FEUILLE 4

Exo 1

(a) 1) domaine $= \mathbb{R}^2$

2) $\frac{\partial f}{\partial x}(x,y) = 3x^2 + 2x\sin(xy) + x^2 y\cos(xy)$

$\frac{\partial f}{\partial y}(x,y) = x^3\cos(xy)$

définies sur $\mathbb{R}^2$

3) $\nabla f(x,y) = (3x^2 + 2x\sin(xy) + x^2 y\cos y \,,\, x^3\cos(xy))$

défini sur $\mathbb{R}^2$

4) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ continues car sin, cos sont continues, et somme et produit de fns continues sont continues

5) $D_w f(v) = \langle \nabla f(v), w \rangle$.

$\nabla f(1,1) = (3 + 2\sin(1) + \cos(1), \cos(1))$

$\langle \nabla f(v), w \rangle = 3 + 2\sin(1) + \cos(1) - \cos(1)$

$= 3 + 2\sin(1)$.

(b) 1) dom $= \mathbb{R}^2$

2) $\frac{\partial f}{\partial x}(x,y) = 4x^3 e^y$ définies sur $\mathbb{R}^2$

$\frac{\partial f}{\partial y}(x,y) = x^4 e^y$

3) $\nabla f(x,y) = (4x^3 e^y, x^4 e^y)$, déf. sur $\mathbb{R}^2$

4) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ continues car e^y est continue, et produit de fns cont. est continu.

5) $\nabla f(0,0) = (0,0)$, $D_w f(v) = 0$

(c) 1) dom $= \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} = B((0,0),1) = B$

2) $\frac{\partial f}{\partial x}(x,y) = \frac{-2x}{1 - x^2 - y^2}$

$\frac{\partial f}{\partial y}(x,y) = \frac{-2y}{1 - x^2 - y^2}$

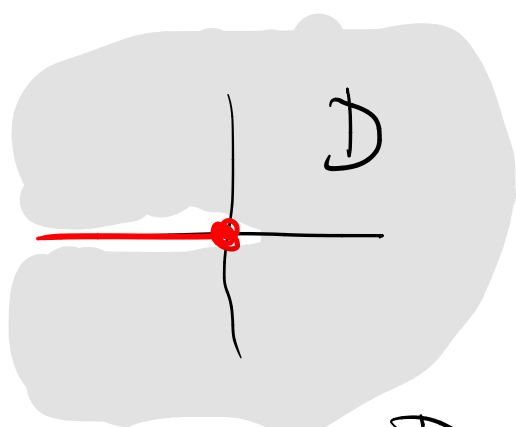
3) $\nabla f(x,y) = -\dfrac{2}{1-x^2-y^2}(x,y)$

4) $1-x^2-y^2 \neq 0$ sur B, quotient de 2 fns continues telles que le dénom. ne s'annule pas est continu

5) $\nabla f(0,\frac{1}{2}) = -\dfrac{2}{1-(\frac{1}{2})^2}(0,\frac{1}{2}) = -\frac{8}{3}(0,\frac{1}{2}) = (0,-\frac{4}{3})$.

$\nabla f(0,5)\cdot(0,2) = -\frac{8}{3} = D_w f(v)$

d) 1) domaine $= \{(x,y)\in\mathbb{R}^2 \mid x+\sqrt{x^2+y^2} > 0\}$



$\sqrt{x^2+y^2} \geq \sqrt{x^2} = |x|$

$x+\sqrt{x^2+y^2} \geq x+|x| \geq 0$

$\to x+\sqrt{x^2+y^2}$ nul ssi ($x\leq 0$ et $y=0$).

$D = \mathbb{R}^2 \setminus \{(x,0) \mid x\leq 0\}$

2) $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{1+\frac{x}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}} \left(= \dfrac{\sqrt{x^2+y^2}+x}{x\sqrt{x^2+y^2}+x^2+y^2}\right)$

$\dfrac{\partial f}{\partial y}(x,y) = \dfrac{\frac{y}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}} \left(= \dfrac{y}{x\sqrt{x^2+y^2}+x^2+y^2}\right)$

définies sur $\mathbb{R}^2\setminus\{(0,0)\}$.

3) $\nabla f(x,y) = \dfrac{1}{x+\sqrt{x^2+y^2}}\left(1+\dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}}\right)$

4) $\sqrt{x^2+y^2} = \|(x,y)\|$ continue

$x+\sqrt{x^2+y^2} \neq 0$ sur $\mathbb{R}^2\setminus\{(0,0)\}$

quotient de fns cont ...

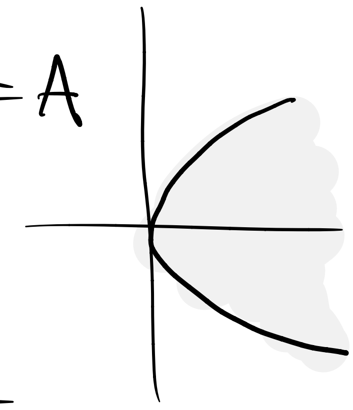
5) $\nabla f(1,0) = \frac{1}{2}(1+1, 0) = (1,0)$

$D_w f(v) = (1,0)\cdot(3,0) = 3.$

e) $f(x,y) = \sqrt{x-y^2}$

1) dom $= \{(x,y) \mid x-y^2 \geq 0\} = A$

$x \geq y^2$



2) $\frac{\partial f}{\partial x}(x,y) = \frac{1}{2}(x-y^2)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x-y^2}}$

$\frac{\partial f}{\partial y}(x,y) = \frac{1}{2}(x-y^2)^{-\frac{1}{2}} \cdot (-2y) = \frac{-y}{\sqrt{x-y^2}}$

domaine $= B = \{(x,y) \in \mathbb{R}^2 \mid x-y^2 > 0\}$

3) $Df(x,y) = \frac{1}{\sqrt{x-y^2}}\left(\frac{1}{2}, -y\right)$.

