Census of the complex hyperbolic sporadic triangle groups

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Abstract

The goal of this paper is to give a conjectural census of complex hyperbolic sporadic triangle groups. We prove that only finitely many of these sporadic groups are lattices.

We also give a conjectural list of all lattices among sporadic groups, and for each group in the list we give a conjectural group presentation, as well as a list of cusps and generators for their stabilisers. We describe strong evidence for these conjectural statements, showing that their validity depends on the solution of reasonably small systems of quadratic inequalities in four variables.

1 Introduction

The motivation for this paper is to construct discrete groups acting on the complex hyperbolic plane $\mathrm{H}^2_{\mathbb{C}}$, more specifically lattices (where one requires in addition that the quotient by the action of the discrete group have finite volume). Complex hyperbolic spaces $\mathrm{H}^n_{\mathbb{C}}$ are a natural generalisation to the realm of Kähler geometry of the familiar non-Euclidean geometry of $\mathrm{H}^n_{\mathbb{R}}$. $\mathrm{H}^n_{\mathbb{C}}$ is simply the unit ball in $\mathbb{C}^n$, endowed with the unique Kähler metric invariant under all biholomorphisms of the ball; this metric is symmetric and has non-constant negative real sectional curvature (holomorphic sectional curvature is constant). The group of holomorphic isometries of $\mathrm{H}^n_{\mathbb{C}}$ is the projectivised group $\mathrm{PU}(n,1)$ of a Hermitian form of Lorentzian signature $(n,1)$.

It is a well known fact due to Borel that lattices exist in the isometry group of any symmetric space, but the general structure of lattices and the detailed study of their representation theory brings forth several open questions. The basic construction of lattices relies on the fact that for any linear algebraic group G defined over $\mathbb{Q}$, the group of integral matrices $G(\mathbb{Z})$ is a lattice in $G(\mathbb{R})$. $G(\mathbb{Z})$ is clearly discrete, and the fact that it is a lattice follows from a theorem of Borel and Harish-Chandra. More generally, to a group defined over a number field (i.e. a finite extension of the rationals), one can associate a group defined over $\mathbb{Q}$ by a process called restriction of scalars. One is naturally led to the general notion of *arithmetic group*, keeping in mind that one would like to push as far as possible the idea of taking integral matrices in a group defined over $\mathbb{Q}$. For the general definition of arithmeticity, we refer the reader to section 2. In the context of the present paper, the arithmeticity criterion in that section (Proposition 2.1) will be sufficient.

It is known since deep work of Margulis that lattices in the isometry group of any symmetric space of higher rank (i.e. rank $\geqslant 2$) are all arithmetic. There are four families of rank 1 symmetric spaces of non-compact type, namely

$$\mathrm{H}^n_{\mathbb{R}}, \mathrm{H}^n_{\mathbb{C}}, \mathrm{H}^n_{\mathbb{H}}, \mathrm{H}^2_{\mathbb{O}}.$$

Lattices in the isometry groups of the last two families (hyperbolic spaces over the quaternions and the octonions) are all known to be arithmetic, thanks to work of Corlette and Gromov-Schoen.

On the other hand, non-arithmetic lattices are known to exist in $\mathrm{PO}(n,1)$ (which is the isometry group of $\mathrm{H}^n_{\mathbb{R}}$) for arbitrary $n \geqslant 2$. A handful of examples coming from Coxeter groups were known in low dimensions before Gromov and Piatetski-Shapiro found a general construction using so-called interbreeding of well-chosen arithmetic real hyperbolic lattices (see [GP]).

The existence of non-arithmetic lattices in $\mathrm{PU}(n,1)$ (the group of holomorphic isometries of $\mathrm{H}^n_{\mathbb{C}}$) for arbitrary n is a longstanding open question. Examples are known only for $n \leqslant 3$, and they are all commensurable to complex reflection groups. More specifically, it turns out that all known non-arithmetic lattices in $\mathrm{PU}(n,1)$ for $n = 2$ or 3 are commensurable to one of the hypergeometric monodromy groups listed in [DM] and [M2] (the same list appears in [T]).

The goal of this paper is to announce (and give outstanding evidence for) results that exhibit several new commensurability classes of non-arithmetic lattices in $\mathrm{PU}(2,1)$. Our starting point was the investigation by Parker and Paupert [ParPau] of *symmetric triangle groups*, i.e. groups generated by three complex reflections of order $p \geqslant 3$ in a symmetric configuration (the case $p = 2$ was studied by Parker in [Par3]).

Writing R_i, $i = 1,2,3$ for the generators, the symmetry condition means that there exists an isometry J of order 3 such that $JR_iJ^{-1} = R_{i+1}$ (indices mod 3). It turns out that conjugacy classes of symmetric triangle groups (with generators of any fixed order $p \geqslant 2$) can then be parametrised by

$$\tau = \mathrm{Tr}(R_1 J),$$

provided we represent isometries by matrices for R_1 and J in $\mathrm{SU}(2,1)$ (see section 3 for basic geometric facts about complex hyperbolic spaces).

Following [ParPau], we denote by $\Gamma(\frac{2\pi}{p}, \tau)$ the group generated by R_1 and J as above. The main problem is to determine the values (p, τ) of the parameters such that $\Gamma(\frac{2\pi}{p}, \tau)$ is a lattice in $\mathrm{PU}(2,1)$. It is a difficult problem to do this in all generality (see the discussion in [M1], [De1] for instance).

To simplify matters, we shall concentrate on a slightly smaller class of groups. The results in [ParPau] give the list of all values of p, τ such that R_1R_2 and R_1J are either parabolic, or elliptic of finite order. When this condition holds, we refer to such a triangle group as *doubly elliptic* (see section 4).

It turns out that the double ellipticity condition is independent of p, and the values of τ that yield doubly elliptic triangle groups come into two continuous 1-parameter families, together with 18 isolated values of the parameter τ.

The continuous families yield groups that are subgroups of so-called Mostow groups, i.e. ones where the generating reflections satisfy the braid relation

$$R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}.$$

In that case, the problem of determining which parameters yield a lattice is completely solved (see [M1], [M3] for the first family and [ParPau] for the second).

The isolated values of τ corresponding to doubly elliptic triangle groups are called *sporadic values*, and the corresponding triangle groups are called *sporadic triangle groups* (the list of sporadic values is given in Table 1, page 7). It has been suspected since [Par3] and [ParPau] that sporadic groups may yield interesting lattices.

In fact, the work in [Pau] shows that only one sporadic triangle group is an *arithmetic* lattice; moreover, most sporadic triangle groups are not commensurable to any of the previously known non-arithmetic lattices (the Picard, Mostow and Deligne-Mostow lattices). The precise statement of what "most sporadic groups" means is given in Theorem 4.3, see also [Pau]. The question left open is of course to determine which sporadic groups are indeed lattices.

To that end, it is quite natural to use the first author's computer program (see [De1]), and to go through an experimental investigation of the Dirichlet domains for sporadic groups. The goal of the present paper is to report on the results of this search, which turn out to be quite satisfactory.

We summarise the results of our computer experimentation in the following (see section 4, Table 1 for the meaning of the parameters $\sigma_1, \ldots, \sigma_9$):

Conjecture 1.1 *The following sporadic groups are non-arithmetic lattices in* $\mathrm{SU}(2,1)$*:*

- *(cocompact):* $\mathbf{\Gamma}(\frac{\mathbf{2\pi}}{\mathbf{5}},\overline{\sigma}_\mathbf{4})$, $\mathbf{\Gamma}(\frac{\mathbf{2\pi}}{\mathbf{8}},\overline{\sigma}_\mathbf{4})$, $\mathbf{\Gamma}(\frac{\mathbf{2\pi}}{\mathbf{12}},\overline{\sigma}_\mathbf{4})$.
- *(non cocompact):* $\mathbf{\Gamma}(\frac{\mathbf{2\pi}}{\mathbf{3}},\sigma_\mathbf{1})$, $\mathbf{\Gamma}(\frac{\mathbf{2\pi}}{\mathbf{3}},\sigma_\mathbf{5})$, $\Gamma(\frac{2\pi}{4},\sigma_1)$, $\mathbf{\Gamma}(\frac{\mathbf{2\pi}}{\mathbf{4}},\overline{\sigma}_\mathbf{4})$, $\mathbf{\Gamma}(\frac{\mathbf{2\pi}}{\mathbf{4}},\sigma_\mathbf{5})$, $\Gamma(\frac{2\pi}{6},\sigma_1)$, $\mathbf{\Gamma}(\frac{\mathbf{2\pi}}{\mathbf{6}},\overline{\sigma}_\mathbf{4})$.

In fact we have obtained outstanding evidence that Conjecture 1.1 is correct, but this evidence was obtained by doing numerical computations using floating point arithmetic, and it is conceivable (though very unlikely) that the results are flawed because of issues of precision, in a similar vein as the analysis in [De1] of the results in [M1]. Instead of arguing that the computer experimentation is not misleading, we will prove Conjecture 1.1 in [DPP] by using more direct geometric methods.

Note that the only part of Conjecture 1.1 that is conjectural is the fact that the groups in question are lattices. The fact that these groups are not arithmetic follows from the results in [ParPau] and [Pau]. The groups indicated in bold are known to not be commensurable to Deligne-Mostow-Picard lattices by [Pau] (in fact, for $\Gamma(\frac{2\pi}{4},\overline{\sigma}_4)$ and $\Gamma(\frac{2\pi}{6},\overline{\sigma}_4)$ this follows from non-cocompactness by the arguments in [Pau]).

Computer experiments also suggest that Conjecture 1.1 is essentially optimal. More specifically, sporadic groups that do not appear in the list seem not to be lattices (most of them are not discrete, a handful seem to have infinite covolume), apart from the following:

$$\Gamma\left(\frac{2\pi}{3},\overline{\sigma}_4\right),\ \Gamma\left(\frac{2\pi}{2},\sigma_5\right),\ \Gamma\left(\frac{2\pi}{2},\overline{\sigma}_5\right). \tag{1.1}$$

These exceptions are in fact completely understood, and they are all arithmetic; the last two groups are both isomorphic to the lattice studied in [De1] (see [Par3]). As for the first group, partly thanks to work in [ParPau], we have:

Theorem 1.1 $\Gamma(\frac{2\pi}{3},\overline{\sigma}_4)$ *is a cocompact arithmetic lattice in* $\mathrm{SU}(2,1)$.

The fact that this group is discrete was proved in [ParPau] (Proposition 6.4), the point being that all non-trivial Galois conjugates of the relevant Hermitian form are definite. In fact it is the only sporadic group that is contained in an arithmetic lattice, by [Pau]. In order to check that it is cocompact, one uses the same argument as in [De2]. More specifically, one needs to verify that the Dirichlet domain is cocompact. This can be done without knowing the precise combinatorics of that polyhedron (it is enough to study a partial Dirichlet domain, and to verify that all the 2-faces of that polyhedron are compact, see [De2]).

The non-discreteness results we prove in section 9 of the paper are close to proving optimality of the statement of the Conjecture, but the precise statement is somewhat lengthy (see Theorem 9.1). For now we simply state the following:

Theorem 1.2 *Only finitely many sporadic triangle groups are discrete.*

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2 Arithmetic lattices arising from Hermitian forms over number fields

For the sake of completeness, we recall in Def. 2.1 the general definition of arithmeticity (see also [Z], chapter 6). For the purposes of the present paper the special case of arithmetic groups arising from Hermitian forms over number fields will be sufficient (see Proposition 2.1 below).

Borel and Harish-Chandra proved that if G is a linear algebraic group defined over $\mathbb{Q}$ then $G(\mathbb{Z})$ is a lattice in $G(\mathbb{R})$. Recall that a **real linear algebraic group defined over** $\mathbb{Q}$ is a subgroup G of $\mathrm{GL}(n,\mathbb{R})$ for some n, such that the elements of G are precisely the solutions of a set of polynomial equations in the entries of the matrices, with the coefficients of the polynomials lying in $\mathbb{Q}$; one denotes $G(\mathbb{R}) = G$ and $G(\mathbb{Z}) = G\cap\mathrm{GL}(n,\mathbb{Z})$. From their result, one can deduce that any real semisimple Lie group contains infinitely many (distinct commensurability classes of) lattices, either cocompact or non cocompact.

