Rational curves on K3 surfaces

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June 15, 2010

DRAFT VERSION
for distribution to the participants of the 2010 Grenoble summer school

Introduction

This document is based on lectures given at the 2007 NATO Advanced Study Institute on 'Higher-Dimensional Geometry over Finite Fields', organized at the University of Göttingen by Yuri Tschinkel. It is also based on lectures given at the 2010 summer school 'Arithmetic Aspects of Rational Curves', organized at the Institut Fourier in Grenoble by Emmanuel Peyre.

This work is supported in part by National Science Foundation Grants #0134259, #0554491, and #0901645.

1 Elements of the geometry of K3 surfaces

Let k be a field.

Definition 1 A *K3 surface* is a smooth projective geometrically integral surface X/k such that the canonical class $K_X = 0$ and $H^1(X, \mathcal{O}_X) = 0$.

A compact complex manifold X with these properties is also called a K3 surface.

Example 2 Equations of low-degree K3 surfaces can be written quite explicitly:

1. Branched double covers: Assume that $\mathrm{char}(k) \neq 2$ and $G_6 \in k[x_0, x_1, x_2]$ is homogeneous of degree 6. The branched double cover of $\mathbb{P}^2$

$$X = \{[w, x_0, x_1, x_2] : w^2 = G_6(x_0, x_1, x_2)\}$$

is a K3 surface if it is smooth.

2. Quartic surfaces: For $F_4 \in k[x_0, x_1, x_2, x_3]$ homogeneous of degree four consider

$$X = \{F_4(x_0, x_1, x_2, x_3) = 0\} \subset \mathbb{P}^3.$$

Then X is a K3 surface if it is smooth.

3. Sextic surfaces: Consider $F_2, F_3 \in k[x_0, x_1, x_2, x_3, x_4]$ Homogeneous of degrees two and three respectively defining a complete intersection surface

$$X = \{F_2 = F_3 = 0\} \subset \mathbb{P}^4.$$

Then X is a K3 surface if it is smooth.

4. Degree-eight surfaces: Consider quadratic polynomials $P, Q, R \in k[x_0, \dots, x_5]$ defining a complete intersection surface

$$X = \{P = Q = R = 0\} \subset \mathbb{P}^5.$$

Again, X is a K3 surface if it is smooth.

From the definition, we can deduce some immediate consequences. Let Ω^1_X, Ω^2_X denote the sheaves of 1-forms and 2-forms on X and $T_X = (\Omega^1_X)^*$ the tangent bundle. Since $K_X = [\Omega^2_X]$ is trivial there exists an everywhere-non-vanishing section

$$\omega \in \Gamma(X, \Omega^2_X).$$

Contraction by ω induces an isomorphism of sheaves

$$\iota_\omega : T_X \xrightarrow{\sim} \Omega^1_X$$

and thus isomorphisms of cohomology groups

$$H^i(X, T_X) \xrightarrow{\sim} H^i(X, \Omega^1_X).$$

Serre duality for K3 surfaces takes the form

$$H^i(X, \mathcal{F}) \simeq H^{2-i}(X, \mathcal{F}^*)^*$$

so in particular

$$H^0(X, \Omega^1_X) \simeq H^2(X, T_X)^* \simeq H^2(X, \Omega^1_X)^*$$

and

$$H^2(X, \mathcal{O}_X) \simeq \Gamma(X, \mathcal{O}_X)^*.$$

The Noether formula

$$\chi(X, \mathcal{O}_X) = \frac{c_1(T_X)^2 + c_2(T_X)}{12}$$

therefore implies that

$$c_2(T_X) = 24.$$

We briefly review the geometric properties of K3 surfaces over $k = \mathbb{C}$.

Computation of Hodge numbers and related cohomology Hodge theory [GH78] gives additional information about the cohomology. We have the decompositions

$$\begin{aligned} H^1(X,\mathbb{C}) &= H^0(X,\Omega^1_X) \oplus H^1(X,\mathcal{O}_X) \\ H^2(X,\mathbb{C}) &= H^0(X,\Omega^2_X) \oplus H^1(X,\Omega^1_X) \oplus H^2(X,\mathcal{O}_X) \end{aligned}$$

where the outer summands are exchanged by complex conjugation. It follows that

$$\Gamma(X,\Omega^1_X) = 0$$

so K3 surfaces admit no vector fields. Furthermore, using the Gauss-Bonnet theorem

$$\chi_{\mathrm{top}}(X) = c_2(T_X) = 24$$

we can compute

$$\dim H^1(X,\Omega^1_X) = 24 - 4 = 20.$$

The Lefschetz Theorem on $(1,1)$-classes describes the Néron-Severi group of X:

$$\mathrm{NS}(X) = H^2(X,\mathbb{Z}) \cap H^1(X,\Omega^1_X).$$

Thus we get a bound on its rank

$$\rho(X) := \mathrm{rank}(\mathrm{Pic}(X)) = \mathrm{rank}(H^1(X,\Omega^1_X) \cap H^2(X,\mathbb{Z})) \leq 20.$$

Another key application is to deformation spaces, in the sense of Kodaira and Spencer. Let $\mathrm{Def}(X)$ denote the deformations of X as a complex manifold. The tangent space

$$T_{[X]}\mathrm{Def}(X) \simeq H^1(X,T_X)$$

and the obstruction space is $H^2(X,T_X)$.

Corollary 3 *If X is a K3 surface then the deformation space* $\mathrm{Def}(X)$ *is smooth of dimension* 20.

However, the general complex manifold arising as a deformation of X has no divisors or non-constant meromorphic functions. If h denotes an ample line bundle, we can consider $\mathrm{Def}(X,h)$, i.e., deformations of X that preserve the polarization h. Its infinitesimal properties are obtained by analyzing cohomology the *Atiyah extension* [Ati57]

$$0 \to \mathcal{O}_X \to \mathcal{E} \to T_X \to 0$$

classified by

$$[h] \in H^1(X,\Omega^1_X) = \mathrm{Ext}^1(T_X,\mathcal{O}_X).$$

We have

$$T_{[X,h]}\mathrm{Def}(X,h) = H^1(X,\mathcal{E}) = \ker(H^1(X,T_X) \stackrel{\cap[h]}{\to} H^2(X,\mathcal{O}_X))$$

and using the contraction $\iota_\omega : T_X \to \Omega^1_X$ we may identify

$$T_{[X,h]}\mathrm{Def}(X,h) \simeq h^\perp \subset H^1(X,\Omega^1_X),$$

i.e., the orthogonal complement of h with respect to the intersection form. The obstruction space

$$H^2(X,\mathcal{E}) \simeq \mathrm{coker}(H^1(X,T_X) \xrightarrow{\cap[h]} H^2(X,\mathcal{O}_X)) = 0$$

because $H^2(X,T_X) = 0$. Thus we have

Corollary 4 *If (X,h) is a K3 surface then the deformation space* $\mathrm{Def}(X,h)$ *is smooth of dimension* 19*.*

