# New examples of damped wave equations with gradient-like structure


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**Abstract :** *This article shows how to use perturbation methods to get new examples of evolutionary partial differential equations with gradient-like structure. In particular, we investigate the case of the wave equation with a variable damping satisfying the geometric control condition only, and the case of the wave equation with a damping of indefinite sign.*



**Keywords : gradient structure, gradient-like systems, perturbation methods, damped wave equation, indefinite damping**

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## 1 Introduction

A large class of physical problems lead to dissipative systems, that is physical systems which admit an energy decreasing along the trajectories and the trajectories of which asymptotically tend to equilibria. Such particular systems have been called gradient systems or gradient-like systems (see Definition 2.1 below). The gradient structure plays an important role in the qualitative study of the dynamics of an equation, since, for example,

a gradient system does not contain any periodic orbit or homoclinic orbit.
Here, we study the damped wave equation on a bounded regular domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2$ or 3):

$$\begin{cases} u_{tt}(x,t) + \gamma(x)u_t(x,t) = \Delta u(x,t) + f(x, u(x,t)) & (x,t) \in \Omega \times \mathbb{R}_+ \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ (u(x,0), u_t(x,0)) \in \mathbb{H}^1_0(\Omega) \times \mathbb{L}^2(\Omega) \end{cases} \tag{1.1}$$

Conditions on the support of the dissipation $\gamma(x) \geq 0$ are known to imply the gradient-like structure for Equation (1.1), see [22] and [16], and also [17] for Neumann boundary conditions. The purpose of this paper is to enhance technics which show that the gradient-like structure is, in some sense, a property which is stable under small perturbations. We briefly illustrate these technics with two examples :
- In Section 2, we prove that, if the support of $\gamma(x)$ satisfies the geometric control condition introduced by C.Bardos, G.Lebeau and J.Rauch in [3], Equation (1.1) generates a gradient-like dynamical system for a generic non-linearity $f(x, u)$ (or $f(u)$).
- In Section 3, we study the case where the damping $\gamma(x)$ of (1.1) can be slightly negative on some part of $\Omega$. Notice that, in this case, no explicit Lyapounov functional is known. However, we can prove the existence of a compact global attractor and exhibit a gradient-like structure for most of the cases.

**Remarks :**
• The damped wave equations are models for the propagation of waves in dissipative media. More generally, they are used to model propagation or invasion phenomena. For example, they arise in biology when studying the evolution of species populations (see [8] and [20]).
• It is very natural to expect that the geometric control condition of [3] is sufficient for the damped wave equation to be a gradient-like system. Indeed, the fact that the trajectory of any wave intersects the support of the damping should imply that any solution relaxes to an equilibrium. Theorem 2.3 is a slight progress in this direction, but the full result, the gradient-like structure without any condition on the nonlinearity, is still a difficult open problem.
• Classically, the damping $\gamma(x)$ of the wave equation is non-negative. In Section 3, the damping has an indefinite sign. This can be explained as follows : the positive part of $\gamma(x)$ is modelling a damping phenomena whereas the negative part is modelling a supply of energy given to the system. Therefore, the indefinite damping is a basic model to study how a small localized supply of energy modifies the dissipative structure of a system. To give a concrete example, in the biological model introduced in [20], a large birth rate of the species compared to the speed of diffusion may be seen as a negative damping.
• Many papers have studied wave or plate equations with indefinite damping (see for example [6], [7] or [18]). However, to our knowledge, it is the first time that the nonlinear equation is considered and that nonlinear properties as existence of a global attractor or gradient structure are obtained.

• The technics used in Section 3 may also be applied to other perturbations of gradient-like dynamical systems. For example, [5] considers the convergence of differential systems with memory to the parabolic equation $u_t = \Delta u + f(x, u)$. Such a study is interesting to justify the models using the parabolic equation. Indeed, the irreversibility of this equation or its infinite propagation speed of informations may be strong limitations in the points of view of physicists and biologists. Therefore, it is interesting to see the parabolic equation as an approximation of more physical systems. In [5], it is considered as a limit of systems having a very short memory. Notice that systems with memory such as the Gurtin-Pipkin model are reversible, admit finite propagation speed of informations and are used to model some physical phenomena as visco-elastic fluids (see [5]). The problem is that, unlike the parabolic equation, systems with memory are not gradient-like in general. Using the same technics as in Section 3, we can show that, under generic hypotheses, a system with memory is gradient-like as soon as the memory concerns sufficiently recent times only.
• In Section 3, technical assumptions on the nonlinearity $f$ are made. These conditions are derived from the ones given by E. Zuazua in [24] and imply that the energy of the trajectories of (1.1) is uniformly exponentially decreasing outside a bounded set of $\mathbb{H}_0^1(\Omega) \times \mathbb{L}^2(\Omega)$. We expect that this exponential decay and the result of [24] hold for all the subcritical nonlinearities satisfying