One obtains the general definition by extending this notion to all groups equivalent to groups of the form $G(\mathbb{Z})$ in the following sense:

Definition 2.1 *Let G be a semisimple Lie group, and Γ a subgroup of G. Then Γ is an* arithmetic lattice *in G if there exist an algebraic group S defined over $\mathbb{Q}$ and a continuous homomorphism $\phi : S(\mathbb{R})^0 \longrightarrow G$ with compact kernel such that Γ is commensurable to $\phi(S(\mathbb{Z}) \cap S(\mathbb{R})^0)$.*

The fact that Γ as in the definition is indeed a lattice follows from the Borel–Harish-Chandra theorem.

Here we focus on the case of *integral groups arising from Hermitian forms over number fields.* This means that we consider groups Γ which are contained in $\mathrm{SU}(H, \mathcal{O}_K)$, where K is a number field, $\mathcal{O}_K$ denotes its ring of algebraic integers, and H is a Hermitian form of signature $(2, 1)$ with coefficients in K. Note that $\mathcal{O}_K$ is usually not discrete in $\mathbb{C}$, so $SU(H, \mathcal{O}_K)$ is usually not discrete in $SU(H)$. Under an additional assumption on the Galois conjugates ${}^{\varphi}H$ of the form (obtained by applying field automorphisms $\varphi \in \mathrm{Gal}(K)$ to the entries of the representative matrix of H), the group $SU(H, \mathcal{O}_K)$ is indeed discrete (see part 1 of Prop. 2.1).

Proposition 2.1 *Let E be a purely imaginary quadratic extension of a totally real field F, and H a Hermitian form of signature (2,1) defined over E.*

1. *$\mathrm{SU}(H; \mathcal{O}_E)$ is a lattice in $SU(H)$ if and only if for all $\varphi \in \mathrm{Gal}(F)$ not inducing the identity on F, the form ${}^{\varphi}H$ is definite. Moreover, in that case, $\mathrm{SU}(H; \mathcal{O}_E)$ is an arithmetic lattice.*

2. *Suppose $\Gamma \subset \mathrm{SU}(H; \mathcal{O}_E)$ is a lattice. Then Γ is arithmetic if and only if $\varphi \in \mathrm{Gal}(F)$ not inducing the identity on F, the form ${}^{\varphi}H$ is definite.*

Part 1 of the Proposition is quite natural (and motivates the formulation of the general definition of arithmeticity). Indeed, it is a general fact that one can embed $\mathcal{O}_K$ discretely into $\mathbb{C}^r$ by

$$x \mapsto (\varphi_1(x), \ldots, \varphi_r(x))$$

where $\varphi_1, \ldots, \varphi_r$ denote the distinct embeddings of K into the complex numbers (up to complex conjugation).

The group $S = \prod_{j=1}^r SU({}^{\varphi_j}H)$ can be checked to be defined over $\mathbb{Q}$ (this is an instance of a general process called *restriction of scalars*). Its integer points correspond to $\prod_{j=1}^r SU({}^{\varphi_j}H, \mathcal{O}_K)$, which is a lattice in $S(\mathbb{R})$ by the theorem of Borel and Harish-Chandra.

Now the key point is that the assumption on the Galois conjugates amounts to saying that the projection

$$\prod_{j=1}^r SU({}^{\varphi_j}H) \to SU({}^{\varphi_1}H)$$

onto the first factor has compact kernel, hence maps discrete sets to discrete sets (compare with Definition 2.1). This implies that $SU(H, \mathcal{O}_K)$ is a lattice in $SU(H)$.

The proof of part 2 of Proposition 2.1 is a bit more sophisticated (see lemma 4.1 of [M1], 12.2.6 of [DM] or Prop. 4.1 of [Pau]). Note that when the group Γ as in the Proposition is non-arithmetic, it necessarily has infinite index in $SU(H, \mathcal{O}_K)$ (which is non-discrete in $SU(H)$).

3 Complex hyperbolic space and its isometries

For the reader's convenience we include a brief summary of key definitions and facts about complex hyperbolic geometry, see [G] for more information.

Let $\langle \cdot, \cdot \rangle$ be a Hermitian form of signature $(n, 1)$ on $\mathbb{C}^{n+1}$, which we can describe in matrix form as

$$\langle v, w \rangle = w^* H v.$$

The unitary group $\mathrm{U}(H)$ is the group of matrices that preserve this inner product, i.e.

$$\mathrm{U}(H) = \{M \in \mathrm{GL}(n+1, \mathbb{C}) : M^* H M = H\}.$$

The signature condition amounts to saying that after an appropriate linear change of coordinates, the Hermitian inner product is the standard Lorentzian Hermitian product

$$-v_0\overline{w}_0 + v_1\overline{w}_1 + \cdots + v_n\overline{w}_n, \tag{3.1}$$

whose unitary group is usually denoted by $\mathrm{U}(n,1)$. For computational purposes, it can be convenient to work with a non-diagonal matrix H (as we do throughout this paper), but of course, under the $(n,1)$ signature assumption, $\mathrm{U}(H)$ is isomorphic to $\mathrm{U}(n,1)$.

As a set, $\mathrm{H}^n_{\mathbb{C}}$ is just the subset of projective space $P^n_{\mathbb{C}}$ corresponding to the set of negative lines in $\mathbb{C}^{n+1}$, i.e. $\mathbb{C}$-lines spanned by a vector $v \in \mathbb{C}^{n+1}$ such that $\langle v, v\rangle < 0$. Working in coordinates where the form is diagonal, any negative line is spanned by a unique vector of the form $(1, v_1, \dots, v_n)$, and negativity translates into

$$|v_1|^2 + \cdots + |v_n|^2 < 1$$

which shows how to describe complex hyperbolic space as the **unit ball** in $\mathbb{C}^n$.

It is often useful to consider the **boundary** of complex hyperbolic space, denoted by $\partial\mathrm{H}^n_{\mathbb{C}}$. This corresponds to the set of null lines, i.e. $\mathbb{C}$-lines spanned by nonzero vectors $v \in \mathbb{C}^{n+1}$ with $\langle v, v\rangle = 0$. In terms of the ball model alluded to in the previous paragraph, the boundary is of course simply the unit sphere in $\mathbb{C}^{n+1}$.

The group $\mathrm{PU}(H)$ clearly acts by biholomorphisms on $\mathrm{H}^n_{\mathbb{C}}$ (the action is effective and transitive), and it turns out that $\mathrm{PU}(H)$ is actually the group of all biholomorphisms of complex hyperbolic space. There is a unique Kähler metric on $\mathrm{H}^n_{\mathbb{C}}$ invariant under the action of $\mathrm{PU}(H)$ (it can be described as the Bergman metric of the ball). We will not need any explicit formula for the metric, all we need is the formula for the distance between two points (this will be enough for the purposes of the present paper). Writing X, Y for negative vectors in $\mathbb{C}^{n+1}$ and x, y for the corresponding $\mathbb{C}$-lines in $\mathrm{H}^n_{\mathbb{C}}$, we have

$$\cosh^2\left(\frac{\rho(x,y)}{2}\right) = \frac{|\langle X, Y\rangle|^2}{\langle X, X\rangle\langle Y, Y\rangle} \tag{3.2}$$

The factor $1/2$ inside the hyperbolic cosine is included for purposes of normalisation only (it ensures that the holomorphic sectional curvature of $\mathrm{H}^n_{\mathbb{C}}$ is -1, rather than just any negative constant).

It is not hard to see that

$$\mathrm{Isom}(\mathrm{H}^n_{\mathbb{C}}) = \mathrm{PU}(n,1) \rtimes \mathbb{Z}/2$$

where the $\mathbb{Z}/2$ factor corresponds to complex conjugation (any involutive antiholomorphic isometry would do).

The usual **classification of isometries** of negatively curved metric spaces, in terms of the analysis of the fixed points in

$$\overline{\mathrm{H}}^n_{\mathbb{C}} = \mathrm{H}^n_{\mathbb{C}} \cup \partial\mathrm{H}^n_{\mathbb{C}},$$

is used throughout in the paper. Any nontrivial $g \in \mathrm{PU}(n,1)$ is of precisely one of the following types:

- *elliptic*: g has a fixed point in $\mathrm{H}^n_{\mathbb{C}}$;
- *parabolic*: g has exactly one fixed point in $\overline{\mathrm{H}}^n_{\mathbb{C}}$, which lies in $\partial\mathrm{H}^n_{\mathbb{C}}$;
- *loxodromic*: g has exactly two fixed points in $\overline{\mathrm{H}}^n_{\mathbb{C}}$, which lie in $\partial\mathrm{H}^n_{\mathbb{C}}$.

In the special case $n = 2$, there is a simple formula involving the trace of a representative $G \in \mathrm{SU}(2,1)$ of $g \in \mathrm{PU}(2,1)$ to determine the type of the isometry g (see [G], p.204).

We will sometimes use a slightly finer classification for elliptic isometries, calling an element **regular elliptic** if any of its representative has pairwise distinct eigenvalues. The eigenvalues of a matrix $A \in \mathrm{U}(n,1)$ representing an elliptic isometry g all have modulus one. Exactly one of these eigenvalues has a eigenvector v with $\langle v, v\rangle < 0$ (the span of v gives a fixed point of g in $\mathrm{H}^n_{\mathbb{C}}$), and such an eigenvalue will be called *of negative type*. Regular elliptic isometries have an isolated fixed point in $\mathrm{H}^n_{\mathbb{C}}$.

Among non-regular elliptic elements, one finds **complex reflections**, whose fixed point sets are totally geodesic copies of $\mathrm{H}^{n-1}_{\mathbb{C}}$ embedded in $\mathrm{H}^n_{\mathbb{C}}$. More specifically, such "complex hyperplanes" can be described by a positive line in $\mathbb{C}^{n+1}$, i.e. a $\mathbb{C}$-line spanned by a vector v with $\langle v, v\rangle > 0$. Given such a vector, the set of $\mathbb{C}$-lines contained in

$$v^\perp = \left\{w \in \mathbb{C}^{n+1} : \langle v, w\rangle = 0\right\}$$

intersects $\mathrm{H}^n_{\mathbb{C}}$ in a copy of $\mathrm{H}^{n-1}_{\mathbb{C}}$. The point in projective space corresponding to v is called *polar* to the hyperplane determined by $v^{\perp}$. In terms of the ball model, these copies of $\mathrm{H}^{n-1}_{\mathbb{C}}$ simply correspond to the intersection with the unit ball of affine hyperplanes in $\mathbb{C}^n$. If v is a positive vector, any isometry of $\mathrm{H}^n_{\mathbb{C}}$ fixing the lines in $v^{\perp}$, can be described as

$$x \mapsto x + (\zeta - 1)\frac{\langle x, v\rangle}{\langle v, v\rangle}v$$

for some $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. The corresponding isometry is called a complex reflection, ζ is called its multiplier, and the argument of ζ is referred to as the rotation angle of the complex reflection.

Note that the respective positions of two complex hyperplanes are easily read off in terms of their polar vectors. Indeed, we have the following (see [G], p.100).

Lemma 3.1 *Let v_1, v_2 be positive vectors in $\mathbb{C}^{n+1}$, and let L_1, L_2 denote the corresponding complex hyperplanes in $\mathrm{H}^n_{\mathbb{C}}$. Let*

$$C = \frac{|\langle v_1, v_2\rangle|^2}{\langle v_1, v_1\rangle\langle v_2, v_2\rangle}.$$

(1) L_1 and L_2 intersect in $\mathrm{H}^n_{\mathbb{C}}$ $\iff$ $C < 1$. In that case the angle θ between L_1 and L_2 satisfies $\cos\theta = C$.

(2) L_1 and L_2 intersect in $\partial\mathrm{H}^n_{\mathbb{C}}$ $\iff$ $C = 1$.

(3) L_1 and L_2 are ultraparallel $\iff$ $C > 1$. In that case the distance ρ between L_1 and L_2 satisfies $\cosh\frac{\rho}{2} = C$.

Lemma 3.1 will be used to get the discreteness test in section 9 (the complex hyperbolic Jørgensen's inequality established in [JKP]).

Parabolic isometries are either **unipotent** or **screw parabolic**; in the former case they are also called **Heisenberg translations** (because the group of unipotent isometries fixing a given point on $\partial\mathrm{H}^n_{\mathbb{C}}$ is isomorphic to the Heisenberg group $\mathcal{H}^{2n-1}$). There are two conjugacy classes of Heisenberg translations, the **vertical translations** (corresponding to the centre, or commutator subgroup, of the Heisenberg group) and the **non-vertical translations** (see [G] for more details on this discussion).