In fact, the Hodge decomposition of a K3 surface completely determines its complex structure

Theorem 5 (Torelli Theorem) *[PŠŠ71] [LP81] Suppose that X and Y are complex K3 surfaces and there exists an isomorphism*

$$\phi : H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(Y,\mathbb{Z})$$

respecting the intersection form

$$\phi(\alpha).\phi(\beta) = \alpha.\beta$$

and the Hodge decomposition

$$(\phi \otimes \mathbb{C})(H^0(X,\Omega^2_X)) = H^0(Y,\Omega^2_Y).$$

Then there exists an isomorphism $X \simeq Y$.

The geometric properties of complex K3 surfaces are thus tightly coupled to its cohomology; most geometric information about X can be read off from $H^2(X)$.

Theorem 6 (Surjectivity of Torelli) *[LP81] [B$^+$85] Let Λ denote the lattice isomorphic to the middle cohomology of a K3 surface under the intersection form. Each Hodge decomposition of $\Lambda \otimes \mathbb{C}$ arises as the complex cohomology of a (not-necessarily algebraic) K3 surface.*

Remark 7 [B$^+$85] [LP81] The cohomology lattice can be explicitly computed

$$\Lambda \simeq U^{\oplus 3} \oplus (-E_8)^{\oplus 2}$$

where

$$U \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and E_8 is the positive definite lattice associated to the Lie group of the same name.

Example 8 Produce an example of a quartic K3 surface $X \subset \mathbb{P}^3$ with three disjoint lines L_1, L_2, L_3.

By the adjunction formula

$$L_i^2 + K_X.L_i = 2g(L_i) - 2$$

we know that $L_i^2 = -2$. Letting h denote the polarization class, the middle cohomology of the desired K3 surface would have a sublattice

$$M = \begin{array}{c|cccc} & h & L_1 & L_2 & L_3 \\ \hline h & 4 & 1 & 1 & 1 \\ L_1 & 1 & -2 & 0 & 0 \\ L_2 & 1 & 0 & -2 & 0 \\ L_3 & 1 & 0 & 0 & -2 \end{array}.$$

Using basic lattice theory, we can embed

$$M \hookrightarrow \Lambda.$$

Surjectivity of Torelli gives the existence of a K3 surface X with

$$\operatorname{Pic}(X) \supset M \supset \mathbb{Z}h;$$

we can even choose h to be a polarization on X. Global sections of $\mathcal{O}_X(h)$ give an embedding [SD74]

$$|h| : X \hookrightarrow \mathbb{P}^3$$

with image having the desired properties.

2 The Mori-Mukai argument

A polarized K3 surface (S, f) consists of a K3 surface and an ample divisor f that is primitive in the Picard group. Its degree is the positive even integer $f \cdot f$. described in Example 2.

Let $\mathcal{K}_g, g \geq 2$ denote the moduli space (stack) of complex polarized K3 surfaces of degree $2g-2$, which is smooth and connected of dimension 19.

The following is attributed to Mumford, although it was known to Bogomolov around the same time:

Theorem 9 *[MM83] Every complex projective K3 surface contains at least one rational curve. Furthermore, suppose $(S, f) \in \mathcal{K}_g$ is very general, i.e., in the complement of a countable union of Zariski-closed proper subsets. Then S contains an infinite number of rational curves.*

Idea: Let N be a positive integer. Exhibit a K3 surface $(S_0, f) \in \mathcal{K}_g$ and smooth rational curves $C_i \to S_0$, with $[C_1 \cup C_2] = Nf$ and $[C_i] \not\sim f$. Deform $C_1 \cup C_2$ to an irreducible rational curve in nearby fibers.

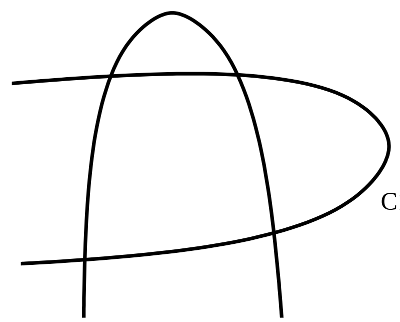

Figure 1: Two smooth rational curves in S_0

Kummer construction (for $N = 1$) We exhibit a K3 surface S_0 containing two smooth rational curves C_1 and C_2 meeting transversally at $g + 1$ points.

Let E_1 and E_2 be elliptic curves admitting an isogeny $E_1 \to E_2$ of degree $2g + 3$ with graph $\Gamma \subset E_1 \times E_2$; $p \in E_2$ a 2-torsion point Take the associated Kummer surface

$$S_0 = (E_1 \times E_2)/\langle \pm 1 \rangle .$$

Γ intersects $E_1 \times p$ transversally in $2g + 3$ points, one of which is 2-torsion in $E_1 \times E_2$. Take C_1 and C_2 to be the images of Γ and $E_1 \times p$ in S_0, smooth rational curves meeting transversally in $g + 1$ points.

The sublattice of $\mathrm{Pic}(S_0)$ determined by C_1 and C_2 is:

	C_1	C_2
C_1	-2	$g+1$
C_2	$g+1$	-2

$f = C_1 + C_2$ is big and nef and has no higher cohomology (by Kawamata-Viehweg vanishing). Deform (S_0, f) to a polarized $(S, f) \in \mathcal{K}_g$

$$\mathcal{S} \to B, \quad \dim(B) = 1,$$

with f ample and indecomposable in the effective monoid.

$H^i(\mathcal{O}_{S_0}(f)) = 0, i > 0$ thus $C_1 \cup C_2$ is a specialization of curves in the generic fiber and $\mathrm{Def}(C_1 \cup C_2 \subset \mathcal{S})$ is smooth of dimension $g + 1$. The locus in $\mathrm{Def}(C_1 \cup C_2 \subset \mathcal{S})$ parametrizing curves with at least ν nodes has dimension $\geq g + 1 - \nu$. When $\nu = g$ the corresponding curves are necessarily rational. The fibers of $\mathcal{S} \to B$ are not uniruled and thus contain a finite number of these curves, so the rational curves deform into nearby fibers.

