0 & 1 & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{\alpha} & 1 \end{pmatrix}.$$

where $\alpha = \frac{1}{\omega - 1}$. This gives matrices with entries in $\mathbb{Z}[\omega]$, so we have:

Proposition 5.18. *The lattices $\mathcal{C}(G_{37}, 3)$ is arithmetic with adjoint trace field $\mathbb{Q}$.*

6. The proof of Theorem 1.2

The fact that $\mathcal{C}(G_{29}, 3)$ is not cocompact is mentioned in the tables in [8]. The adjoint trace field was determined in section 5, where we also proved non-arithmeticity, see Proposition 5.6. The only thing that is left to prove is the fact that it is not commensurable to the Deligne-Mostow group Γ_μ with $\mu = (3, 3, 3, 3, 5, 7)/12$.

This is not obvious, since both groups have the same rough commensurability invariants (both are non-uniform, have non-arithmeticity index one, and they have the same adjoint trace field).

We will argue by comparing the cusps in both groups. It is known that both groups have a single orbit of cusps, but we will show that the corresponding cusps are not commensurable. In section 6.3 and 6.2 we will describe their respective cusps, and in 6.4 we will use this to show that they are incommensurable (see Proposition 6.8). We start with a general descriptions of cusps in 3-dimensional complex hyperbolic space (most of this can be found in chapter 4 of [18]).

6.1. **Cusps and the Heisenberg group.** When studying a cusp, we will write the Hermitian form in block form as

$$H = \left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & K & 0 \\ \hline 1 & 0 & 0 \end{array}\right), \tag{6.1}$$

where K is a positive definite 2×2 Hermitian form.

It is easy to see that any parabolic transformation in $PU(H)$ fixing $(1, 0, 0, 0)$ can be written as

$$P(B, w, t) = \left(\begin{array}{c|c|c} 1 & -w^* K B & -\frac{1}{2} w^* K w + it \\ \hline 0 & B & w \\ \hline 0 & 0 & 1 \end{array}\right), \tag{6.2}$$

where $B \in U(K)$, $w \in \mathbb{C}^2$ and $t \in \mathbb{R}$.

Moreover, one has

$$P(B,w,t)P(B',w',t') = P(BB', Bw'+w, t+t'+\mathfrak{Im}\,(w'^*B^*Kw)),$$

and

$$P(B,w,t)^{-1} = P(B^{-1}, -B^{-1}w, -t).$$

In the special case $B = Id$, we get unipotent elements $U(w,t) = P(Id,w,t)$ that satisfy

$$U(w,t)U(w',t') = U(w+w', t+t'+\mathfrak{Im}\,(w'^*Kw)), \quad U(w,t)^{-1} = U(-w,-t). \tag{6.3}$$

The corresponding group law $(w,t)\star(w',t) = (w+w', t+t'+\mathfrak{Im}\,(w'^*Kw))$ is called the (5-dimensional) Heisenberg group law, and the corresponding isometry of complex hyperbolic space is called a Heisenberg translation. When $w = 0$, the corresponding translation is called a *vertical* translation, and these are precisely the elements that are central in the Heisenberg group.

From equation (6.3), one easily checks that the commutator of $U(w,t)$ and $U(w',t')$ is given by

$$[U(w,t), U(w',t')] = U(0, 2\mathfrak{Im}\,(w'^*Kw)),$$

which is a vertical translation.

The parabolic stabilizer of $(1,0,0,0)$ has a projection onto the complex unitary affine group $\mathbb{C}^2 \ltimes U(K)$, given by keeping only the lower-right 3×3 block of the above matrices $P(B,w,t)$, the kernel consisting of the vertical translations in the group.

We will use this description to describe a cusp Γ_∞ of a non-cocompact lattice Γ. We may assume that the corresponding ideal fixed point is given by $(1,0,0,0)$, and that the Hermitian form is as in equation (6.1). Since a discrete group cannot have both parabolic and loxodromic elements fixing the same ideal point, the cusps will only contain parabolic elements.

The projection of Γ_∞ to the affine group $\mathbb{C}^2 \ltimes U(K)$ is then a complex crystallographic group, i.e. it must act cocompactly (which amounts to requiring that its translation subgroup has rank 4), and the vertical part is then an infinite cyclic group, commensurable to the one obtained by taking a single non-trivial commutator of Heisenberg translations.