4 Sporadic groups

In this section we setup some notation and recall the main results from [ParPau] and [Pau].

Definition 4.1 *A* symmetric triangle group *is a group generated by two elements R_1, $J \in SU(2,1)$ where R_1 is a complex reflection of order p and J is a regular elliptic isometry J of order 3.*

The reason we call this a triangle group is that it is a subgroup of index at most three in the group generated by three complex reflections R_1, R_2 and R_3, defined by

$$R_2 = JR_1J^{-1}, \quad R_3 = JR_2J^{-1}, \tag{4.1}$$

and we think of their three mirrors as describing a "triangle" of complex lines (however the mirrors of the various R_j need not intersect in general).

The basic observation is that symmetric triangle groups can be parameterised up to conjugacy by the order of p of R_1 and

$$\tau = \mathrm{Tr}(R_1J).$$

We denote by $\psi = 2\pi/p$ the rotation angle of R_1, and by

$$\Gamma(\psi, \tau)$$

the group generated by a complex reflection R_1, J as above.

The generators for this group can be described explicitly by matrices of the form

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{4.2}$$

$$R_1 = \begin{bmatrix} e^{2i\psi/3} & \tau & -e^{i\psi/3}\overline{\tau} \\ 0 & e^{-i\psi/3} & 0 \\ 0 & 0 & e^{-i\psi/3} \end{bmatrix} \tag{4.3}$$

These preserve the Hermitian form $\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* H_\tau \mathbf{z}$ where

$$H_\tau = \begin{bmatrix} 2\sin(\psi/2) & -ie^{-i\psi/6}\tau & ie^{i\psi/6}\overline{\tau} \\ ie^{i\psi/6}\overline{\tau} & 2\sin(\psi/2) & -ie^{-i\psi/6}\tau \\ -ie^{-i\psi/6}\tau & ie^{i\psi/6}\overline{\tau} & 2\sin(\psi/2) \end{bmatrix}. \tag{4.4}$$

The above matrices always generate a subgroup Γ of $\mathrm{GL}(3,\mathbb{C})$, but the signature of H_τ depends on the values of ψ and τ. For any fixed value of ψ, the parameter space for τ is described in Sections 2.4 and 2.6 of [ParPau].

Definition 4.2 *The symmetric triangle group generated by R_1 and J as in* (4.2) *and* (4.3) *is called* hyperbolic *is H_τ has signature* $(2,1)$.

In order to get a tractable class of groups, we shall assume that R_1J is elliptic, and that $R_1R_2 = R_1JR_1J^{-1}$ is either elliptic or parabolic. The motivation for this condition is explained in [Par3], [ParPau] (it is quite natural in the context of the search for *lattices*, rather than discrete groups of possibly infinite covolume).

A basic necessary condition for a subgroup of $\mathrm{PU}(2,1)$ to be discrete is that all its elliptic elements must have finite order, hence we make the following definition.

Definition 4.3 *A symmetric triangle group is called* doubly elliptic *if R_1J is elliptic of finite order and $R_1R_2 = R_1JR_1J^{-1}$ is either elliptic of finite order or parabolic.*

The list of parameters that yield double elliptic triangle groups was obtained in [Par3] (see also [ParPau]), by using a result of Conway and Jones on sums of roots of unity. We recall the result in the following.

Theorem 4.1 *Let Γ be a symmetric triangle group such that R_1J is elliptic and R_1R_2 is either elliptic or parabolic. If R_1J and R_1R_2 have finite order (or are parabolic), then one of the following holds:*

- *Γ is one of Mostow's lattices ($\tau = e^{i\phi}$ for some ϕ).*
- *Γ is a subgroup of one of Mostow's lattices ($\tau = e^{2i\phi} + e^{-i\phi}$ for some ϕ).*
- *Γ is one of the sporadic triangle groups, i.e $\tau \in \{\sigma_1, \overline{\sigma}_1, ..., \sigma_9, \overline{\sigma}_9\}$ where the σ_j are given in Table 1.*

$\sigma_1 = e^{i\pi/3} + e^{-i\pi/6}\,2\cos(\pi/4)$	$\sigma_2 = e^{i\pi/3} + e^{-i\pi/6}\,2\cos(\pi/5)$	$\sigma_3 = e^{i\pi/3} + e^{-i\pi/6}\,2\cos(2\pi/5)$
$\sigma_4 = e^{2\pi i/7} + e^{4\pi i/7} + e^{8\pi i/7}$	$\sigma_5 = e^{2\pi i/9} + e^{-i\pi/9}\,2\cos(2\pi/5)$	$\sigma_6 = e^{2\pi i/9} + e^{-\pi i/9}\,2\cos(4\pi/5)$
$\sigma_7 = e^{2\pi i/9} + e^{-i\pi/9}\,2\cos(2\pi/7)$	$\sigma_8 = e^{2\pi i/9} + e^{-i\pi/9}\,2\cos(4\pi/7)$	$\sigma_9 = e^{2\pi i/9} + e^{-i\pi/9}\,2\cos(6\pi/7)$

Table 1: The 18 sporadic values are given by σ_j or $\overline{\sigma}_j$, $j = 1, \ldots, 9$. They correspond to isolated values of the parameter τ for which any $\Gamma(\frac{2\pi}{p}, \tau)$ is doubly elliptic, i.e. R_1R_2 and R_1J are either parabolic or elliptic of finite order.

Therefore, for each value of $p \geqslant 3$, we have a finite number of new groups to study, the $\Gamma(2\pi/p, \sigma_i)$ and $\Gamma(2\pi/p, \overline{\sigma_i})$ which are hyperbolic. The list of sporadic groups that are hyperbolic is given in the table of Section 3.3 of [ParPau] (and we give them below in Table 6); for the sake of brevity we only recall the following:

Proposition 4.1 *For $p \geqslant 4$ and $\tau = \sigma_1, \sigma_2, \sigma_3, \overline{\sigma}_4, \sigma_5, \sigma_6, \sigma_7, \overline{\sigma}_8$ or σ_9, $\Gamma(2\pi/p, \tau)$ is hyperbolic.*

It was shown in [ParPau] that some of the hyperbolic sporadic groups are non-discrete (see Corollary 4.2, Proposition 4.5 and Corollary 6.4 of [ParPau]), essentially by using the lists of discrete triangle groups on the sphere, the Euclidean plane and the hyperbolic plane (this list is due to Schwarz in the spherical case, and to Knapp in the hyperbolic case). For the convenience of the reader, we recall the main non-discreteness results from [ParPau] in the following:

Proposition 4.2 *For $p \geqslant 3$ and (τ or $\overline{\tau} = \sigma_3, \sigma_8$ or σ_9), $\Gamma(2\pi/p, \tau)$ is not discrete. Also, for $p \geqslant 3$, $p \neq 5$ and (τ or $\overline{\tau} = \sigma_6$), $\Gamma(2\pi/p, \tau)$ is not discrete.*

The new non-discreteness results contained in section 9 push the same idea much further, by a series of technical algebraic manipulations (in some places we use Jørgensen's inequality and a complex hyperbolic version of Shimizu's lemma due to the second author, see Theorem 9.5);

The main results of [Pau] are the following two statements. The first result was obtained by applying the arithmeticity criterion from Proposition 2.1. The second result was obtained by finding a commensurability invariant which distinguishes the various groups Γ, namely the field $\mathbb{Q}[\mathrm{TrAd}\Gamma]$ (the trace field of the adjoint representation of Γ).

Theorem 4.2 *Let $p \geqslant 3$ and $\tau \in \{\sigma_1, \overline{\sigma}_1, ..., \sigma_9, \overline{\sigma}_9\}$, and suppose that the triangle group $\Gamma(2\pi/p, \tau)$ is hyperbolic, and that it is a lattice in $SU(H_\tau)$. Then $\Gamma(2\pi/p, \tau)$ is arithmetic if and only if $p = 3$ and $\tau = \overline{\sigma}_4$.*

Theorem 4.3 *The sporadic groups $\Gamma(2\pi/p, \tau)$ ($p \geqslant 3$ and $\tau \in \{\sigma_1, \overline{\sigma}_1, ..., \sigma_9, \overline{\sigma}_9\}$) fall into infinitely many distinct commensurability classes. Moreover, they are not commensurable to any Picard or Mostow lattice, except possibly when:*

- $p = 4$ or 6
- $p = 3$ and $\tau = \sigma_7$
- $p = 5$ and τ or $\overline{\tau} = \sigma_1, \sigma_2$
- $p = 7$ and $\tau = \overline{\sigma}_4$
- $p = 8$ and $\tau = \sigma_1$
- $p = 10$ and $\tau = \sigma_1, \sigma_2, \overline{\sigma}_2$
- $p = 12$ and $\tau = \sigma_1, \sigma_7$
- $p = 20$ and $\tau = \sigma_1, \sigma_2$
- $p = 24$ and $\tau = \sigma_1$

5 Dirichlet domains

Given a subgroup Γ of $\mathrm{PU}(2, 1)$, the Dirichlet domain for Γ centred at p_0 is the set:

$$F_\Gamma = \left\{ x \in \mathrm{H}^2_{\mathbb{C}} : d(x, p_0) \leqslant d(x, \gamma p_0), \forall \gamma \in \Gamma \right\}.$$

A basic fact is that Γ is discrete if and only if F_Γ has nonempty interior, and in that case F_Γ is a fundamental domain for Γ modulo the action of the (finite) stabiliser of p_0 in Γ.

The simplicity of this general notion, and its somewhat canonical nature (it only depends on the choice of the centre p_0), make Dirichlet domains convenient to use in computer investigation as in [M1], [Ri], [De1] and [De2]. Note however that there is no algorithm to decide whether the set F_Γ has non-empty interior, and the procedure we describe below may never end (this is already the case in the constant curvature setting, i.e. in real hyperbolic space of dimension at least 3, see for instance [EP]).

Our computer search is quite a bit more delicate than the search for fundamental domains in the setting of arithmetic groups. The recent announcement that Cartwright and Steger have been able to find presentations for the fundamental groups of all so-called *fake projective planes* mentions the use of massive computer calculations in the same vein as our work (see [CS]), but there are major differences however.

They use Dirichlet domains, but their task is facilitated by the fact that the fundamental groups of fake projective planes are known to be *arithmetic* subgroups of $\mathrm{PU}(2, 1)$ (see [Kl] and [Y]). In particular, all the groups they consider are known to be discrete *a priori* (which is certainly not the case for most complex hyperbolic sporadic groups). Cartwright and Steger also use the knowledge of the volumes of the corresponding fundamental domains (the list of arithmetic lattices that could possibly contain the fundamental group of a fake projective plane is brought down to a finite list by using Prasad's volume formula [Pra]). This allows one to check whether a partial Dirichlet domain

$$F_W = \left\{ x \in \mathrm{H}^2_{\mathbb{C}} : d(x, p_0) \leqslant d(x, \gamma p_0), \forall \gamma \in W \right\}$$

determined by a given finite set $W \subset \Gamma$ is actually equal to F_Γ.

For an arbitrary discrete subgroup $\Gamma \subset \mathrm{PU}(2,1)$ and an arbitrary choice of the centre p_0, the set F_Γ is a polyhedron bounded by *bisectors* (see [M1] and [G]), but it may have infinitely many faces, even if Γ is geometrically finite (see [B]).

Moreover, the combinatorics of Dirichlet domains tend to be unnecessarily complicated, and one usually expects that simpler fundamental domains can be obtained by suitable clever geometric constructions. This general idea is illustrated by Dirichlet domains for lattices in $\mathbb{R}^2$: when the group is not a rectangular lattice, i.e. not generated by two translations along orthogonal axes, the Dirichlet domain centred at any point is a hexagon (rather than a parallelogram).

In $\mathrm{H}^2_{\mathbb{C}}$, Dirichlet domains typically contain digons (pairs of vertices connected by distinct edges), see Figure 1. In particular the 1-skeleton is not piecewise totally geodesic. One can also check that the 2-faces of a Dirichlet domain can never be contained in a totally real totally geodesic copy of $\mathrm{H}^2_{\mathbb{R}}$, which makes this notion a little bit unnatural (this was part of the motivation behind the constructions of [DFP], where fundamental domains with simpler combinatorics that those in [M1] were obtained).

6 Experimental results

6.1 The G-procedure

In order to sift through the complex hyperbolic sporadic groups, we have run the procedures explained in [De1] and [De2] in order to explore the Dirichlet domains centred at the centre of mass of the mirrors of the three generating reflexions.