Conclusion For $(S, f) \in \mathcal{K}_g$ generic, there exist rational curves in the linear series $|f|$. However, rational curves can only specialize to unions of rational curves (with multiplicities), thus *every* K3 surface in $\mathcal{K}_g$ contains at least one rational curve

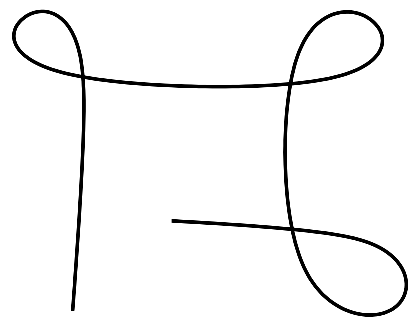

Figure 2: Deformation of $C_1 \cup C_2$ in nearby K3 surface

Remark 10 Yau-Zaslow [YZ96], Beauville [Bea99], Bryan-Leung [BL00], Xi Chen [Che99, Che02], etc. have beautiful *enumerative results* on the rational curves in $|f|$

Generalized construction (for arbitrary N) Let (S_0, f) be a polarized K3 surface of degree $2g-2$ with

$$\mathrm{Pic}(S_0)_{\mathbb{Q}} = \mathbb{Q}C_1 + \mathbb{Q}C_2$$

with C_1 and C_2 smooth rational curves and

$$Nf = C_1 + C_2$$

where:

	C_1	C_2
C_1	-2	$N^2(g-1)+2$
C_2	$N^2(g-1)+2$	-2

The existence of these can be deduced from surjectivity of Torelli, i.e., take a general lattice polarized K3 as above.

Deform (S_0, f) to a polarized $(S, f) \in \mathcal{K}_g$ as above

$$\mathcal{S} \to B, \quad \dim(B) = 1.$$

$\mathrm{Def}(C_1 \cup C_2 \subset \mathcal{S})$ is smooth of dimension $N^2(g-1)+2$; the locus parametrizing curves with at least $N^2(g-1)+1$ nodes (i.e., the rational curves) has dimension ≥ 1. There are a finite number in each fiber, thus we obtain *irreducible* rational curves in $|Nf|$ for generic K3 surfaces in $\mathcal{K}_g$.

This argument proves that *very general* K3 surfaces admit irreducible rational curves in $|Nf|$ for each $N \in \mathbb{N}$. In particular, they have admit infinitely many rational curves. Conceivably, for special K3 surfaces these might coincide, i.e., so that the infinite number of curves *all* specialize to cycles

$$m_1 C_1 + \ldots + m_r C_r$$

supported in a *finite* collection of curves.

Remark 11 Lee-Leung [LL05] and Li-Wu [LW06, Wu07] have enumerated curves in $|2f|$, analyzing the contributions of reducible and non-reduced rational curves

A. Klemm, D. Maulik, R. Pandharipande, and E. Scheidegger [KDPS08] have recently shown that the BPS count of rational curves in $|Nf|$ (i.e., the number from Gromov-Witten theory taking the multiple cover formula into account) depends only on the self intersection

$$(Nf) \cdot (Nf) = N^2(2g-2),$$

not on the divisibility N.

3 Questions on rational curves

3.1 Conjectures and how they are related

Let K be algebraically closed field of characteristic zero and T/K projective K3 surface. The following is well-known but hard to trace in the literature:

Question 12 (Main conjecture) There exist an infinite number of rational curves on T.

The following extreme version is more easily attributed:

Conjecture 13 (Bogomolov 1981) Let S be a K3 surface defined over number field F. Then each point $s \in S(F)$ is contained in a rational curve $C \subset S$ defined over $\overline{\mathbb{Q}}$.

It would follow that $S_{\overline{\mathbb{Q}}}$ has an infinite number of rational curves, because $S(\overline{\mathbb{Q}})$ is Zariski dense in S. Moreover, we can easily reduce the Main Conjecture to the case of number fields.

Proposition 14 *[Blo72, Ran95, Voi92] Let B be a smooth complex variety, $\pi : \mathcal{T} \to B$ a family of K3 surfaces, and $\mathcal{D}$ a divisor on $\mathcal{T}$. Then the set*

$$V := \{b \in B : \textit{there exists a rational curve } C \subset \mathcal{T}_b = \pi^{-1}(b) \\ \textit{with } [C] = \mathcal{D}_b\}.$$

is open. More precisely, any generic immersion

$$f_b : \mathbb{P}^1 \to \mathcal{T}_b, \quad f_{b*}[\mathbb{P}^1] = \mathcal{D}_b,$$

can be deformed to nearby fibers.

Thus rational curves deform provided their homology classes remain of type $(1,1)$. Note the use of Hodge theory!

Proof: Main Conjecture/$\overline{\mathbb{Q}}$ $\Rightarrow$ Main Conjecture/K

Suppose there exists a K3 surface T over K with a finite number of rational curves. We may assume that K is the function field of some variety $B/\overline{\mathbb{Q}}$.

'Spread out' to get some family $\mathcal{T} \to B$, and choose a point $b \in B(\overline{\mathbb{Q}})$ such that the fiber $\mathcal{T}_b$ has general Picard group

$$\mathrm{Pic}(\mathcal{T}_b) = \mathrm{Pic}(T_{\overline{K}}).$$

Since $\mathcal{T}_b$ has a infinite number of rational curves, the same holds for T.

3.2 Rational curves on special K3 surfaces

Theorem 15 *[BT00] Let S be a complex projective K3 surface admitting either*

1. *a non-isotrivial elliptic fibration; or*
2. *an infinite group of automorphisms.*

Then S admits an infinite number of rational curves.

The argument actually goes through for all but the most degenerate elliptic K3 surfaces, which turn out to be either Kummer elliptic surfaces or to have Néron-Severi group of rank twenty.

Proof: in the case $|\mathrm{Aut}(S)| = \infty$

Consider the monoid of effective divisors on S Each nonzero indecomposable element D contains rational curves by the Mori-Mukai argument (when $D \cdot D > 0$) or direct analysis (when $D \cdot D = 0, -2$) It suffices to show there must be infinitely many such elements This is clear, because otherwise the image of

$$\mathrm{Aut}(S) \to \mathrm{Aut}(\mathrm{Pic}(S))$$

would be finite, so the kernel would have to be infinite, which is impossible

Example 16 Let Λ be a rank-two even lattice of signature $(1, 1)$ that does not represent -2 or 0, and (S, f) polarized K3 surface with $\mathrm{Pic}(S) = \Lambda$.

The positive cone

$$\mathcal{C}_S := \{D \in \Lambda : D \cdot D > 0, D \cdot f > 0\}$$

equals the ample cone and is bounded by irrational lines. Infinitely-many indecomposable effective divisors implies infinitely-many rational curves in S.

