

NON-ARITHMETIC LATTICES AND THE KLEIN QUARTIC

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ABSTRACT. We give an algebro-geometric construction of some of the non-arithmetic ball quotients constructed by the author, Parker and Paupert. The new construction reveals a relationship between the corresponding orbifold fundamental groups and the automorphism group of the Klein quartic, and also with groups constructed by Barthel-Hirzebruch-Höfer and Couwenberg-Heckman-Looijenga.

1. INTRODUCTION

It is by now well known that there exist non-arithmetic lattices in $\mathrm{PU}(2, 1)$, the group of biholomorphisms of the complex hyperbolic plane \mathbb{H}^2 . The first examples were due to Mostow [23], who gave explicit sets of matrices that generate non-arithmetic lattices (the fact that these groups are indeed lattices was proved by constructing explicit fundamental domains for their action on the complex hyperbolic plane).

A similar approach (using different kinds of fundamental domains) was used by the author in joint work with Parker and Paupert [14], and this allowed us to increase the number of known commensurability classes of non-arithmetic lattices in $\mathrm{PU}(2, 1)$ (there are currently 22 known classes, see [12]).

The main goal of the present paper is to give an alternate construction of the lattices $\mathcal{S}(p, \bar{\sigma}_4)$ of [14], that avoids fundamental domains altogether. For a description of these groups, see section 2 (or [12] for much more detail). Recall that, for $p = 4, 5, 6, 8, 12$, the groups $\mathcal{S}(p, \bar{\sigma}_4)$ give pairwise incommensurable non-arithmetic lattices, that are not commensurable to any Deligne-Mostow lattice.

Our results give a connection between these lattices and the automorphism group of the Klein quartic, which is the closed Riemann surface of genus 3 with largest possible automorphism group, i.e. such that equality holds in the Hurwitz bound $|\mathrm{Aut}| \leq 84(g - 1) = 168$. It is well known that the corresponding automorphism group is the unique simple group of order 168, and that it is also isomorphic to the general linear group $GL_3(\mathbb{F}_2)$ in three variables over the finite field with two elements. This group also turns out to be the projectivization of a unitary group generated by complex reflections, which appears as the group G_{24} in the Shephard-Todd list. The quotient \mathbb{P}^2/G_{24} is isomorphic as a normal complex space to the weighted projective plane $X = \mathbb{P}(2, 3, 7)$ (this can be checked by deriving invariant polynomials from the equation of the Klein quartic, see section 3 for details). The branch locus of the corresponding quotient map $\mathbb{P}^2 \rightarrow \mathbb{P}^2/G_{24}$ is an irreducible curve M of degree 21 in $\mathbb{P}(2, 3, 7)$, which we refer to as the Klein discriminant.

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This can be rephrased to the statement that the pair $(X, (1 - \frac{1}{2})M)$ is a complex projective orbifold, i.e. an orbifold uniformized by \mathbb{P}^2 , which we refer to as the Klein orbifold.

We will see that the pair $(X, (1 - \frac{1}{3})M)$ obtained from the Klein orbifold by changing the multiplicity of M from 2 to 3 is also an orbifold, but now uniformized by the complex hyperbolic plane \mathbb{H}^2 . More generally, our main result will state that, for $p = 3, 4, 5, 6, 8, 12, \infty$, the pair $(X, (1 - \frac{1}{p})M)$ *can be modified* to a complex hyperbolic orbifold. Indeed, for $p > 3$, changing the multiplicity of the ramification divisor is not enough, much more drastic modifications are needed.

In order to explain these modifications, we start by establishing notation. Let us denote by \mathcal{H} the line arrangement in \mathbb{P}^2 given by the mirrors of complex reflections in G_{24} , which has 28 triple points and 21 quadruple points. The action of G_{24} is transitive on the set of triple intersections, and also on the set of quadruple intersections. We denote by s_3 (resp. s_4) the image in $\mathbb{P}(2, 3, 7)$ of any triple (resp. quadruple) intersection.

Denote by \widehat{Y} the blow-up of \mathbb{P}^2 at the 21 quadruple intersections of \mathcal{H} . Similarly, \widehat{Z} denotes the blow-up at all 49 (triple or quadruple) singular points of \mathcal{H} . We denote by $X = \mathbb{P}^2/G_{24} = \mathbb{P}(2, 3, 7)$, $Y = \widehat{Y}/G_{24}$ and $Z = \widehat{Z}/G_{24}$. With a slight abuse of notation, we denote by M either the Klein discriminant in X or its strict transform in Y or Z . Similarly, E denotes the exceptional locus above s_4 either in Y or in Z , and F denotes the exceptional locus in Z above s_3 .

We consider the pairs $(X^{(p)}, D^{(p)})$ given in Table 4.1.

Theorem 1. *For $p = 3, 5, 8$ and 12 , the pairs $(X^{(p)}, D^{(p)})$ are compact complex hyperbolic orbifolds. In other words, there exists a uniform lattice $G^{(p)}$ such that $G^{(p)} \backslash \mathbb{H}^2$ is isomorphic to $X^{(p)}$ as a normal complex space, and the ramification divisor of the quotient map is given by $D^{(p)}$.*

For $p = 4$ and 6 , the singularities of $(X^{(p)}, D^{(p)})$ fail to be log-terminal, so one cannot hope for these to be orbifold pairs (see section 4.1). In fact, the pair $(X^{(4)}, D^{(4)})$ is log-terminal at every point apart from s_4 (see Proposition 4.1), so we consider $X_0^{(4)} = X \setminus \{s_4\}$. Similarly, $(X^{(6)}, D^{(6)})$ is log-terminal away from s_3 , so we consider $X_0^{(6)} = Y \setminus \{s_3\}$. One more pair can be used in order to construct ball quotients, namely $(X^{(\infty)}, D^{(\infty)})$. In that case we take $X_0^{(\infty)} = Z \setminus M$. In all three cases ($p = 4, 6$ and ∞), we write $D_0^{(p)} = X_0^{(p)} \cap D$. We will prove the following.

Theorem 2. *For $p = 4, 6$, and ∞ the pair $(X_0^{(p)}, D_0^{(p)})$ is a non-compact complex hyperbolic orbifold of finite volume with one end. In other words, there exists a 1-cusped lattice $G^{(p)}$ such that $G^{(p)} \backslash \mathbb{H}^2$ is isomorphic to $X_0^{(p)}$ as a normal complex space, and the ramification divisor of the quotient map is given by $D_0^{(p)}$.*

Theorems 1 and 2 will be proved by showing that equality holds in the logarithmic Bogomolov-Miyaoka-Yau inequality. More precisely, we use a result due to Kobayashi, Nakamura and Sakai (see Theorem 4.1), and a significant part of our paper is devoted to checking that their result applies (the fact that the hypotheses are indeed satisfied is our Proposition 5.1).

From the proof of the Kobayashi-Nakamura-Sakai result, it may seem difficult to gather arithmetic information about the groups $G^{(p)}$. Indeed, the existence is shown by solving the appropriate Monge-Ampère equation to produce a Kähler-Einstein metric with negative Einstein constant. Under the assumption that equality holds in the Bogomolov-Miyaoka-Yau inequality, one shows that the Kähler-Einstein metric actually has constant holomorphic sectional curvature. From there, it is not at all obvious how to obtain an explicit description of the corresponding lattices in $U(2, 1)$, say in terms of explicit matrix generators.

In order to identify the lattices, we will use a description due to Naruki of the fundamental group of the complement of the Klein configuration of mirrors in \mathbb{P}^2 , see [25]. This gives enough explicit information about the orbifold fundamental group of $(X^{(p)}, D^{(p)})$ to show the following (see section 6).

Theorem 3. *For $p = 3, 4, 5, 6, 8$ or 12 , the lattices $G^{(p)}$ are conjugate in $\mathrm{PU}(2, 1)$ to the lattices $\mathcal{S}(p, \bar{\sigma}_4)$. In particular, the lattices $G^{(p)}$, $p = 4, 5, 6, 8, 12$, are pairwise incommensurable non-arithmetic lattices, not commensurable to any Deligne-Mostow lattice.*

The case $p = \infty$ turns out to be less interesting, since the corresponding lattice is arithmetic, see Proposition 6.3. The proof of Theorem 3 relies in part on previous joint work of the author with Parker and Paupert, namely some of the non-discreteness results in [13] (which we review in section 2).

Putting together Theorems 1, 2 and 3, we get a new proof that the groups $\mathcal{S}(p, \bar{\sigma}_4)$ are discrete, a fact which was known so far only by using heavy computer work, see [14] (and also [12]).

The construction of ball quotients via uniformization (exploiting the equality case in the Bogomolov-Miyaoka-Yau inequality) appears in several places in the literature, notably in work of Barthel, Hirzebruch and Höfer [4]. In fact, after proving Theorems 1 and 2, we realized that the lattices $\mathcal{S}(p, \bar{\sigma}_4)$ with even values of p ($p = 4, 6, 8$) appear explicitly on p. 215 of the book [4]. Moreover, that table suggested to us to include the value $p = \infty$.

The Barthel-Hirzebruch-Höfer construction was later reinterpreted and extended by Couwenberg, Heckman and Looijenga, and it turns out that all lattices $\mathcal{S}(p, \bar{\sigma}_4)$, $p = 3, 4, 5, 6, 8, 12$ appear in their paper (see Table 8.5 of [8]). We refer to these groups as CHL lattices of Shephard-Todd type G_{24} . The isomorphism of Theorem 3 also implies the following.

Theorem 4. *The CHL lattice of Shephard-Todd type G_{24} for $p = 4, 5, 6, 8, 12$ are not arithmetic. They are pairwise incommensurable, and they are not commensurable to any Deligne-Mostow lattice.*

To the author's knowledge, the arithmetic structure of the CHL ball quotients was never worked out in the literature, nor was their commensurability relations with other known ball quotients. It seems to have been known to Deligne and Mostow, at least conjecturally, see page 181 in [10] (Deligne and Mostow also refer to the work of Yoshida [32]).

Note that part of our proof (the existence of a complex hyperbolic uniformization for the pairs in Table 4.1) could in fact be avoided, if we were to simply quote the results of Couwenberg, Heckman, Looijenga. However we believe our proof, which is fairly short

and elementary, gives an interesting different perspective on this special case of the results in [8].

More of the lattices that appear in [12] can be treated with techniques similar to the ones in this paper, see for instance [11].

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2. DESCRIPTION OF THE GROUPS $\mathcal{S}(p, \bar{\sigma}_4)$

In order to describe the group $\mathcal{S}(p, \bar{\sigma}_4)$, rather than using the original description, we use the characterization of this lattice given in Proposition 2.1. The characterization follows immediately from the results in [13] (see also [12]), we briefly review the argument.

In the next statement, we take R_1, J to be elements of $SU(2, 1)$. Suppose that R_1 is a complex reflection with eigenvalues u^2, \bar{u}, \bar{u} , where $u = e^{2\pi i/3p}$ and J is a regular elliptic element of order 3. We write $R_2 = JR_1J^{-1}$, $R_3 = J^{-1}R_1J$, and G for the group generated by R_1 and J .