In terms of the notation in Section 4, we take p_0 to be the unique fixed point in $\mathrm{H}^2_{\mathbb{C}}$ of the regular elliptic element J (this point is given either by $(1,1,1)$, $(1,\omega,\overline{\omega})$ or $(1,\overline{\omega},\omega)$ for $\omega = (-1+i\sqrt{3})/2$, depending on the parameters p and τ).

We start with the generating set $W_0 = \{R_1^{\pm 1}, R_2^{\pm 1}, R_3^{\pm 1}\}$ for Γ, and construct an increasing sequence of sets $W_0 \subset W_1 \subset W_2 \subset \dots$ by the G-procedure (named after G. Giraud, see [De1] for the explanation of this terminology).

First define a *G-step* of the procedure by:

$$G(W) = W \cup \{\alpha^{-1}\beta : \alpha, \beta \in W \text{ yield a non-empty generic 2-face of } F_W\}$$

Here "yielding a non-empty 2-face of F_W" means that the set of points of F_W that are equidistant from p_0, αp_0 and βp_0 has dimension two (i.e. it has non-empty interior in the corresponding intersection of two bisectors). "Generic" means that this 2-face is not contained in a complex geodesic (see [De1]).

Definition 6.1 *The set W is called G-closed if $G(W) = W$.*

The sequence W_k is defined inductively by

$$W_{k+1} = G(W_k).$$

The hope is that this sequence stabilises to a G-closed set $W = W_N$ after a finite number of steps. In particular, this procedure is probably suitable only for the search for lattices (not for discrete groups with infinite covolume).

6.2 Issues of precision

The determination of the sequence of sets W_k described in Section 6.1 depends on being able to determine the precise list of all nonempty 2-faces of the polyhedron W, for a given finite set $W \subset \Gamma$. The difficult part is to prove that two bisectors really yield a subset of F_W of dimension smaller than 2, when they appear to do so numerically.

Recall that the polyhedron F_W is described by a (possibly large) set of quadratic inequalities in 4 variables (the real and imaginary parts of the ball coordinates, for instance), where the coefficients of the quadratic polynomials are obtained from matrices which are possibly very long words in the generators R_1, R_2, R_3.

The computation of these matrices can be done without loss of precision, since it can be reduced to arithmetic in the relevant number field (see Section 2.5 of [ParPau]).

It is not clear how to solve the corresponding system of quadratic inequalities. In order to save computational time, and for the lack of having better methods, we have chosen to do all the computations numerically, with a fixed (somewhat rough) precision, essentially the same way as described in [De2]. We now briefly summarise what our computer program does.

For a given (coequidistant) bisector intersection B, we need a method to test whether $B \cap F_W$ has dimension two. In order to do this, we work in spinal coordinates (see [De1]), and fit the disk B into a rectangular $N \times N$ grid. The 2-face is declared non-empty whenever we find more than one point in a given horizontal and in a given vertical line in the grid. For the default version of the program, we take $N = 1000$.

In particular, the above description suggests that whenever the polyhedron F_W becomes small enough, our program will not find any 2-face whatsoever. If this happens at some stage k, the program will consider W_k as being G-closed and stop.

When fed a group that has infinite covolume, one expects that the program would often run forever, since in that case Dirichlet domains have tend to infinitely many faces. In practice, after a certain number of steps, the sets W_k are too large for the computer's capacity, and the program will crash.

For the groups we have tested (namely all sporadic groups with $p \leqslant 24$), we have found these three behaviours:

A: The program finds a G-closed set $W_N = G(W_N)$, and the set of numerically non-empty two-faces is non-empty.

B: The program finds a set W_N for which it does not find any nonempty 2-face whatsoever (in particular W_N is Giraud closed, so the program stops).

C: The program exceeds its capacity in memory and crashes.

As a working hypothesis, we shall interpret Behaviour B as meaning that the group is not discrete, and Behaviour C meaning that the group has infinite covolume (the latter behaviour is of course also conceivable when the group is actually not discrete, or when we make a bad choice of the centre of the Dirichlet domain).

6.3 Census of sporadic groups generated by reflections of small order

The computer program available on the first author's webpage at

`http://www-fourier.ujf-grenoble.fr/~deraux/java`

was run for all sporadic groups (see Section 4) with $2 \leqslant p \leqslant 24$.

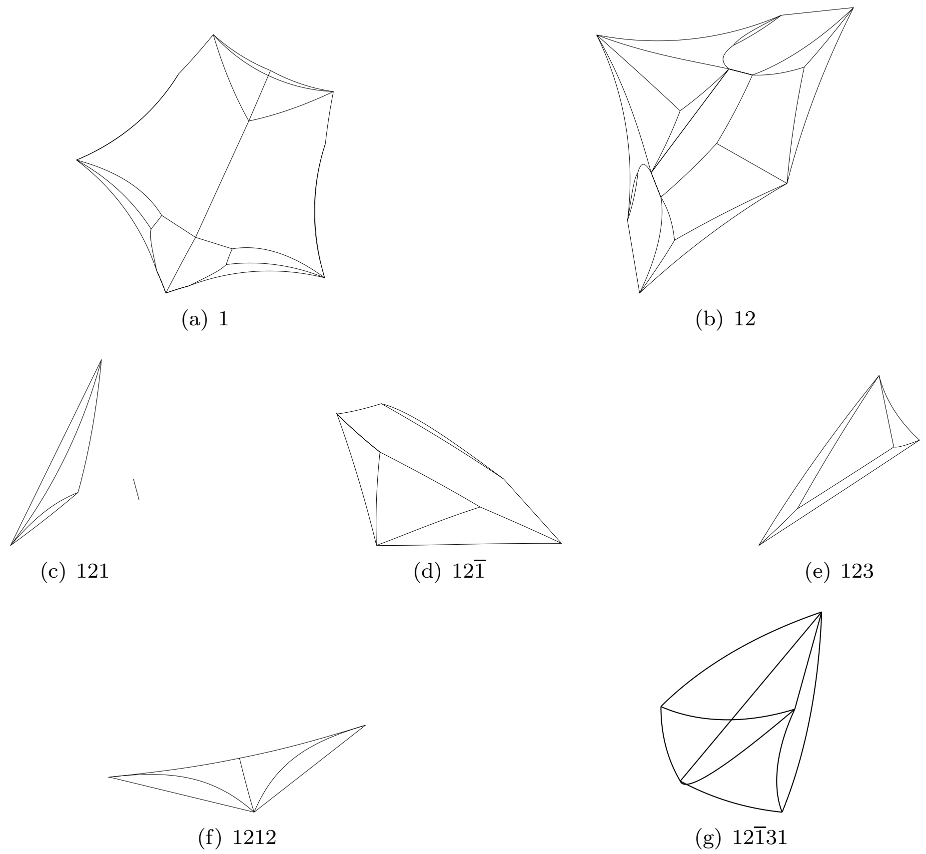
The groups with $p = 2$ were analysed by Parker in [Par3], and our program confirms his results; in that case τ and $\overline{\tau}$ give the same groups, and only $\tau = \sigma_5$ or σ_7 appear to be discrete. Both exhibit Behaviour A, but the first one gives a compact polyhedron; as mentioned in the introduction, this lattice is actually the same as the $(4, 4, 4; 5)$-triangle group, i.e. the group that is studied in [De2], see [Par3] and [Sc]. The Giraud-closed polyhedron obtained for σ_7 has infinite volume.

For $3 \leqslant p \leqslant 24$, there are few groups that exhibit Behaviour A (as defined in Section 6.2), namely: all groups with $\tau = \overline{\sigma}_4$, those with $\tau = \sigma_1$, $p = 3, 4, 5, 6$, and finally those with $\tau = \sigma_5$, $p = 3, 4$ or 5.

Pictures of the (isometry classes of) 3-faces of the Dirichlet domain for $\Gamma(2\pi/3, \overline{\sigma}_4)$ are given in Figure 1. We chose to display the faces for that specific group because its combinatorics are particularly simple among all sporadic groups (Dirichlet domains for sporadic lattices can have about a hundred faces).

In case of Behaviour A, the program provides a list of faces for the polyhedron F_W, and checks whether it has side-pairings in the sense of the Poincaré polyhedron theorem (once again, we choose to check this only numerically). There is a minor issue of ambiguity between the side pairings, due to the fact that most groups $\Gamma(\frac{2\pi}{p}, \tau)$ actually contain J, which means that the centre of the Dirichlet domain has non-trivial stabiliser. Possibly after adjusting the side-pairings by pre-composing them with J or J^{-1}, all the groups exhibiting Behaviour A turn out to have side-pairings (or at least they appear to, numerically). Another way to take care of the issue of non-trivial stabiliser for the centre of the Dirichlet domain is of course simply to change the centre (within reasonably small distance to the centre of mass of the mirrors, since we want the side-pairings obtained from the Dirichlet domain to be related in simple terms to the original generating reflections).

In either case, either after adjusting the side-pairings by elements of the stabiliser, or after changing the centre, we are in a position to check the cycle conditions of the Poincaré polyhedron theorem. The general



(a) 1 (b) 12

(c) 121 (d) $12\overline{1}$ (e) 123

(f) 1212 (g) $12\overline{1}31$

Figure 1: Faces of the Dirichlet domain for $\Gamma(2\pi/3, \overline{\sigma}_4)$, drawn in spinal coordinates.

philosophy that grew out of [De1] (see also [M1], or even [Pic]) is that the only cycle conditions that need to be checked are those for complex totally geodesic 2-faces, where the cycle transformations are simply complex reflections. Our program goes through all these complex 2-faces, and computes the rotation angle of the cycle transformations (as well as the total angle inside the polyhedron along the cycle).

Table 2 gives the list of sporadic groups that exhibit Behaviour A and whose complex cycles rotate by an integer part of 2π (for $\tau = \overline{\sigma}_4$, $p = 8$, one needs to use a centre for the Dirichlet domain other than the centre of mass of the mirrors of the three reflections).

τ	p
σ_1	$p = 3, 4, 6$
$\overline{\sigma}_4$	$p = 3, 4, 5, 6, 8, 12$
σ_5	$p = 3, 4$

Table 2: Sporadic groups with $3 \leqslant p \leqslant 24$ whose Dirichlet domain satisfies the hypotheses of the Poincaré polyhedron theorem, at least numerically.

For groups that exhibit behaviour A but whose cycle transformations rotate by angles that are not integer parts of 2π, all one can quickly say is that the G-closed polyhedron cannot be a fundamental domain for their action (even modulo the stabiliser of p_0), but the group may still be a lattice. This issue is related to the question of whether the integrality condition of [DM] is close to being necessary and sufficient for the corresponding reflection group to be a lattice (see the analysis in [M3]).

There is a natural refinement of the procedure described in Section 6.1 to handle this case. Suppose a given cycle transformation g rotates by an angle α, and $2\pi/\alpha$ is not an integer. If that number is not rational, the group is not discrete (the irrationality can of course be difficult to actually prove). If $\alpha = 2\pi m/n$ for $m, n \in \mathbb{Z}$, then some power $h = g^k$ rotates by an angle $2\pi/n$, and it is natural to replace the G-closed set of group elements W by

$$W \cup hWh^{-1}. \tag{6.1}$$

One then starts over with the G-procedure as described in Section 6.1, starting from $W_0 = W \cup hWh^{-1}$.

The groups with problematic rotation angles are

$$\Gamma\left(\frac{2\pi}{5}, \sigma_1\right), \Gamma\left(\frac{2\pi}{5}, \sigma_5\right),$$

and all groups with $\tau = \overline{\sigma}_4$, $p \neq 3, 4, 5, 6, 8, 12$. The ones with $\tau = \overline{\sigma}_4$ are known to be non-discrete, see Theorem 9.1. The groups $\Gamma(\frac{2\pi}{5}, \sigma_1)$ and $\Gamma(\frac{2\pi}{5}, \sigma_5)$ do not seem to be discrete. Indeed, their Giraud-closed sets have problematic rotation angles, see Table 3. In both cases, after implementing the refinement of (6.1), the G-procedure exhibits Behaviour B.

Group	cycle transformation	angle
$\Gamma(\frac{2\pi}{5}, \sigma_1)$	$(R_1R_2)^2$	$4\pi/5$
$\Gamma(\frac{2\pi}{5}, \sigma_5)$	$((R_1J)^5R_2^{-1})^2$	$4\pi/15$

Table 3: Some problematic rotation angles in Giraud-closed polyhedra.

