Numeration Systems and associated Tilings Dynamical Systems I

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A **Pisot number** is an algebraic integer > 1 such that all conjugates other than itself has modulus strictly less than 1.

A well known property: if β is a Pisot number, then $d(\beta^n, \mathbb{Z}) \to 0$ as $n \to \infty$.

A partial converse is shown by Hardy:

Let $\beta > 1$ be an algebraic number and $x \neq 0$ is a real number. If $d(x\beta^n, \mathbb{Z}) \to 0$ then β is a Pisot number.

Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure preserving transformation. Then $(X, \mathcal{B}, \mu, T)$ forms a measure theoretical dynamical system. By Poincaré's recurrence theorem, for a set $Y \in \mathcal{B}$ with $\mu(Y) > 0$, almost all T-orbit from Y is **recurrent**. The first return map on Y is defined by:

$$\hat{T} = T^{m(x)}(x)$$

where $m(x) = \min\{m \in \mathbb{Z}_{>0} \mid T^m(x) \in Y\}$. This gives the **induced system**:

$$(Y, \mathcal{B} \cap Y, \frac{1}{\mu(Y)}\mu, \hat{T}).$$

From now on let $X \subset \mathbb{R}^d$. The system $(X, \mathcal{B}, \mu, T)$ is **self-inducing** if there is a Y such that $(Y, \mathcal{B} \cap Y, \frac{1}{\mu(Y)}\mu, \hat{T})$ is isomorphic to the original dynamics by the affine isomorphism map ϕ:

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \phi \big\downarrow & & \big\downarrow \phi \\ Y & \xrightarrow[\hat{T}]{} & Y \end{array}$$

Motivation:

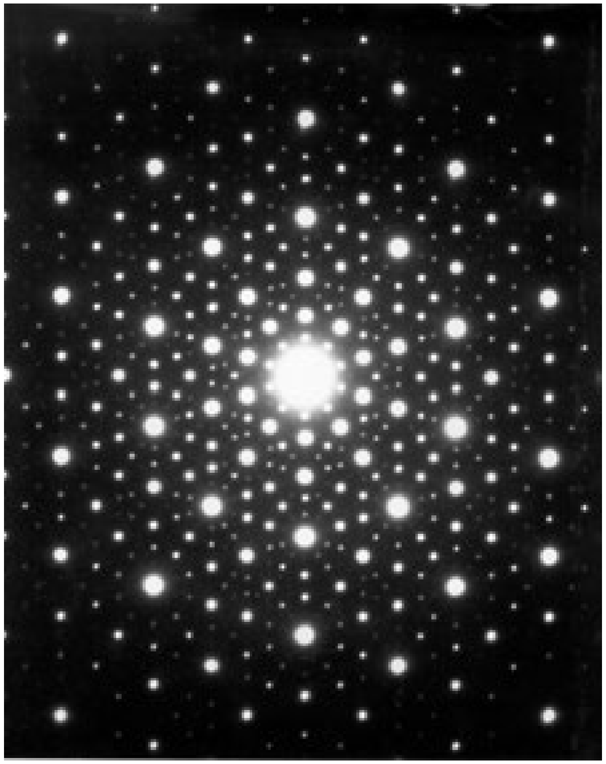
The self-inducing structure corresponds to pure periodic expansion in arithmetic algorithms. The **scaling constant** (the maximal eigenvalue of the matrix of ϕ^{-1}) often becomes a Pisot number, moreover a Pisot unit.

Many examples: irrational rotation and continued fraction, interval exchange, piecewise isometry, outer billiard, etc.

We wish to know why the Pisot number plays the role. Self-inducing structure is modeled by **Substitutive dynamical system**.

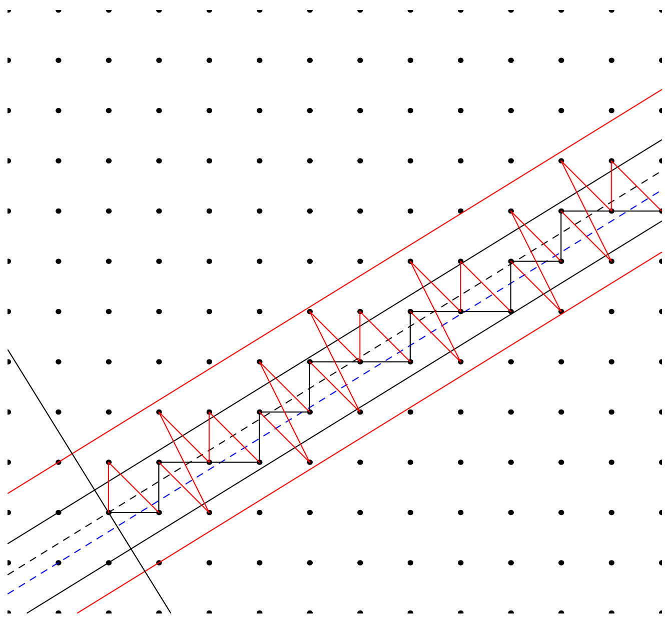
Another motivation comes from the study of quasi-crystal.

Mathematics of Aperiodic Order is continuously motivated by the **quasi-crystals** found by Shechtman [8] in 1984. Diffraction pattern is considered as an image of Fourier transform of the correlation of point sets. As a primitive model of quasicrystal configuration, Penrose tilings attracted great interest of researchers. Spectrum of translation dynamics of substitutive point sets and self-similar tilings are studied in detail along with this study.