Later in the paper, we will need to understand how the horizontal (w) and vertical (t) components of $U(w,t)$ behave under isometric changes of coordinates. If Q is a general isometry of the form H in equation (6.1), then it can be written as

$$Q = \left(\begin{array}{c|c|c} \alpha & -\alpha v^*KC & -\frac{1}{2}\alpha w^*Kw + is \\ \hline 0 & C & v \\ \hline 0 & 0 & \frac{1}{\alpha} \end{array}\right), \tag{6.4}$$

where $C \in U(K)$, $\alpha, t \in \mathbb{R}$ and $v \in \mathbb{C}^2$. One verifies by direct computation that

$$QU(w,t)Q^{-1} = U(\alpha Cw, \alpha^2(t + 2\Im(w^*C^*Kv))). \tag{6.5}$$

In particular, the square norm (with respect to K) of the horizontal part gets multiplied by α^2. In the special case $w = 0$, we have $QU(0,t)Q^{-1} = U(0,\alpha^2 t)$ for every $t \in \mathbb{R}$, so the vertical component of vertical translations also gets multiplied by α^2.

6.2. **The cusp of the Deligne-Mostow non-arithmetic lattice in $PU(3,1)$.** We now review some facts about the cusp of the group $\Gamma = \Gamma_{\mu,\Sigma}$ where $\mu = (3,3,3,3,5,7)/12$, and $\Sigma \simeq S_4$ permutes the first four weights. It is well known from Deligne-Mostow theory that the quotient has a single cusp, and that the corresponding cusp stabilizer can be described using hypergeometric functions for weights $(3,3,3,3,5+7)/12 = (1,1,1,1,4)/4$, and this monodromy group is described in section §15.20 of [11] (it corresponds to a parabolic case in Deligne-Mostow).

We give another argument, based on the description of $\Gamma = \Gamma_{\mu,\Sigma}$ as $\mathcal{C}(B_4,3,4)$ (see section 4 in [14]). The point is that the relevant braid group is generated by three reflections R_2, R_3, R_4 that correspond to half-twists between points with equal weights (these have order $2(1-2\mu_1)^{-1} = 4$) and a complex reflection R_1 corresponding to a full-twist between a point with weight μ_4 and one with weight μ_5 (this has order $(1-\mu_4-\mu_5)^{-1} = 3$).

We then have $\mathrm{br}(R_2,R_3) = \mathrm{br}(R_3,R_4) = 3$, $\mathrm{br}(R_1,R_2) = 4$, and other pairs of reflections among the R_j commute. These braid relations determine the group up to conjugation, using the fact that R_1 (resp. R_j for $j = 2,3,4$) has multiplier $e^{2\pi i/3} = \omega$ (resp. $e^{2\pi i/4} = i$).

We describe the group in coordinates that make the structure of the cusp group visible. Consider the Hermitian form

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & -1-i & 0 \\ 0 & -1+i & 2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{6.6}$$

and the reflections $R_1 = R_{v_1,\omega}$, $R_j = R_{v_j,i}$ for $j = 2,3,4$, where we use the notation in equation (2.1), and

$$v_1 = (2,0,0,\zeta^2+\zeta+1), \quad v_2 = (2,i-1,0,0), \quad v_3 = (0,0,1,0), \quad v_4 = (0,i+1,2,0). \tag{6.7}$$

Here we denote by $\zeta = e^{2\pi i/12}$. One easily verifies that these satisfy the correct braid relations, and the corresponding complex Coxeter diagram has no loop, so we get:

Proposition 6.1. *The group generated by $R_1,\dots,R_4$ is conjugate to $\mathcal{C}(B_4,3,4)$, which is also $\Gamma_{\mu,\Sigma}$ for $\mu = (3,3,3,3,5,7)/12$ and $\Sigma = S_4$.*

Note that the non-trivial Galois conjugate of $\mathcal{C}(B_4,3,4)$ is also generated by complex reflections satisfying the same relations, but the multipliers of the complex reflections are different; indeed, it corresponds to the automorphism φ of $\mathbb{Q}(\zeta)$ induced by $\varphi(\zeta) = \zeta^5$ (or its complex conjugate), and one then has $\varphi(i) = \varphi(\zeta^3) = \zeta^{15} = \zeta^3 = i$, and similarly $\varphi(\omega) = \varphi(\zeta^4) = \zeta^{20} = \bar{\omega}$.