Proposition 2.1. *Suppose G is a lattice, $(R_j R_k)^2 = (R_k R_j)^2$ for all $j \neq k$, and $R_1 J$ has order 7. Then G is conjugate to $\Gamma_p = \mathcal{S}(p, \bar{\sigma}_4)$ (and p is equal to 3, 4, 5, 6, 8, or 12).*

Proof: From the braid relation $(R_j R_k)^2 = (R_k R_j)^2$, it follows that the eigenvalues of $R_j R_k$ are roots of unity, namely $\bar{u}^2, \pm iu$ (see for instance Proposition 2.3 of [12]). Clearly the eigenvalues of $R_1 J$ are roots of unity as well.

It follows that the pair of matrices (R_1, J) can be simultaneously conjugated to standard generators of a complex hyperbolic sporadic group, i.e. up to complex conjugation and multiplication by a cube root of unity, $\text{tr}(R_1 J)$ can be assumed to be one of the 10 values σ_j , $j = 1, \dots, 10$ listed in [12].

Inspection of the values of the order of $R_1 J$ and the braid length $\text{br}(R_j, R_k)$ for these values of $\text{tr}(R_1 J)$ show that we may assume $\text{tr}(R_1 J) = \sigma_4$ or $\bar{\sigma}_4$. Now the group $\mathcal{S}(p, \sigma_4)$ is a subgroup of $PU(2, 1)$ only for $p = 4, 5$ or 6 , and it is known to be non-discrete for each of these three values. The groups $\mathcal{S}(p, \bar{\sigma}_4)$ are subgroups of $PU(2, 1)$ for $p > 2$, and if so they are not discrete for $p \neq 3, 4, 5, 6, 8, 12$. Both non-discreteness statements just mentioned follow from Theorem 9.1 of [13]. \square

3. WEIGHTED PROJECTIVE PLANE

The group $\Gamma_2 = \mathcal{S}(2, \bar{\sigma}_4)$ was not studied in [13], because it does not act on the complex hyperbolic plane. On the other hand, the matrices given there make sense for $p = 2$, and in that case the group generated by R_1 and J preserves a definite Hermitian form (which is unique up to scaling). In other words, the group can be seen as a discrete (in fact finite) group of isometries of the Fubini-Study metric on \mathbb{P}^2 .

We omit the proof of the following Proposition, since it is not needed anywhere in the paper (it is included only for motivation).

Proposition 3.1. *The group Γ_2 is isomorphic to the projective automorphism group of the Klein quartic, which is also the 24-th Shephard-Todd group G_{24} .*

From this point on, we work only with the automorphism group of the Klein quartic, and we write it as G (and we write $\mathbb{P}G$ for its projectivization).

Proposition 3.2. *The group $\mathbb{P}G$ is the finite simple group of order 168. The quotient \mathbb{P}^2/G is isomorphic as complex analytic orbifold to the pair $(X, \frac{1}{2}M)$ where $X = \mathbb{P}(2, 3, 7)$ and M is the image of the union of mirrors of reflections in $\mathbb{P}G$. Moreover,*

- (i) *M is an irreducible curve of degree 21 in $\mathbb{P}(2, 3, 7)$, with equation (1) in a natural set of coordinates,*
- (ii) *M contains precisely one singular point of $\mathbb{P}(2, 3, 7)$, which is its A_1 singularity,*
- (iii) *M has two singular points s_3 and s_4 in the smooth part of $\mathbb{P}(2, 3, 7)$, where it has (analytic) local equations of the form $w_1^3 = w_2^2$ and $w_1(w_1 - w_2^2) = 0$ respectively.*

We refer to the curve M described in the proposition as the *Klein discriminant*. Recall that $\mathbb{P}(2, 3, 7)$ is the quotient of \mathbb{C}^3 by the equivalence relation $(z_1, z_2, z_3) \sim (\lambda^2 z_1, \lambda^3 z_2, \lambda^7 z_3)$ for all $\lambda \in \mathbb{C}^*$. It has three singular points, given in (weighted) homogeneous coordinates z_1, z_2, z_3 by the coordinate axes, in other words $(z_1, z_2, z_3) = (1, 0, 0), (0, 1, 0), (0, 0, 1)$. The corresponding singularities are of type A_1 , A_2 and $\frac{1}{7}(2, 3)$, respectively.

One way to describe the curve M is to give an explicit equation, which in a suitable set (z_1, z_2, z_3) of coordinates can be taken to be

$$(1) \quad z_3^3 + 27 \cdot 64z_2^7 - 88z_1^2z_2z_3^2 + 16 \cdot 63z_1z_2^4z_3 + 17 \cdot 64z_1^4z_2^2z_3 - 256z_1^7z_3 - 128 \cdot 469z_1^3z_2^5 + 43 \cdot 512z_1^6z_2^3 - 2048z_1^9z_2 = 0.$$

This curve may seem very mysterious, and its structure (irreducibility, nature of singular points) is not completely obvious. To prepare for the proof of Proposition 3.2, we first briefly review some classical facts about the automorphism group of the Klein quartic.

The Klein quartic is the complex curve given in homogeneous coordinates (x_1, x_2, x_3) for \mathbb{P}^2 by $f(x_1, x_2, x_3) = 0$ where

$$(2) \quad f(x_1, x_2, x_3) = x_1^3x_2 + x_2^3x_3 + x_3^3x_1.$$

It is well known that its automorphism group is the only simple group of order 168, and that it can be written out explicitly as the (projective) unitary group generated by the following three matrices:

$$T = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^4 \end{pmatrix} \quad J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad R = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix},$$

where $\zeta = e^{2\pi i/7}$, $a = h(\zeta^4 - \bar{\zeta}^4)$, $b = h(\zeta^2 - \bar{\zeta}^2)$, $c = h(\zeta - \bar{\zeta})$, and $h = i/\sqrt{7} = 1/(\zeta^3 + \zeta^5 + \zeta^6 - \bar{\zeta}^3 - \bar{\zeta}^5 - \bar{\zeta}^6)$.

Remark 3.1. The group $\langle R, J, T \rangle$ is not quite generated by complex reflections, but it is a subgroup of index two in a reflection group, obtained from it by adjoining $-I$. The corresponding group of order 336 is the 24-th group in the Shephard-Todd list [28].

Note that R has order 2, and it is (the opposite of) a complex reflection (it has eigenvalues 1, -1 , -1). We write v_1 for a unit eigenvector with eigenvalue 1, which is orthogonal to the mirror of R .

Proposition 3.3. *The matrices T and J generate a group H of order 21. The orbit of the mirror of R under this group has 21 elements, and the stabilizer of $\mathbb{C}v_1$ in G has order 8. The group G has order 168, and the conjugates of R by elements of H are the only reflections in G .*

Proof: The first statement is obvious, since J permutes the eigenspaces of T cyclically. Let us denote by V the orbit of v_1 under H . One easily checks that no element of H fixes v_1 , so V has 21 elements.

Among these 21 vectors, one checks that 4 are orthogonal to v_1 , and these come in two pairs $\{a_1, a_2\}$, $\{a_3, a_4\}$ of orthogonal vectors (a_1 and a_2 are orthogonal to each other, but they are not orthogonal to a_3 nor to a_4). Explicitly, $a_1 = JT v_1$, $a_2 = J^{-1}T^3 v_1$, $a_3 = J^{-1}T^4 v_1$, $a_4 = JT^{-1} v_1$.

This implies that the stabilizer of v_1 in G has order dividing 8. One checks that $-TRTJRJ$ fixes v_1 and acts by switching a_1 and a_2 , and $-T^2JTRT$ fixes v_1, a_1 and changes a_2 to $-a_2$. Since these two transformations generate a group of order 8, the stabilizer of v_1 has order 8.

This implies that G has order $21 \cdot 8 = 168$. □

Proof: (of Proposition 3.2) Part (i) follows from the explicit determination of generators for the ring of invariants for G . This was worked out in the nineteenth century by Klein [18], see also [31], [29].

It is not completely obvious that the equation (2) of the Klein quartic is invariant under R , but it can be checked directly. This gives an invariant of degree 4. One then constructs another invariant from the Hessian $H(f) = \left(\frac{\partial^2 f}{\partial x_j \partial x_k} \right)_{j,k}$, namely $\Delta = \frac{1}{9} \det(H(f))$, which is homogeneous of degree 6, then

$$C = \frac{1}{9} \det \begin{pmatrix} H(f) & \nabla(\Delta) \\ \nabla(\Delta)^t & 0 \end{pmatrix},$$

which is homogeneous of degree 14, and finally

$$K = \frac{1}{14} \det (\nabla f, \nabla \Delta, \nabla C),$$

which has degree 21.

Note that the last polynomial is not invariant under $-I$ (but its square is). The ring of invariants for the group $\langle G, -I \rangle$, which is a reflection group, is generated by f, Δ, C , so the quotient of \mathbb{P}^2 is $\mathbb{P}(4, 6, 14)$ which is isomorphic to $\mathbb{P}(2, 3, 7)$.

In particular, K^2 can be expressed as a polynomial in f, Δ, C , which was also computed in the nineteenth century by Gordan [17]. This gives an equation for D , which is equation (1), in the coordinates $(z_1, z_2, z_3) = (f, \Delta, C)$ (see [29] or p.529 of [31]).

The irreducibility of M follows from the fact that G acts transitively on the set of mirrors of reflections in G (see Proposition 3.3). Part (ii) is obvious from equation (1).

The proof of part (iii) relies on the detailed study of fixed points in \mathbb{P}^2 of elements in $\mathbb{P}G$. One checks that there are 171 fixed points of regular elements in the group, and these fixed points come in 5 orbits with various stabilizers, as listed in Table 3.1 (see §136 of [31]). When the stabilizer is generated by complex reflections, we list its Shephard-Todd notation in the last row (in that case the corresponding orbit gives a smooth point of $\mathbb{P}(2, 3, 7)$). In particular, the group generated by R and J is generated by R and JR , and these two are indeed complex reflections. We denote by M_1 and M_2 the matrices that already appeared (up to their sign) in the proof of Proposition 3.3, namely $M_1 = TRTJRJ$ and $M_2 = T^2JTRT$. Note that the configuration of mirrors of reflections in G has 28 triple

Notation	s_3	s_4	t_2	t_3	t_7
# mirrors	3	4	1	0	0
# stab	6	8	4	3	7
# orbit	28	21	42	56	24
Gens	J, R	M_1, M_2	TR	J	T
ST	$G(3, 3, 2)$	$G(2, 1, 2)$	\times	\times	\times

TABLE 3.1. The list of orbits of fixed points of regular elliptic elements in the group. For each point, we give the number of mirrors through that point, the order of and generators for each stabilizer in G , and if applicable the Shephard-Todd notation, if the stabilizer is generated by complex reflections.

points, and 21 quadruple points, see page 96 of [4].

The local analytic structure of the singularities of M can be understood by studying invariants for the $G(m, p, n)$ groups that appear here in Table 3.1. In suitable coordinates for \mathbb{C}^2 , the group $G(3, 3, 2)$ is generated by $(z_1, z_2) \mapsto (z_2, z_1)$ and $(z_1, z_2) \mapsto (\omega z_1, \bar{\omega} z_2)$. The ring of invariants is the polynomial ring in $u_1 = z_1 z_2$, $u_2 = z_1^3 + z_2^3$, in other words the map $(z_1, z_2) \mapsto (u_1, u_2) = (z_1 z_2, z_1^3 + z_2^3)$ gives the quotient map $\mathbb{C}^2 \rightarrow \mathbb{C}^2$ by the group action of $G(3, 3, 2)$.