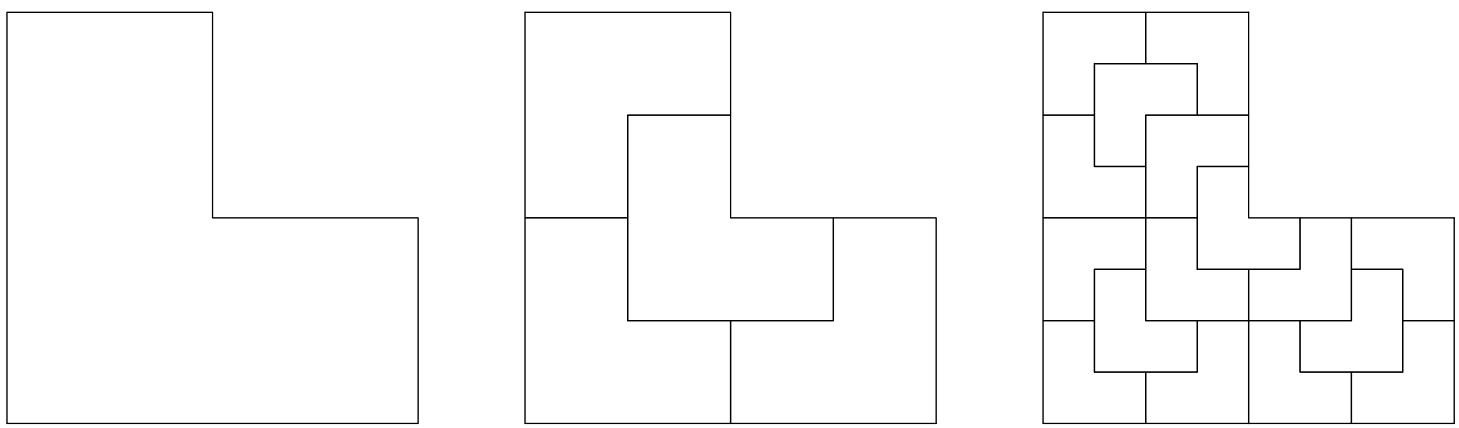
Diffractive Point Set

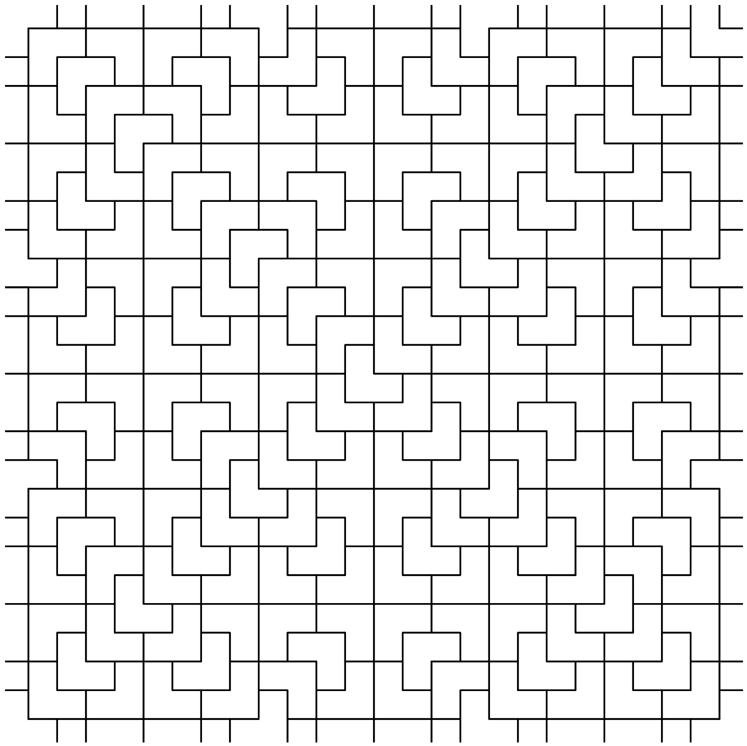
A point set showing Bragg peaks must include a lot of repetitions of local patterns. It is well known that cut and projection gives a **model set** which shows pure point diffraction. In this talk, we focus on less known constructions.

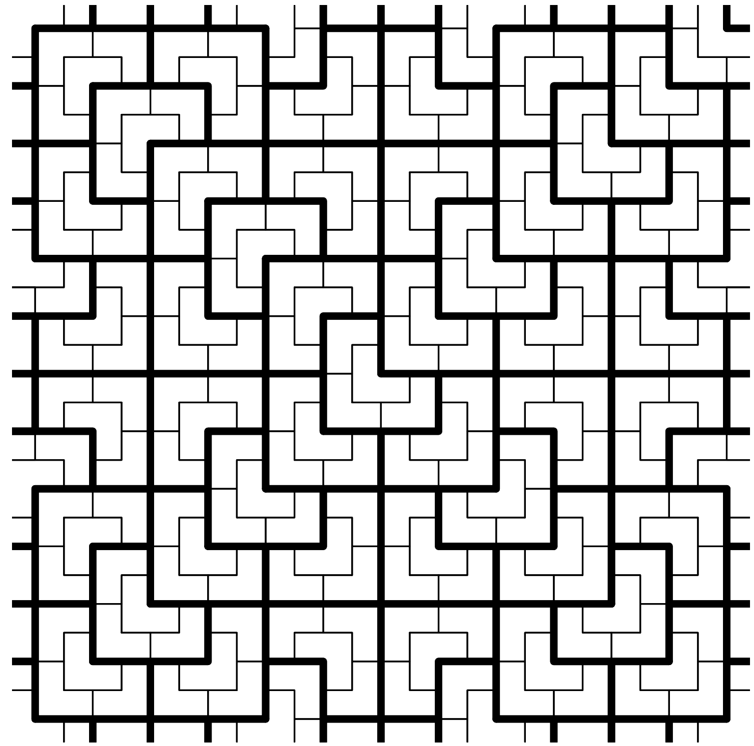


Rep-tiles

Let us start with an easy example of substitution tiling. A **rep-tile** is a tile composed of similar copies of itself which used to appear in elementary puzzles. Iterating this we have







