# Numeration Systems and associated Tilings Dynamical Systems II 

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Number Theory and Dynamics, Summer School 2013:
Grenoble, June 20

## Why self-similar tilings ?

- Self-inducing structures: Substitution dynamical system.
- Special scaling constants (Pisot, Salem number) appear.
- Explicit construction of Markov partition.
- Diophantine approximation, Discrete geometry.
- Mathematical models of quasicrystals.

We are interested in the structures with pure pointed diffraction. Self-affine tilings give nice mathematical models of quasi-crystals. They are created by substitution rule and have self-affine structure.

By using duality of tilings and point sets, we transfer the problem to the spectral property of tiling dynamical systems. We are thus interested in tiling dynamical system with pure discrete spectrum.


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Figure 1: Rauzy Fractal


Figure 2: Penrose Tiling by fat and thin rhombus


Figure 3: Substitution rule and point sets

Self-affine tiling $\mathcal{T}$ is the tiling of $\mathbb{R}^{d}$ having

- Finite local complexity (FLC),
- Repetitivity
- Inflation-subdivision structure: $Q T_{j}=\bigcup_{i} T_{i}+D_{i j}$
by an expanding matrix $Q$ and finite sets $D_{i j}$ of translations. We consider the translation action by $\mathbb{R}^{d}$ and the orbit of $\mathcal{T}$. By introducing natural topology and taking its closure, we arrive at the tiling dynamical system $\left(X_{\mathcal{T}}, \mathbb{R}^{d}\right)$.
- Finite Local Complexity (FLC) $\Leftrightarrow$ Compactness
- Repetitivity $\Leftrightarrow$ Minimality
- Uniform patch frequency $\Leftrightarrow$ Unique ergodicity
- Assuming substitution rule, primitivity of $\left(\# D_{i j}\right)$ implies minimality and unique ergodicity.

By unique ergodicity, there is a unique translation invariant measure $\mu$. The isometry $U_{a}(f(x))=f(x-a)$ is defined on $L^{2}\left(X_{\mathcal{T}}, \mu\right)$. We say that $a$ is an eigenvalue with respect to an
eigenfunction $f$, if

$$
U_{z}(f)=\exp (2 \pi \sqrt{-1} a \cdot z) f
$$

for $z \in \mathbb{R}^{d}$ where $a \cdot z$ is the Euclidean inner product.

## Rigidity of self-similar tilings

Following observation is a clue.
Theorem 1 (Dekking-Keane [7]). ( $X_{\sigma}, s$ ) is not mixing.
In the tribonacci fixed point $x=0102010010201 \ldots$, take a word $w=010$. The return word of is 0102 . Considering

$$
x=\sigma^{n}(010) \sigma^{n}(2) \sigma^{n}(010) \ldots,
$$

for ANY subword $v$, we find a lot of patterns of the shape $[v] \cap s^{-\left|\sigma^{n}(0102)\right|}[v]$. We can show that

$$
\mu\left([v] \cap s^{-\left|\sigma^{n}(0102)\right|}[v]\right)>\text { const } \cdot \mu([v]) .
$$

Taking $v$ long enough, we see that the system is not mixing.
Pursuing this discussion we arrive at:
Theorem 2 (Bombieri-Taylor [4], Solomyak [11]). $\left(X_{\mathcal{T}}, \mathbb{R}\right)$ is not weakly mixing if and only if the Perron Frobenius root of $M_{\sigma}$ is a Pisot number.

This is a dynamical version of Hardy's theorem. Let $\ell(w)$ the suspension length of a word $w$. Then $f\left(x-\beta^{n} \ell(0102)\right)=$ $\lambda^{\beta^{n} \ell(0102)} f$ must be close to $f$ for $f=\chi_{\text {[tile for } 010]}$, in the above example. Thus we must have $\lambda^{\beta^{n} \ell(w)} \rightarrow 1$, for a return word $w$. Thus $\beta$ must be a Pisot number.