The mirrors of reflections in the group are $z_1 = z_2$, $z_1 = \omega z_2$, $z_1 = \bar{\omega} z_2$, so the union of the mirrors has equation $z_1^3 - z_2^3 = 0$, which has the same zero set as the invariant polynomial $(z_1^3 - z_2^3)^2$. In terms of our invariant generators, the union of the mirrors is given by $u_2^2 - 4u_1^3 = 0$. This also gives the branch locus of the branched covering of the quotient map.

Similarly, the group $G(2, 1, 2)$ is generated by $(z_1, z_2) \mapsto (z_2, z_1)$ and $(z_1, z_2) \mapsto (-z_1, z_2)$. The corresponding ring of invariants is the polynomial ring in $u_1 = z_1^2 z_2^2$, $u_2 = z_1^2 + z_2^2$. The mirrors of reflections in the group are $z_1 = \pm z_2$, $z_1 = 0$, $z_2 = 0$, so the union of the mirrors has equation $z_1^2 z_2^2 (z_1 - z_2)^2 = 0$, or equivalently $u_1(u_2^2 - 4u_1) = 0$. \square

One easily verifies that a given mirror of any reflection in the group contains precisely 10 points that are fixed by elements in the group other than the reflection itself and the central element. We deduce the following result, which will be used later in the paper.

Proposition 3.4. *Let $X = \mathbb{P}(2, 3, 7)$, and let $X_0 = X \setminus (M \cup \{t_2, t_3, t_7\})$. Then $\chi(X_0) = 0$.*

Proof: This follows by computing the Euler characteristic of the complement \tilde{X}_0 in \mathbb{P}^2 of the union of all fixed point sets of non-trivial elements in the group $\mathbb{P}G$, where the action of $\mathbb{P}G$ is free. X_0 is obtained from \mathbb{P}^2 by removing 21 copies of $\mathbb{P}^1 \setminus \{10\text{pts}\}$, as well as the 171 fixed points of regular elements (these include intersections of mirrors). This gives

$$\chi(\mathbb{P}^2) = 3 = \chi(\tilde{X}_0) + 21 \cdot (2 - 10) + 171,$$

so $\chi(\tilde{X}_0) = 0$, and $\chi(X_0) = \chi(\tilde{X}_0)/168 = 0$. \square

In order to compute orbifold Chern numbers, we will need some basic algebro-geometric properties of weighted projective space $\mathbb{P}(2, 3, 7)$, all of which can be derived from the general theory of toric varieties (see §3.4 and §4.3 of [16], or [9]).

Proposition 3.5. *Let $X = \mathbb{P}(a_1, a_2, a_3)$ be a well-formed weighted projective space, i.e. such that every pair of integers in $\{a_1, a_2, a_3\}$ is relatively prime.*

- (1) *The Picard group $\text{Pic}(X)$ is isomorphic to \mathbb{Z} , we denote by H its positive generator.*
- (2) *The canonical divisor K_X is equivalent to $-(a_1 + a_2 + a_3)H/a_1a_2a_3$.*
- (3) *$H^2 = a_1a_2a_3$.*

4. BALL QUOTIENTS VIA UNIFORMIZATION

In order to construct orbifolds uniformized by the ball, we would like to consider pairs $(X, (1 - \frac{1}{p})M)$ for integers $p \geq 2$, where $X = \mathbb{P}(2, 3, 7)$ and M is the Klein discriminant (see Proposition 3.2). This idea is somewhat natural given the families of lattices studied in [12], and also in view of the work by Couwenberg, Heckman and Looijenga. Indeed, they constructed 1-parameter families of complex hyperbolic structures on $\mathbb{P}^2 \setminus \mathcal{H}$, where the parameter corresponds to the common rotation angle of the holonomy around all lines in \mathcal{H} . Moreover, their structures actually descend to the quotient by the action of the finite group G_{24} (see section 6 of [8]).

We have already seen that $(X, (1 - \frac{1}{2})M)$ is an orbifold uniformized by \mathbb{P}^2 (see Proposition 3.2). Perhaps surprisingly, we will see that the pair $(X, (1 - \frac{1}{3})M)$ is an orbifold as well, now uniformized by \mathbb{H}^2 . For higher values of p , more drastic modifications of X need to take place in order to make the orbifold uniformizable. In order to explain these modifications, we start by reviewing the logarithmic Bogomolov-Miyaoka-Yau inequality.

4.1. The logarithmic Bogomolov-Miyaoka-Yau inequality. We review some results about identifying ball quotients by computing ratios of Chern classes. It is well known that a smooth compact complex surface of general type X is a ball quotient if and only if $c_1^2(X) = 3c_2(X)$. Note that the latter relation between Chern numbers holds for $X = \mathbb{P}^2$, so it also holds for compact ball quotients by the Hirzebruch proportionality principle.

The fact that a surface of general type whose Chern numbers satisfy this relation is indeed a ball quotient follows from the solution of the Calabi conjecture by Aubin and Yau. Indeed, on a surface of general type, there exists a Kähler-Einstein metric with negative constant, and a classical computation shows that if $c_1^2(X) = 3c_2(X)$, then that metric actually has constant holomorphic sectional curvature (and the constant can be taken to be equal to -1). It then follows that X can be written as $\Gamma \backslash \mathbb{H}^2$ for some *torsion-free* lattice Γ in $\text{Bihol}(\mathbb{H}^2) = \text{PU}(2, 1)$.

Ball quotients $\Gamma \backslash \mathbb{H}^2$ where Γ is a lattice that is not torsion-free are in general not smooth surfaces, but they are smooth as orbifolds. The corresponding map $\mathbb{H}^2 \rightarrow \Gamma \backslash \mathbb{H}^2$ is then a branched cover, with branch locus given by the set of fixed points of non-trivial elements in Γ . The quotient is smooth near the orbit $\Gamma \cdot x$ if and only if the stabilizer of x in Γ is generated by complex reflections, by a theorem of Chevalley [6].

One naturally expects that the above characterization of ball quotients should hold for orbifold logarithmic pairs (X, D) (here D plays is a \mathbb{Q} -divisor that corresponds the ramification divisor of the quotient map from the ball), provided we replace the usual Chern numbers by orbifold Chern numbers. A statement along those lines was indeed proved by Kobayashi, Nakamura and Sakai [20], see also [19], [4] and [30].

The setting is as follows. Let (X, D) be a pair with X a compact complex normal surface, and D is a \mathbb{Q} -divisor of the form $D = \sum_j (1 - \frac{1}{b_j})D_j$, each D_j being distinct irreducible curves and $b_j = 2, 3, \dots$ or ∞ (in the latter case, we set $1/\infty = 0$). We assume that the pair (X, D) has at worst log-canonical singularities, and that $K_X + D$ ample. Then the Kodaira dimension $\kappa(X, D)$ is equal to two, the log-canonical ring $R(X, D) = \bigoplus_{m \geq 0} H^0(X, m(K_X + D))$ is finitely generated, and $X = \text{Proj}(R(X, D))$ is its own log-canonical model.

Let X_0 be obtained from X by removing the components D_j such that $b_j = \infty$, as well as the singular points of (X, D) that are *not* log-terminal, and let $D_0 = D \cap X_0$. The main result prove by Kobayashi, Nakamura and Sakai in [20] is the following.

Theorem 4.1. *Under the above hypotheses, the pair (X_0, D_0) is an orbifold, $c_1(X_0, D_0)^2 \leq 3c_2(X_0, D_0)$ and equality holds if and only if (X_0, D_0) is a quotient $\Gamma \backslash \mathbb{H}^2$ for some lattice $\Gamma \subset \text{PU}(2, 1)$.*

Note that $c_2(X_0, D_0)$ makes sense because the pair (X_0, D_0) is an orbifold, and it is often called the orbifold Euler characteristic. The left hand side $c_1(X_0, D)$ can also be computed as the self-intersection $(K_X + D)^2$ of the orbifold canonical divisor (we will also refer to the latter as the log-canonical divisor of the pair (X, D)).

We briefly recall the definition of log-canonical and log-terminal singularities. Let p be a singular point of (X, D) , and let $\mu : \tilde{X} \rightarrow X$ be a log-resolution of that singularity. Here \tilde{D} denotes the strict transform of D , and we write \tilde{E} for the exceptional set, and write E_α for its irreducible components. We assume $K_X + D$ is \mathbb{Q} -Cartier, and that the resolution is “good”, i.e. that G has normal crossings, where the divisor G obtained by writing

$$K_{\tilde{X}} + \tilde{D} = \mu^*(K_X + D) + G$$

The divisor G can be written as $\sum_\alpha g_\alpha E_\alpha$, and then the singularity is called log-canonical (resp. log-terminal) if $g_\alpha \geq -1$ for all α (resp. $g_\alpha > -1$ for all α , and $b_i < \infty$ for all i). This condition is known to be independent of the good resolution μ .

As mentioned above, in order to apply the Kobayashi-Nakamura-Sakai result, there are three things to check, each handled in one of the following sections. In section 5.1, we study the singularities of the pairs. In section 5.2 we verify that equality holds in the orbifold Bogomolov-Miyaoka-Yau inequality. In section 5.3 we check that the log-canonical divisors of the relevant pairs are ample.

4.2. Description of the logarithmic pairs. The pairs $(X, (1 - \frac{1}{p})M)$ do not usually satisfy the technical hypotheses of the result Kobayashi, Nakamura and Sakai (see section 4.1). The pairs we will use to produce ball quotients are obtained from $(X, (1 - \frac{1}{p})M)$ by suitable birational modifications.

Later, we will describe the relevant birational transformations directly on the level of $\mathbb{P}(2, 3, 7)$, see section 5.1, but for now we give an equivariant description on the level of \mathbb{P}^2 , seen as a branched covering of $\mathbb{P}(2, 3, 7)$ of degree 168. It is well known that the configuration of mirrors of the group G in \mathbb{P}^2 has 49 singular points, that come in 21 quadruple points (all in the same G -orbit), and 28 triple points (also forming a G -orbit), see p. 211 in [4], for instance.

Definition 4.1. Let \tilde{X} be \mathbb{P}^2 , and denote by \tilde{Y} the blow-up of \tilde{X} at every point of the G -orbit of quadruple points, and by \tilde{Z} the blow-up of \tilde{X} at all singular points of the union of mirrors in G (all quadruple and triple points). Let $X = \tilde{X}/G$, $Y = \tilde{Y}/G$ and $Z = \tilde{Z}/G$. We denote by E the image in Y (or Z) of any exceptional divisor in \tilde{Y} (or \tilde{Z}) corresponding to quadruple mirror intersections. We denote by F the image in Z of any exceptional divisor in \tilde{Z} corresponding to triple mirror intersections. The divisor E (resp. F) lies over a point of $\mathbb{P}(2, 3, 7)$ which we denote by s_4 (resp. s_3). We denote by M the image in X (or Y , or Z) of the union of mirrors in \mathbb{P}^2 .

Of course, X is just $\mathbb{P}(2, 3, 7)$ (see Proposition 3.2). Note that Y (resp. Z) is not the usual blow-up of X at s_3 (resp. at s_3 and s_4). Indeed, the image of the blown-up divisors are smooth points of $\mathbb{P}(2, 3, 7)$, but Y has a singular point in the image of the exceptional locus in \tilde{Y} . Similarly, Z has a singular point in the image of each G -orbit of exceptional divisors in \tilde{Z} .