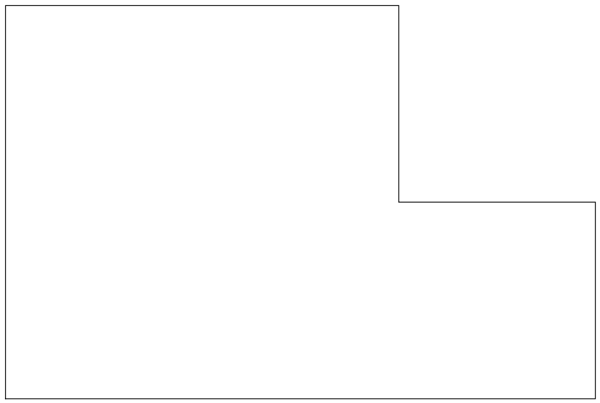
We can confirm a set equation:

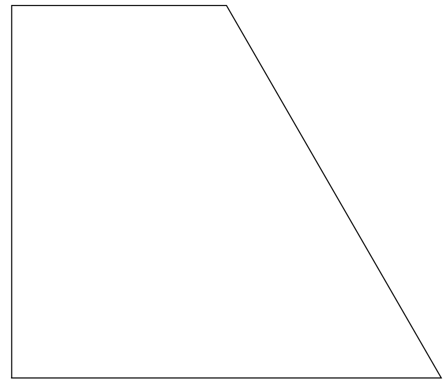
$$QT_j = \bigcup_{i=0}^{3} T_i + D_{ij}$$

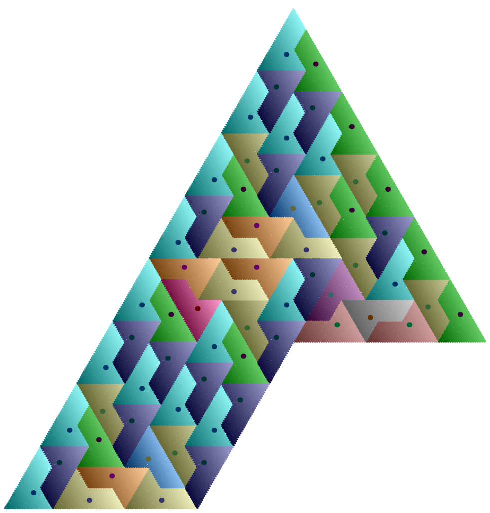
with a matrix is $Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. D_{ij} is equal to

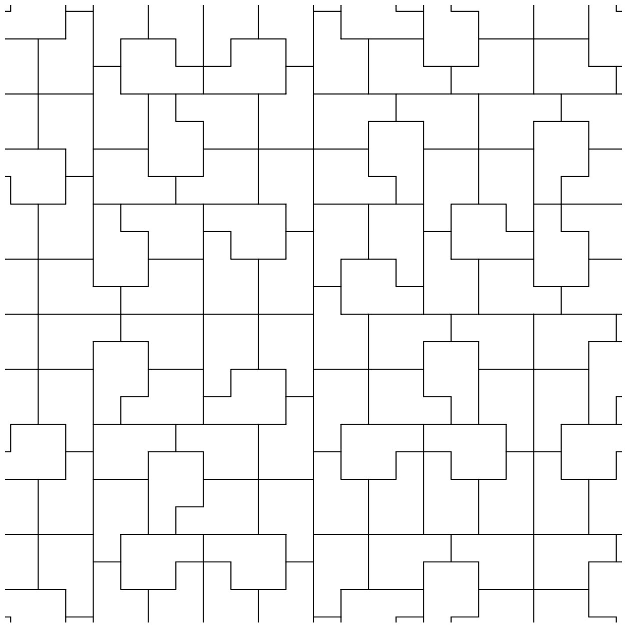
$$\begin{pmatrix} \{(0,0),(1,1)\} & \{(-4,0)\} & \emptyset & \{(0,-4)\} \\ \{(4,0)\} & \{(0,0),(-1,1)\} & \{(0,-4)\} & \emptyset \\ \emptyset & \{(0,4)\} & \{(0,0),(-1,-1)\} & \{(4,0)\} \\ \{(0,4)\} & \emptyset & \{(-4,0)\} & \{(0,0),(1,-1)\} \end{pmatrix}$$

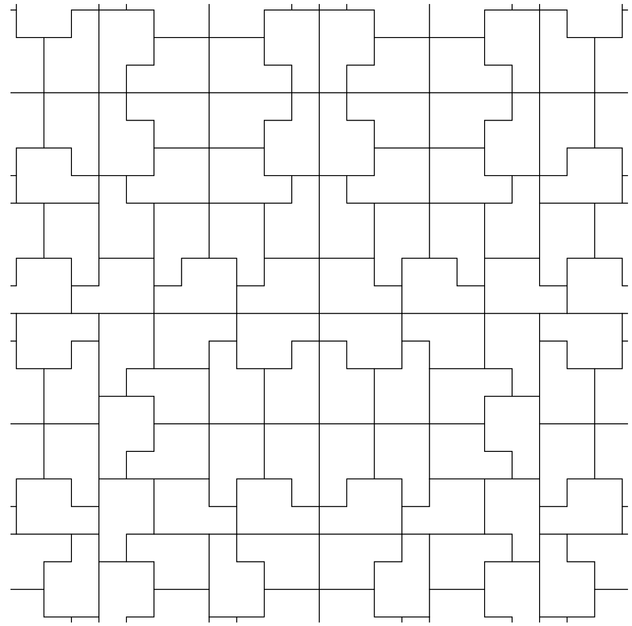
Try several examples:

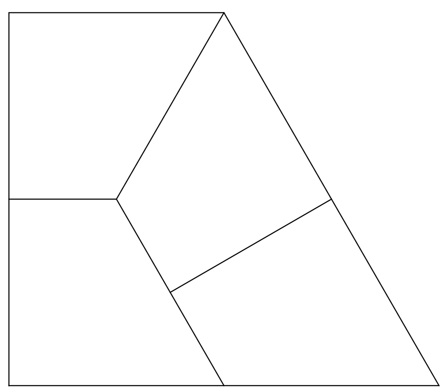


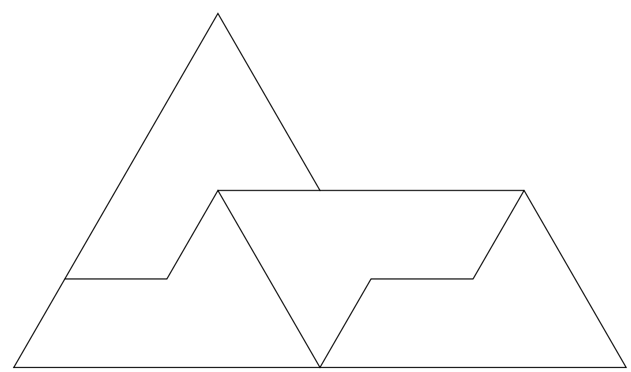












Delone sets and Meyer sets

Substitutive Delone sets give an easy way to model point configurations in quasi-crystals.

In this talk, X will be a Euclidean space $\mathbb{R}^d$ or positive real numbers $\mathbb{R}_+$.

A subset Y of a space X is called **relatively dense** if there exists $r > 0$ that any ball $B(x, r)$ must intersects Y.

A subset Y of a space X is called **uniformly discrete** if there exists $R > 0$ that any ball $B(x, R)$ contains at most one point in Y.

A set Y is a **Delone set** if both relatively dense and uniformly discrete in X.

A set Y is a **Meyer set** if Y is a Delone set and there exists a finite set $F \subset X$ such that $Y - Y \subset Y + F$.

Lagarias [4] showed that this condition is equivalent to that fact that both Y and $Y - Y$ are Delone sets.

- WLOG, we assume $0 \in Y$.

- For a pair $(x, y) \in Y^2$, we prepare a step-stone sequence (x_i, y_i) to $(v, 0)$ with $v = x - y$ of bounded distance R, where R is a relatively dense constant of Y.

- Approximate (x_i, y_i) by $(p_i, q_i) \in Y^2$. Then (p_i, q_i)'s are in $B(v, 2R)$. There are finitely many ways of steps.

- We need a 'uniform' bound of the number of steps. We use the criterion: given a connected graph of size m, any path can be chosen to be less than m in length.

The definition of Meyer set suggests that the set has a structure like an additive group.

Meyer set provides a good framework for quasi-crystal structure. It is also intimately related to the model set generated by cut and project scheme. In this lecture, I wish to talk on Meyer sets which have self-similar structure, which allows handy construction.

Let Q be a $d \times d$ expanding real matrix. $\Lambda_i \in \mathbb{R}^d$ $(i = 0, 1, \ldots, m-1)$ are **substitutive Delone sets** if each Λ_i is

Delone and satisfies a set equation:

$$\Lambda_i = \bigcup_{j=0}^{m-1} Q\Lambda_j + D_{ij}$$

for some translation sets $D_{ij} \subset \mathbb{R}^d$ and the right sides are disjoint. If each Λ_i is Meyer, we call it **substitutive Meyer set**. Substitutive Meyer set is a nice quasi-periodic structure. The matrix $M = (\#D_{ij})$ is called the **substitution matrix**. Throughout this talk we assume that M is **primitive**, i.e., there is $n > 0$ that all the entries of M^n is positive.

Self-affine tiling

Self-affine tiling is a **dual** object of substitutive Delone set. A **tile** is a compact set in $\mathbb{R}^d$ which coincides with the closure of its interior. A **tiling** $\mathcal{T}$ is a collection of tiles which covers $\mathbb{R}^d$ without overlaps of interior points. A patch is a finite subset of $\mathcal{T}$. We assume that $\mathcal{T}$ has **finite local complexity** (FLC), that is, there are only finitely many patches up to translation. Under FLC, there are only finitely many tiles up to translation. The representative of tiles $\mathcal{A} = \{T_0, T_1, \ldots, T_m\}$ is called **alphabets**. $\mathcal{T}$ is **repetitive** if any patch P must reappear in a sufficiently large ball regardless of its location.

We assume that $\mathcal{A}$ satisfies a set equation:

$$QT_j = \bigcup_{i=0}^{m-1} T_i + D_{ij}$$

This gives rise to a **substitution rule** ω of the alphabet by inflation subdivision:

$$\omega(T_j) = \{T_i + d_i \; | d_i \in D_{ij}\}$$

The substitution rule ω is **primitive** if the substitution matrix $(\# D_{ij})$ is primitive. This means for any i, j, the tile T_i must appear in $\omega^k(T_j)$ for some k. A patch P is **legal** if there exist

$i, k \in \mathbb{N}, t \in \mathbb{R}^d$ and $P + t$ appears in $\omega^k(T_i)$. A repetitive tiling $\mathcal{T}$ with FLC is called **self-affine** if every patch P of $\mathcal{T}$ is legal.

Duality

Lagarias-Wang [3] discussed the duality of point sets and tilings. An important necessary condition for the duality is

$$\text{Max eigen value of}(\#D_{ij}) = |\det(Q)|$$

which is called *Lagarias-Wang condition*.

- Max eigen value of$(\#D_{ij}) < |\det(Q)|$, then Tiles can not have d-dim Lebesgue measure.
- Max eigen value of$(\#D_{ij}) \quad > \quad \det(Q)|$, then we have 'overlaps' caused too many digits. Uniformly discreteness of point sets is impossible.