Higher dimensional cases are extensively studied by

Solomyak, Lee, Moody. The expanding matrix $Q$ satisfies Pisot family condition, if a conjugate $\gamma$ of an eigenvalue of $Q$ has modulus $\geq 1$, then it is also an eigenvalue of $Q$. Under a mild condition that $Q$ is diagonalizable and eigenvalues have same multiplicity, then following statements are all equivalent:

- $\left(X_{\mathcal{T}}, \mathbb{R}^{d}\right)$ has $d$-linearly independent eigenvectors.
- $Q$ is of Pisot family.
- Associated substitutive Delone set $\Lambda$ is Meyer.

Solomyak showed an impressive result: $\left(X_{\mathcal{T}}, \mathbb{R}\right)$ is pure discrete iff

$$
\text { density } \mathcal{T} \cap\left(\mathcal{T}-\beta^{n} v\right) \rightarrow 1
$$

where $\beta$ is the Perron-Frobenius root of $M_{\sigma}$ and $v$ be a return vector. We easily see from where the shape $\beta^{n} v$ comes! This is equivalent to a combinatorial condition: Overlap coincidence.

The dynamical system $\left(X_{\mathcal{T}}, \mathbb{R}^{d}\right)$ is pure discrete if there are dense measurable eigenfunctions of $L^{2}\left(X_{\mathcal{T}}, \mu\right)$.

By von-Neumann's theorem, the dynamical system is pure
discrete if and only if the system if conjugate to the translation action of a compact abelien group.

Lee [10] showed that pure discreteness is also equivalent to the fact that the reference point set is an inter model set, so such set is generated by cut and projection.

## Example: Substitution dynamical system

Let $\sigma$ be a substitution over the monoid $\{0,1\}^{*}$ defined by $\sigma(0)=01, \sigma(1)=0$. Iteration of $\sigma$ to 0 yields:

$$
\begin{aligned}
\sigma^{2}(0) & =010 \\
\sigma^{3}(0) & =01001 \\
\sigma^{4}(0) & =01001010 \\
\sigma^{5}(0) & =0100101001001 \\
\sigma^{\infty}(0) & =010010100100101001010 \ldots
\end{aligned}
$$

The substitution matrix of $\sigma$ is $M_{\sigma}=\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$. Associate entries of the left Perron Frobenius eigenvalue of $M_{\sigma}$, i.e., $\omega=(1+\sqrt{5}) / 2$ to the letter 1 and 1 to 0 , we obtain a self-similar tiling of the positive real axis.

Figure 4: Fibonacci Tiling
By repetitivity, we can extend it to a self-similar tiling of $\mathbb{R}$. For 1-dim substitution tiling dynamics, it is not weakly mixing if and only if the Perron Frobenius root of $M_{\sigma}$ is a Pisot number.

## Overlap algorithm by Solomyak [11]

Take a return vector $v$ from $\left\{v_{1}, v_{2}, \ldots, v_{d}\right\}$ and denote by $T, U$ the tiles in $\mathcal{T}$. We say that $(T, v, U)$ is an overlap if

$$
(\stackrel{\circ}{T}-v) \bigcap \stackrel{\circ}{U} \neq \emptyset .
$$

We say that the overlap is a coincidence if it is of the form $(T, 0, T)$. Using inflation-subdivision, $(Q T, Q v, Q U)$ is subdivided into overlaps and we draw edges from $(T, v, U)$ to new overlaps. If $Q$ satisfies Pisot family condition then the number of overlaps generated by this iteration is finite. Eventually we get a finite graph. From each overlaps there is a
path to a coincidence, then the system is pure discrete and the converse is also true.

Overlap coincidence gives an algorithm to determine pure discreteness. However this is difficult to compute, because it is hard to tell whether $T-v$ and $U$ have an inner point in common. One can do this for 1-dim substitutive tiling, but it is already difficult in Penrose tiling to implement. Moreover, self-affine tiles often show fractal shapes !