7 Group presentations

From the geometry of the Dirichlet domains for sporadic lattices, one can infer explicit group presentations. Indeed one knows that the side-pairings generate the group, and the relations are normally generated by the cycle transformations, see [EP] for instance.

Given that there are many faces, it is of course quite prohibitive to write down such a presentation by hand. It is reasonably easy however to have a computer do this. Our program produces files that can be passed to GAP in order to simplify the presentations (it is quite painful, even though not impossible, to do these simplifications by hand). It turns out that the presentations coming from the Dirichlet domains can all be reduced to quite a simple form (see Table 4).

Note that the results of this section are just as conjectural as the statement of Conjecture 1.1, since they depend on the accuracy of the combinatorics of the Dirichlet domains.

8 Description of the cusps of the non-compact examples

The geometry of the Dirichlet domains for sporadic lattices gives information about the isotropy groups of any vertex. Rather than giving a whole list, we gather information about the cusps in the Dirichlet domain and in $M = \Gamma \setminus \mathrm{H}^2_{\mathbb{C}}$, by giving the number of cusps, as well as generators and relations for their stabilisers (see Table 5).

Once again, the results of this section are conjectural (they depend on the accuracy of the combinatorics of the Dirichlet domains).

9 Non-discreteness results

In this section we prove some restrictions on the parameters for the group $\Gamma(2\pi/p, \tau)$ to be discrete, aiming to show the optimality of the statement of Conjecture 1.1. More specifically, we will show the following.

Theorem 9.1 *Only finitely many of the sporadic triangle groups are discrete. More precisely:*

- *For* $p \geqslant 7$, $\Gamma(\frac{2\pi}{p}, \sigma_1)$ *is not discrete.*
- *For* $p = 3, 5, 6, 7$, $\Gamma(\frac{2\pi}{p}, \overline{\sigma}_1)$ *is not discrete.*
- *For* $p \geqslant 6$, $\Gamma(\frac{2\pi}{p}, \sigma_2)$ *is not discrete.*
- *For* $6 \leqslant p \leqslant 19$, $\Gamma(\frac{2\pi}{p}, \overline{\sigma}_2)$ *is not discrete.*
- *For* $p = 4, 5, 6$, $\Gamma(\frac{2\pi}{p}, \sigma_4)$ *is not discrete.*
- *For* $p \neq 2, 3, 4, 5, 6, 8, 12$, $\Gamma(\frac{2\pi}{p}, \overline{\sigma}_4)$ *is not discrete.*
- *For* $p \neq 2, 3, 4, 5, 6, 8, 12$, $\Gamma(\frac{2\pi}{p}, \sigma_5)$ *is not discrete.*
- $\Gamma(\frac{2\pi}{4}, \overline{\sigma}_5)$ *is not discrete.*
- $\Gamma(\frac{2\pi}{5}, \sigma_6)$ *and* $\Gamma(\frac{2\pi}{5}, \overline{\sigma}_6)$ *are not discrete.*
- *For* $p \neq 2, 3, 4, 7, 14$, $\Gamma(\frac{2\pi}{p}, \sigma_7)$ *is not discrete.*

The proofs are slightly different for each part of the statement, as detailed in Table 6. Since all of them are based either on Knapp's theorem or on Jørgensen's inequality, we shall briefly review these results in section 9.1.

9.1 Knapp, Jørgensen and Shimizu

Knapp's theorem gives a necessary and sufficient condition for a two-generator subgroup of $\mathrm{PU}(1,1)$ to be discrete, assuming both generators as well as their product are elliptic. The reference for Knapp's theorem is [Kna], see also [KS]. The full list of possible rotation angles for A, B and AB will not be needed here. In fact we shall only use the following special case of Knapp's theorem, that applies to isosceles triangles.

Theorem 9.2 *(Knapp) Consider a triangle in* $\mathrm{H}^2_{\mathbb{R}}$ *with angles* α, α, β, *and let* Δ *be the group generated by the reflections in its sides. If* Δ *is discrete then one of the following holds:*

- $\alpha = \frac{\pi}{q}$ *and* ($\beta = \frac{2\pi}{r}$ *or* $\frac{4\pi}{q}$) *with* $q, r \in \mathbb{N}^*$
- $\alpha = \frac{2\pi}{r}$ *and* $\beta = \frac{2\pi}{r}$ *with* $r \in \mathbb{N}^*$.

Remark 9.3 In a few cases, we also use the spherical version of Knapp's theorem, which is a result of Schwarz (see [ParPau]).

$$\Gamma(\tfrac{2\pi}{3},\sigma_1): \quad \begin{array}{c} J = 12312312 = 23123123 = 31231231; \\ 1^3 = Id; \quad (123)^8 = Id; \\ (12)^3 = (21)^3; \quad [1(23\overline{2})]^2 = [(23\overline{2})1]^2; \quad 1(232\overline{32})1 = (232\overline{32})1(232\overline{32}). \end{array}$$

$$\Gamma(\tfrac{2\pi}{4},\sigma_1): \quad \begin{array}{c} J = 12312312 = 23123123 = 31231231; \\ 1^4 = Id; \quad (123)^8 = Id; \quad (12)^{12}; \\ (12)^3 = (21)^3; \quad [1(23\overline{2})]^2 = [(23\overline{2})1]^2; \quad 1(232\overline{32})1 = (232\overline{32})1(232\overline{32}). \end{array}$$

$$\Gamma(\tfrac{2\pi}{6},\sigma_1): \quad \begin{array}{c} J = 12312312 = 23123123 = 31231231; \\ 1^6 = Id; \quad (123)^8 = Id; \quad (12)^6; \quad [1(23\overline{2})]^{12} = Id; \\ (12)^3 = (21)^3; \quad [1(23\overline{2})]^2 = [(23\overline{2})1]^2; \quad 1(232\overline{32})1 = (232\overline{32})1(232\overline{32}); \end{array}$$

$$\Gamma(\tfrac{2\pi}{3},\overline{\sigma}_4): \quad \begin{array}{c} J^{-1} = 1231231 = 2312312 = 3123123; \\ 1^3 = Id; \quad (123)^7 = Id; \\ (12)^2 = (21)^2; \end{array}$$

$$\Gamma(\tfrac{2\pi}{4},\overline{\sigma}_4): \quad \begin{array}{c} J^{-1} = 1231231 = 2312312 = 3123123; \\ 1^4 = Id; \quad (123)^7 = Id; \\ (12)^2 = (21)^2; \end{array}$$

$$\Gamma(\tfrac{2\pi}{5},\overline{\sigma}_4): \quad \begin{array}{c} J^{-1} = 1231231 = 2312312 = 3123123; \\ 1^5 = Id; \quad (123)^7 = Id; \quad (12)^{20}; \\ (12)^2 = (21)^2; \end{array}$$

$$\Gamma(\tfrac{2\pi}{6},\overline{\sigma}_4): \quad \begin{array}{c} J^{-1} = 1231231 = 2312312 = 3123123; \\ 1^6 = Id; \quad (123)^7 = Id; \quad (12)^{12}; \\ (12)^2 = (21)^2; \end{array}$$

$$\Gamma(\tfrac{2\pi}{8},\overline{\sigma}_4): \quad \begin{array}{c} J^{-1} = 1231231 = 2312312 = 3123123; \\ 1^8 = Id; \quad (123)^7 = Id; \quad (12)^8; \quad [1(23\overline{2})]^{24}; \\ (12)^2 = (21)^2; \end{array}$$

$$\Gamma(\tfrac{2\pi}{12},\overline{\sigma}_4): \quad \begin{array}{c} J^{-1} = 1231231 = 2312312 = 3123123; \\ 1^{12} = Id; \quad (123)^7 = Id; \quad (12)^6; \quad [1(23\overline{2})]^{12}; \\ (12)^2 = (21)^2; \end{array}$$

$$\Gamma(\tfrac{2\pi}{3},\sigma_5): \quad \begin{array}{c} J^3 = Id; \quad J1J^{-1} = 2; \quad J2J^{-1} = 3; \quad J3J^{-1} = 1; \\ 1^3 = Id; \quad (123)^{10}; \\ (12)^2 = (21)^2; \quad 1(23\overline{2})1(23\overline{2})1 = (23\overline{2})1(23\overline{2})1(23\overline{2}). \end{array}$$

$$\Gamma(\tfrac{2\pi}{4},\sigma_5): \quad \begin{array}{c} J^3 = Id; \quad J1J^{-1} = 2; \quad J2J^{-1} = 3; \quad J3J^{-1} = 1; \\ 1^4 = Id; \quad (123)^{10}; \quad (1\overline{3}23\overline{1}23\overline{2})^{12}; \\ (12)^2 = (21)^2; \quad 1(23\overline{2})1(23\overline{2})1 = (23\overline{2})1(23\overline{2})1(23\overline{2}). \end{array}$$

Table 4: Conjectural presentations for the groups that appear in Conjecture 1.1. The groups with $\tau = \sigma_1, \overline{\sigma}_4$ are generated by R_1, R_2 and R_3, that is J can be expressed as a product of the R_j's. For $\tau = \sigma_5$ this is not the case, and $\langle R_1, R_2, R_3 \rangle$ has index 3 in $\langle J, R_1 \rangle$.

p	τ	# cusps	# cusps in M	Generators	Relations	
3	σ_1	3	1	1, 2	$1^3 = 2^3 = Id,$	$(12)^3 = (21)^3$
4	σ_1	6	1	1, $23\overline{2}$	$1^4 = (23\overline{2})^4 = Id,$	$[1(23\overline{2})]^2 = [(23\overline{2})1]^2$
6	σ_1	6	2	1, $232\overline{3}\,\overline{2}$	$1^6 = (232\overline{32})^6 = Id,$	$1(232\overline{32})1 = (232\overline{32})1(232\overline{32})$
				1, $\overline{3}\,\overline{2}323$	$1^6 = (\overline{32}323)^6 = Id,$	$1(\overline{32}323)1 = (\overline{32}323)1(\overline{32}323)$
4	$\overline{\sigma}_4$	3	1	1, 2	$1^4 = 2^4 = Id,$	$(12)^2 = (21)^2$
6	$\overline{\sigma}_4$	6	1	1, $23\overline{2}$	$1^6 = (23\overline{2})^6 = Id,$	$1(23\overline{2})1 = (23\overline{2})1(23\overline{2})$
3	σ_5	3	1	$23\overline{2}$, $(1J)^5$	$(23\overline{2})^3 = [(IJ)^5]^6 = Id,$	$[(23\overline{2})(1J)^{-5}]^2 = [(1J)^{-5}(23\overline{2})]^2$
4	σ_5	3	1	1, 2	$1^4 = 2^4 = Id,$	$(12)^2 = (21)^2$

Table 5: Conjectural list of cusps for the non-cocompact examples from Conjecture 1.1. All of the relations follow directly from the conjectural presentations given in Table 4; some follow directly but others with slightly more work.

τ	p with H_τ hyperbolic	p where non-discrete	Result used
σ_1	$[3, \infty)$	$7, 8, [10, \infty)$	Proposition 9.4
		9	Proposition 9.9
$\overline{\sigma}_1$	$[3, 7]$	$5, 7$	Proposition 9.5
		$3, 6, 7$	Proposition 9.10
σ_2	$[3, \infty)$	$[6, 9], [11, \infty)$	Proposition 9.6
		10	Proposition 9.11
$\overline{\sigma}_2$	$[3, 19]$	$[6, 9], [11, 19]$	Proposition 9.7
		10	Proposition 9.11
σ_3	$[3, \infty)$	$[3, \infty)$	Proposition 4.5 of [ParPau]
$\overline{\sigma}_3$	$[3, 6]$	$[3, 6]$	Proposition 4.5 of [ParPau]
σ_4	$[4, 6]$	$[4, 6]$	Proposition 9.3
$\overline{\sigma}_4$	$[3, \infty)$	$7, [9, 11], [13, \infty)$	Proposition 9.1
σ_5	$[2, \infty)$	$7, [9, 11], [13, \infty$	Proposition 9.2
$\overline{\sigma}_5$	$\{2,\ 4\}$	4	Proposition 9.3
σ_6	$[3, \infty)$	$3, 4, [6\infty)$	Proposition 4.5 of [ParPau]
		5	Proposition 9.3
$\overline{\sigma}_6$	$[3, 29]$	$3, 4, [6\infty)$	Proposition 4.5 of [ParPau]
		5	Proposition 9.3
σ_7	$[2, \infty)$	$5, 6, [8, 13], [15, \infty)$	Proposition 9.8
$\overline{\sigma}_7$	$\{2\}$		
σ_8	$[4, 41]$	$[4, 41]$	Corollary 4.2 of [ParPau]
$\overline{\sigma}_8$	$[4, \infty)$	$[4, \infty)$	Corollary 4.2 of [ParPau]
σ_9	$[3, \infty)$	$[3, \infty)$	Corollary 4.2 of [ParPau]
$\overline{\sigma}_9$	$[4, 8]$	$[4, 8]$	Corollary 4.2 of [ParPau]

Table 6: Values of the parameter where Knapp or Jørgensen show non-discreteness. The second column gives the values of p for which the Hermitian form H_τ has signature $(2, 1)$ (taken from [ParPau]). The third and fourth columns give values of p for which a well chosen subgroup fails the Knapp test or the Jørgensen test (and hence the group is not discrete). If this was done in [ParPau] we give the reference. For some values of τ we apply Knapp and Jørgensen to two different complex reflections in the group (in which case the results are listed on two separate lines).