A **cluster** is a subset of substitutive Delone set. One can define legality of the cluster as well. It is $legal$ if its translation appears as an image of iterated substitution of one point. Under this, it is shown that substitution Delone set is realized as a reference point set of a self-affine tiling if and only if all the cluster is legal (c.f. Lee-Moody-Solomyak [6]).

Dynamical Spectrum

Primitivity of ω ensures that the self-affine tiling is repetitive. **Tiling dynamical system** is a topological dynamical system generated as the orbit closure by translation of $\mathcal{T}$ under natural local topology: two tilings are close if big patches around the origin agree up to a small translation. Primitivity of ω also guarantees that this translation dynamics of $\mathcal{T}$ is minimal and uniquely ergodic. So we can discuss spectral properties of this system. The spectrum of tiling dynamical system is intimately related to the diffraction pattern generated by point sets in $\mathbb{R}^d$, representing atomic configuration. Especially it is known that pure discreteness of the tiling dynamics is equivalent to pure pointedness of the diffraction in a pretty general setting (e.g. Baake-Lenz [7], Lee-Moody-Solomyak).

Several methods to construct self-affine tilings:

1. Rep-tiles

2. Number system and related tilings.

3. Boundary endomorphism.

4. Substitutive dynamical system and its geometric realizations.

The advantage of these constructions is that they have self-similar structures from the beginning. However unlike usual cut and projection scheme, it remains to discuss whether this system shows **purely discrete spectrum**.

β-expansion and β-integers

Fix $\beta > 1$ and consider a map T_β from $[0,1)$ to itself:

$$T_\beta : x \mapsto \beta x - \lfloor \beta x \rfloor.$$

Record the trajectory:

$$x = x_1 \xrightarrow{a_{-1}} x_2 \xrightarrow{a_{-2}} x_3 \xrightarrow{a_{-3}} x_4 \xrightarrow{a_{-4}} \dots$$

by symbols $a_{-i} = \lfloor \beta x_i \rfloor$ in $[0, \beta) \cap \mathbb{Z}$. This is a generalization of decimal expansion and gives an expansion:

$$x = \frac{a_{-1}}{\beta} + \frac{a_{-2}}{\beta^2} + \frac{a_{-3}}{\beta^3} + \cdots = \bullet a_{-1}a_{-2}a_{-3}a_{-4}\ldots$$

For any positive x, there is an integer N such that $x/\beta^N \in [0,1)$. Therefore there is a way to expand into:

$$x = \sum_{i=-\infty}^{N} a_i\beta^i = a_N a_{N-1}\ldots a_1 a_0 \bullet a_{-1}a_{-2}a_{-3}\ldots$$

which is called the β-**expansion** of x. The β-**integer part** of x is $\sum_{i=0}^{N} a_i\beta^i = a_N a_{N-1}\ldots a_1 a_0\bullet$ and the β-**fractional part** is $\sum_{i=-\infty}^{-1} a_i\beta^i = \bullet a_{-1}a_{-2}a_{-3}\ldots$.

A β-**integer** is a positive number whose β-fractional part is zero. Denote by $\mathbb{Z}_\beta$ the set of β-integers. Then $\mathbb{Z}_\beta$ is relatively dense in $\mathbb{R}_+$.

An algebraic integer $\beta > 1$ is a **Pisot number** if all other conjugates of β is smaller than one in modulus.

If β is a Pisot number, then $\mathbb{Z}_\beta$ is a Meyer set. Conversely if $\mathbb{Z}_\beta$ is a Meyer set then β is a Pisot number or a Salem number (c.f. Lagarias [5]).

The most famous example: **Fibonacci chain** is viewed as number system. Let $\tau = (1+\sqrt{5})/2$, a root of the polynomial $x^2 - x - 1$. τ is a Pisot number since the conjugate $\tau' = (1-\sqrt{5})/2$ is less than one in modulus.

Exercise

The number system in base τ expresses numbers by 0 and 1. By the greedy property, we can not have 11 in its expression. It is convenient to think that $11 = 100$.

$$
\begin{aligned}
1 &= 1 \\
2 &= 10.01 \\
3 &= ? \\
4 &= ? \\
5 &= ? \\
6 &= ? \\
7 &= ? \\
8 &= ?
\end{aligned}
$$

$$
\begin{aligned}
1 &= 1 \\
2 &= 10.01 \\
3 &= 100.01 \\
4 &= 101.01 \\
5 &= 1000.1001 \\
6 &= 1010.0001 \\
7 &= 10000.0001 \\
8 &= 10001.0001
\end{aligned}
$$

$\mathbb{Z}_\tau$ is a Meyer set of $\mathbb{R}_+$.

$\mathbb{Z}_\tau \cup (-\mathbb{Z}_\tau)$ is a Meyer set of $\mathbb{R}$.

$\mathbb{Z}_\tau$ has a Fibonacci expression:

$$\left\{ \sum_{i \geq 0} a_i \tau^i \;\middle|\; a_i \in \{0, 1\}, \quad a_i a_{i+1} = 0 \right\}$$

Let us try to see why $\mathbb{Z}_\tau$ is uniformly discrete. The product:

$$\left(\sum_{i\geq 0}(a_i-b_i)\tau^i\right)\left(\sum_{i\geq 0}(a_i-b_i)\tau'^i\right)$$

is a symmetric polynomial of $\tau+\tau'(=1), \tau\tau'(=-1)$. Thus it must be an integer. Because it is non-zero, we have an inequality:

$$\left|\sum_{i\geq 0}(a_i-b_i)\tau^i\right| > 1/\left(\frac{2}{1-|\tau'|}\right)$$

which gives a minimum separation of adjacent points. The proof readily applies to all Pisot numbers.

One can see that $\mathbb{Z}_\tau$ forms a self-similar tiling of $\mathbb{R}_+$.