4) $\sqrt{\cdot}$ continue sur $\mathbb{R}_+^* =]0,+\infty[$.

$\sqrt{x-y^2}$ cont sur B, et non nul

$\frac{1}{2}, -y$ sont cont

quotient de 2 fonctions continues t.q. donc f est continue.

5) $Df(1,0) = \frac{1}{\sqrt{1}}\left(\frac{1}{2}, 0\right)$

$D_v f(v) = \left(\frac{1}{2}, 0\right) \cdot (1,1) = \frac{1}{2}$.

Exo 2 :

1) $J_f(x,y) = \begin{pmatrix} y\cos(xy) & x\cos(xy) \\ e^{-(x^2+y^2)}(1-2x^2) & -2xy\, e^{-(x^2+y^2)} \end{pmatrix}$

$f(x,y) = (f_1(x,y), f_2(x,y))$

$\frac{\partial f_2}{\partial x}(x,y) = e^{-(x^2+y^2)} + x\, e^{-(x^2+y^2)} \cdot (-2x)$ (etc)

$J_f(0,0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

2) $J_f(x,y,z) = \begin{pmatrix} 2x & -z & -y \\ yz & x z & xy \end{pmatrix}$

$J_f(1,1,1) = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

3) $J_f(x,y) = \begin{pmatrix} (y+2x)e^{xy+x^2} & x e^{xy+x^2} \\ -\sin(x+2y) & -2\sin(x+2y) \\ 1 & 2y \end{pmatrix}$

$J_f(0,1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \end{pmatrix}$

4) $J_f(x,y,z) = \begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{pmatrix}$

$J_f(1,0,3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 3 & 0 \\ 3 & 0 & 1 \end{pmatrix}$

Exo 3

a) $\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$

$\nabla f(x,y) = (2x+y, x+2y)$

$J_g(s,t) = \begin{pmatrix} 1 & \cos t \\ -1 & e^t \end{pmatrix}$

$h = f \circ g$

$Dh(v) =$

$Df_{g(v)} \circ Dg(v)$

$g(v) = g(0,0) = (0,1)$

$J_g(0,0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$\nabla f(g(0,0)) = \nabla f(0,1) = (1,2)$

$\nabla h(v) = (1\ 2)\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = (-1,3).$

b) $\nabla f(x,y) = (-\sin(x+4y), -4\sin(x+4y))$

$J_g(s,t) = \begin{pmatrix} 0 & 20t^3 \\ 6s & 0 \end{pmatrix}$

$g(1,1) = (5,3)$

$\nabla f(5,3) = (-\sin(17), -4\sin(17))$

$J_g(1,1) = \begin{pmatrix} 0 & 20 \\ 6 & 0 \end{pmatrix}$

$\nabla h(1,1) = (-\sin(17) \quad -4\sin(17)) \begin{pmatrix} 0 & 20 \\ 6 & 0 \end{pmatrix}$

$= (-24\sin(17), -20\sin(17)).$

c) $\nabla f(x,y) = \frac{1}{\sqrt{1+x^2+y^2}}(x,y)$ $\qquad \nabla f(0,-1) = \frac{1}{\sqrt{2}}(0,-1).$

$J_g(s,t) = \begin{pmatrix} \cos s & 0 \\ -\sin s & 0 \end{pmatrix}$ $\qquad J_g(\pi,0) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$

$(\pi,0) \xrightarrow{g} g(\pi,0) = (0,-1) \xrightarrow{f} \sqrt{2}$

$\nabla h(\pi,0) = \frac{1}{\sqrt{2}}(0 \quad -1)\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = (0 \quad 0).$

2) $h = f \circ g$

$J_h(u) = J_f(g(u)) \cdot J_g(u).$

$J_f(x,y) = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}, \quad J_g(s,t) = \begin{pmatrix} t & s \\ 2s & 2t \end{pmatrix}$

$J_g(f(x,y)) = \begin{pmatrix} 3x+4y & 2x \\ 4x & 6x+8y \end{pmatrix}$

$J_h(x,y) = \begin{pmatrix} 3x+4y & 2x \\ 4x & 6x+8y \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 12x+8y & 8x \\ 26x+24y & 24x+32y \end{pmatrix}$

$$J_f(x,y) = \begin{pmatrix} \cos x & 0 \\ y e^{xy} & x e^{xy} \end{pmatrix}$$

$$J_g(s,t) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$J_h(x,y) = J_g(f(x,y)) \cdot J_f(x,y)$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \cos x & 0 \\ y e^{xy} & x e^{xy} \end{pmatrix}$$

$$= \begin{pmatrix} \cos x - y e^{xy} & -x e^{xy} \\ \cos x + y e^{xy} & x e^{xy} \end{pmatrix}.$$

Exo 4

$$\nabla f(x,y) = \left(-y^2 e^{-xy}, (1-xy)\, e^{-xy}\right)$$

$$\nabla f(0,2) = (-4, 1)$$

$$(-4,1)\cdot(v_1, v_2) = 1$$

$$\|(v_1, v_2)\| = 1$$

$$\begin{cases} v_1^2 + v_2^2 = 1 \\ -4v_1 + v_2 = 1 \end{cases}$$

$$\Rightarrow v_1^2 + (1+4v_1)^2 = 1$$

$$17 v_1^2 + 8 v_1 = 0$$

$$v_1(17 v_1 + 8) = 0$$

$v = (0, 1)$ et $v = \left(-\frac{8}{17}, -\frac{15}{17}\right)$