Basic hyperbolic trigonometry gives a relationship between the angles and the length of the base of the triangle (see Figure 2). Indeed, if the length of the base is 2δ, then

$$\cosh\delta\ \sin\alpha = \cos\frac{\beta}{2} \tag{9.1}$$

This gives a practical computational way to check whether the conditions of Knapp's theorem hold.

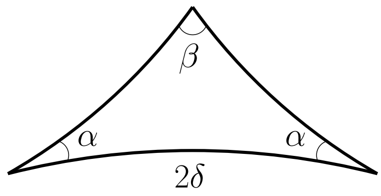


Figure 2: We shall apply Knapp's theorem in the special case of isosceles triangles, see formula (9.1) for the relationship between angles and distances.

Note also that the statement of Knapp's theorem implies that if $\alpha = \pi/q$ for $q \in \mathbb{N}^*$, and if the angle β is larger than $2\pi/3$, then the group cannot be discrete. In view of formula (9.1), the latter statement is the same as one would obtain from Jørgensen's inequality (see [JKP]):

Theorem 9.4 *(Jiang-Kamiya-Parker) Let A be a complex reflection through angle $2\alpha = \frac{2\pi}{q}$ with $q \in \mathbb{N}^*$, with mirror the complex line L_A. Let $B \in \mathrm{PU}(2,1)$ be such that $B(L_A)$ and L_A are ultraparallel, and denote their distance by 2δ. If*

$$|\cosh\delta\ \sin\alpha| < \frac{1}{2} \tag{9.2}$$

then $\langle A, B\rangle$ is non-discrete.

In certain cases we need to deal with groups generated by vertical Heisenberg translations (see definition in section 3). In this case we need results that generalise the above version of Jørgensen's inequality and Knapp's theorem. These results are complex hyperbolic versions of Shimizu's lemma, Proposition 5.2 of [Par1] and a lemma of Beardon, Theorem 3.1 of [Par2]. We combine them in the following statement which is equivalent to the statements given in [Par1] and [Par2].

Theorem 9.5 *(Parker) Let $A \in \mathrm{SU}(2,1)$ be a parabolic map conjugate to a vertical Heisenberg translation with fixed point z_A. Let $B \in \mathrm{SU}(2,1)$ be a map not fixing z_A. If $\langle A, B\rangle$ is discrete then either $\mathrm{tr}(ABAB^{-1}) \leqslant -1$ or $\mathrm{tr}(ABAB^{-1}) = 3 - 4\cos^2(\pi/r)$ for some $r \in \mathbb{N}$ with $r \geqslant 3$. In particular, if*

$$2 < \mathrm{tr}(ABAB^{-1}) < 3 \tag{9.3}$$

then $\langle A, B\rangle$ is non-discrete.

9.2 Using Knapp and Jørgensen with powers of R_1R_2

9.2.1 The general set up

Recall from [ParPau] that for any sporadic value τ, there is a positive rational number r/s so that

$$|\tau|^2 = 2 + 2\cos(r\pi/s), \tag{9.4}$$

which corresponds to the fact that R_1R_2 should have finite order. The values of these r and s are clearly the same for σ_j and $\overline{\sigma}_j$, and are given by

τ	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9
r/s	1/3	1/5	3/5	1/2	1/2	1/2	1/7	5/7	3/7

(9.5)

Straightforward calculation shows

$$R_1R_2 = \begin{bmatrix} e^{2i\pi/3p}(1-|\tau^2|) & e^{4i\pi/3p}\tau & \tau^2-\overline{\tau} \\ -\overline{\tau} & e^{2i\pi/3p} & e^{-2i\pi/3p}\tau \\ 0 & 0 & e^{-4i\pi/3p} \end{bmatrix}$$

which has eigenvalues $-e^{2i\pi/3p}e^{ri\pi/s}$, $-e^{2i\pi/3p}e^{-ri\pi/s}$, $e^{-4i\pi/3p}$. Therefore $(R_1R_2)^s$ has a repeated eigenvalue. An $e^{-4i\pi/3p}$-eigenvector of R_1R_2 is given by

$$\mathbf{p}_{12} = \begin{bmatrix} e^{-2i\pi/3p}\tau^2 + e^{4i\pi/3p}\overline{\tau} - e^{-2i\pi/3p}\overline{\tau} \\ e^{2i\pi/3p}\overline{\tau}^2 + e^{-4i\pi/3p}\tau - e^{2i\pi/3p}\tau \\ 2\cos(2\pi/p) + 2\cos(r\pi/s) \end{bmatrix}.$$

For most values of p and τ this vector is negative, in which case its orthogonal complement (with respect to H_τ) gives a complex line in the ball. Hence (in most cases) it is a complex reflection, and one checks easily that it commutes with both R_1 and R_2.

Likewise, for most values of p, τ, $(R_2R_3)^s$ is a complex reflection that commutes with R_2 and R_3, and it fixes a complex line whose polar vector is $\mathbf{p}_{23} = J(\mathbf{p}_{12})$. If the distance between these two lines is $2\delta_p$ then from Lemma 3.1:

$$\cosh^2(\delta_p) = \frac{\langle \mathbf{p}_{12}, \mathbf{p}_{23}\rangle\langle \mathbf{p}_{23}, \mathbf{p}_{12}\rangle}{\langle \mathbf{p}_{12}, \mathbf{p}_{12}\rangle\langle \mathbf{p}_{23}, \mathbf{p}_{23}\rangle} = \frac{\left|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\right|^2}{\left(2\cos(2\pi/p) + 2\cos(r\pi/s)\right)^2}. \tag{9.6}$$

The eigenvalues of $(R_1R_2)^s$ are $(-1)^{s+r}e^{2is\pi/3p}$, $(-1)^{s-r}e^{2is\pi/3p}$, $e^{-4is\pi/3p}$. Therefore the rotation angle of $(R_1R_2)^s$ is $(r+s)\pi + 2s\pi/p$. This may or may not be of the form $2\pi/c$. When it is not, we can find a positive integer k so that $(R_1R_2)^{sk}$ is a complex reflection whose angle has the form $2\pi/c$. We define $2\alpha_p$ to be the smallest positive rotation angle among all powers of $(R_1R_2)^s$.

Assuming that the parameter τ is fixed, the group $\Gamma(2\pi/p, \tau)$ is indiscrete thanks to the Jørgensen inequality, for the values of p satisfying:

$$\cosh\delta_p \sin\alpha_p = \frac{\left|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\right|\sin\alpha_p}{\left|2\cos(2\pi/p) + 2\cos(r\pi/s)\right|} < \frac{1}{2}. \tag{9.7}$$

Likewise, in order to prove non-discreteness using Knapp's theorem we seek values of p for which

$$\cosh\delta_p \sin\alpha_p = \frac{\left|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\right|\sin\alpha_p}{\left|2\cos(2\pi/p) + 2\cos(r\pi/s)\right|} \neq \cos(\pi/q) \text{ or } \cos(2\alpha_p) \tag{9.8}$$

for a natural number q.

Since $(R_1R_2)^s$ is a complex reflection that rotates through angle $(r+s)\pi + 2s\pi/p$, we can apply the test of Jørgensen's inequality simply to $(R_1R_2)^s$. As $\left|\sin\big((r+s)\pi + 2s\pi/p\big)\right| = \left|2\sin(2s\pi/p)\right|$ this involves finding values of p for which

$$\left|\cosh(\delta_p)\sin(2s\pi/p)\right| = \frac{\left|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\right|\ |\sin(2s\pi/p)|}{|2\cos(2\pi/p) + 2\cos(r\pi/s)|} < \frac{1}{2}. \tag{9.9}$$

For fixed r and s, as p tends to infinity, the left hand side tends to zero. This shows at once that there can be only finitely many discrete groups among all sporadic groups; the rest of the paper is devoted to the proof of Theorem 9.1, which is a vast refinement of that statement.

In the next few sections, we shall apply Knapp or Jørgensen to various powers of R_1R_2 (other elements in the group as well) in order to get the better non-discreteness results.

9.2.2 Cases where $|\tau|^2 = 2$

From (9.4), for any τ with $|\tau|^2 = 2$, we have $r/s = 1/2$ and so

$$\cosh\delta_p = \frac{\left|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\right|}{\left|2\cos(2\pi/p)\right|}.$$

This happens for $\tau = \sigma_4$, $\overline{\sigma}_4$, σ_5, $\overline{\sigma}_5$, σ_6 or $\overline{\sigma}_6$. For all these values we have

$$(R_1R_2)^2 = \begin{bmatrix} -e^{4i\pi/3p} & 0 & e^{2i\pi/3p}\overline{\tau} + e^{-4i\pi/3p}\tau^2 - e^{-4i\pi/3p}\overline{\tau} \\ 0 & -e^{4i\pi/3p} & e^{-2i\pi/p}\tau + \overline{\tau}^2 - \tau \\ 0 & 0 & e^{-8i\pi/3p} \end{bmatrix},$$

which is a complex reflection commuting with both R_1 and R_2, and whose rotation angle is $(p-4)\pi/p$. Note that $(p-4)\pi/p = 2\pi/c$ for some $c \in \mathbb{Z} \cup \{\infty\}$ if and only if p and c are as given in the following table:

p	2	3	4	5	6	8	12
c	-2	-6	∞	10	6	4	3

(When $p = 4$, and hence $c = \infty$, we find that $(R_1R_2)^2$ is parabolic.) For other values of p, by choosing an appropriate power k, we can arrange that $(R_1R_2)^{2k}$ rotates by a smaller angle than $(R_1R_2)^2$:

Lemma 9.1 *Let $(R_1R_2)^2$ be as above. There exists $k \in \mathbb{Z}$ so that $(R_1R_2)^{2k}$ has rotation angle $2\alpha_p$ where*

$$\alpha_p = \frac{\gcd(p-4, 2p)\pi}{2p}.$$

In particular

- *If $p \equiv 1$ (mod 2), then $\gcd(p-4,2p) = 1$, and so $\alpha_p = \frac{\pi}{2p}$.*
- *If $p \equiv 2$ (mod 4), then $\gcd(p-4,2p) = 2$, and so $\alpha_p = \frac{\pi}{p}$.*
- *If $p \equiv 4$ (mod 8), then $\gcd(p-4,2p) = 8$, and so $\alpha_p = \frac{4\pi}{p}$.*
- *If $p \equiv 0$ (mod 8), then $\gcd(p-4,2p) = 4$, and so $\alpha_p = \frac{2\pi}{p}$.*

Proof. We want to find k so that $k(p-4)\pi/p$ reduced modulo 2π is "minimal". More precisely, we write this as

$$k(p-4)\pi/p - 2\pi l = 2\alpha_p$$

for $k \in \mathbb{N}^*$, $l \in \mathbb{N}$, and we want to find α_p of the form π/c for some $c \in \mathbb{N}$. The optimal value of k depends on arithmetic properties of p. Let $d = \gcd(p-4, 2p)$ then we can find integers k and l so that $k(p-4) - l(2p) = d$. This means that $k(p-4)\pi/p - 2\pi l = d\pi/p$ and so $\alpha_p = d\pi/2p$. This proves the first assertion.