Remark 4.1. The singularities of Y and Z (see Definition 4.1) can be studied by using explicit coordinates on the blow-up of \mathbb{C}^2 at a given point, and looking at the linearized action at every fixed point of the action of $G(2, 1, 2)$ (resp. of $G(3, 3, 2)$) on the blow-up. Since it is slightly cumbersome, we will bypass this verification and define explicit birational maps $Y \rightarrow X$ and $Z \rightarrow X$, see section 5.1.

In order to construct ball quotients, we consider the pairs given in Table 4.1. In section 5.1, we will see that these pairs have at worst log-canonical singularities. As mentioned in section 4.1, it is also important to describe the locus $X_0^{(p)}$ of points where each

- $X^{(3)} = X, D^{(3)} = (1 - \frac{1}{3})M.$
- $X^{(4)} = X, D^{(4)} = (1 - \frac{1}{4})M.$
- $X^{(5)} = Y, D^{(5)} = (1 - \frac{1}{5})M + (1 - \frac{1}{10})E.$
- $X^{(6)} = Y, D^{(6)} = (1 - \frac{1}{6})M + (1 - \frac{1}{6})E.$
- $X^{(8)} = Z, D^{(8)} = (1 - \frac{1}{8})M + (1 - \frac{1}{4})E + (1 - \frac{1}{8})F.$
- $X^{(12)} = Z, D^{(12)} = (1 - \frac{1}{12})M + (1 - \frac{1}{3})E + (1 - \frac{1}{4})F.$
- $X^{(\infty)} = Z, D^{(\infty)} = M + (1 - \frac{1}{2})E + (1 - \frac{1}{2})F.$

TABLE 4.1. The above pairs will be shown to give orbifolds that are uniformized by the ball (possibly after removing log-canonical singularities that are not log-terminal).

pair $(X^{(p)}, D^{(p)})$ is actually log-terminal, since it gives the open set uniformized by the ball in Theorem 4.1. We will prove the following.

Proposition 4.1. *The pairs $(X^{(p)}, D^{(p)})$ are log-terminal for $p = 3, 5, 8, 12$. The log-terminal locus of $(X^{(4)}, D^{(4)})$ is given by $X_0^{(4)} = X \setminus \{s_4\}$. The log-terminal locus of $(X^{(6)}, D^{(6)})$ is given by $X_0^{(6)} = Y \setminus \{s_3\}$. The log-terminal locus of $(X^{(\infty)}, D^{(\infty)})$ is $X_0^{(\infty)} = Z \setminus M$.*

5. BALL QUOTIENTS

The main goal of this section is to prove Proposition 5.1 which, by Theorem 4.1, implies Theorems 1 and 2.

Proposition 5.1. *Let $(V, D) = (X^{(p)}, D^{(p)})$ be as in Table 4.1 for $p = 3, 4, 5, 6, 8, 12$ or ∞ . Then*

- (1) *The pair (V, D) has at worst log-canonical singularities, and these are log-terminal singularities, except at s_4 when $p = 4$, at s_3 when $p = 6$, and along the divisor M when $p = \infty$.*
- (2) *We have the equality $c_1(V, D)^2 = 3c_2(V_0, D_0)$, where $(V_0, D_0 = D \cap V_0)$ is the complement in (V, D) of the non log-terminal locus.*
- (3) *The log-canonical divisor $K_V + D$ is ample, and the pair (V, D) is its own log-canonical model.*

Among the items in Proposition 5.1, the first item guarantees that (V_0, D_0) is an orbifold, thanks to the classification of log-canonical singularities of pairs (we will also describe the orbifold structure explicitly, since this is needed in order to compute $c_2(V_0, D_0)$, see section 5.2.2). The last two items guarantee that the corresponding orbifold is uniformized by the ball, by Theorem 4.1.

Each part of Proposition 5.1 will be treated in a separate section, see sections 5.1, 5.2, 5.3.

5.1. Study of the singularities.

5.1.1. *Singularities of X .* Throughout this section, we write $\lambda = 1 - \frac{1}{p}$ (this is the coefficient of M in the orbifold canonical divisor). We first consider the case $p = 3$, and the singularities of (X, D) , where $X = \mathbb{P}(2, 3, 7)$, and $D = (1 - \frac{1}{3})M$. The singular points outside M are log-terminal, since they are quotient singularities (see Proposition 4.18 in [21], for instance).

There are three points to consider, namely t_2 , which is the A_1 singularity of $\mathbb{P}(2, 3, 7)$, and s_3 and s_4 , which are smooth points of X . Near the A_1 -singularity, there is nothing to show since the pair $(X, \lambda M)$ is isomorphic to the quotient of \mathbb{C}^2 by a cyclic group generated by a diagonal map whose square is a complex reflection, so this fits in the classification of log-canonical singularities (see part (1) of Theorem 5 in [20]).

Near s_3 (resp. s_4), the curve M has a local analytic equation of the form $z_1^2 = z_2^3$ (resp. $z_1(z_1 - z_2^2) = 0$). One way to check the log-canonical character of these singularities is to identify these curves as branch loci of suitable complex reflection groups (see [28] and [3]). We give a direct proof, and along the way we derive formulas that will be used when computing $c_1^2(X^{(p)}, D^{(p)})$.

We consider a local resolution of the pair (X, D) at s_4 , which is given by blowing up s_4 , so that the strict transform of M has two transverse branches, and then blowing up that transverse intersection once more. We denote by $\pi_4 : X_4 \rightarrow X$ the corresponding composition of blow-ups, by E_2 the strict transform of the first exceptional divisor (which is a (-2) -curve), and by E_1 the second exceptional divisor (which is a (-1) -curve).

Since $K_{X_4} = \pi_4^*K_X + 2E_1 + E_2$ and $\pi_4^*M = \widetilde{M} + 4E_1 + 2E_2$ (where \widetilde{M} denotes the strict transform of M), we have

$$(3) \quad K_{X_4} + \lambda \widetilde{M} = \pi_4^*(K_X + \lambda M) + (2 - 4\lambda)E_1 + (1 - 2\lambda)E_2,$$

so the pair is log-canonical at s_4 if and only if $\lambda \in [0, \frac{3}{4}]$, which is the case for $p \leq 4$. For $p = 4$, we get a log-canonical singularity which is not log-terminal.

We now consider a local resolution of the pair (X, D) at s_3 . In this case, three successive blow-ups are needed; we blow-up the cusp of M , the proper transform is then tangent to the exceptional divisor. We then blow-up the point of tangency, which makes the intersection transverse, and then blow-up that transverse intersection. We denote by $\pi_3 : X_3 \rightarrow X$ the corresponding sequence of blow-ups.

The exceptional set is a chain of three copies of \mathbb{P}^1 , with self-intersections $-2, -1, -3$, we denote the corresponding curves by F_2, F_1, F_3 respectively. The corresponding formula is

$$(4) \quad K_{X_3} + \lambda \widetilde{M} = \pi_3^*(K_X + \lambda M) + (4 - 6\lambda)F_1 + (2 - 3\lambda)F_2 + (1 - 2\lambda)F_3,$$

so we get a log-canonical singularity for $p \leq 6$ (which is not log-terminal for $p = 6$).

5.1.2. *Singularities of Y .* The space Y has a birational map $Y \rightarrow X$ which we now describe. One simply does the same blow-up of s_4 as described in section 5.1.1, and then contract the (-2) -curve. Note that this contraction produces an A_1 -singularity (this follows from the uniqueness of the minimal resolution of surface singularities).

As before, we denote by $\pi : X_4 \rightarrow X$ the composition of three blow-ups described above, and we write $\gamma_4 : X_4 \rightarrow Y$ for the contraction, $E = \gamma_4(E_1)$ and $\varphi_4 : Y \rightarrow X$ for the corresponding birational map, see the left diagram in (5).

$$(5) \quad \begin{array}{ccc} \widetilde{M} \subset X_4 & & X_3 \\ \downarrow \gamma_4 & \searrow \pi_4 & \downarrow \gamma_3 \\ M' \subset Y & \xrightarrow{\varphi_4} & X \supset M \\ & & \downarrow \varphi_3 \\ & Z_3 & \end{array}$$

By construction, γ_4 gives a good resolution of the pair (Y, D) , where $D = \lambda M' + \mu E$. Here we denote by \widetilde{M} the strict transform of M in X_4 , and by M' the image of \widetilde{M} under γ_4 . We also write $\lambda = 1 - \frac{1}{p}$ (resp. $\mu = 1 - \frac{1}{m}$) for the coefficient of \widetilde{M} (resp. E) in the relevant log-canonical divisor, see Table 4.1 for $p \geq 5$.

Note that $\gamma_4^* E = E_1 + \frac{1}{2} E_2$. Indeed, $\gamma_4^* E = E_1 + a E_2$ for some $a \in \mathbb{Q}$, but $0 = \gamma_4^* E \cdot E_2 = (E_1 + a E_2) \cdot E_2 = 1 - 2a$, so $a = 1/2$. In particular, we get

$$(6) \quad E^2 = E_1 \cdot \gamma_4^* E = E_1 \cdot (E_1 + \frac{1}{2} E_2) = -1/2.$$

We have $\gamma_4^* M' = \widetilde{M} + b E_2$ for some $b \in \mathbb{Q}$. Intersecting both sides with E_2 , we get $\widetilde{M} \cdot E_2 - 2b = 0$, but $\pi_4^* M = \widetilde{M} + 4E_1 + 2E_2$, so $\widetilde{M} \cdot E_2 = 0$, which gives $b = 0$. Finally, we note that $K_{X_4} = \gamma_4^* K_Y$, since A_1 singularities are crepant. For completeness, we mention that $K_{X_4} = \pi^* K_X + 2E_1 + E_2$, and $K_Y = \varphi_4^* K_X + 2E$.

Now we have

$$K_{X_4} + \lambda \widetilde{M} + \mu E_1 = \gamma_4^* (K_Y + \lambda M' + \mu E) - \frac{\mu}{2} E_2,$$

so the pair is log-terminal provided $\lambda, \mu \in [0, 1[, -\frac{\mu}{2} > -1$ (only log-canonical if equality holds in some of these strict inequalities).

This shows that pairs the (Y, D) corresponding to $p = 5, 6$ have log-terminal singularities above s_4 (in fact this remains true for the pairs (Z, D) corresponding to $p = 8, 12$, that will be introduced in the next section, since they have the same local structure near s_4).

5.1.3. Singularities of Z . In a similar way, the space Z has a birational map $\varphi_3 : Z \rightarrow X$. Near s_4 , we perform the same sequence of blow-ups then contraction as in section 5.1.2.

Near s_3 , we perform three successive blow-ups as in section 5.1.1, then contract the (-2) -curve (producing an A_1 singularity) and (-3) -curve (producing a singularity of type $\frac{1}{3}(1, 1)$). Once again, the identification of the type of singularities after contraction follows from the uniqueness of the minimal resolution of surface singularities, and the knowledge of a suitable model resolution. As a model, one can take the resolution of the cone over the rational normal curve of degree $d = 2$ or 3 , which gives the Hirzebruch surface \mathbb{F}_d , the exceptional locus being a $(-d)$ -curve.