Note: K3 surfaces with $\mathrm{Aut}(S)$ infinite or admitting an elliptic fibration have

$$\mathrm{rank}(\mathrm{Pic}(S)) \geq 2.$$

Thus these techniques do not apply to 'most' K3 surfaces. Indeed, I know no example before 2009 of a K3 surface $S/\overline{\mathbb{Q}}$ with $\mathrm{Pic}(S) = \mathbb{Z}$ admitting infinitely many rational curves This is entirely consistent with the possibility that the Mori-Mukai argument might break down over a countable union of subvarieties in $\mathcal{K}_g$.

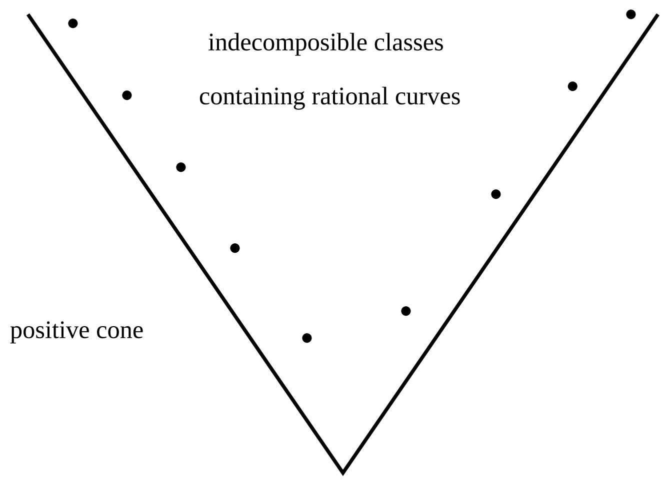

Figure 3: 'Typical' rank-two K3 surfaces have infinitely many curve classes containing rational curves

4 K3 surfaces in positive characteristic

4.1 What goes wrong in characteristic p?

In characteristic zero, K3 surfaces are never unirational (or even uniruled). Indeed, if there were a dominant map

$$\mathbb{P}^2 \dashrightarrow X$$

then we can resolve indeterminacy to a morphism from a smooth projective rational surface

$$\phi : S \to X.$$

Since the derivative of ϕ is non-vanishing at the generic point, $\phi^*\omega$ would be a nonzero twoform on S.

In characteristic p the derivative of a map *can* vanish everywhere. This happens when the associated extension of function fields

$$\begin{array}{c} k(S) \\ | \\ k(X) \end{array}$$

has inseparability.

Example 17 (Shioda's example [Shi74]) Consider the Fermat hypersurface

$$X = \{x_1^{q+1} - x_2^{q+1} = x_3^{q+1} - x_4^{q+1}\}$$

over a field of characteristic $p \neq 2$, where $q = p^e$. (Our main interest is the Fermat quartic K3 surface over a field of characteristic 3.) This is unirational.

Setting

$$x_1 = y_1 + y_2,\ x_2 = y_1 - y_2,\ x_3 = y_3 + y_4,\ x_4 = y_3 - y_4$$

we can rewrite our equation as

$$y_1 y_2 (y_1^{q-1} + y_2^{q-1}) = y_3 y_4 (y_3^{q-1} + y_4^{q-1}).$$

Dehomogenize by setting $y_4 = 1$ and write

$$y_2 = y_1 u, \quad y_3 = uv$$

so we obtain

$$y_1^{q+1}(1 + u^{q-1}) = v(u^{q-1}v^{q-1} + 1).$$

Take the *inseparable* field extension $t = y_1^{1/q}$ so we have

$$t^{(q+1)q}(1 + u^{q-1}) = v(u^{q-1}v^{q-1} + 1)$$

or

$$u^{q-1}(t^{q+1} - v)^q = v - t^{q(q+1)}.$$

Setting $s = u(t^{q+1} - v)$ we get

$$s^{q-1}(t^{q+1} - v) = v - t^{q(q+1)}$$

whence

$$v = t^{q+1}(s^{q-1} + t^{q^2-1})/(s^{q-1} + 1).$$

Thus the function field $k(X)$ admits an extension equal to $k(t, s)$ and so there is a degree-q dominant map

$$\mathbb{P}^2 \dashrightarrow X.$$

Exercise 18 Show that

$$X = \{x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0\}$$

is unirational over any field of characteristic p with $p \equiv 3 \pmod 4$.

Shioda also showed that the Néron-Severi groups of these surfaces behave quite strangely. Recall that the Néron-Severi group $\mathrm{NS}(X)$ of a smooth projective surface X over an algebraically closed field is the Picard group modulo 'algebraic equivalence': $D_1 \equiv D_2$ if there is a connected family of divisors containing D_1 and D_2.

Proposition 19 *Let X be a smooth projective unirational (or even uniruled) surface over a field of characteristic p. Suppose that X arises as the reduction* $\pmod p$ *of a surface S defined over a field of characteristic zero. Then we have*

$$\rho(X) = \mathrm{rank}(\mathrm{NS}(X)) = \mathrm{rank}(H^2(S, \mathbb{Z})).$$

Thus our Fermat quartic surface

$$X = \{x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0\}$$

has $\rho(X) = 22$!

K3 surfaces with $\rho = 22$ are said to be *supersingular* in the sense of Shioda [RS81, §5]. Artin has a different definition of supersingularity [RS81, §9, Prop. 2] [Art74] that is implied by Shioda's definition; the converse remains open. Artin [Art74, p. 552] and Shioda [Shi77, Ques. 11] have conjectured that supersingular K3 surfaces (in either sense) are all unirational.

4.2 What goes right in characteristic p?

Assume now that k is a field with $\mathrm{char}(k) = p$.

Theorem 20 *[RŠ78] [LN80] [Nyg79] K3 surfaces have no vector fields, i.e., if X is a K3 surface then $\Gamma(X, T_X) = 0$.*

Unfortunately, there appears to be no really simple proof of this important theorem. All the proofs I have seen start the same way: K3 surface with vector fields are unirational.

Given this, we can recover most of the deformation-theoretic results that make complex K3 surfaces so attractive:

Theorem 21 (Smoothness theorem) *[Del81] Suppose k is algebraically closed with $\mathrm{char}(k) = p$ and let X be a K3 surface defined over k. Then the formal deformation space $\mathrm{Def}(X)$ is smooth of dimension 20 over k. If h is a primitive polarization of X then $\mathrm{Def}(X, h)$ is flat of dimension 19 and arises from an algebraic scheme over k.*

The argument uses the Chern-class formalism described above and formal deformation theory of Schlessinger [Sch68]: For each Artinian local k-algebra A consider flat proper morphisms

$$\mathcal{X} \to \mathrm{Spec}\, A$$

with closed fiber $\mathcal{X}_0 = X$. The formal deformation space is obtained by taking the inverse limit of all such families over all Artinian k-algebras.