We now study the structure of the cusp group using the notation in section 6.1. It follows from CHL theory that $\mathcal{C}(B_4,3,4)$ has a single cusp, represented by the group generated by R_2, R_3 and R_4.

Indeed, the cusps of the quotient are in one-to-one correspondence with the G-orbits of irreducible mirror intersections L with $\kappa_L = 1$, in the notation of [8]. Recall that

$$\kappa_L = \frac{\sum_{H\supset L}\kappa_H}{\mathrm{codim} L},$$

L	L_1	L_2	L_{12}	L_{23}	L_{123}	L_{234}
Type	$(1,0)$	$(0,1)$	$(2,2)$	$(0,3)$	$(3,6)$	$(0,6)$
κ_L	$1-\frac{2}{p_1}$	$1-\frac{2}{p_2}$	$2-\frac{2}{p_1}-\frac{2}{p_2}$	$\frac{3}{2}(1-\frac{2}{p_2})$	$3-\frac{2}{p_1}-\frac{4}{p_2}$	$2(1-\frac{2}{p_2})$

TABLE 1. Irreducible mirror intersections for $W(B_4)$. The type of an irreducible mirror intersection L is the number of mirrors in each G-orbit of mirrors containing L, for instance an L of type (3,6) is contained in 3 mirrors in the G-orbit of L_1 and 6 mirrors in the G-orbit of L_2.

where the sum ranges over all mirrors in the arrangement.

We denote by $r_1, \dots, r_4$ the generators of $G = W(B_4)$, by L_j the mirror of r_j. A list of representatives for all (non-trivial, pairwise distinct) irreducible mirror intersections is given in Table 1 (compare with the tables in [14]). In the table, L_{jk} stands for $L_j \cap L_k$, and L_{jkl} for $L_j \cap L_k \cap L_l$. For the group $\mathcal{C}(B_4, 3, 4)$, we take $(p_1, p_2) = (3, 4)$, and only one irreducible mirror intersection gives $\kappa_L = 1$, namely $L = L_{234}$.

In other words, the lattice $\Gamma = \mathcal{C}(B_4, 3, 4)$ has a single Γ-orbit of cusps, which is generated by R_2, R_3 and R_4. Note also that the corresponding cusp group is a CHL group of type $\mathcal{C}(A_3, 4)$, and it fits in their framework as a parabolic case (see section 5.3 of [8]).

We now work out a detailed description of the cusp group. The vector $e_1 = (1, 0, 0, 0)$ is a null vector for the Hermitian form H, and $\langle e_1, v_2\rangle = \langle e_1, v_3\rangle = \langle e_1, v_4\rangle = 0$ (see the definition of the v_j in equation (6.7)). This implies that e_1 gives the global ideal fixed point of the group generated by R_2, R_3, R_4.

We have chosen the Hermitian form so that the corresponding matrices for R_2, R_3, R_4 have Gaussian integers entries, i.e. entries in $\mathbb{Z}[i]$ (the entries of R_1 are not algebraic integers, but this is irrelevant). In fact, the corresponding reflections read

$$R_2 = \begin{pmatrix} 1 & 2 & -1-i & -1+i \\ 0 & i & 1 & -i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -i & i & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.8}$$

. As mentioned in the general description of cusps given in section 6.1, the projection onto the complex affine group $U(K) \ltimes \mathbb{C}^2$ is given by the lower-right 3×3 submatrices of R_2, R_3 and R_4.