## Potential Overlap Algorithm

Recently with Jeong-Yup Lee, we invented an easy practical algorithm [1].

By duality of tiling and point sets, we choose a reference point $c(T)$ for each $T$ such that all points of $T$ is within a ball of radius $R$. We say that $(T, v, U)$ is a potential overlap, if

$$
|c(T)-v-c(U)| \leq 2 R
$$

Clearly, an overlap must be a potential overlap. Then we apply the same overlap algorithm for potential overlaps.

If $(T-v) \cap U=\emptyset$, then from such potential overlap there are
no infinite paths, because the distance between them becomes larger by the application of $Q$ and eventually all descendants becomes non potential overlaps.

However, if $T-v$ and $U$ are touching at their boundaries, there should be an infinite path from this potential overlap, our main problem is to say that such contribution is small. In a potential overlap graph $\mathcal{G}$, take a subset of vertices which leads to coincidences and take an induced graph $\mathcal{G}_{\text {coin }}$ and we also take $\mathcal{G}_{\text {res }}$ the induced graph to the reminder vertices. We have proved
Theorem 3 (A. \& J.Y.Lee [1]). $\left(X_{\mathcal{T}}, \mathbb{R}^{d}\right)$ is pure discrete if and only if $\rho\left(G_{\text {res }}\right)<\rho\left(G_{\text {coin }}\right)$

To see this, we have to simultaneously prove, a folklore conjecture:

Theorem 4 (A. \& J.Y.Lee [1]). The boundary of tiles of selfaffine tiling $\mathcal{T}$ has Hausdorff dimension less than $d$. However we use a slightly modified Hausdorff dimension by quasi norm.

We also implemented a Mathematica program.
http://mathweb.sc.niigata-u.ac.jp/~akiyama/Research1.html
It works for all self-affine tilings, including non-unit scaling and non lattice-based tilings.

## Implementation 1: Data Structure

To compute coincidences, we need exact computation. All computation are executed in a fixed algebraic number field. In Mathematica, we use AlgebraicNumberField package. For e.g., AlgebraicNumber[Root[\#^3-\#^2-1, 2], \{0, -1, 1\}]
stands for $\theta^{2}-\theta$ for a root of $x^{3}-x^{2}-1$ by the 2 nd embedding to $\mathbb{C}$. First we find an appropriate decomposition field to express all numbers in the fixed field with a fixed embedding. This is sometimes laborious and not so easy to find an appropriate field.

## Implementation 2: Graph theory functions

In the final stage of the computation, we use efficiently many functions from 'Combinatorica' package to deal with directed graphs. For e.g., adjacency matrix, outdegree, transitive closure, induced subgraph, spectrum. We do not know which language is suitable other than Mathematica.

## Implementation 3: Initial Overlaps

Collecting initial overlaps by $v_{i}$ is the bottle neck of our algorithm.

Our method is to collect overlaps in a ball of radius $R$ and then compare with a ball of radius $C R$ by a constant $C>1$ computed from the data. If they are the same the computation is finished. If not we need to go to $C^{i} R$ for $i=2,3, \ldots$ and compare results for $C^{i-1} R$ and $C^{i} R$.

This computation takes long time especially in dimension larger than 2.

## Example: Penrose Tiling

We confirmed directly that

is pure discrete. Computation takes half a day by our program by a machine equipped with large memory.

## Example: Einstein problem

Recently J. Socolar and J. Taylor discovered an aperiodic monotile, which solves the famous 'Einstein' problem.

This is just a hexagonal tile with matching condition. This can tiles the plane with its rotated and reflected pieces, but only in aperiodic way.

We used an equivalent half-hexagonal tiling by 168 different protiles. One week computation said it is pure discrete. Currently many people are analyzing this construction.