Even more remarkably, we can use the vanishing of vector fields to show that *every* K3 surface in characteristic p is obtained as the reduction $\pmod p$ of a K3 surface in characteristic zero!

For each algebraically closed (or perfect) field k with $\mathrm{char}(k) = p$ let $W(k)$ denote the *Witt-vectors* with components in k. For example, if $k = \mathbb{F}_p$ then $W(k) = \mathbb{Z}_p$, the p-adic integers. When $k = \bar{\mathbb{F}}_p$ then $W(\bar{\mathbb{F}}_p)$ is the unique complete unramified extension of $\mathbb{Z}_p$ with algebraically closed residue class field. This can be obtained by adjoining all n-th roots of unity with $(n, p) = 1$.

Theorem 22 (Deligne's Lifting Theorem [Del81]) *Let (X, L) be a K3 surface over an algebraically closed field k of characteristic p. Consider the deformations spaces over the Witt-vectors*

$$\mathrm{Def}(X/W(k)), \mathrm{Def}((X, L)/W(k)) \to \mathrm{Spec}\, W(k)$$

i.e., the space associated to taking flat proper morphisms

$$\mathcal{X} \to \mathrm{Spec}\, A, \ \mathcal{X}_0 = X$$

where A is an Artinian module over $W(k)$. Then $\mathrm{Def}(X/W(k))$ is smooth over $\mathrm{Spec}\, W(k)$ and $\mathrm{Def}((X, L)/W(k))$ is flat over $\mathrm{Spec}\, W(k)$.

This uses the Schlessinger formalism for Artinian $W(k)$-algebras rather than k-algebras. For the finite field $k = \mathbb{F}_p$, this is the difference between considering schemes over $\mathrm{Spec}\, \mathbb{F}_p[t]/\langle t^n \rangle$ versus schemes over $\mathrm{Spec}\, \mathbb{Z}/\langle p^n \rangle$. In both contexts, the obstructions to lifting to order $n+1$ lie in a $\mathbb{F}_p$-vector space.

Corollary 23 *Suppose X is a K3 surface over an algebraically closed field k with $\mathrm{char}(k) = p$. Then there exists a finite extension T of $W(k)$ and a flat projective scheme*

$$\mathcal{X} \to \mathrm{Spec}\, T$$

such that X is isomorphic to the fiber over the closed point.

Remark 24 The question of whether we can lift X to a flat projective scheme over the Witt vectors

$$\mathcal{X} \to \mathrm{Spec}\, W(k)$$

is quite subtle, especially in characteristic two. We refer the interested reader to Ogus' work [Ogu79] for details.

Let S be the surface appearing as the generic fiber of $\mathcal{X} \to \mathrm{Spec}\, T$, which is defined over a field of characteristic zero. We know that

- S is smooth, because X is smooth;
- $K_S = 0$, because $K_X = 0$ and $H^1(X, \mathcal{O}_X) = 0$ and canonical sheaf commutes with base extension;
- $H^1(S, \mathcal{O}_S) = 0$ by semicontinuity.

Thus S is a complex K3 surface and we can apply everything we know about its cohomology. Using the comparison theorem (relating complex and étale cohomology) and smooth basechange (relating the cohomology of the generic and special fibers) we find

$$H^2_{et}(X, \mu_{\ell^n}) = H^2(S, \mu_{\ell^n}) \simeq H^2(S, \mathbb{Z}/\ell^n\mathbb{Z}),$$

for each prime ℓ different from p and $n \in \mathbb{N}$. Furthermore, this is compatible with cup products.

Corollary 25 *Let X be a K3 surface over an algebraically closed field. The middle ℓ-adic cohomology of X*

$$H^2(X, \mathbb{Z}_\ell(1)) = \varprojlim_{n \in \mathbb{N}} H^2_{et}(X, \mu_{\ell^n})$$

is given by the $\mathbb{Z}_\ell$-lattice

$$\Lambda \simeq U^{\oplus 3} \oplus (-E_8)^{\oplus 2}.$$

The odd-dimensional cohomologies of X vanish.

Remark 26 Each smooth projective surface X admits a $\mathbb{Z}$-valued intersection form on its Picard group. The induced nondegenerate form on the Néron-Severi group has signature $(1, 21)$ by the Hodge index theorem.

Suppose that k is algebraically closed and X/k is a unirational K3 surface with $\rho(X) = 22$, i.e., all the middle cohomology is algebraic. How can the signature of Λ be $(3, 19)$? The point is that

$$\operatorname{Pic}(X) \to H^2(X, \mathbb{Z}_\ell(1)) \simeq \Lambda \otimes \mathbb{Z}_\ell$$

as a lattice in an ℓ-*adic* quadratic form. It only makes sense to compare the ℓ-adic invariants, not the $\mathbb{R}$-invariants!

4.3 Frobenius and the Weil conjectures

Let X be a K3 surface defined over a finite field $\mathbb{F}_q$ and $\bar{X}$ the basechange to the algebraic closure $\bar{\mathbb{F}}_q$. Consider Frobenius $x \mapsto x^q$ which induces a morphism

$$\begin{array}{ccc} X & \stackrel{\mathrm{Fr}}{\to} & X \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} \mathbb{F}_q & \end{array}$$

whose fixed-poits are precisely $X(\mathbb{F}_q)$. We get an induced action on the ℓ-adic cohomology groups

$$\mathrm{Fr}^* : H^i(\bar{X}, \mathbb{Z}_\ell) \to H^i(\bar{X}, \mathbb{Z}_\ell)$$

where ℓ is a prime not dividing q. We have the Lefschetz trace formula (due to Grothendieck!)

$$\begin{aligned} \#X(\mathbb{F}_q) &= \textstyle\sum_{i=0}^{4} (-1)^i \mathrm{tr} F^* | H^i(\bar{X}, \mathbb{Q}_\ell) \\ &= 1 + \mathrm{tr}\mathrm{Fr}^* | H^2(\bar{X}, \mathbb{Q}_\ell) + q^2. \end{aligned}$$

Here we are using the vanishing of the odd-dimensional cohomology groups. The fundamental class $[X]$ and the point class give the 1 and q^2 contributions.