The linear parts are given by

$$B_2 = \begin{pmatrix} i & 1 \\ 0 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1 & 0 \\ -i & i \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1 & -1 \\ 0 & i \end{pmatrix}, \tag{6.9}$$

and the translation parts are given by

$$w_2 = \begin{pmatrix} -i \\ 0 \end{pmatrix}, w_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, w_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.10}$$

Proposition 6.2. *The matrices B_2, B_3, B_4 generate a copy of the Shephard-Todd group G_8, which has order 96 (and center of order 4).*

L	L_1	L_{12}	L_{24}	L_{123}, L_{12343}	L_{124}	L_{234}
# mirrors	1	3	4	6	9	12
κ_L	$1-\frac{2}{p}$	$\frac{3}{2}(1-\frac{2}{p})$	$2(1-\frac{2}{p})$	$2(1-\frac{2}{p})$	$3(1-\frac{2}{p})$	$4(1-\frac{2}{p})$

TABLE 2. Irreducible mirror intersections for G_{29}.

Proof: The B_j are complex reflections of order 4, and $\mathrm{br}(R_2, R_3) = 3$. This implies that B_2 and B_3 generate a copy of G_8, and one easily verifies that B_4 is in the group generated by B_2 and B_3 (in fact $B_4 = B_3B_2B_3B_2B_3$). □

This gives the first part of the following proposition.

Proposition 6.3. *The projection to the complex affine group $U(K) \ltimes \mathbb{C}^2$ has linear part G_8, and translation subgroup given by $\mathbb{Z}[i] \times \mathbb{Z}[i]$.*

Proof: It is clear that the translation subgroup is contained in $\mathbb{Z}[i] \times \mathbb{Z}[i]$. We claim that it is precisely equal to it, which can be checked by computing the following,

$$\begin{aligned} T_{(1,0)} &= A_2A_3^2A_4^{-1}A_3^2 \\ T_{(i,0)} &= A_3^2A_4^{-1}A_3^2A_2 \\ T_{(0,1)} &= A_2^{-1}A_3A_2A_4^{-1}A_3A_2^{-1} \\ T_{(0,i)} &= A_2A_3^{-1}A_4^{-1}A_3A_2^{-1}A_3. \end{aligned} \tag{6.11}$$

Here we denote by T_w the vertical translation $U(Id, 0, w)$. □

In particular, given the shape of the 2×2 Hermitian matrix K in equation (6.6), we get that $w^*Kw \in \mathbb{Z}$ for every translation.

The following follows from the fact that the matrices in equation (6.8) have entries in $\mathbb{Z}[i]$.

Proposition 6.4. *The vertical translations in the cusp have translation length in $\mathbb{Z}$.*

In other works, the vertical translations are of the form $P(Id, 0, t)$ for some $t \in \mathbb{Z}$.

6.3. **The cusp of $\mathcal{C}(G_{29}, 3)$.** It follows from the analysis in [8] that the lattice $\mathcal{C}(G_{29}, 3)$ has a single conjugacy class of cusps. As mentioned in the previous section, in order to see this, we need to consider G_{29}-orbits of strata of the mirror arrangement given by (irreducible) mirror intersections, and compute the number κ_L (see p. 88 of [8]). The cusps correspond to strata where $\kappa_L = 1$.

A list of representatives for all the G-orbits of irreducible mirror intersections is given in Figure 2. Here, apart from L_{12343}, we denote by $L_{j_1,\dots,j_r}$ the intersection of the mirrors of $r_{j_1}, \dots, r_{j_r}$, and we use the numbering given in the diagram on page 28. For L_{12343}, we take the intersection of the mirrors of r_1, r_2 and $r_3r_4r_3 = r_3r_4r_3^{-1}$. For $p = 3$, the only G-orbit with $\kappa_L = 1$ is the G-orbit of L_{124}. This means that the group $\Gamma = \mathcal{C}(G_{29}, 3)$ has a single Γ-orbit of cusps, represented by the group generated by R_1, R_2 and R_4. Note that this cusp group is a CHL group of the form $\mathcal{C}(B_3, 3)$, which fits in their analysis of parabolic cases.

We now work out the detailed structure of the cusp, using the general framework of section 6.1. First note that the reflections R_1, R_2 and R_4 fix a common point in the ideal boundary $\partial_\infty H^3_{\mathbb{C}}$, given in the basis used in section 5.2 by the vector

$$v = (\zeta^2, \zeta^2+1, 0, \zeta^2+\zeta-1).$$

We use this as the first vector, and (a suitable multiple of) $R_3 v$ as the last basis vector. As the second vector, we use (a suitable multiple of) the polar vector to the mirror of R_1, and as the third one we take one that is orthogonal to both v and $R_3 v$ (and that makes the matrix of R_2 as simple as we could make it).