If we write $(p-4) = ad$ and $2p = bd$ then, eliminating p, we have $2ad + 8 = bd$ and so $d = 1$, 2, 4 or 8. It is easy to check which values of p correspond to which value of d. □

In the case where $c = \infty$ the map $(R_1R_2)^2$ is parabolic. Up to multiplying by a cube root of unity, we have

$$\mathrm{tr}\Big((R_1R_2)^2 J (R_1R_2)^2 J^{-1}\Big) = \mathrm{tr}\Big((R_1R_2)^2(R_2R_3)^2\Big) = 3 - \big|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\big|^2.$$

Thus applying Theorem 9.5 with $A = (R_1R_2)^2$ and $J = B$ we see can prove non-discreteness by showing that

$$\big|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\big| < 1 \quad \text{or} \quad \big|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\big| \neq 2\cos(\pi/r) \tag{9.10}$$

with r a natural number at least 3.

Checking (9.7), (9.8) or (9.10) is best done by a computer.

Proposition 9.1 *Let $\tau = \overline{\sigma}_4 = (-1 - i\sqrt{7})/2$ and so $r/s = 1/2$. Then:*

- *If p is odd then (9.7) holds for $p \geqslant 7$;*
- *If $p \equiv 2$ (mod 4) then (9.7) holds for $p \geqslant 10$;*
- *If $p \equiv 4$ (mod 8) then (9.8) holds for $p = 20$ and (9.7) holds for $p \geqslant 28$;*

- *If $p \equiv 0$ (mod 8) then (9.7) holds for $p \geqslant 16$.*

Thus for all the values of p given above $\langle (R_1R_2)^2, J\rangle$ and hence $\Gamma(\frac{2\pi}{p}, \overline{\sigma}_4)$ is not discrete.

Proof. For the sake of concreteness, we list some of the values in the following table

p	α_p	$\cosh(\delta_p)\sin(\alpha_p)$	
7	$\pi/14$	$0.4257\ldots$	(9.7)
9	$\pi/18$	$0.2650\ldots$	(9.7)
10	$\pi/10$	$0.4423\ldots$	(9.7)
14	$\pi/14$	$0.2774\ldots$	(9.7)
20	$\pi/5$	$0.6748\ldots$	(9.8)
28	$\pi/7$	$0.4754\ldots$	(9.7)
16	$\pi/8$	$0.4601\ldots$	(9.7)
24	$\pi/12$	$0.2889\ldots$	(9.7)

□

Proposition 9.2 *Let $\tau = \sigma_5 = e^{2i\pi/9} + e^{-i\pi/9} 2\cos(2\pi/5)$ and so $r/s = 1/2$. Then:*

- *If p is odd then (9.7) holds when $p \geqslant 7$;*
- *If $p \equiv 2$ (mod 4) then (9.7) holds when $p \geqslant 10$;*
- *If $p \equiv 4$ (mod 8) then (9.8) holds when $p = 20$ and (9.7) holds when $p \geqslant 28$;*
- *If $p \equiv 0$ (mod 8) then (9.7) holds when $p \geqslant 16$.*

Thus for all the values of p given above $\langle (R_1R_2)^2, J\rangle$ and hence $\Gamma(\frac{2\pi}{p}, \sigma_5)$ is not discrete.

Proof. Some values are given in the following table:

p	α_p	$\cosh(\delta_p)\sin(\alpha_p)$	
7	$\pi/14$	$0.4977\ldots$	(9.7)
9	$\pi/18$	$0.3011\ldots$	(9.7)
10	$\pi/10$	$0.4974\ldots$	(9.7)
14	$\pi/14$	$0.3032\ldots$	(9.7)
20	$\pi/5$	$0.7202\ldots$	(9.8)
28	$\pi/7$	$0.4988\ldots$	(9.7)
16	$\pi/8$	$0.4980\ldots$	(9.7)
24	$\pi/12$	$0.3053\ldots$	(9.7)

□

Recall from [ParPau] that $\Gamma(\frac{2\pi}{p}, \sigma_4)$ has signature $(2,1)$ exactly when $4 \leqslant p \leqslant 6$; that $\Gamma(\frac{2\pi}{p}, \overline{\sigma}_5)$ has signature $(2,1)$ exactly when $p = 2$ or 4, and that $\Gamma(\frac{2\pi}{p}, \sigma_6)$ and $\Gamma(\frac{2\pi}{p}, \overline{\sigma}_6)$ are not discrete except possibly when $p = 5$. Hence for each of these values of τ we only have finitely many things to check. We gather these cases into a single result.

Proposition 9.3

- *If $\tau = \sigma_4 = (-1 + i\sqrt{7})/2$, and so $r/s = 1/2$, and $p = 4$ then (9.10) holds;*
- *If $\tau = \sigma_4 = (-1 + i\sqrt{7})/2$, and so $r/s = 1/2$, and $p = 5$ then (9.7) holds;*
- *If $\tau = \sigma_4 = (-1 + i\sqrt{7})/2$, and so $r/s = 1/2$, and $p = 6$ then (9.8) holds;*
- *If $\tau = \overline{\sigma}_5 = e^{-2i\pi/9} + e^{i\pi/9} 2\cos(2\pi/5)$, and so $r/s = 1/2$, and $p = 4$ then (9.10) holds;*
- *If $\tau = \sigma_6 = e^{2i\pi/9} + e^{-i\pi/9} 2\cos(4\pi/5)$, and so $r/s = 1/2$, and $p = 5$ then (9.8) holds;*

- *If $\tau = \overline{\sigma}_6 = e^{-2i\pi/9} + e^{i\pi/9} 2\cos(4\pi/5)$, and so $r/s = 1/2$, and $p = 5$ then (9.8) holds.*

Thus for these values of τ and p then $\langle (R_1R_2)^2, J\rangle$ and hence $\Gamma(\frac{2\pi}{p}, \tau)$ is not discrete.

Proof. Suppose $\tau = \sigma_4 = (-1 + i\sqrt{7})/2$. If $p = 4$ we have

$$\left|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\right| = \sqrt{3 - \sqrt{7}} = 0.595\ldots$$

If $p = 5$ then $\alpha_p = \pi/10$ and $\cosh(\delta_p)\sin(\alpha_p) = 0.445\ldots$. If $p = 6$ then $\alpha_p = \pi/6$ and $\cosh(\delta_p)\sin(\alpha_p) = 0.550\ldots \in \big(\cos(\pi/3), \cos(\pi/4)\big)$.

If $\tau = \overline{\sigma}_5 = e^{-2i\pi/9} + e^{i\pi/9} 2\cos(2\pi/5)$ and $p = 4$ then

$$\left|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\right| = \sqrt{\frac{7 + \sqrt{5} - 3\sqrt{3} - \sqrt{15}}{2}} = 0.289\ldots.$$

If $\tau = \sigma_6 = e^{2i\pi/9} + e^{-i\pi/9} 2\cos(4\pi/5)$ and $p = 5$ then $\cosh(\delta_p)\sin(\alpha_p) = 0.937\ldots \in \big(\cos(\pi/8), \cos(\pi/9)\big)$.

If $\tau = \overline{\sigma}_6 = e^{-2i\pi/9} + e^{i\pi/9} 2\cos(4\pi/5)$ and $p = 5$ then $\cosh(\delta_p)\sin(\alpha_p) = 0.750\ldots \in \big(\cos(\pi/4), \cos(\pi/5)\big)$. □

9.2.3 Cases where $|\tau|^2 = 3$

We now consider the case $|\tau|^2 = 3$, which happens for $\tau = \sigma_1$ or $\overline{\sigma}_1$. In this case $r/s = 1/3$ and

$$(R_1R_2)^3 = \begin{bmatrix} e^{2i\pi/p} & 0 & (e^{-2i\pi/p} - 1)\big(e^{-2i\pi/3p}\tau^2 + e^{4i\pi/3p}\overline{\tau} - e^{-2i\pi/3p}\overline{\tau}\big) \\ 0 & e^{2i\pi/p} & (e^{-2i\pi/p} - 1)\big(e^{2i\pi/3p}\tau^2 + e^{-4i\pi/p}\tau - e^{2i\pi/3p}\tau\big) \\ 0 & 0 & e^{-4i\pi/p} \end{bmatrix}.$$

This is a complex reflection commuting with both R_1 and R_2, with angle $6\pi/p$. As above, we want to check whether (9.7) holds for α_p the smallest possible rotation angle of powers of $(R_1R_2)^3$.

- If $p \equiv 1$ or $2 \pmod 3$, then we can find $k, l \in \mathbb{N}$ so that $6k\pi/p - 2\pi l = 2\pi/p$. Hence $\alpha_p = \pi/p$.
- If 3 divides p then $6\pi/p$ is already in the form $2\pi/c$, hence $\alpha_p = 3\pi/p$.

Proposition 9.4 *Let $\tau = \sigma_1 = e^{i\pi/3} + e^{-i\pi/6} 2\cos(\pi/4)$ and so $r/s = 1/3$. Then*

- *If $p \equiv 1$ or $2 \pmod 3$ then (9.8) holds when $p = 7$ and (9.7) holds when $p \geqslant 8$;*
- *If p is divisible by 3 then (9.8) holds when $p = 12$, 15 or 18 and (9.7) holds when $p \geqslant 21$.*

Thus for all the values of p given above $\langle (R_1R_2)^3, J\rangle$ and hence $\Gamma(\frac{2\pi}{p}, \sigma_1)$ is not discrete.

Proof. Some values are given in the following table

p	α_p	$\cosh(\delta_p)\sin(\alpha_p)$	
7	$\pi/7$	$0.6510\ldots$	(9.8)
8	$\pi/8$	$0.4969\ldots$	(9.7)
12	$\pi/4$	$0.8134\ldots$	(9.8)
15	$\pi/5$	$0.6510\ldots$	(9.8)
18	$\pi/6$	$0.5416\ldots$	(9.8)
21	$\pi/7$	$0.4631\ldots$	(9.7)

□

From [ParPau] we know that if $\tau = \overline{\sigma}_1 = e^{-i\pi/3} + e^{i\pi/6} 2\cos(\pi/4)$ then the only values of p that give signature $(2, 1)$ are those with $3 \leqslant p \leqslant 7$.

Proposition 9.5 *Let $\tau = \overline{\sigma}_1 = e^{-i\pi/3} + e^{i\pi/6} 2\cos(\pi/4)$ and so $r/s = 1/3$.*

- *If $p = 5$ then (9.8) holds;*
- *If $p = 7$ then (9.7) holds.*

Thus for $p = 5$ or 7 we see that $\langle (R_1R_2)^3, J\rangle$ and hence $\Gamma(\frac{2\pi}{p}, \overline{\sigma}_1)$ is not discrete.

9.2.4 Cases where $|\tau|^2 = 2 + 2\cos(\pi/5)$

This happens for $\tau = \sigma_2$ or $\overline{\sigma}_2$. In that case $r/s = 1/5$ and $(R_1R_2)^5$ is a complex reflection with eigenvalues $e^{10i\pi/3p}$, $e^{10i\pi/3p}$, $e^{-20i\pi/3p}$, thus it has rotation angle $10\pi/p$.

- If p is not divisible by 5, then we can find $k, l \in \mathbb{N}$ such that $10k\pi/p - 2\pi l = 2\pi/p$. Hence $\alpha_p = \pi/p$.
- If p is divisible by 5, then $10\pi/p$ is already in the form $2\pi/c$, hence $\alpha_p = 5\pi/p$.

Proposition 9.6 *Let $\tau = \sigma_2 = e^{i\pi/3} + e^{-i\pi/6}2\cos(\pi/5)$ and so $r/s = 1/5$.*

- *If p is not divisible by 5 then (9.8) holds when $p = 6$ or 7 and (9.7) holds when $p \geqslant 8$;*
- *If p is divisible by 5 then (9.8) holds when $15 \leqslant p \leqslant 30$ and (9.7) holds when $p \geqslant 35$.*

Thus for for these values of p we see that $\langle (R_1R_2)^5, J\rangle$, and hence, $\Gamma(\frac{2\pi}{p}, \sigma_2)$ is not discrete.

Proof. Some values are given in the following table

p	α_p	$\cosh(\delta_p)\sin(\alpha_p)$	
6	$\pi/6$	$0.631\ldots$	(9.8)
7	$\pi/7$	$0.516\ldots$	(9.8)
8	$\pi/8$	$0.438\ldots$	(9.7)
15	$\pi/3$	$0.908\ldots$	(9.8)
20	$\pi/4$	$0.729\ldots$	(9.8)
25	$\pi/5$	$0.601\ldots$	(9.8)
30	$\pi/6$	$0.508\ldots$	(9.8)
35	$\pi/7$	$0.440\ldots$	(9.7)

□.