The Weil conjectures were proven for K3 surfaces before they were established in general:

Theorem 27 *[Del72] Let X be a K3 surface defined over a finite field with Frobenius endomorphism* Fr. *The characteristic polynomial*

$$p_{X,\mathrm{Fr}^*}(t) = \det(tI - \mathrm{Fr}^*)|H^2(\bar{X}, \mathbb{Q}_\ell)$$

is integral and its complex roots α satisfy $|\alpha| = q$.

The proof uses Clifford algebras: The middle integral cohomology of a polarized complex K3 surface (S, L) carries an integral quadratic form $\langle , \rangle$. Consider the orthogonal complement

$$V_{\mathbb{Z}} = L^{\perp} \subset H^2(S, \mathbb{Z})$$

which also inherits a quadratic form of signature $(2, 19)$. Set $V_{\mathbb{Q}} = V \otimes \mathbb{Q}$ and let $C(V, \langle , \rangle)$ denote the *Clifford algebra* of this quadratic form. This is a 2^{21}-dimensional associative $\mathbb{Q}$-algebra, admitting a linear injection $i : V \hookrightarrow C(V, \langle , \rangle)$, and determined by the following universal property: Given another $\mathbb{Q}$-algebra and a linear map $f : V \to A$ with $f(v)^2 = \langle v, v\rangle$, there exists a unique $\mathbb{Q}$-algebra homomorphism $g : C(Q, \langle , \rangle) \to A$ with $f = g \circ i$. If $e_1, \ldots, e_{21}$ is a basis for V then the Clifford algebra has basis $e_{j_1} \ldots e_{j_r}, j_1 < j_2 < \ldots < j_r$. We can decompose this into parts with even and odd degrees

$$C(V, \langle , \rangle) = C^+(V, \langle , \rangle) \oplus C^-(V, \langle , \rangle)$$

each of dimension 2^{20}.

Here is the first marvelous insight of Deligne:

> The Hodge decomposition on S
>
> $$H^2(S, \mathbb{C}) = H^0(S, \Omega^2_S) \oplus H^1(S, \Omega^1_S) \oplus H^2(S, \mathcal{O}_S)$$
>
> induces a Hodge decomposition on
>
> $$C^+(V, \langle , \rangle) \otimes \mathbb{C} = H^{01} \oplus H^{01}.$$
>
> Moreover, there is an isogeny-class of abelian varieties A of dimension 2^{19} such that
>
> $$H^1(A, \mathbb{C}) \simeq C^+(V, \langle , \rangle) \otimes \mathbb{C}$$
>
> as Hodge structures. Moreover, these abelian varieties come with a huge number of endomorphisms, e.g., the elements of the Clifford algebra.

This construction is usually known as the Kuga-Satake construction. See [vG00] for a user-friendly introduction to these Hodge-theoretic techniques.

The second step is to do this over the whole moduli space of K3 surfaces in such a way that everything is defined over a number field. Suppose that $\mathcal{S} \to B$ is a single family (say over a 19-dimensional base) containing every complex K3 surface of degree $\langle L, L\rangle$. Then after finite basechange $B' \to B$ we want a family of 2^{19}-dimensional abelian varieties $\mathcal{A} \to B'$ such that, fiber-by-fiber, they are related to $\mathcal{S}' \times_B B' \to B'$ via the Kuga-Satake construction.

Remark 28 Note that we have no *algebro-geometric* connection between S and A. We only have a connection between their cohomologies. The Hodge conjecture predicts [vG00, 10.2] the existence of a correspondence

$$\begin{array}{ccc} Z & \to & A \times A \\ \downarrow & & \\ S & & \end{array}$$

inducing the cohomological connection. However, these are known to exist only in special cases. [Voi96]

Finally, the truly miraculous part: The universal construction relating $\mathcal{S} \times_B B'$ and $\mathcal{A}$ can be reduced $\mod p$, so as to allow the Weil conjectures for $X = S \pmod p$ to be deduced from the Weil conjectures for the reductions of the fibers of $\mathcal{A} \to B$. □

4.4 On the action of Frobenius

Again, let X be a K3 surface over a finite field $\mathbb{F}_q$ with polarization L. What can we say *structurally* about the action of Fr^* on $H^2(\bar{X}, \mathbb{Q}_\ell)$?

First, it will make our analysis easier if we replace

$$H^2(\bar{X}, \mathbb{Z}_\ell) = \varprojlim_{n \in \mathbb{N}} H^2_{et}(\bar{X}, \mathbb{Z}/\ell^n\mathbb{Z})$$

with the twist

$$H^2(\bar{X}, \mathbb{Z}_\ell(1)) = \varprojlim_{n \in \mathbb{N}} H^2_{et}(\bar{X}, \mu_{\ell^n}).$$

The reason is that the Kummer sequence

$$0 \to \mu_{\ell^n} \to \mathbb{G}_m \stackrel{\times \ell^n}{\to} \mathbb{G}_m \to 0$$

gives connecting homomorphisms

$$\mathrm{Pic}(\bar{X}) = H^1(\bar{X}, \mathbb{G}_m) \to H^2_{et}(\bar{X}, \mu_{\ell^n})$$

inducing the cycle class map

$$\mathrm{Pic}(\bar{X}) \to H^2(\bar{X}, \mathbb{Z}_\ell(1))$$

and

$$\mathrm{Pic}(X) \to H^2(\bar{X}, \mathbb{Z}_\ell(1))^\Gamma$$

where

$$\Gamma = \langle \mathrm{Fr} \rangle = \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q).$$

Since Frobenius acts via multiplication by q on μ_{ℓ^n} we can show that the action of Fr^* on our original cohomology group and its twist differ by a factor of q.

Proposition 29 *If X is a K3 surface over $\mathbb{F}_q$ then*

$$\rho(X) \leq \dim\{\xi \in H^2(\bar{X}, \mathbb{Q}_\ell) : \mathrm{Fr}^*\xi = q\xi\},$$

i.e., the multiplicity of q as an eigenvalue of $p_{X,\mathrm{Fr}^*}(t)$.

Applying a similar analysis to the cup-product operation yields:

Proposition 30 Fr^* *respects the intersection form, i.e.,*

$$\langle \mathrm{Fr}^*\xi_1, \mathrm{Fr}^*\xi_2 \rangle = q^2 \langle \xi_1, \xi_2 \rangle .$$

If α is a root of $p_{X,\mathrm{Fr}^*}(t)$ then q^2/α is also a root.