Concretely, we take

$$Q = \begin{pmatrix} -\bar\omega & \bar\omega - 1 & i\sqrt{3}(1-\zeta) & i(3-2\sqrt{3}) \\ 1-\bar\omega & 0 & i(\sqrt{3}-3) & 3\zeta(\zeta-1)^2) \\ 0 & 0 & 0 & (\omega-1)(1+\omega\zeta) \\ \zeta+\omega & 0 & \zeta(3-\sqrt{3}) & i-\bar\omega-5\zeta+4 \end{pmatrix}.$$

Writing $S_j = Q^{-1} R_j Q$, we get

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & (\zeta-1)(1-\omega) & 3(1-\zeta) & -2+(2-3\zeta)\omega \\ 0 & 1 & 0 & 0 \\ 0 & -\omega & \omega & i-\omega \\ 0 & 0 & 0 & 1 \end{pmatrix}, S_4 = \begin{pmatrix} 1 & 0 & \zeta(\sqrt{3}-2) & -2+\omega(2-3\zeta) \\ 0 & 1 & -1 & \frac{1}{2}(i\sqrt{3}\zeta-\omega) \\ 0 & 0 & \omega & \frac{1}{2}i(3-\sqrt{3}) \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which preserve the Hermitian form

$$Q^* H Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 3 & \bar\omega - 1 & 0 \\ 0 & \omega-1 & 3 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{6.12}$$

In the notation of section 6.1, we have $S_j = P(B_j, w_j, t_j)$ for

$$B_1 = \begin{pmatrix} \omega & 1 \\ 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ -\omega & \omega \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1 & -1 \\ 0 & \omega \end{pmatrix}, \tag{6.13}$$

and the translation parts are given by

$$w_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, w_2 = \begin{pmatrix} 0 \\ i-\omega \end{pmatrix}, w_4 = \begin{pmatrix} \frac{1}{2}(i\sqrt{3}\zeta-\omega) \\ \frac{1}{2}i(3-\sqrt{3}) \end{pmatrix}. \tag{6.14}$$

One checks that the matrices B_1, B_2, B_4 generate a group isomorphic to the Shephard-Todd group G_5 (which has order 72). Indeed, the matrices B_2 and B_4 are reflections of order 3 and $\mathrm{br}(B_2, B_4) = 4$, so they generate a copy of G_5 (see [6] for instance). One then checks that B_1 is in the group generated by B_2 and B_4, for instance $B_1 = B_2[B_4^{-1}, B_2]$.

This proves the following.

Proposition 6.5. *The cusp of $\mathcal{C}(G_{29}, 3)$ is a central extension of a 2-dimensional affine crystallographic group generated by reflections, with linear part G_5.*

We denote by $\mathcal{T}$ the subgroup of translations (i.e. the unipotent subgroup) in the cusp. The first observation is that it consists of matrices of the form $U(w,t) = P(Id, w, t)$ (see the notation in section 6.1) such that w has entries in $Z[\omega]$.

This is not obvious, since the matrices S_j have entries in $\mathbb{Z}[\zeta]$. We start by enumerating some translations (that we obtained by computing all words in S_1, S_2, S_4 of length at most 6). The following can be checked by direct computation.

$$\begin{aligned} T_1 &= U\left((1,1), 3(1-\tfrac{\sqrt{3}}{2})\right) = S_1S_2S_1^{-1}S_4^{-1}S_2^{-1}S_4 \\ T_2 &= U\left((1,0), -\tfrac{\sqrt{3}}{2}\right) = T_{(1,0)} = S_1S_2S_4^{-1}S_2^{-1}S_4S_2^{-1} \\ T_3 &= U\left((\bar{\omega},\bar{\omega}), -3(1-\tfrac{\sqrt{3}}{2})\right) = S_4^{-1}S_2^{-1}S_4S_1S_2S_1^{-1} \\ T_4 &= U\left((0,\omega), -\tfrac{\sqrt{3}}{2}\right) = S_1S_4S_2S_1^{-1}S_4^{-1}S_2^{-1} \end{aligned} \tag{6.15}$$

Note also that

$$[T_2, T_1] = U(0, \sqrt{3}),$$

we denote the vertical translation by V.