Figure 5: Half-Hex Tiling by Monotile

Example: endomorphisms of free group (Dekking [5, 6], Kenyon [8])

We consider a self similar tiling is generated a boundary substitution:

$$
\begin{aligned}
\theta(a) & =b \\
\theta(b) & =c \\
\theta(c) & =a^{-1} b^{-1}
\end{aligned}
$$

acting on the boundary word $a b a^{-1} b^{-1}, a c a^{-1} c^{-1}, b c b^{-1} c^{-1}$,
representing three fundamental parallelogram. The associated tile equation is

$$
\begin{aligned}
& \alpha A_{1}=A_{2} \\
& \alpha A_{2}=\left(A_{2}-1-\alpha\right) \cup\left(A_{3}-1\right) \\
& \alpha A_{3}=A_{1}-1
\end{aligned}
$$

with $\alpha \approx 0.341164+i 1.16154$ which is a root of the polynomial $x^{3}+x+1$.


Figure 6: Tiling by boundary endomorphism
The tiling dynamical system has pure discrete spectrum.

## Kenyon-Vershik's sofic cover

Kenyon-Vershik [9] introduced a geometric realization of hyperbolic toral automorphism. This construction gives a sofic system that has the toral automorphism as a factor. Taizo Sadahiro gave me an example of such construction in 4-dim case which is one to one whose characteristic polynomial is:

$$
x^{4}-x^{3}-x^{2}+x+1
$$

with 27 states and digits in $\{0,1\}$.


Figure 7: Sofic cover by Kenyon-Vershik
Our program said this system is pure discrete.

## Example: Arnoux-Furukado-Harriss-Ito tiling

Arnoux-Furukado-Harriss-Ito [2] recently gave an explicit Markov partition of the toral automorphism for the matrix:

$$
\left(\begin{array}{cccc}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{array}\right)
$$

which has two dim expanding and two dim contractive planes.
They defined $2-\mathrm{dim}$ substitution of 6 polygons. Let $\alpha=$ $-0.518913-0.66661 \sqrt{-1}$ a root of $x^{4}-x^{3}+1$. The multi
colour Delone set is given by $6 \times 6$ matrix:

$$
\left(\begin{array}{cccccc}
\} & \{z / \alpha\} & \{z / \alpha\} & \} & \} & \} \\
\} & \} & \} & \{z / \alpha\} & \{z / \alpha\} & \} \\
\} & \} & \} & \} & \} & \{z / \alpha\} \\
\{z / \alpha\} & \} & \} & \} & \} & \} \\
\} & \{z / \alpha+1-\alpha\} & \} & \} & \} & \} \\
\} & \} & \} & \{(z-1) / \alpha+\alpha\} & \} & \}
\end{array}\right)
$$

and the associated tiling for contractive plane is:


Figure 8: AFHI Tiling
Our program said it is purely discrete.

## Non pure example: Fractal Chair.

Christoph Bandt discovered a non-periodic tiling in [3] whose setting comes from crystallographic tiles. This is a 3-reptile defined by:

$$
-I \omega \sqrt{3} A=A \cup(A+1) \cup(\omega A+\omega)
$$

where $\omega=(1+\sqrt{-3}) / 2$ is the 6 -th root of unity.


Figure 9: Fractal chair tiling

Fractal chair tiling is not purely discrete! An overlap creates new overlaps without any coincidence. One can draw an overlap graph.


Figure 10: Overlap graph of fractal chair

However, from this overlap graph, one can construct another tiling which explains well this non pureness.


Figure 11: Tiling from overlaps

The 2nd tiling associates different colors to translationally equivalent tiles. Forgetting colors of tiles, it is periodic.

## Pisot Conjecture

Assuming two conditions:

1. $Q$ satisfies Pisot family condition.
2. Characteristic polynomial of substitution matrix is irreducible.

Then the tiling dynamical system $\left(X_{\mathcal{T}}, \mathbb{R}^{d}\right)$ may be pure discrete. It is an open question even in 1-d substitution.

The 2nd condition seems to be too restrictive and many important examples do not satisfy this. We are seeking for the correct formulation.

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