Since $\langle , \rangle$ is nondegenerate, after passing to a field extension and changing coordinates we may assume the corresponding symmetric matrix is the identity. Let Φ denote the matrix of Fr^* in these coordinates, which satisfies

$$\Phi^t\Phi = q^2 I.$$

It follows that

$$\begin{aligned} p_{X,\mathrm{Fr}^*}(t) &= \det(tI - \Phi) \\ &= (-t)^{\deg(p)} \det(\Phi) \det(t^{-1}I - \Phi^{-1}) \\ &= (-t)^{\deg(p)} \det(\Phi) \det(t^{-1}I - q^{-2}\Phi^t) \\ &= (-t/q^2)^{\deg(p)} \det(\Phi) \det(q^2t^{-1}I - \Phi^t) \\ &= (-t/q^2)^{\deg(p)} \det(\Phi) p_{X,\mathrm{Fr}^*}(q^2/t). \end{aligned}$$

Corollary 31 *Suppose X is a polarized K3 surface. Then the distinguished subspaces*

$$\{\xi \in H^2(\bar{X}, \mathbb{Q}_\ell) : \mathrm{Fr}^*\xi = q\xi\}, \quad \{\xi \in H^2(\bar{X}, \mathbb{Q}_\ell(1)) : \mathrm{Fr}^*\xi = \xi\}$$

are even dimensional.

4.5 Tate conjecture for K3 surfaces

The decomposition of the cohomology under Frobenius should strongly reflect the geometry:

Conjecture 32 Let X be a K3 surface over a finite field. All the Galois-invariant cycles come from divisors, i.e.,

$$\mathrm{Pic}(X) \otimes \mathbb{Q}_\ell \to H^2(X, \mathbb{Q}_\ell(1))^\Gamma$$

is surjective.

This is the finite-field analog of the Lefschetz $(1,1)$ theorem on the Néron-Severi group.

For most K3 surfaces, the conjecture is known to be true: [NO85]

Theorem 33 *The Tate conjecture holds for $K3$ surfaces of finite height over finite fields of characteristic ≥ 5.*

The height of a K3 surface is computed from its formal Brauer group, which is associated to the system

$$\varprojlim_{A/k \text{Artinian}} \mathrm{Br}(X \times_k A).$$

The unirational K3 surfaces we constructed have infinite height.

The following consequence is well-known to experts (and was ascribed to Swinnerton-Dyer in [Art74, p. 544]) but we are not aware of a reference:

> The rank of the Néron-Severi group of a finite-height K3 surface over a finite field is always even. In particular, it is at least two.

An especially nice special case of the Tate conjecture is K3 surfaces with elliptic fibrations [ASD73]

$$X \to \mathbb{P}^1.$$

Here the Tate conjecture is related to proving finiteness of the Tate-Shafarevich group of the associated Jacobian fibration

$$J(X) \to \mathbb{P}^1.$$

Remark 34 For an analysis of the eigenvalues of Frobenius on the *non-algebraic* cohomology of a K3 surface, we refer the reader to [Zar93].

5 Mori-Mukai in mixed characteristic

This is joint work with F. Bogomolov and Y. Tschinkel.

Let S be a projective K3 surface over a number field F with $\overline{S} = S_{\overline{\mathbb{Q}}}$. Let $\mathfrak{o}_F$ be the ring of integers with spectrum $B = \mathrm{Spec}\,(\mathfrak{o}_F)$ and $\pi : \mathcal{S} \to B$ a flat projective model for S. Fix $\mathfrak{p} \in B$ a prime of good reduction for S, i.e., $\mathcal{S}_{\mathfrak{p}} = \pi^{-1}(\mathfrak{p})$ is a smooth K3 surface over a *finite* field. Let k be a finite field with algebraic closure $\overline{k}$, $p = \mathrm{char}(k)$ and X/k K3 surface and $\overline{X} = X_{\overline{k}}$.

Let Fr denote the Frobenius endomorphism on $\bar{X}$ acting on ℓ-adic cohomology

$$\mathrm{Fr}^* : H^2(\overline{X}, \mathbb{Q}_\ell) \to H^2(\overline{X}, \mathbb{Q}_\ell);$$

X is *ordinary* if $p \nmid \mathrm{Trace}(\mathrm{Fr})$. Joshi-Rajan [JR01] and Bogomolov-Zarhin [BZ09] have shown

$$\{\mathfrak{p} \in B : \mathcal{S}_{\mathfrak{p}} \text{ ordinary }\}$$

has positive Dirichlet density; we call these places of *excellent reduction.*

Key properties: Assume X/k is an ordinary K3 surface

1. X is *not* uniruled, i.e., rational curves in X cannot deform;

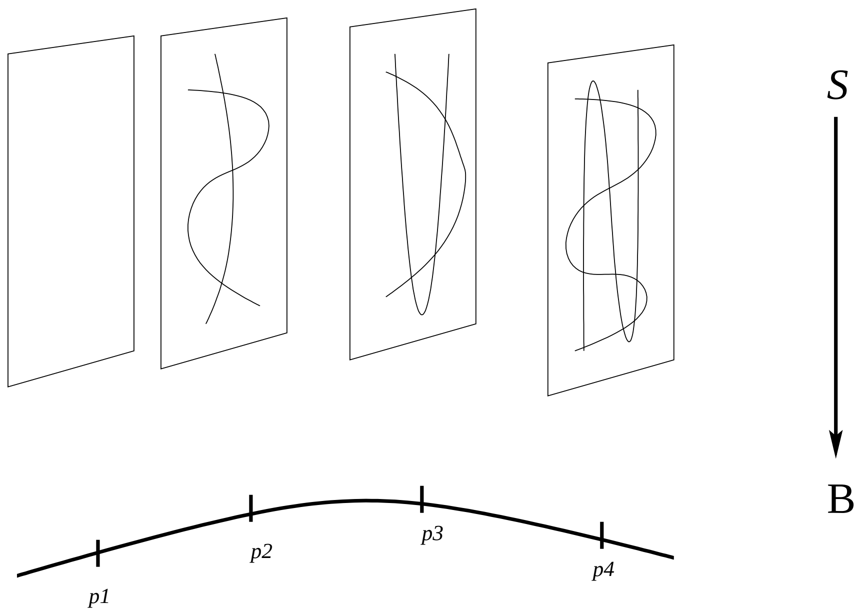

Figure 4: Reductions of a K3 surface mod $\mathfrak{p}$ have extra curve classes

2. the Tate conjecture holds for X, thus the image of

$$\mathrm{Pic}(\overline{X}) \to H^2(\overline{X}, \mathbb{Q}_\ell)$$

consists of the elements fixed by Fr^m for some $m \in \mathbb{N}$;

3. $\mathrm{Pic}(\overline{X})$ has rank ≥ 2.