Consider the abstract group with presentation

$$G = \langle s_1, s_2, s_4 \mid s_1^3, s_2^3, s_4^3, \mathrm{br}_3(s_1,s_2), \mathrm{br}_4(s_2,s_4), [s_1,s_4], s_1s_4s_2s_1^{-1}s_4^{-1}s_2^{-1}\rangle$$

where the last two relations come from the right hand side of the first and fourth row in equation (6.15). We could have included the other two relations coming from the second and third rows, but it is easy to see that we would get the same abstract group if we did.

One checks (most conveniently with a computer algebra system, say GAP) that the abstract group G has order 72, hence it is isomorphic to G_5, which is the linear part of the complex crystallographic group obtained by killing off the center of our cusp group.

As a consequence, we have the following.

Proposition 6.6. *The translation subgroup $\mathcal{T}$ of the cusp group generated by S_1, S_2, S_4 is generated by the translations T_1, T_2, T_3, T_4 in equation* (6.15).

Proof: Let Γ_∞ denote the group generated by S_1, S_2, S_4 and let $\mathcal{T}_0$ denote the subgroup generated by T_1, T_2, T_3, T_4.

The projection $\varphi : \Gamma_\infty \to U(K)$ given by mapping $P(B, w, t)$ to B is a homomorphism onto G_5, and its kernel is the full translation subgroup $\mathcal{T} \subset \Gamma_\infty$.

Every element in the kernel of φ can be written as a product of conjugates of S_1^3, $S_1S_2S_1(S_2S_1S_2)^{-1}$, $(S_2S_4)^2(S_2S_4)^{-2}$, $S_1S_4S_1^{-1}S_4^{-1}$, $S_1S_4S_2S_1^{-1}S_4^{-1}S_2^{-1}$. Only the last element gives a non-trivial translation.

Now one computes

$$\begin{aligned} S_1T_4S_1^{-1} &= (T_1T_3)^{-1}V \\ S_2T_4S_2^{-1} &= T_4^{-1}T_1^{-1}T_2V^{-1} \\ S_4T_4S_4^{-1} &= T_2T_3V^{-1}, \end{aligned} \tag{6.16}$$

which shows that the above conjugates are in fact in $\mathcal{T}_0$. In other words $\mathcal{T} = \mathcal{T}_0$. □

We now have the following.

Proposition 6.7. *The complex crystallographic group obtained by projecting Γ_∞ to $U(K) \rtimes \mathbb{C}^2$ has translation part $\mathbb{Z}[\omega] \times \mathbb{Z}[\omega]$. Its vertical translation group is generated by $V = U(0, \sqrt{3})$.*

Proof: One easily checks that the vectors $u_1 = (1,1), u_2 = (1,0), u_3 = (\bar{\omega}, \bar{\omega}), u_4 = (0, \omega)$ generate $\mathbb{Z}[\omega] \times \mathbb{Z}[\omega]$. □

6.4. **Incommensurability.** We give an argument that was suggested by the referree, which replaces our previous computational argument by a more geometric one.

Proposition 6.8. *The lattices $\Gamma_1 = \mathcal{C}(B_4, 3, 4)$ and $\Gamma_2 \mathcal{C}(G_{29}, 3)$ are not commensurable.*

Proof: Suppose they were commensurable, then there exists a $g \in GL(4, \mathbb{C})$ such that $g\Gamma_1 g^{-1}$ and Γ_2 have a common finite index subgroup Γ. By irreducibility, this implies that $g^* H_1 g = \lambda H_2$ for some $\lambda > 0$, where H_1 and H_2 are the Hermitian forms from equations (6.6) and (6.12), respectively.

We may assume that Γ has a cusp represented by the standard basis vector e_1 and H_2 has the shape given in equation (6.12). The group Γ contains both a nontrivial Heisenberg translation $gU_1(w_1, t_1)g^{-1}$ and a nontrivial vertical translation $gU_1(0, t_1')g^{-1}$, which can also be written as $U_2(w_2, t_2)$ and $U_2(0, t_2')$ (we use U_j for the Heisenberg description using the Hermitian form H_j).