Let

$$A = \left\{ \sum_{i=1}^{\infty} c_i \tau^{-i} \;\middle|\; c_i \in \{0,1\}, c_i c_{i+1} = 0 \right\}$$

and

$$B = \{x \in A \,|c_1 = 0\}$$

Then we see

$$\tau A = A \bigcup B + 1 \qquad \tau B = A$$

Forgetting translation, it is produced by a substitution rule:

$$A \to AB, \quad B \to A$$

The fixed point is

$$ABAABABAABAA\ldots$$

and regard A a tile of length 1 and B of length $1/\tau$. Then we have:

$$[0,1] \cup [1,\tau] \cup [\tau,\tau^2] \cup [\tau^2, 1+\tau^2] \cup [1+\tau^2, \tau^3] \cup \ldots\ldots$$

Rewritten in a form

$$[0,1]\cup[1,10]\cup[10,100]\cup[100,101]\cup[101,1000]\cup\dots$$

Therefore the end points of this tiling is nothing but $\mathbb{Z}_\tau$.

Probably the second simplest example is the **Tribonacci Pisot number**, a positive root of x^3-x^2-x-1. Then

$\mathbb{Z}_\theta$ is a Meyer set of $\mathbb{R}_+$.

$\mathbb{Z}_\theta\cup(-\mathbb{Z}_\theta)$ is a Meyer set of $\mathbb{R}$.

$\mathbb{Z}_\theta$ has a tribonacci expression:

$$\left\{ \sum_{i \geq 0} a_i \theta^i \;\middle|\; a_i \in \{0,1\}, \quad a_i a_{i+1} a_{i+2} = 0 \right\}$$

The corresponding tiling of $\mathbb{R}_+$ is

$$[0,1] \cup [1,10] \cup [10,11] \cup [11,100] \cup [100,101] \cup [101,110] \cup [110,1000] \cup \ldots$$

Dual tiling due to Thurston

Let β-expansion by a Pisot number β of degree d which has r_1 real conjugates and $2r_2$ complex conjugates. We assume that β is a **unit**, which means that $1/\beta$ is also an algebraic integer. Consider a canonical embedding

$$\Phi : \mathbb{Q}(\beta) \to \mathbb{R}^{r_1-1} \times \mathbb{C}^{r_2} \simeq \mathbb{R}^{d-1}$$

defined by $x \mapsto (x^{(2)}, \ldots, x^{(r_1)}, x^{(r_1+1)}, \ldots, x^{(r_1+r_2)})$ where $x^{(i)}$ are the non trivial Galois conjugates of x. As β is Pisot, the set $\overline{\Phi(\mathbb{Z}_\beta)}$ is compact, which is called the **central tile**.

Example: Denote by $\tau' = (1-\sqrt{5})/2 = -1/\tau$.

$$\begin{aligned} \overline{\Phi(\mathbb{Z}_\tau)} &= \left\{ \sum_{i=0}^{\infty} a_i(\tau')^i \middle| a_i \in \{0,1\}, a_i a_{i+1} = 0 \right\} \\ &= [-1, \tau] := A \end{aligned}$$

gives an interval of length τ^2. Dividing by τ', we have

$$\begin{aligned} \beta'^{-1}A &= \left[-\tau^2, -1\right] \cup [-1, \tau] \\ &= BA \end{aligned}$$

with B, an interval of length τ. This growing rule naturally leads us to define an **anti homomorphism** on the word monoid generated by A, B:

$$\sigma(A) = BA, \quad \sigma(B) = A$$

with $\sigma(xy) = \sigma(y)\sigma(x)$ for any words x, y.

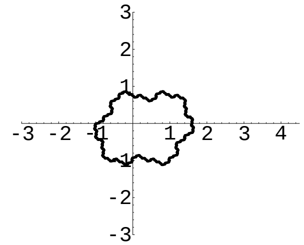
$$\begin{array}{rl} B & A \\ B & AA \\ BAB & AA \\ BAB & AABAA \\ BABAABAB & AABAA \\ BABAABAB & AABAABABAABAA \end{array}$$

We see $\sigma^\infty(A)$ gives a tiling of $\mathbb{R}$.

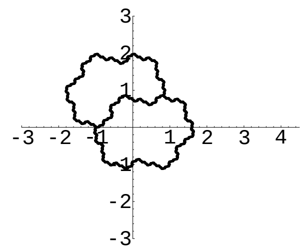
Let us do the same game with $A = \overline{\Phi(\mathbb{Z}_\theta)}$, that is,

$$\left\{ \sum_{i=0}^{\infty} a_i(\theta')^i \;\middle|\; a_i \in \{0,1\}, a_i a_{i+1} a_{i+2} = 0 \right\}.$$

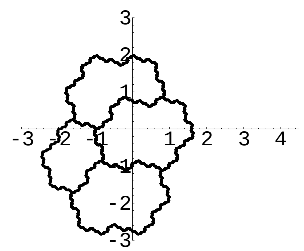
which is a central tile. In this case, it is a compact set in $\mathbb{C}$. The substitution rule becomes two dimensional.



