Distortion function and the heat kernel of a positive line bundle

by

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Introduction

Let X be a compact, connected, complex manifold of dimension n, endowed with a hermitian metric ω. Let E be a hermitian line bundle over X, and D its Chern connection (i.e. D is compatible with the metric of E, and its $(0,1)$ part is $\bar{\partial}$). We denote $D^2 = c(E)$ the curvature tensor of E, considered as a $(1,1)$-form on X. Furthermore, let $\alpha_1 \leq \ldots \leq \alpha_n$ be the eigenvalues of $ic(E)$ with respect to ω. When E is positive (i.e. $\alpha_1 > 0$), we can construct a family of new metrics on E called the Fubini-Study metrics. Namely, the diagram

$$\begin{array}{ccc} E^k & \longrightarrow & \mathcal{O}(1) \\ \downarrow & & \downarrow \\ X & \stackrel{\pi}{\longrightarrow} & \mathbb{P}(V_k^*) \end{array} \qquad V_k = H^0(X, E^k)$$

(where π is an embedding for k sufficiently large (Kodaira theorem)) induces on E^k the pull-back metric of the Fubini-Study metric on $\mathcal{O}(1)$. If $(s_0, \ldots, s_{N_k})$ is an orthonormal basis of $H^0(X, E^k)$ (with respect to the global L^2 norm), these s_j define coordinates on V_k^* so that the Fubini-Study metric of E^k has the expression $|\xi|^2_{F-S} = (|s_0(x)|^2 + \ldots + |s_{N_k}(x)|^2)^{-1}\ |\xi\ |^2_k$ in terms of the initial metric of E^k for some $\xi \in E^k_x$. We call b_k the *distortion function* between these two metrics, which was introduced by Kempf in the case of invertible sheaves over Abelian varieties. Obviously,

$$b_k(x) = |s_0(x)|^2 + \ldots + |s_{N_k}(x)|^2.$$

Kempf [1] proved the following inequality $ak^{2n} \leq b_{k^2} \leq ck^{2n}$ for some positive constants a, c. Later, Ji [4] gave a less natural analoguous inequality for all k in the same context. Separately, Tian [7] has constructed so-called peak sections of

E^k of unit L^2 norm, which lead to the estimate $b_k(x) = k^n + O(k^{n-1})$ over any Kähler variety endowed with the metric $\omega = ic(E)$. We shall prove here, by means of a heat kernel estimate the following theorem :

Theorem 1.— If ω is any metric over X (not necessarily Kähler), call $\det ic_x(E) = \prod_{j=1}^{n} \alpha_j(x)$*; then the following equivalent holds*

$$b_k(x) \sim k^n \det ic_x(E) \quad as \quad k \to +\infty$$

uniformly with respect to x.

Theorem 1 & 2 below have been lectured late summer by J.-P. Demailly in Santa-Cruz [3]. The complete proofs are available in [2].

1. The heat kernel method

Define $\square_k^q = (\bar{\partial}_k^* + \bar{\partial}_k)^2$ where $\bar{\partial}_k$ acts on $(0,q)$-forms with values in E^k. This is an elliptic operator so that the operator $e^{-\frac{2t}{k}\square_k^q}$ posseses a Schwartz kernel, called the heat kernel of E^k with the following expression: for $j = 0, 1, \ldots, n$, let λ_j^k be the eigenvalues of $\square_k^q$, and ψ_j be an orthonormal L^2 basis of eigenforms associated to λ_j^k, then

$$e_k^q(t,x,x) = \sum_{j=0}^{+\infty} e^{-\frac{2t}{k}\lambda_j^k} |\psi_j(x)|^2. \tag{1}$$

Remark that b_k is exactly the zero-eigenvalue part of e_k^q, and write $e_k^q(t,x,x) = b_k(x) + r_k(t,x)$. We are going to use the next theorem to compute an equivalent for e_k^q when $t = k^\varepsilon$ tends to infinity, and to prove that r_k tends to zero.

Theorem 2.— For general E (no positivity asumption), let $\overline{\alpha_J} = \sum_{j \notin J} \alpha_j - \sum_{j \in J} \alpha_j$, then $\exists \varepsilon > 0$ such that

$$e_k^q(t,x,x) = (4\pi)^{-n} \Big(\sum_{|J|=q} e^{t\overline{\alpha_J}} \Big) \prod_{j=1}^{n} \frac{\alpha_j(x)}{\sinh \alpha_j(x) t} \big(k^n + o(k^n)\big)$$

uniformly with respect to $x \in X$ and $t \in [t_0, k^\varepsilon]$.