Recall that the results of section 6.2 imply that we may assume $w_1^* K_1 w_1 \in \mathbb{Z}$ and $t_1' \in i\mathbb{Z}$, and the results of section 6.3 imply that we may assume (possibly after post-composing g with some isometry of H_2), that $w_2^* K_2 w_2 \in \mathbb{Z}$ and $t_2' \in i\sqrt{3}\mathbb{Z}$.

Moreover, the observation of equation (6.5) says that there is a $\lambda > 0$ such that $w_2^* K_2 w_2 = \lambda w_1^* K_1 w_1$ and $t_2' = \lambda t_1'$.

The first equation implies $\lambda \in \mathbb{Q}$, whereas the second implies $\lambda \notin \mathbb{Q}$, contradiction. □

Appendix A. Coxeter diagrams

For the reader's convenience, we gather Coxeter diagrams for the Shephard-Todd groups. These were worked out by Coxeter [9] and Shephard-Todd [26], see also the Appendix 2 in [6] for a convenient list.

Appendix B. Rough commensurability invariants

In this section, we gather rough commensurability invariants for the Couwenberg-Heckman-Looijenga lattices in $PU(n,1)$ that correspond to the exceptional Shephard-Todd groups. For each such Shephard-Todd group, we list values of the order of reflections that yield lattices, and mention whether the corresponding lattices are cocompact (C/NC) and arithmetic (A/NA). We also give their adjoint trace field $\mathbb{Q}(\mathrm{tr} Ad\Gamma)$.