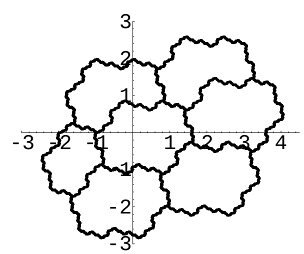
A



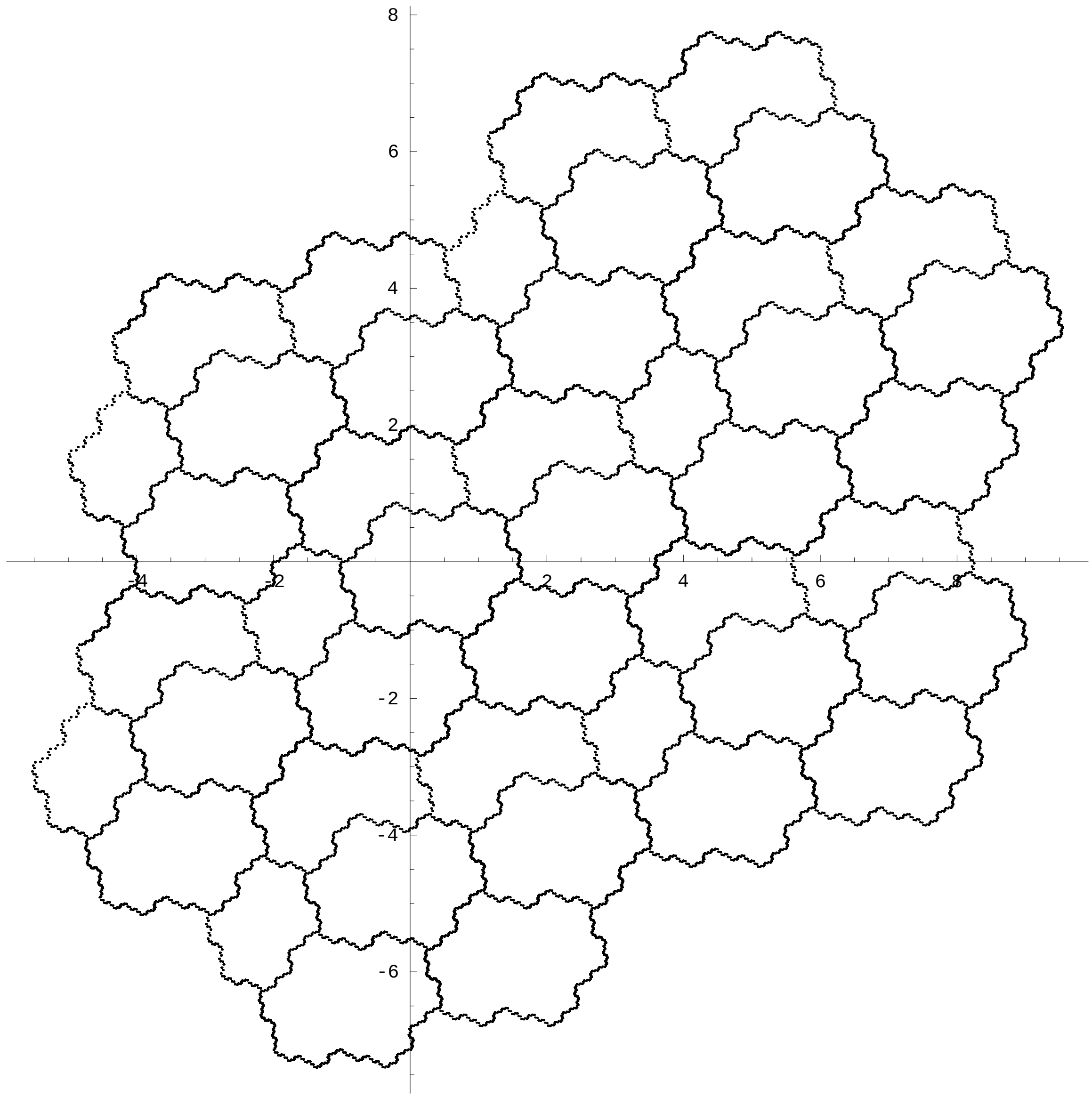
$\beta'^{-1}A$



$\beta'^{-2}A$



$\beta'^{-3}A$


8
6
4
2
-2
-4
-6
-4
-2
2
4
6
8

Each tile corresponds to the subset of beta expansions of $\mathbb{Z}[\beta] \cap \mathbb{R}_+$ having a fixed β-fractional part.

Substitution Tiling

A **substitution** σ is a non-erasing homomorphism of word monoid generated by finite letters, say $\{0,1,2\}$:

$$\sigma : 0 \to 01, \quad 1 \to 02, \quad 2 \to 0$$

with $\sigma(xy) = \sigma(x)\sigma(y)$ for any words $x, y \in \{0,1,2\}^*$.

Substitutions provide us the simplest examples of dynamical systems with self-inducing structures.

One can associate a matrix L_σ:

$$L_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

which counts the number of occurrence of letters after substituted.

If this matrix is primitive, then σ is a **primitive substitution**. The substitution is called **unimodular**, if $\det L_\sigma = \pm 1$.

We may assume that σ has a **fixed point**, a right infinite word fixed by σ which is successively approximated by $\sigma^i(0)$:

$$\sigma(010201001020101020\,1\ldots) = 0102010010201010201\ldots$$

Primitive substitution is called a **Pisot substitution** if the Perron-Frobenius root of L_σ is a Pisot number.

Geometric realization of a fixed point: 1-st

Let $v = (v_0, v_1, v_2)$ be the left eigenvector of L_σ. Identify $\{0, 1, 2\}$ with intervals of length v_0, v_1, v_2 respectively.

$$010201\cdots = [0, v_0] \cup [v_0, v_0 + v_1] \cup [v_0 + v_1, 2v_0 + v_1] \cup \ldots$$

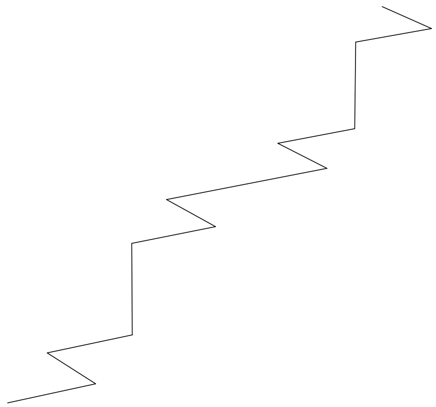
gives a self similar tiling of $\mathbb{R}_+$ which is invariant under multiplication of the Perron-Frobenius root β of L_σ. The end points forms a Delone set provided that σ is a primitive substitution. If β is a Pisot number, then it forms a Meyer set (c.f. Bombieri-Taylor [2]).

