Exo 11

1) $f(x) = \dfrac{\ln(1-x)}{x-1}$

$$f'(x) = \frac{\frac{-1}{1-x}(x-1) + \ln(1-x)}{(x-1)^2}$$

$$= \frac{1}{(x-1)^2} + \frac{1}{(x-1)} f(x)$$

$$(x-1) f'(x) - f(x) = \frac{1}{x-1}.$$

2) a) $S = \sum_{n=0}^{\infty} a_n x^n$

$$S' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$(1-x) \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} x^n$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} x^n$$

$$-a_0 + a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} a_n x^n = \dots$$

$$-a_0 + a_1 + \sum_{n=1}^{\infty} \left((n+1) a_{n+1} - (n+1) a_n \right) x^n = \sum_{n=0}^{\infty} x^n$$

$a_1 - a_0 = 1$, $\forall n \geq 1$, $a_{n+1} - a_n = \dfrac{1}{n+1}$ (cf (a)).

(ok pour $n = 0 \to a_1 - a_0 = 1$)

(b) $S(0) = f(0) = 0 \to a_0 = 0.$??

$a_1 = 1$, $a_n = \sum_{k=1}^{n} \dfrac{1}{k}$ $\left(\xrightarrow[n\to\infty]{} \infty \right)$.

(c) $$\frac{a_{n+1}}{a_n} = \frac{a_n + \frac{1}{n+1}}{a_n} \xrightarrow{n\to\infty} 1$$

$R = 1.$

3) $(1-x)f' - f = \frac{1}{1-x}$ $\qquad g = S - f$

$(1-x)S' - S = \frac{1}{1-x}$

$((1-x)g)' = (1-x)g' - g = 0$

$(1-x)g = C^{ste}$

$g = \frac{C}{1-x}$

$g(0) = 0 \to C = 0,\ S = f$

4) $-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$

$\frac{1}{1-x} = \sum_{\ell=0}^{\infty} x^\ell$

$\sum_{k=1}^{\infty} \frac{x^k}{k} \cdot \sum_{\ell=0}^{\infty} x^\ell =$

$\sum_{p=0}^{\infty} \left(\sum_{q=0}^{p-1} \frac{1}{p-q} \right) x^p =$

$\sum_{p=0}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{p}\right) x^p$