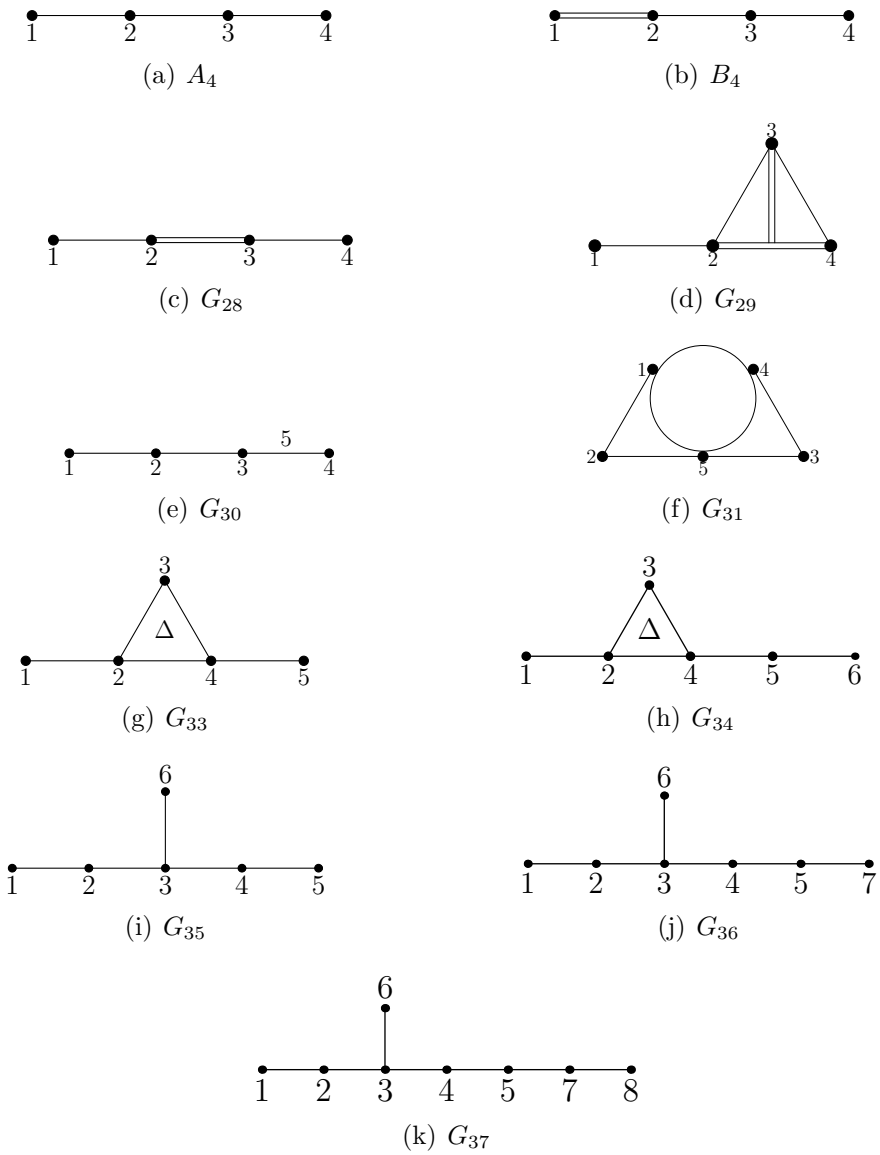

FIGURE 2. Coxeter diagrams for exceptional Shephard-Todd groups. The symbol Δ in the diagrams for G_{33} and G_{34} stand for the relation $(R_2R_4R_3)^2 = (R_4R_3R_2)^2 = (R_3R_2R_4)^2$.

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Shephard-Todd	Other description	p
G_{23}	Coxeter H_3, $\mathcal{S}(\sigma_{10}, p)$	$\mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{10}$
G_{24}	$\mathcal{S}(\overline{\sigma}_4, p)$	$\mathbf{3}, 4, \mathbf{5}, 6, \mathbf{8}, \mathbf{12}$
G_{27}	$\mathcal{T}(\mathbf{S_2}, p)$	$\mathbf{3}, 4, \mathbf{5}$

TABLE 3. 2-dimensional CHL lattices. Note that G_{25} and G_{26} are not listed because they yield Deligne-Mostow groups. Values of p that appear in boldface correspond to cocompact lattices, the ones that appear in red give non-arithmetic lattices.

Shephard-Todd	Dimension	Other description	p or (p_1, p_2)	A?	C?	$\mathbb{Q}(\mathrm{tr}Ad\Gamma)$
G_{28}	3	Coxeter F_4	$(2,4)$	A	NC	$\mathbb{Q}$
			$(\mathbf{2},\mathbf{5})$	A	C	$\mathbb{Q}(\sqrt{5})$
			$(2,6)$	A	NC	$\mathbb{Q}$
			$(\mathbf{2},\mathbf{8})$	A	C	$\mathbb{Q}(\sqrt{2})$
			$(\mathbf{2},\mathbf{12})$	A	C	$\mathbb{Q}(\sqrt{3})$
			$(3,3)$	A	NC	$\mathbb{Q}$
			$(\mathbf{3},\mathbf{4})$	A	C	$\mathbb{Q}(\sqrt{3})$
			$(3,6)$	A	NC	$\mathbb{Q}$
			$(\mathbf{3},\mathbf{12})$	A	C	$\mathbb{Q}(\sqrt{3})$
			$(4,4)$	A	NC	$\mathbb{Q}$
			$(6,6)$	A	NC	$\mathbb{Q}$
G_{29}	3		3	NA	NC	$\mathbb{Q}(\sqrt{3})$
			4	A	NC	$\mathbb{Q}$
G_{30}	3	Coxeter H_4	$\mathbf{3}$	A	C	$\mathbb{Q}(\sqrt{5})$
			$\mathbf{5}$	A	C	$\mathbb{Q}(\sqrt{5})$
G_{31}	3		3	A	NC	$\mathbb{Q}$
			$\mathbf{5}$	A	C	$\mathbb{Q}(\sqrt{5})$
G_{33}	4		3	A	NC	$\mathbb{Q}$
G_{34}	5		3	A	NC	$\mathbb{Q}$
G_{35}	5	$W(E_6)$	3	A	NC	$\mathbb{Q}$
			4	A	NC	$\mathbb{Q}$
G_{36}	6	$W(E_7)$	3	A	NC	$\mathbb{Q}$
G_{37}	7	$W(E_8)$	3	A	NC	$\mathbb{Q}$

TABLE 4. Rough commensurability invariants for CHL lattices. The group G_{32} is not listed, since it is also A_3 and yields Deligne-Mostow groups.

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