The proof of theorem 2 involves four steps:

1.— By the Bochner-Kodaira-Nakano identity, $\frac{2}{k}\overline{\square}_k^q$ can be seen as a Schrödinger operator with magnetic field $ic(E)$ and electric field $\bar{A}$, a diagonal matrix with entries $\overline{\alpha}_J$ in the $d\bar{z}_J \otimes e_k$ basis of $\Lambda^q \overline{T^*X} \otimes E^k$.

2.— We thus study a uniform equivalent for a general Schrödinger operator. First, we show that the asymptotic behavior of e_k^q only depends on the local value of $\overline{\square}_k^q$. This is obtained by means of the Kato inequality and the maximum principle for the eigenfunctions of a parabolic operator.

3.— We compute explicitly the heat kernel associated to $\widetilde{\square}_k^q$ (i.e. the Schrödinger operator defined by the datas of $\overline{\square}_k^q$ frozen at x^0). This is a straightforward consequence of the Mehler formula.

4.— At last, we show that e_k^q can be expressed as an infinite sum only depending on $\widetilde{\square}_k^q$ on a ball of radius r_k. By 3., we have an explicit computation of this sum, thus a good control of convergence, and, by 2., we know it is consistant if r_k is chosen such that $kr_k^2 \to +\infty$ and $kr_k^3 \to 0$.

2. The proof of theorem 2, with some remarks & consequences

As theorem 1 is a very easy consequence of theorem 2, we shall give the complete proof here. First, as $\sinh \alpha_j t \sim {}^1\!/_2 \quad e^{|\alpha_j|t}$, when $t \to +\infty$, we see that $e^{\overline{\alpha_J}(x)t}/\prod_{j=1}^n \sinh \alpha_j(x)t$ tends to 0 unless x is a point where $ic(E)$ is of index q, and J is precisely the multiindex of those j's for which α_j is negative. Therefore, if we denote $X(q)$ the open subset of X where $ic(E)$ is nondegenerate and has q negative eigenvalues, we have

$$e_k^q(k^\varepsilon, x, x) \underset{k\to+\infty}{\longrightarrow} \mathbf{1}_{X(q)} k^n \det \frac{i}{2\pi} c_x(E). \tag{2}$$

When E is positive, $X = X(0)$, and so we only have to prove that $r_k(k^\varepsilon, x) \to 0$. On one hand, observe that theorem 2 implies, for k sufficiently large :

$$e_k^0(t, x, x) \leq Ck^n \qquad \text{for} \quad t \in [2t_0, k^\varepsilon]$$

so, (1) yields

$$|\psi_j(x)|^2 \leq Ck^n e^{\frac{t\lambda_j^k}{k}}$$

Now, if ψ_j is an eigenfunction of $\overline{\Box}_k^0$ for a nonzero eigenvalue λ_j^k, $\bar{\partial}_k\psi_j$ is a nonzero eigenform of $\overline{\Box}_k^1$ for the same eigenvalue λ_j^k ; which implies that

$$\sum_{\lambda_j^k\neq 0} e^{-\frac{2t\lambda_j^k}{k}} \frac{|\bar{\partial}_k\psi_j(x)|^2}{\|\bar{\partial}_k\psi_j\|^2} \leq e_k^1(t,x,x)$$

$$\leq Ck^n e^{-\alpha_1 t}$$

for k large and $t \in [t_0, k^\varepsilon]$. Integrating both sides, we obtain

$$\sum_{\lambda_j^k\neq 0} e^{-\frac{t\lambda_j^k}{k}} \leq Ck^n e^{-\frac{\alpha_0 t}{2}}. \tag{3}$$

Now,

$$r_k(k^\varepsilon, x) = \sum_{\lambda_j^k>0} e^{2k^{\varepsilon-1}\lambda_j^k}|\psi_j(x)|^2 \leq Ck^{2n}e^{-\frac{\alpha_0 k^\varepsilon}{2}}.$$

Q.E.D.

The first consequence of theorem 1 is that the Fubini-Study metrics converge to the initial one on E, because $b_k^{1/k}$ tends to 1. Tian proved that it is also true for the Bergmann metrics of X (i.e. the curvature forms of the F.-S. metrics), which involves higher derivatives estimates.

An other consequence of formula 2 is Demailly's holomorphic Morse inequalities (of which theorem 2 can be seen as a *pointwise (*or *local* version (in the sense there are local index theorems) although previous approaches were *global* ones). This is quite straightforward by means of this

Lemma (Bismut [1]).— *Using the previous notations*

$$\sum_{\nu=0}^{q}(-1)^{q-\nu}\dim H^\nu(X,E^k) \leq \sum_{\nu=0}^{q}(-1)^{q-\nu}\int_X e_k^\nu(t,x,x)\frac{\omega^n}{n!}$$

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and letting $t = k^\varepsilon$ go to infinity (see [3] for complete statements & references on this subject).

References

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