Indeed, since Fr respects the intersection form, if α is a root of the characteristic polynomial of Fr then so is q^2/α

Let S be a K3 surface over a number field with $\mathrm{Pic}(\overline{S}) = \mathbb{Z}f$. Our goal is to produce irreducible rational curves in $\overline{S}$ with class Nf for unbounded N.

Strategy: Part I Choose an excellent place $\mathfrak{p}$ such that we can write

$$Nf|\mathcal{S}_\mathfrak{p} \equiv m_1C_1 + \ldots + m_rC_r \tag{5.1}$$

where the $C_i \subset \mathcal{S}_\mathfrak{p}$ are rational curves and each effective sub-cycle

$$n_1C_1 + \ldots + n_rC_r \prec m_1C_1 + \ldots + m_rC_r$$

is not proportional to f.

Any curve $C \subset \overline{S}$ specializing to $m_1C_1 + \ldots + m_rC_r$ would have to be irreducible. For example, take $N \in \mathbb{N}$ *minimal* such that $Nf|\mathcal{S}_\mathfrak{p}$ can be expressed as a positive sum of indecomposable curves $C_i \subset \mathcal{S}_\mathfrak{p}$ *not* proportional to f. We prove that $N \to \infty$ as $\mathfrak{p} \to \infty$.

Given $d > 0$,

$$\{\mathfrak{p} \in B : \mathcal{S}_\mathfrak{p} \supset C \text{ with } C \cdot f \leq d, C \not\sim f\}$$

is *finite*. Indeed, there are a finite-number of Noether-Lefschetz divisors corresponding to K3 surfaces T with

$$\mathrm{Pic}(T) \supset \Lambda, \qquad \Lambda = \begin{array}{c|cc} & f & C \\ \hline f & 2g-2 & d \\ C & d & C \cdot C \end{array},$$

and $B = \mathrm{Spec}\,(\mathfrak{o}_F)$ meets these in a finite number of primes. Consequently, as $\mathfrak{p} \to \infty$ both $d \to \infty$ and $N \to \infty$.

Strategy, Part II: Emulate the Mori-Mukai argument.

Try to show that the cycle $m_1C_1 + \ldots + m_rC_r \subset \mathcal{S}$ deforms in a one-parameter family of *rational curves* in $\mathcal{S}$. This is relatively straightforward when

$$m_1 = \ldots = m_r = 1,$$

since we can apply the deformation theory arguments above with minor technical modifications. Note that the formal deformation space of $X = \mathcal{S}_\mathfrak{p}$ is smooth of dimension twenty over the Witt vectors; deformations of a polarized K3 surface are flat of dimension nineteen over the Witt vectors.

Since $\mathcal{S}_\mathfrak{p}$ is ordinary it is not uniruled, so the one-parameter family of rational curves in $\mathcal{S}$ cannot all lie in $\mathcal{S}_\mathfrak{p}$. Thus this family dominates B, which tells us that the rational cycle lifts to a rational curve in characteristic zero. Standard algebraization arguments yield a rational curve in $\overline{S}$ with class Nf.

A sample theorem:

Theorem 35 *Let S be a K3 surface defined over a number field with* $\mathrm{Pic}(\overline{S}) = \mathbb{Z}f$ *and $f \cdot f = 2$. Then S contains an infinite number of rational curves.*

In other words, S is a double cover of $\mathbb{P}^2$ branched over a very general plane sextic curve. Here the associated involution greatly simplifies the multiplicity analysis.

6 Evaluating the Picard group in practice

Our analysis of the Frobenius action on K3 surfaces over finite fields pays dividends back to characteristic zero. (Thus our discussion turns full circle!)

In the analysis of complex K3 surfaces S, we used the Hodge decomposition

$$H^2(S, \mathbb{C}) = H^0(S, \Omega^2_S) \oplus H^1(S, \Omega^1_S) \oplus H^2(S, \mathcal{O}_S)$$

and the description

$$\mathrm{NS}(S) = H^2(S,\mathbb{Z}) \cap H^1(S,\Omega^1_S)$$

Our discussion may have left with you with the impression that these objects are well-known, but much remains mysterious:

Problem 36 Let (S, L) be a polarized K3 surface over a number field k. Give an algorithm to compute

$$\rho(\bar{S}) = \mathrm{rank}\mathrm{NS}(\bar{S}).$$

In particular, is there an effective test for deciding whether $\mathrm{Pic}(\bar{S}) = \mathbb{Z}[L]$?

There is one obvious constraint:

Proposition 37 *Suppose S is a K3 surface over $\mathbb{Q}$, p a prime, and assume that the reduction $X = S \pmod p$ is a smooth K3 surface. Then there is a restriction map*

$$\mathrm{Pic}(\bar{S}) \to \mathrm{Pic}(\bar{X})$$

compatible with the isomorphism on cohomology groups

$$H^2(\bar{S}, \mathbb{Z}_\ell(1)) \to H^2(\bar{X}, \mathbb{Z}_\ell(1))$$

arising from smooth base change.

Corollary 38 *If some reduction $X = S \pmod p$ has Néron-Severi group of rank $2m$ then*

$$\rho(\bar{S}) \leq 2m.$$

Unfortunately, the Tate conjecture precludes using reduction $\mod p$ to prove that $\rho(\bar{S}) = 1$! Terasoma [Ter85], Ellenberg [Ell04], van Luijk [vL05], and Elsenhans-Jahnel [EJ08a, EJ08b, EJa, EJb] have demonstrated that we often can show this be reducing modulo multiple primes, and then comparing the various restrictions

$$\mathrm{Pic}(\bar{S}) \to \mathrm{Pic}(\bar{S} \pmod{p_i}).$$

Example 39 (van Luijk's example) This an a quartic K3 surface $S \subset \mathbb{P}^3$ over $\mathbb{Q}$ with

$$\mathrm{NS}(\bar{S} \pmod 2) = \begin{array}{c|cc} & h & C \\ \hline h & 4 & 2 \\ C & 2 & -2 \end{array}$$

and

$$\mathrm{NS}(\bar{S} \pmod 3) = \begin{array}{c|cc} & h & L \\ \hline h & 4 & 1 \\ L & 1 & -2 \end{array}.$$

In geometric terms, the reduction $\mod 2$ contains a conic C and the reduction mod 3 contains a line L. The first lattice has discriminant -12 and the second lattice has discriminant −9, so these cannot both be specializations of rank-two sublattice of $\mathrm{NS}(\bar{S})$.

Of course, the key questions is: How many primes must we check to determine the rank of $\mathrm{NS}(\bar{S})$? Can this be bounded in terms of arithmetic invariants of S?

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