We denote by \widehat{Z} the space X blown-up twice at s_4 and three times at s_3 (in the precise way that we just described), and by Z the space obtained from \widehat{Z} by contracting the exceptional curves with self-intersection (-2) or (-3) . We denote by $\pi : \widehat{Z} \rightarrow X$ the composition of

blow-ups, by $\gamma : \widehat{Z} \rightarrow Z$ the contraction, and by $\varphi : Z \rightarrow X$ the corresponding morphism. The corresponding situation is illustrated in Figure 5.1. As mentioned in the end of the

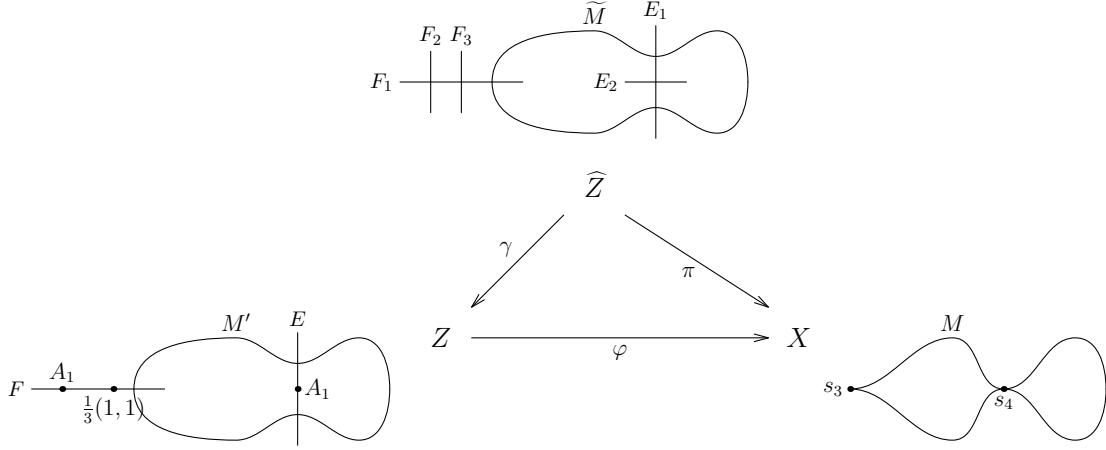


FIGURE 5.1. Schematic picture of the space Z , which is obtained from X by performing successive blow-ups at s_3 and s_4 , then contracting the curves with self-intersection (-2) or (-3) . This picture corresponds to the cases $p = 8$ and 12 (for $p = 5$ or 6 , s_3 does not get blown-up).

previous section, the pair (Z, D) is log-canonical above s_4 , so we need only work locally (in the analytic topology) around s_3 . Hence we will work with X_3 rather than \widehat{Z} .

We denote by $\pi_3 : X_3 \rightarrow X$ the sequence of blow-ups above s_3 , and we write $\gamma_3 : X_3 \rightarrow Z_3$ for the contraction, $F = \gamma_3(F_1)$ and $\varphi_3 : Z_3 \rightarrow X$ for the corresponding map (see the right part of diagram (5)). As in section 5.1.2, we denote by \widetilde{M} (resp. M') the proper transform of M in X_3 (resp. its push-forward in Z_3). Again, γ_3 gives a good resolution of the pair (Z_3, D) near s_3 , where $D = \lambda M' + \nu F$. Here $\nu = 1 - \frac{1}{n}$ is the coefficient of F in the relevant pair, see Table 4.1 for $p = 8, 12$.

We start by computing F^2 , since it will be needed later when computing c_1^2 . We write $\gamma_3^* F = F_1 + aF_2 + bF_3$ for some $a, b \in \mathbb{Q}$, and intersect both sides with F_2 or F_3 , to get $1 - 2a$ and $1 - 3b = 0$. Then by the projection formula,

$$(7) \quad F^2 = F_1 \cdot \gamma_3^* F = -1 + a + b = -\frac{1}{6}.$$

The other relevant formulas are the following:

$$\gamma_3^* F = F_1 + \frac{1}{2}F_2 + \frac{1}{3}F_3,$$

$$\pi_3^* M = \widetilde{M} + 6F_1 + 3F_2 + 2F_3,$$

and

$$K_{X_3} = \pi_3^* K_X + 4F_1 + 2F_2 + F_3.$$

Pushing the last formula forward by γ_3 , we get $K_{Z_3} = \varphi_3^*K_X + 4F$, which gives

$$K_{X_3} = \gamma_3^*K_{Z_3} - \frac{1}{3}F_3.$$

One checks that M' does not go through the singular points of Z_3 on F , so $\gamma_3^*M' = \widetilde{M}$ and

$$K_{X_3} + \lambda\widetilde{M} + \nu F_1 = \gamma_3^*(K_{Z_3} + \lambda M' + \nu F) - \frac{\nu}{2}F_2 - \frac{1+\nu}{3}F_3.$$

Hence the pair is log-terminal at points above s_3 if and only if $\lambda, \nu \in [0, 1[$, $\frac{\nu}{2} > -1$ and $-\frac{1+\nu}{3} > -1$, but only log-canonical if equality holds in some of these inequalities.

The result is that for $p > 6$, all of the relevant pairs are log-terminal above s_3 .

5.2. Equality holds in the Bogomolov-Miyaoka-Yau inequality.

5.2.1. *Computation of $c_1^2(X^{(p)}, D^{(p)})$.* For $p = 3$ or 4 , by Proposition 3.5, we have $K_X = -12(H/42)$, where H is the positive generator of the Picard group of X . Since M has weighted degree 21, $\lambda D = 21\lambda(H/42)$, and

$$(K_X + \lambda M)^2 = \frac{1}{42}(-12 + 21\lambda)^2.$$

For higher values of p , we push the formulas (3) and (4) to Y or Z , and we use the computation of E^2 and F^2 from section 5.1 (see equations (6) and (7)).

For $p = 5$ or 6 , we use the map $\varphi_4 : Y \rightarrow X$ from section 5.1.2. Equation (3) gives $K_Y + \lambda M' + \mu E = \varphi_4^*(K_X + \lambda M) + (2 - 4\lambda + \mu)E$, hence

$$(K_Y + \lambda M' + \mu E)^2 = \frac{1}{42}(-12 + 21\lambda)^2 + (-\frac{1}{2})(2 - 4\lambda + \mu)^2.$$

For $p = 8, 12$ or ∞ , we use the map $\varphi : Z \rightarrow X$ from section 5.1.3. Equations (3) and (4) give $K_Z + \lambda M' + \mu E + \nu F = \varphi^*(K_X + \lambda M) + (2 - 4\lambda + \mu)E + (4 - 6\lambda + \nu)F$, hence

$$(K_Z + \lambda M' + \mu E + \nu F)^2 = \frac{1}{42}(-12 + 21\lambda)^2 + (-\frac{1}{2})(2 - 4\lambda + \mu)^2 + (-\frac{1}{6})(4 - 6\lambda + \nu)^2.$$

We gather the corresponding numerical values, obtained from the above formulas by taking specific values of p, m, n corresponding to Table 4.1. Recall that $\lambda = 1 - \frac{1}{p}$, $\mu = 1 - \frac{1}{m}$, $\nu = 1 - \frac{1}{n}$.

p	3	4	5	6	8	12	∞
m			10	6	4	3	2
n					8	4	2
$c_1^2(X^{(p)}, D^{(p)})$	$\frac{2}{21}$	$\frac{75}{224}$	$\frac{141}{280}$	$\frac{25}{42}$	$\frac{297}{448}$	$\frac{221}{336}$	$\frac{3}{7}$

TABLE 5.1. Numerical values of $c_1^2(X^{(p)}, D^{(p)})$, for various values of p .

5.2.2. *Computation of $c_2(X_0^{(p)}, D_0^{(p)})$.* Recall that we need to consider the pairs $(X^{(p)}, D^{(p)})$ for $p = 3, 5, 8, 12$, and $(X_0^{(p)}, D_0^{(p)})$ for $p = 4, 6, \infty$.

In order to compute $c_2(X_0^{(p)}, D_0^{(p)})$, we compute the orbifold Euler characteristic separately on each stratum with constant isotropy group and sum the corresponding values. More precisely, we write $X_0^{(p)}$ as a disjoint union $\sqcup_{S \in \mathfrak{S}} S$ where the S have constant isotropy group, and compute

$$(8) \quad \chi^{orb}(X_0^{(p)}, D_0^{(p)}) = \sum_{S \in \mathfrak{S}} \frac{\chi(S)}{|I_S|},$$

where $\chi(S)$ is the usual topological Euler characteristic, and I_S is the isotropy group of an arbitrary point of S , see [27] (and also [2], [22]).

Note that this formula is not exactly the same as the formula on the right hand side of the inequality that appears in Theorem 12 of [20], which is closer to a Riemann-Hurwitz formula. One reason why we use our formulation is that the Kobayashi-Nakamura-Sakai formula is slightly ambiguous in our situation (the d_i that is stated in [20] to be a number of singularities of a certain type should be counted with multiplicity).

We break up the sum in equation (8) into summands corresponding to each complex dimension $k = 0, 1, 2$, and write

$$\chi^{orb} = \chi_0^{orb} + \chi_1^{orb} + \chi_2^{orb}.$$

By Proposition 3.4, the stratum with trivial isotropy group has Euler characteristic 0, hence for every value of p we have

$$\chi_2^{orb}(X_0^{(p)}, D_0^{(p)}) = 0.$$

For χ_0^{orb} and χ_1^{orb} , we will get different formulas depending on p . In fact the general form of our formulas depends on whether $X^{(p)}$ is equal to X , Y or Z (see section 5.1). We suggest the reader to keep Figure 5.1 in mind, since it helps keep track of the combinatorics and topology of the strata with constant isotropy groups.

The most complicated isotropy groups are the isotropy groups at the two singular points s_3 and s_4 of the Klein discriminant. Since $\mathbb{P}(2, 3, 7)$ is smooth at those points, the corresponding isotropy groups must be 2-dimensional finite groups generated by complex reflections, which were tabulated by Shephard-Todd, see [28].

For each such singular point s_k ($k = 3$ or 4), we consider a “small” contractible neighborhood U_k of s_k in X .

Proposition 5.2. *For $k = 3$ or 4 , and for U_k small enough, we have $\pi_1(U_k \setminus M) = \langle a, b \mid (ab)^{k/2} = (ba)^{k/2} \rangle$.*

Recall that, for an odd integer n , $(xy)^{n/2}$ stands for an alternating product $xyx \cdots yx$ with n factors, and we call the relation $(xy)^{n/2} = (yx)^{n/2}$ a braid relation of length n . For short, when x and y braid with length n , we write $\text{br}_n(x, y)$.