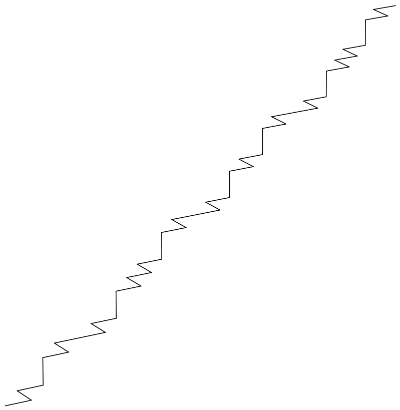
Geometric realization of a fixed point: 2-nd

P.Arnoux and Sh.Ito [1] gave a nicer way to realize unimodular Pisot substitutions, extending the idea of G.Rauzy. Regard $\{0, 1, 2\}$ as unit line segments parallel to the coordinate axis. We realize the action of substitution to give an infinite broken line in $\mathbb{R}^d$ whose broken end points are in $\mathbb{Z}^d$:

$$010201 \cdots = \vec{e}_0 + \vec{e}_1 + \vec{e}_0 + \vec{e}_2 + \vec{e}_0 + \vec{e}_1 + \ldots$$

where each $\vec{e}_i$ is a unit coordinate arrow whose right end is connected to the left end of the next arrow.



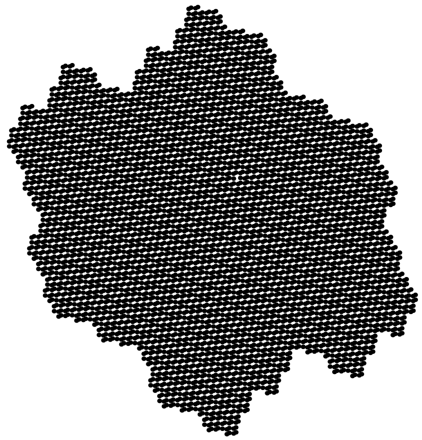


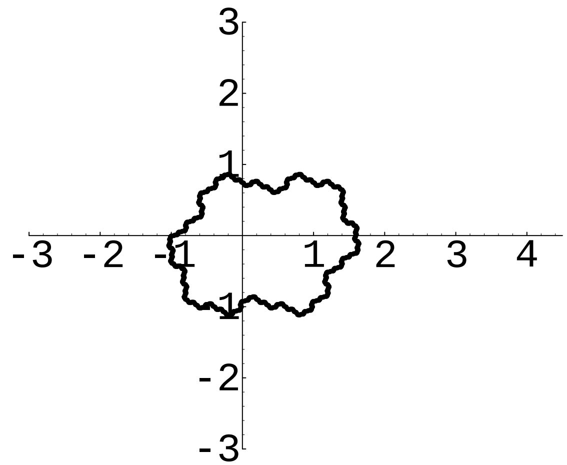
By Pisot property, this broken line gives a good approximation of the 1-dimensional expanding eigen line of L_σ.

Two projections of broken end points

The end points of the broken line are projected along the contractive eigenspace to the expanding line. This gives a tiling of $\mathbb{R}_+$, which is the same as the 1-st construction.

We have a different way. The end points of the broken line can be projected along the expanding line to the contractive plane. Then we have a compact set which is our **atomic surface** X:



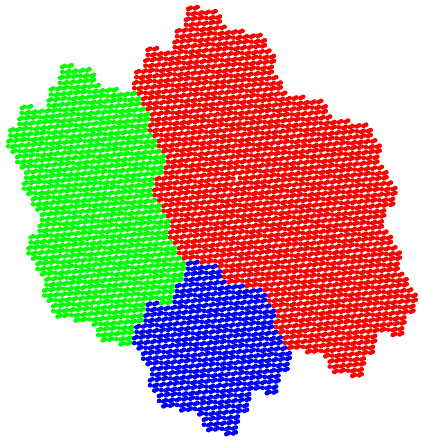


In this case, the figure is essentially the same as the central tile of θ up to some affine transformation.

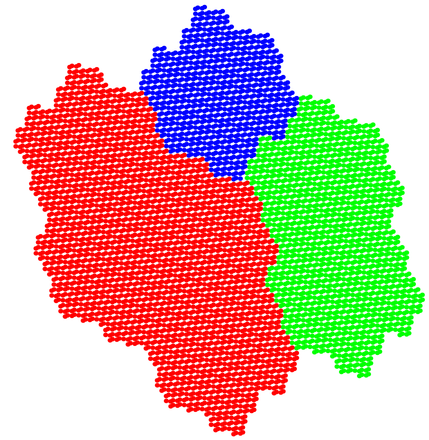
The atomic surface X is subdivided into cylinder sets in two different ways:

$$X = X_0 \cup X_1 \cup X_2 = Y_0 \cup Y_1 \cup Y_2$$

where X_i corresponds to the **left** end points of the broken arrows **start with** the letter i and Y_i corresponds to the **right** end points of the broken arrow **followed by** the letter i. The odometer which sends each end point to the next one, is realized as a domain exchange.



$\rightarrow$



$$\ldots a_2 a_1 a_0 \quad \rightarrow \quad \ldots a_3 a_2 a_1$$

This means that the 'shift' is realized as a domain exchange. This fact has important applications in number theory: construction of low discrepancy sequences.

Arnoux-Ito also defined the dual map of above geometric substitution in a natural way under the duality of the Euclidean norm. This **dual substitution** completes the whole picture like the one for β-expansion. They showed that under a certain **coincidence condition**, the domain exchange is measure theoretically isomorphic to the substitutive dynamical system, and moreover it is semi-conjugate to the rotation of the torus.

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