From [ParPau] we know that if $\tau = \overline{\sigma}_2 = e^{-i\pi/3} + e^{i\pi/6}2\cos(\pi/5)$ then the Hermitian form has signature $(2,1)$ only when $3 \leq p \leq 19$.

Proposition 9.7 *Let $\tau = \overline{\sigma}_2 = e^{-i\pi/3} + e^{i\pi/6}2\cos(\pi/5)$ and so $r/s = 1/5$.*

- *If p is not divisible by 5 then (9.8) holds when $p = 6$ and (9.7) holds when $7 \leqslant p \leqslant 19$;*
- *If p is divisible by 5 then (9.8) holds when $p = 15$.*

Thus for for these values of p we see that $\langle (R_1R_2)^5, J\rangle$, and hence $\Gamma(\frac{2\pi}{p}, \overline{\sigma}_2)$, is not discrete.

Proof. Some values are given below

p	α_p	$\cosh(\delta_p)\sin(\alpha_p)$	
6	$\pi/6$	0.5660	(9.8)
7	$\pi/7$	0.4713	(9.7)
15	$\pi/5$	0.8718	(9.8)

□

9.2.5 Cases where $|\tau|^2 = 2 + 2\cos(\pi/7)$

This happens for $\tau = \sigma_7$ or $\overline{\sigma}_7$. In this case $r/s = 1/7$. The only group with $\tau = \overline{\sigma}_7$ and signature $(2,1)$ is $p = 2$. This group is a relabelling of the group with $\tau = \sigma_7$ and $p = 2$. It is discrete. So for the remainder of this section we consider the case when $\tau = \sigma_7 = e^{2i\pi/9} + e^{-i\pi/9}2\cos(2\pi/7)$.

Then $(R_1R_2)^7$ is a complex reflection with eigenvalues $e^{14i\pi/3p}$, $e^{14i\pi/3p}$, $e^{-28i\pi/3p}$; thus it has rotation angle $14\pi/p$.

- If p is not divisible by 7, then we can find $k, l \in \mathbb{N}$ so that $14k\pi/p - 2\pi l = 2\pi/p$. Hence $\alpha_p = \pi/p$.
- If p is divisible by 7, then $14\pi/p$ is already in the form $2\pi/c$, hence $\alpha_p = 7\pi/p$.

Proposition 9.8 *Let $\tau = \sigma_7 = e^{2i\pi/9} + e^{-i\pi/9} 2\cos(2\pi/7)$ and so $r/s = 1/7$.*

- *If p is not divisible by 7 then (9.8) holds for $p = 5$ or 6 and (9.7) holds when $p \geqslant 8$;*
- *If p is divisible by 7 then (9.8) holds for $21 \leqslant p \leqslant 42$ and (9.7) holds when $p \geqslant 49$; with $p \geqslant 49$.*

Thus for for these values of p we see that $\langle (R_1R_2)^7, J\rangle$, and hence $\Gamma(\frac{2\pi}{p}, \sigma_7)$, is not discrete.

Proof. Some values are given below

p	α_p	$\cosh(\delta_p)\sin(\alpha_p)$	
5	$\pi/5$	$0.929\ldots$	(9.8)
6	$\pi/6$	$0.702\ldots$	(9.8)
8	$\pi/8$	$0.476\ldots$	(9.7)
21	$\pi/3$	$0.921\ldots$	(9.8)
28	$\pi/4$	$0.739\ldots$	(9.8)
35	$\pi/5$	$0.608\ldots$	(9.8)
42	$\pi/6$	$0.514\ldots$	(9.8)
49	$\pi/7$	$0.444\ldots$	(9.7)

□

9.3 Using Knapp and Jørgensen with powers of $R_1R_2R_3R_2^{-1}$

9.3.1 The general set up

A starightforward calculation shows that:

$$R_1R_2R_3R_2^{-1} = \begin{bmatrix} e^{2i\pi/3p}(1 - |\tau^2 - \overline{\tau}|^2) & e^{-2i\pi/3p}(\tau - (\tau^2 - \overline{\tau})\overline{\tau}) & -\tau^2 + (\tau^2 - \overline{\tau})(|\tau|^2 - e^{2i\pi/p}) \\ \tau(\tau - \overline{\tau}^2) & e^{-4i\pi/3p}(1 - |\tau|^2) & e^{-2i\pi/3p}\tau(|\tau|^2 - 1 + e^{2i\pi/p}) \\ e^{-2i\pi/3p}(\tau - \overline{\tau}^2) & -e^{-2i\pi/p}\overline{\tau} & e^{2i\pi/3p} + e^{-4i\pi/3p}|\tau|^2 \end{bmatrix},$$

hence $\mathrm{tr}(R_1R_2R_3R_2^{-1}) = e^{2i\pi/3p}(2 - |\tau^2 - \overline{\tau}|^2) + e^{-4i\pi/3p}$. An $e^{-4i\pi/3p}$ eigenvector of $R_1R_2R_3R_2^{-1}$ is given by

$$\mathbf{p}_{1232} = \begin{bmatrix} e^{-4i\pi/3p}\big(\tau(1 - e^{2i\pi/p}) - (\tau^2 - \overline{\tau})\overline{\tau}\big) \\ |\tau|^2(1 - e^{-2i\pi/p}) - \overline{\tau}(\overline{\tau}^2 - \tau) - |1 - e^{2i\pi/p}|^2 \\ e^{-2i\pi/p}\big(\overline{\tau}(1 - e^{-2i\pi/p}) - (\overline{\tau}^2 - \tau)\tau\big) \end{bmatrix}.$$

Suppose that $|\tau^2 - \overline{\tau}|^2 = 2 + 2\cos(r'\pi/s')$. Then $(R_1R_2R_3R_2^{-1})^s$ is a complex reflection. The values of r' and s' are clearly the same for σ_j and $\overline{\sigma}_j$. They are (see [ParPau]):

τ	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
r'/s'	1/2	1/3	1/3	2/3	2/5	4/5

(9.11)

Let $2\delta'_p$ denote the distance from its mirror to the image of its mirror under J (with polar vector $\mathbf{p}_{2313} = J(\mathbf{p}_{1232})$). Then, from Lemma 3.1:

$$\cosh(\delta'_p) = \frac{|\langle \mathbf{p}_{2313}, \mathbf{p}_{1232}\rangle|}{|\langle \mathbf{p}_{1232}, \mathbf{p}_{1232}\rangle|} = \frac{\big|(1 - e^{2i\pi/p})\tau + |\tau|^2(\overline{\tau}^2 - 2\tau) + e^{-2i\pi/p}\overline{\tau}^2\big|}{2\cos(2\pi/p) + 2\cos(r'\pi/s')}.$$

Let α'_p be the smallest non-zero angle that a power of $(R_1R_2R_3R_2^{-1})^s$ rotates by. Let δ'_p, r' and s' be as above. In order to prove non-discreteness using the Jørgensen inequality, we need to find values of p such that

$$\cosh\delta'_p \sin\alpha'_p = \frac{\big|(1 - e^{2i\pi/p})\tau + |\tau|^2(\overline{\tau}^2 - 2\tau) + e^{-2i\pi/p}\overline{\tau}^2\big|\sin\alpha'_p}{\big|2\cos(2\pi/p) + 2\cos(r'\pi/s')\big|} < \frac{1}{2}. \tag{9.12}$$

In order to prove non-discreteness using Knapp's theorem we must find values of p for which

$$\cosh\delta'_p \sin\alpha'_p = \frac{\left|(1-e^{2i\pi/p})\tau + |\tau|^2(\overline{\tau}^2 - 2\tau) + e^{-2i\pi/p}\overline{\tau}^2\right| \sin\alpha'_p}{\left|2\cos(2\pi/p) + 2\cos(r'\pi/s')\right|} \neq \cos(\pi/q) \text{ or } \cos(2\alpha_p) \tag{9.13}$$

for a natural number q.

9.4 When $|\tau^2 - \overline{\tau}|^2 = 2$

In this case $\tau = \sigma_1$ or $\overline{\sigma}_1$, and $r'/s' = 1/2$. Moreover, $(R_1R_2R_3R_2^{-1})^2$ is a complex reflection with angle $(p-4)\pi/p$. So we proceed as in the section with $|\tau|^2 = 2$. In particular, α'_p is given by Lemma 9.1.

Using Proposition 9.4, we already know that when $p = 7$, 8 or $p \geq 10$ then $\Gamma(\frac{2\pi}{p}, \sigma_1)$ is not discrete. Therefore, we restrict our attention to $p \leq 9$.

Proposition 9.9 *Let $\tau = \sigma_1 = e^{i\pi/3} + e^{-i\pi/6}2\cos(\pi/4)$ and so $r'/s' = 1/2$. Then* (9.13) *holds for $p = 9$. Thus $\langle (R_1R_2R_3R_1^{-1})^2, J\rangle$, and hence also $\Gamma(\frac{2\pi}{p}, \sigma_1)$, is not discrete.*

Proof. In this case $\alpha'_p = \pi/18$ and $\cosh(\delta'_p)\sin(\alpha'_p) = 0.686\ldots \in \big(\cos(\pi/3), \cos(\pi/4)\big)$. □

For $\overline{\sigma}_1$, recall from [ParPau] that $\Gamma(\frac{2\pi}{p}, \overline{\sigma}_1)$ has signature $(2,1)$ exactly when $3 \leqslant p \leqslant 7$.

Proposition 9.10 *Let $\tau = \overline{\sigma}_1 = e^{-i\pi/3} + e^{+i\pi/6}2\cos(\pi/4)$ and so $r'/s' = 1/2$. Then*

- *If $p = 3$ or $p = 6$ then* (9.13) *holds;*
- *If $p = 7$ then* (9.12) *holds.*

Thus for $p = 3$, 6 or 7 the group $\langle (R_1R_2R_3R_1^{-1})^2, J\rangle$, and hence also $\Gamma(\frac{2\pi}{p}, \overline{\sigma}_1)$, is not discrete.

Proof. The values of $\cosh(\delta'_p)\sin(\alpha'_p)$ are:

p	α_p	$\cosh(\delta'_p)\sin(\alpha'_p)$	
3	$\pi/6$	$0.982\ldots$	(9.13)
7	$\pi/14$	$0.269\ldots$	(9.12)
6	$\pi/6$	$0.859\ldots$	(9.13)

□

9.5 Cases where $|\tau^2 - \overline{\tau}|^2 = 3$

We only consider the case $\tau = \sigma_2$ or $\overline{\sigma}_2$ (since σ_3 or $\overline{\sigma}_3$ were already handled in [ParPau]). In this case $r'/s' = 1/3$. Then $(R_1R_2R_3R_1^{-1})^3$ is a complex reflection with angle $6\pi/p$. So we proceed as in the section with $|\tau|^2 = 3$. Namely,

- if p is not divisible by 3, some power gives an angle $\alpha_p = \pi/p$;
- if p is divisible by 3, some power gives $\alpha_p = 3\pi/p$.

In order to use Jørgensen, we check whether $\cosh\delta'_p \sin\alpha_p < \frac{1}{2}$.

For $\tau = \sigma_2$, using Propositions 9.6 and 9.7 we only need to consider the cases where $p \leq 5$ or $p = 10$. This method yields nothing new for $p \leq 5$.

Proposition 9.11 *Let $p = 10$.*

- *If $\tau = \sigma_2 = e^{i\pi/3} + e^{-i\pi/6}2\cos(\pi/5)$, and so $r'/s' = 1/3$, then* (9.13) *holds;*

- *If $\tau=\overline{\sigma}_2=e^{-i\pi/3}+e^{i\pi/6}2\cos(\pi/5)$, and so $r'/s'=1/3$, then (9.12) holds.*

Thus for $p=10$ and $\tau=\sigma_2$ or $\overline{\sigma}_2$, the group $\langle (R_1R_2R_3R_2^{-1})^3, J\rangle$ is not discrete. Hence $\Gamma(\frac{2\pi}{10},\sigma_2)$ and $\Gamma(\frac{2\pi}{10},\overline{\sigma}_2)$ are not discrete.

Proof. When $p=10$ and $\tau=\sigma_2$ we have

$$\cosh(\delta_p')\sin(\alpha_p')=0.6181\ldots\in\big(\cos(\pi/3),\cos(\pi/4)\big).$$

When $p=10$ and $\tau=\overline{\sigma}_2$ we have $\cosh(\delta_p')\sin(\alpha_p')=0.3871\ldots<1/2$. $\square$

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