Proof: This can be seen from standard arguments using projection onto one of the axes in \mathbb{C}^2 and studying the monodromy. Alternatively, one can use the description of the

finite reflection stabilizers in the automorphism group of the Klein quartic, see the proof of Proposition 3.2 and Table 3.1. The corresponding local fundamental groups are tabulated in [3]. \square

The local orbifold fundamental group at s_3, s_4 for the orbifold $(X^{(p)}, D^{(p)})$ are obtained from the groups in proposition 5.2 by adding the relations $a^p = b^p = id$. Denote by $I_n(p)$ the group

$$I_n(p) = \langle a, b \mid a^p, b^p, \text{br}_n(a, b) \rangle.$$

Proposition 5.3. *The groups $I_3(3), I_3(4), I_3(5), I_4(3)$ are finite, of orders given by $|I_3(3)| = 24$, $|I_3(4)| = 96$, $|I_3(5)| = 600$, $|I_4(3)| = 72$. Each of these four $I_n(p)$ admits a faithful representation in $U(2)$ such that a and b are represented by complex reflections of order p . In particular, they are isomorphic to specific 2-dimensional primitive Shephard-Todd groups, namely $I_3(3) = G_4$, $I_3(4) = G_8$, $I_3(5) = G_{16}$ and $I_4(3) = G_5$.*

Proof: This is explained in section 2.2 of [23] for instance, see also the computation of the local fundamental group of the branch locus for 2-dimensional Shephard-Todd groups given in [3]. \square

The cases $p = 3, 4$

For $p = 3$ or 4 , there is a unique stratum of dimension 1, namely $M_0 = M \setminus \{t_2, s_3, s_4\}$, where the isotropy group is cyclic of order p . This is a \mathbb{P}^1 with four points removed (note that s_4 is a double point of M , see Figure 5.1), hence $\chi(M_0) = 2 - 4 = -2$ and

$$\chi_1^{orb} = -\frac{2}{p}.$$

In order to compute χ_0^{orb} , we describe the local structure of the corresponding orbifolds near points with isolated isotropy type (i.e. points such that any neighboring point has a different isotropy group).

LOCAL STRUCTURE NEAR s_4 : The local analytic structure of the pair $(X, \frac{1}{3}M)$ near s_4 is given by the quotient of \mathbb{C}^2 by the Shephard-Todd group G_5 , see Proposition 5.3, the corresponding isotropy group has order 72.

For $p = 4$, recall from section 5.1 that the singularity of the pair $(X^{(4)}, D^{(4)})$ at s_4 is only log-canonical, not log-terminal. So only $X_0^{(4)} = X^{(4)} \setminus \{s_4\}$ carries an orbifold structure.

LOCAL STRUCTURE NEAR s_3 : The local analytic structure of the pair $(X^{(p)}, D^{(p)})$ near s_3 is given by the quotient of \mathbb{C}^2 by the Shephard-Todd group G_4 for $p = 3$, and by G_8 for $p = 4$ (see Poposition 5.3). These give isotropy groups of order 24 and 96, respectively.

LOCAL STRUCTURE NEAR t_2 : Near the A_1 -singularity, the local model given by a regular elliptic element whose square is a complex reflection of order p , hence we get isotropy group of order $2p$.

LOCAL STRUCTURE NEAR t_3, t_7 : These two points are not on the Klein discriminant curve, so we keep the orbifold structure given by the quotient $\mathbb{P}(2, 3, 7) = \mathbb{P}^2/G_{24}$.

The statements in the previous paragraphs are summarized in Table 5.2. We get

$$\chi_0^{orb}(X^{(3)}, D^{(3)}) = \frac{1}{2 \cdot 3} + \frac{1}{24} + \frac{1}{72} + \frac{1}{3} + \frac{1}{7},$$

p	t_2	t_3	t_7	s_3	s_4
2	\mathbb{Z}_4	\mathbb{Z}_3	\mathbb{Z}_7	$ G(3, 3, 2) = 6$	$ G(2, 1, 2) = 8$
3	\mathbb{Z}_6	\mathbb{Z}_3	\mathbb{Z}_7	$ G_4 = 24$	$ G_5 = 72$
4	\mathbb{Z}_8	\mathbb{Z}_3	\mathbb{Z}_7	$ G_8 = 96$	∞

TABLE 5.2. Order of isotropy groups contributing to χ_0^{orb} , for $p = 2, 3, 4$.

$$\chi_1^{orb}(X^{(3)}, D^{(3)}) = \frac{-2}{3},$$

and finally

$$c_2(X^{(3)}, D^{(3)}) = \chi_0^{orb}(X^{(3)}, D^{(3)}) + \chi_1^{orb}(X^{(3)}, D^{(3)}) = \frac{2}{63}.$$

Similarly,

$$c_2(X_0^{(4)}, D_0^{(4)}) = \left(\frac{1}{2 \cdot 4} + \frac{1}{96} + \frac{1}{3} + \frac{1}{7}\right) + \left(\frac{-2}{4}\right) = \frac{25}{224}.$$

Comparing with the formulas in section 5.2.1, we see that

$$c_1^2(X_0^{(p)}, D_0^{(p)}) = 3c_2(X_0^{(p)}, D_0^{(p)})$$

for both cases $p = 3, 4$.

The cases $p = 5, 6$

Here and in the remainder of section 5.2.2, with a slight abuse of notation, we use the same notation for M in X and its proper transform in Y (or Z).

For $p = 5, 6$, there are two 1-dimensional strata, consisting of generic points of M and E , respectively. We denote by M_0 and E_0 the corresponding non-compact curves. Just as in the cases $p = 3, 4$ we have $\chi(M_0) = -2$.

Recall from section 5.1.2 that Y is obtained from X by blowing up the point s_4 twice, then contracting the (-2) -curve in the exceptional locus. This produces an A_1 singularity on E , which we denote by e_0 , and the divisor E has two transverse intersection points with M , which we denote by e_1 and e_2 . Now E_0 is a copy of \mathbb{P}^1 with three points removed (for a picture of this, contract the exceptional divisor F in Figure 5.1).

In other words, we get

$$(9) \quad \chi_1^{orb}(X_0^{(p)}, D_0^{(p)}) = \frac{-2}{p} + \frac{-1}{m},$$

where $m = 10$ (resp. $m = 6$) if $p = 5$ (resp. $p = 6$).

In order to compute χ_0^{orb} , we describe local models near each point with special isotropy.

LOCAL STRUCTURE NEAR s_3 : For $p = 5$, the local model is given the group generated by two complex reflections a, b of order 5 that satisfy the braid relation $aba = bab$. This is the Shephard-Todd group G_{16} , which has order 600. For $p = 6$, the corresponding group would be infinite. In fact we have seen in section 5.1 that the pair $(X^{(6)}, D^{(6)})$ is not log-terminal at s_3 , so the corresponding pair is not an orbifold, and we need to remove that point in order to get a ball quotient. In other words, $X_0^{(6)} = X^{(6)} \setminus \{s_3\}$, $D_0^{(6)} = D^{(6)} \cap X_0^{(6)}$.

p	t_2	t_3	t_7	s_3	e_0	e_1, e_2
5	\mathbb{Z}_{10}	\mathbb{Z}_3	\mathbb{Z}_7	$ G_{16} = 600$	$2m = 20$	$mp = 50$
6	\mathbb{Z}_{12}	\mathbb{Z}_3	\mathbb{Z}_7	∞	$2m = 12$	$mp = 36$

TABLE 5.3. Order of isotropy groups contributing to χ_0^{orb} , for $p = 5, 6$.

LOCAL STRUCTURE NEAR e_1, e_2 : These have abelian isotropy groups generated by two complex reflections of order p and m , where $m = 10$ (resp. $m = 6$) if $p = 5$ (resp. $p = 6$), see Table 4.1.

LOCAL STRUCTURE NEAR e_0 : Recall that this is an A_1 singularity on E . The isotropy group at this point is a regular elliptic element whose square has order m , hence it has order $2m$.

LOCAL STRUCTURE NEAR t_2 : The isotropy group at this point is the same as for $p = 3, 4$ (regular elliptic element whose square is a complex reflection of order p), it has order $2p$.

LOCAL STRUCTURE NEAR t_3, t_7 : The order of these groups do not change with p , they have order 3, 7 respectively.

We summarize the result of the previous paragraphs in Table 5.3. We then have

$$\chi_0^{orb}(X^{(5)}, D^{(5)}) = \frac{1}{2 \cdot 5} + \frac{1}{3} + \frac{1}{7} + \frac{1}{m} \left(\frac{2}{5} + \frac{1}{2} \right) + \frac{1}{600},$$

and

$$\chi_0^{orb}(X^{(5)}, D^{(5)}) = \frac{1}{2 \cdot 6} + \frac{1}{3} + \frac{1}{7} + \frac{1}{m} \left(\frac{2}{6} + \frac{1}{2} \right).$$

Combining this with equation (9), we get

$$c_2(X^{(5)}, D^{(5)}) = \frac{47}{280},$$

$$c_2(X_0^{(6)}, D_0^{(6)}) = \frac{25}{126}.$$

Comparing with the computations in section 5.2.1, we see that $c_1^2 = 3c_2$ for both $p = 5, 6$.

The cases $p = 8, 12$

Recall that to go from X to Z , we do the same blow up and contraction above s_4 as for $p < 8$, and also a sequence of three blow-ups above s_3 , then contract a (-2) and a (-3) -curves. This produces two singular points, an A_1 -singularity and a singularity of type $\frac{1}{3}(1, 1)$. We denote by E the \mathbb{P}^1 that goes through the singular point corresponding to s_4 , and F the \mathbb{P}^1 that goes through the two singular points corresponding to s_3 (see Figure 5.1).

We use the same notation e_0, e_1, e_2 for special points on E as in the cases $p = 5, 6$, and we denote by f_0 the A_1 singularity on f , by f_1 the $\frac{1}{3}(1, 1)$ singularity, and by f_2 the intersection point $F \cap M$.

Since the set of E_0 (resp. F_0) of generic points of E (resp. F) is a \mathbb{P}^1 with three points removed, we have $\chi(E_0) = \chi(F_0) = -1$. The generic isotropy groups on M are cyclic of

p	t_2	t_3	t_7	f_0	f_1	f_2	e_0	e_1, e_2
8	\mathbb{Z}_{16}	\mathbb{Z}_3	\mathbb{Z}_7	$2n = 16$	$3n = 24$	$np = 64$	$2m = 8$	$mp = 24$
12	\mathbb{Z}_{24}	\mathbb{Z}_3	\mathbb{Z}_7	$2n = 8$	$3n = 12$	$np = 48$	$2m = 6$	$mp = 36$

TABLE 5.4. Order of isotropy groups contributing to χ_0^{orb} , for $p = 8, 12$.

p , cyclic of order m on E and cyclic of order n on F , where $(m, n) = (4, 8)$ for $p = 8$, $(m, n) = (3, 4)$ for $p = 12$ (see Table 4.1). Hence we have

$$\chi_1^{orb} = \frac{\chi(E_0)}{m} + \frac{\chi(F_0)}{n} + \frac{\chi(M_0)}{p} = \frac{-1}{m} + \frac{-1}{n} + \frac{-2}{p}.$$

The isotropy groups with special isotropy, corresponding to the 0-dimensional strata are listed in Table 5.4. We then get

$$\chi_0^{orb} = \frac{1}{2 \cdot p} + \frac{1}{3} + \frac{1}{7} + \frac{1}{m} \left(\frac{2}{p} + \frac{1}{2} \right) + \frac{1}{n} \left(\frac{1}{p} + \frac{1}{2} + \frac{1}{3} \right),$$

which gives

$$c_2(X^{(8)}, D^{(8)}) = \frac{99}{448}, \quad c_2(X^{(12)}, D^{(12)}) = \frac{221}{1008}.$$

The case $p = \infty$

In that case, we remove from $Z = X^{(\infty)}$ the curve M above M , and write $X_0^{(\infty)} = X^{(\infty)} \setminus M$, $D_0^{(\infty)} = D^{(\infty)} \cap X_0^{(\infty)}$. In this case the formulas are the same as for the cases $p = 8, 12$, but we set $p = \infty$ and $m = n = 2$ (see Table 4.1). This gives

$$\chi_1^{orb}(X_0^{(\infty)}, D_0^{(\infty)}) = \frac{-1}{m} + \frac{-1}{n},$$

and

$$\chi_0^{orb}(X_0^{(\infty)}, D_0^{(\infty)}) = \frac{1}{3} + \frac{1}{7} + \frac{1}{m} \left(\frac{1}{2} \right) + \frac{1}{n} \left(\frac{1}{2} + \frac{1}{3} \right),$$

and we get

$$c_2(X_0^{(\infty)}, D_0^{(\infty)}) = \frac{1}{7}.$$

Once again, this shows $c_1^2 = 3c_2$.

5.3. Ample ness. The goal of this section is to prove that the pairs $(X^{(p)}, D^{(p)})$ as above have log-general type, i.e. that $K_{X^{(p)}} + D^{(p)}$ is ample.

Note that this (would be false for $p = 2$ and it) is easy for $p = 3$ or 4 . Indeed, in that case, since M has degree 21,

$$K_X + \lambda M = (-12 + 21\lambda)H/42,$$

where H is the positive generator of $\text{Pic}(X)$. The coefficient $-12 + 21 \cdot (1 - \frac{1}{p})$ is > 0 for $p \geq 3$ (whereas $-12 + 21 \cdot (1 - \frac{1}{2}) = -\frac{3}{2} < 0$). For $p \geq 5$ the question is slightly more subtle, since $X^{(p)}$ is not simply given by X .

We first assume $p \geq 8$. Consider the map $\varphi : Z \rightarrow X$ constructed in section 5.1.3. Recall that

$$(10) \quad K_Z + \lambda M' + \mu E + \nu F = \varphi^*(K_X + \lambda M) + (4 - 6\lambda + \mu)E + (2 - 4\lambda + \nu)F.$$

Note that the coefficients of E and F in the right hand side are negative.

We have $\varphi^*M = M' + 6E + 4F$, so $K_Z + \lambda M' + \mu E + \nu F$ is ample if and only if $\frac{21\lambda-12}{21}M' + AE + BF$ is ample, where

$$A = 6 \frac{21\lambda - 12}{21} + (4 - 6\lambda + \mu)$$

and

$$B = 4 \frac{21\lambda - 12}{21} + (2 - 4\lambda + \nu).$$

For $p = 5$ or 6 , we do the same with $\varphi_4 : Y \rightarrow X$, but now

$$(11) \quad K_Y + \lambda M' + \mu E = \varphi^*(K_X + \lambda M) + (4 - 6\lambda + \mu),$$

and we get a similar expression without F (or in other words $B = 0$).

The values of A, B are listed in Table 5.5 (last three columns), note in particular that $A, B > 0$.

p	5	6	8	12	∞
A	$\frac{103}{70}$	$\frac{59}{42}$	$\frac{37}{28}$	$\frac{26}{21}$	$\frac{15}{4}$
B			$\frac{33}{56}$	$\frac{13}{28}$	$\frac{3}{14}$

TABLE 5.5. Values of A and B for relevant values of p .

We now prove

Proposition 5.4. *The divisor $K_{X^{(p)}} + D^{(p)}$ is ample for $p = 3, 4, 5, 6, 8, 12$ and ∞ .*

Proof: By the Nakai-Moishezon criterion, it is enough to check that the intersection $(K^{(p)} + D^{(p)}) \cdot W$ is > 0 for every irreducible curve W .

Since $(K^{(p)} + D^{(p)})$ is numerically equivalent to $\frac{21\lambda-12}{21}M' + AE + BF$ (or $\frac{21\lambda-12}{21}M' + AE$ if $p = 5, 6$), and the coefficients in this expression are all positive, it is enough to check that the intersection with M' , E and F are all > 0 .

Indeed, if W is distinct from M' , E and F , its intersection with $(K^{(p)} + D^{(p)})$ is ≥ 0 . If it is 0, then W projects to a curve in X disjoint from M , but this is impossible since $X = \mathbb{P}(2, 3, 7)$ has Picard number one (see Proposition 3.5).

For the intersection with E or F , we use equation (10) to get $(K_S + \lambda M' + \mu E) \cdot E = (4 - 6\lambda + \mu)E^2$ and $(K_S + \lambda M' + \mu E) \cdot F = (2 - 4\lambda + \nu)F^2$ (where S is either Y or Z , depending on the value of p). These numbers are both positive, since $E^2 = -\frac{1}{2} < 0$, $F^2 = -\frac{1}{6} < 0$ (see equations (6) and (7)) and for relevant values of p , we get $(2 - 4\lambda + \nu) < 0$ and $(4 - 6\lambda + \mu) < 0$, as shown in Table 5.6.

p	5	6	8	12	∞
$2 - 4\lambda + \mu$	$-\frac{3}{10}$	$-\frac{1}{2}$	$-\frac{3}{4}$	-1	$-\frac{3}{2}$
$4 - 6\lambda + \nu$			$-\frac{3}{8}$	$-\frac{3}{4}$	-2

TABLE 5.6. Values of the coefficients of E and F , which show that the intersection of the log-canonical divisor with E and F are positive.

Using $E^2 = -\frac{1}{2}$ and $F^2 = -\frac{1}{6}$ once again, we have that the intersection with M' , which is the same as the intersection with $\varphi^*M - 6E - 4F$ gives

$$(21\lambda - 12) \cdot 21 \cdot \frac{1}{42} + (4 - 6\lambda + \mu) + 2(2 - 4\lambda + \nu),$$

which is positive for $p = 8, 12$ and ∞ (the respective values are $23/16, 23/24$ and $1/3$).

For $p = 5, 6$, there is only one exceptional divisor E , so one simply needs to check $(2 - 4\lambda + \nu) < 0$ (see the first two columns of Table 5.6) and

$$(21\lambda - 12) \cdot 21 \cdot \frac{1}{42} + (4 - 6\lambda + \mu) > 0,$$

and this does indeed hold for $p = 5, 6$ (one gets $37/20$ and $17/12$, respectively). \square

6. IDENTIFICATION OF THE LATTICES

In this section we prove Theorem 3. The main ingredient is a computation due to Naruki [25], which describes the fundamental group of the complement in \mathbb{P}^2 of the union of mirrors of the reflections in the automorphism group of the Klein quartic. Naruki actually computes the fundamental group of the complement of the Klein discriminant in $\mathbb{P}(2, 3, 7)$, which is given as follows (see Proposition 3.6 of [25]).

Proposition 6.1. *The fundamental group of $X_0 = X \setminus (M \cup \{t_2, t_3, t_7\})$ has a presentation of the form*

$$(12) \quad \langle \alpha, \delta \mid (\alpha\delta)^7, br_3(\alpha, \delta^2), br_4(\alpha, \delta^{-1}\alpha\delta) \rangle.$$

Here we use the notation from [12] and write $br_n(a, b)$ for the relation $(ab)^{n/2} = (ba)^{n/2}$. For odd n , $(ab)^{n/2}$ stands for a product of the form $aba \dots ba$ with n factors. In particular, $br_3(a, b)$ stands for $aba = bab$, and this implies that a and b are conjugate, since $a = (ba)b(ba)^{-1}$.

Following the proof given by Naruki, one easily sees that α corresponds to a loops that winds once around the Klein discriminant.

The inclusion $\iota : X_0 \rightarrow X$ induces a surjective homomorphism ι_* on the level of orbifold fundamental groups, such that $\iota_*(\alpha)$ is a complex reflection of order p . It follows that $\iota_*(\delta^2)$ is also a reflection of order p (note that α and δ^2 are conjugate, since $br_3(a, \delta^2)$).

In the remainder of this section, for the sake of readability, we abuse notation and simply write γ for $\iota_*(\gamma)$. We also write $\bar{\alpha}, \bar{\delta}$ for α^{-1}, δ^{-1} , respectively.

We will chose an isomorphism such that δ maps to the element S_1 mentioned in equation (10) of [14] (which is a squareroot of R_1), and α to $R_3^{-1}R_2R_3$.

Proposition 6.2. *The elements $J = (\alpha\delta)^2\alpha\bar{\delta}$ and $R_1 = \delta^2$ generate a group isomorphic to $\mathcal{S}(p, \bar{\sigma}_4)$.*

Proof: One checks using the presentation (12) that J^3 has order 3. Indeed,

$$\begin{aligned} ((\alpha\delta)^2\alpha\bar{\delta})^2 &= \alpha\delta^2(\bar{\delta}\alpha\delta\alpha)^2\delta\alpha\bar{\delta} = \alpha\delta^2(\alpha\bar{\delta}\alpha\delta)^2\delta\alpha\bar{\delta} = (\alpha\delta^2\alpha)\bar{\delta}\alpha\delta\alpha\bar{\delta}(\alpha\delta^2\alpha)\bar{\delta} \\ &= (\delta^2\alpha\delta^2)\bar{\delta}\alpha\delta\alpha\bar{\delta}(\delta^2\alpha\delta^2)\bar{\delta} = \delta^2(\alpha\delta)^4 = \delta^2(\bar{\delta}\bar{\alpha})^3 = ((\alpha\delta)^2\alpha\bar{\delta})^{-1}. \end{aligned}$$

In the orbifold fundamental group seen as a subgroup of $PU(2, 1)$, this element must be a regular elliptic element (this follows from the action of the group G_{24} on \mathbb{P}^2 , recall that every complex reflection in that group has order 2). Since $\alpha\delta$ has order 7, so does $P = R_1J = \delta(\delta\alpha)^3\delta^{-1}$.

We get a group generated by a complex reflection $R_1 = PJ^{-1} = \delta^2$ of angle $2\pi/p$ and a regular elliptic element J of order 3, which satisfy the relations of Proposition 2.1. Discreteness then implies that the group generated by R_1 and J is isomorphic to $\mathcal{S}(4, \bar{\sigma}_4)$. \square

Proof: (of Theorem 3) A priori the group generated by R_1 and J as in Proposition 6.2 is only a subgroup of the one generated by α and δ , we show that these groups are equal, using the relations given in the presentation (12). As before, we write $R_2 = JR_1J^{-1}$, $R_3 = J^{-1}R_1J$, $P = R_1J$. We then have

$$R_2 = \delta\bar{\alpha}\bar{\delta}\alpha\delta\alpha\bar{\delta}, \quad R_3 = \alpha\delta\alpha\bar{\delta}\bar{\alpha},$$

and this implies $R_3^{-1}R_2R_3 = \alpha$. Indeed,

$$R_3^{-1}R_2R_3 = \alpha\delta\bar{\alpha}\bar{\delta}\bar{\alpha} \cdot \delta\bar{\alpha}\bar{\delta}\alpha\delta\alpha\bar{\delta} \cdot \alpha\delta\alpha\bar{\delta}\bar{\alpha} = \alpha\delta\bar{\alpha}\bar{\delta}\bar{\alpha} \cdot \delta\bar{\alpha}\bar{\delta}(\alpha\delta\alpha\bar{\delta})^2\bar{\alpha} = \alpha\delta\bar{\alpha}\bar{\delta}\bar{\alpha} \cdot \delta\bar{\alpha}\bar{\delta}(\delta\alpha\bar{\delta}\alpha)^2\bar{\alpha} = \alpha.$$

One can also write δ in terms of R_1 and J , namely $\delta = P^2R_1P^{-2}R_1P^2$ (the right hand side of this equation may seem complicated, but it appears in previous work of the author, see p. 708 in [14]). Indeed, $P^2 = \delta(\delta\alpha)^6\bar{\delta} = \delta\bar{\alpha}\bar{\delta}^2$, since $\alpha\delta$ has order 7. This gives

$$P^2R_1P^{-2}R_1P^2 = \delta\bar{\alpha}\bar{\delta}^2 \cdot \delta^2 \cdot \delta^2\alpha\bar{\delta} \cdot \delta^2 \cdot \delta\bar{\alpha}\bar{\delta}^2 = \delta\bar{\alpha}(\delta^2\alpha\delta^2)\bar{\alpha}\bar{\delta}^2 = \delta\bar{\alpha}(\alpha\delta^2\alpha)\bar{\alpha}\bar{\delta}^2 = \delta$$

\square

For completeness, we mention the following.

Proposition 6.3. *The lattice $G^{(\infty)}$ is an arithmetic group commensurable with $PU(2, 1, \mathcal{O}_7)$, where \mathcal{O}_7 is the ring of integers in $\mathbb{Q}(i\sqrt{7})$.*

Proof: Proposition 6.2 shows that $G^{(\infty)}$ is generated by a parabolic element R_1 and a regular elliptic element J such that the relations in Proposition 2.1 hold, i.e. it is isomorphic to $\mathcal{S}(\infty, \bar{\sigma}_4)$. Using the (obvious extension to the case $p = \infty$ of the) description of the groups $\mathcal{S}(p, \tau)$ given in section 2.5 of [26], we may write

$$R_1 = \begin{pmatrix} 1 & -1-i\sqrt{7} & \frac{1-i\sqrt{7}}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & \mu & \bar{\mu} \\ \bar{\mu} & 0 & \mu \\ \mu & \bar{\mu} & 0 \end{pmatrix},$$

where $\mu = -i\sqrt{7}\bar{\sigma}_4 = (-7 + i\sqrt{7})/2$. This exhibits $\mathcal{S}(\infty, \bar{\sigma}_4)$ as a subgroup of $U(H, \mathcal{O}_7)$, where H is a form with entries in \mathcal{O}_7 . Since $\mathbb{Q}(i\sqrt{7})$ is a quadratic number field, \mathcal{O}_7 is discrete and $U(H, \mathcal{O}_7)$ is arithmetic.

It follows from the classification of arithmetic subgroups of $PU(2, 1)$ that $G^{(\infty)}$ is commensurable to $PU(2, 1, \mathcal{O}_7)$ (see e.g. [15]). \square

7. CONGRUENCE SUBGROUPS

As mentioned in the introduction, the lattices $G^{(p)}$ for $p = 4, 6, 8, \infty$ appear in work of Barthel, Hirzebruch and Höfer, see [4]. The reason why only these values of p appear there is that Barthel, Hirzebruch and Höfer worked on the level of \mathbb{P}^2 , not on the level of the finite quotient $\mathbb{P}^2/G_{24} = \mathbb{P}(2, 3, 7)$. Since the corresponding quotient map $\mathbb{P}^2 \rightarrow \mathbb{P}^2/G_{24}$ branches with order 2 on the mirrors of reflections in G_{24} , the orbifold structures we consider on \mathbb{P}^2/G_{24} (possibly blown-up) do not lift to an orbifold structure on \mathbb{P}^2 (possibly blown-up). Indeed, generic points on the Klein discriminant curve, which have integer multiplicity p , would lift to points with multiplicity $p/2$ in \mathbb{P}^2 (which is in general not an integer). Similarly, for $p = 12$, the curve E has odd multiplicity, see Table 4.1, and the orbifold structure does not lift to \mathbb{P}^2 either.

The difference between even or odd weights is closely related to the distinction between the INT and the Σ -INT condition for the hypergeometric monodromy groups of Deligne-Mostow, see [24].

In this section, we interpret the coverings corresponding to pulling-back the orbifold structure via $\mathbb{P}^2 \rightarrow G_{24} \setminus \mathbb{P}^2$ for $p = 4, 6, 8, \infty$ as explicit congruence subgroups.

In order to get explicit linear groups $\tilde{\Gamma}_p$ in $U(2, 1)$ (rather than the projectivized $PU(2, 1)$), we use specific matrices R_1, R_2, R_3 as generators of the sporadic group $\tilde{\Gamma}_p$, namely:

$$(13) \quad R_1 = \begin{pmatrix} a & \tau & -\bar{\tau} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ -a\bar{\tau} & a & \tau \\ 0 & 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a\tau & -a\bar{\tau} & a \end{pmatrix}.$$

where $\tau = -\frac{1+i\sqrt{7}}{2}$, $a = e^{2\pi i/p}$. Note that in $PU(2, 1)$, we have $J = (R_1 R_2 R_3 R_1 R_2 R_3 R_1)^{-1}$, so the matrices in equation (13) generate the same group as R_1 and J .

We denote by $\phi_p^{(n)} : \tilde{\mathcal{S}}_p \rightarrow \mathbb{F}_{n^r}$ reduction modulo some prime factor of n in the field $\mathbb{K} = \mathbb{Q}(a, \tau)$. The kernel $\tilde{\Gamma}_p(n) = \text{Ker}(\phi_p^{(n)})$ is a congruence subgroup of $\tilde{\mathcal{S}}_p$. We will prove the following.

Theorem 7.1. (1) $\text{Im}(\phi_4^{(2)})$, $\text{Im}(\phi_8^{(2)})$, $\text{Im}(\phi_\infty^{(2)})$ are all isomorphic to $GL_3(\mathbb{F}_2)$, which is the unique simple group of order 168.

(2) $\text{Im}(\phi_6^{(3)})$ is a subgroup of order 336 in $GL_3(\mathbb{F}_9)$, with center of order 2. The quotient of this subgroup by its center is the simple group of order 168.

In particular, the lattices Γ_p have normal subgroups of index 168 for $p = 4, 6, 8, \infty$. For $p = 3, 5, 12$, there is no such normal subgroup, as can be verified fairly easily using computational group theory software.

Proof: We give a proof for each relevant value of p .

$p = 4$

Here and in what follows, we write $\tau = -(1+i\sqrt{7})/2$. Consider the field $\mathbb{K} = \mathbb{Q}(i\tau)$ and the ideal generated by $\zeta = i - \tau$. Note that ζ is a factor of 2 in $\mathcal{O}_{\mathbb{K}}$, since $(i - \tau)(i - \bar{\tau}) = 1 + i$, which gives

$$(i - \tau)^2(i - \bar{\tau})^2 = 2i.$$

An integral basis for K is given by $1, i(\tau - 1), 2 + \tau, i(\tau - 2)$. One easily checks that $\mathcal{O}_{\mathbb{K}}/(\zeta)$ is a field with two elements, which we simply denote by \mathbb{F}_2 , and i and τ reduce to 1, while $\bar{\tau}$ reduces to 0.

In particular, the matrices R_1, R_2 and R_3 from equation (13) reduce to

$$(14) \quad \tilde{R}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{R}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{R}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

It is well known that $GL_3(\mathbb{F}_2) \cong PSL_2(\mathbb{F}_7)$, and that it is the unique simple group of order 168, see the Atlas of Finite Groups [7] for instance. For an elementary proof, as well as an overview of other proofs in the literature, see also [5].

Using Gaussian elimination, it is easy to see that $GL_3(\mathbb{F}_2)$ is generated by the above three matrices together with two permutation matrices T_{12} and T_{23} corresponding to a transposition of two standard basis vectors.

One can recover such transpositions by verifying that

$$\tilde{R}_1(\tilde{R}_2\tilde{R}_3)^2\tilde{R}_1 = T_{12}, \quad \tilde{R}_2(\tilde{R}_3\tilde{R}_1)^2\tilde{R}_2 = T_{23}.$$

$p = 6$

For $p = 6$, we consider $\mathbb{K} = \mathbb{Q}(\omega\tau)$, where $\omega = (-1 + i\sqrt{3})/2$, and the ideal generated by $\eta = \omega - 1$, which is a prime factor of 3 in \mathcal{O}_K .

One checks that the residue field is \mathbb{F}_9 , which we write as a vector space over \mathbb{F}_3 generated by 1 and u . For a suitable choice of u , the reduction of the matrices are given by

$$(15) \quad \tilde{R}_1 = \begin{pmatrix} 2 & 2u & 2u+1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{R}_2 = \begin{pmatrix} 1 & 0 & 0 \\ u+2 & 2 & 2u \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{R}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & u+2 & 2 \end{pmatrix}.$$

The study of the group generated by these matrices is slightly more subtle than for the case $p = 4$, since $GL_3(\mathbb{F}_9)$ is quite large. We will use the group presentation for G_{24} given by Shephard and Todd, see p. 299 of [28]. We define $A_1 = \tilde{R}_1$, $A_2 = \tilde{R}_2$, $A_3 = \tilde{R}_2\tilde{R}_3\tilde{R}_2$, and verify that $A_1^2 = A_2^3 = A_3^2 = (A_1A_2)^4 = (A_2A_3)^4 = (A_3A_1)^3 = (A_1A_2A_1A_3)^3 = Id$. This implies that $Im(\phi_6^{(3)})$ is a homomorphic image of G_{24} .

Note that G_{24} has an index two subgroup which is the simple group of order 168, and the simplicity of that subgroup implies that the above homomorphism has trivial kernel.

$p = 8$

For $p = 8$, the number field $\mathbb{K} = \mathbb{Q}(\zeta_8, \tau)$ has degree 8. The minimal polynomial of a primitive generator, say $\alpha = i\sqrt{7} + (1+i)/\sqrt{2}$ is given by $x^8 + 28x^6 + 294x^4 + 1288x^2 + 2500$.

The computations are clearly very intricate, and they are best achieved with a computer. One gets a prime factorization of the form $(2) = I_1^4 I_2^4$, where I_1 is the two generator ideal

$$(2, -19/15600\alpha^7 - 7/780\alpha^6 - 49/1950\alpha^5 - 59/312\alpha^4 - 1337/7800\alpha^3 - 1057/780\alpha^2 - 833/3900\alpha - 241/156).$$

One checks that the corresponding residue field is \mathbb{F}_2 , and that the reduction mod I_1 gives the same matrices as in (14). Note also that taking I_2 instead of I_1 would give an isomorphic reduction, since \mathbb{K} is a Galois extension of \mathbb{Q} .

$p = \infty$

For $p = \infty$, the proof is essentially the same as for $p = 4$, with the simpler number field $\mathbb{K} = \mathbb{Q}(i\sqrt{7})$. Note that $\mathcal{O}_{\mathbb{K}}$ is then simply the \mathbb{Z} -module generated by $1, \tau$; we take the ideal generated by τ , which is a prime factor of 2, since $-\tau(\tau + 1) = 2$. The residue field is \mathbb{F}_2 , and one gets the same reduced matrices as in (14).

□

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