$$\limsup_{u \longrightarrow \pm\infty} \sup_{x \in \Omega} \frac{f(x, u)}{u} < \lambda_1 , \tag{1.2}$$

where $\lambda_1$ is the first eigenvalue of the Laplacian operator with Dirichlet boundary conditions. Notice that (1.2) is the natural condition to avoid the blow-up of the energy and thus to ensure global existence of the trajectories of the dynamical system. However, to our knowledge, the exponential decay of the energy for the nonlinear damped wave equation, assuming (1.2) only, is still open.

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# 2 A generic gradient-like structure for effective dissipations

## 2.1 Statement of the result

Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^d$ with $d = 2, 3$. We set $X = \mathbb{H}_0^1(\Omega) \times \mathbb{L}^2(\Omega)$. Let $\mathfrak{G}^k$ be the space of the functions of class $\mathcal{C}^k(\Omega \times \mathbb{R}, \mathbb{R})$, $k \geq 1$ which are subcritical and dissipative that is, there exists $\alpha \in ]0, \frac{2}{d-2}[$ such that

$$\forall (u_1, u_2) \in \mathbb{R}^2, \ \sup_{x\in\Omega} |f(x,u_1) - f(x,u_2)| \le C(1 + |u_1|^\alpha + |u_2|^\alpha)|u_1 - u_2| \tag{2.1}$$

$$\text{and} \quad \limsup_{u\longrightarrow\pm\infty} \ \sup_{x\in\Omega} \frac{f(x,u)}{u} < \lambda_1 \ , \tag{2.2}$$

where $\lambda_1$ is the first eigenvalue of the Laplacian operator $-\Delta_D$ with Dirichlet boundary conditions. We endow the space $\mathfrak{G}^k$ of non-linearities with the Whitney topology, that is the topology generated by the open sets

$$\mathcal{V} = \{g \in \mathfrak{G}^k \ / \ |D^i f(x,u) - D^i g(x,u)| \le \delta(u), \ i = 0, ..., k, \ (x,u) \in \overline{\Omega} \times \mathbb{R}\} \ , \tag{2.3}$$

where $f$ belongs to $\mathfrak{G}^k$, $D^i f$ is the differential of order $i$ and $\delta$ is a positive continuous function on $\mathbb{R}$.

Let $\gamma \in \mathbb{L}^\infty(\Omega)$ be a nonnegative function. We consider the damped wave equation

$$\left\{ \begin{array}{ll} u_{tt}(x,t) + \gamma(x) u_t(x,t) = \Delta u(x,t) + f(x, u(x,t)) & (x,t) \in \Omega \times \mathbb{R}_+ \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ (u(x,0), u_t(x,0)) = (u_0, u_1) \in X & \end{array} \right. \tag{2.4}$$

It is well-known that Equation (2.4) generates a local dynamical system $S(t)$ on $X$. The linear operator associated with (2.4) is

$$A = \begin{pmatrix} 0 & Id \\ \Delta & -\gamma(x) \end{pmatrix} \qquad D(A) = (\mathbb{H}^2(\Omega) \cap \mathbb{H}^1_0(\Omega)) \times \mathbb{H}^1_0(\Omega) \ ,$$

and we denote $e^{At}$ the linear semigroup generated by $A$. We assume that $\gamma(x)$ satisfies the following property : there is a length $L$ such that all geodesics on $\Omega$ associated with the operator $\partial^2_{tt} - \Delta$ and of length greater than $L$ meet the support of $\gamma$. We recall that the geodesics are in fact straight lines which rebound on the boundary according to the laws of reflexion. Such a dissipation $\gamma$ will be called here an effective dissipation. This geometric control condition was first introduced by C.Bardos, G.Lebeau and J.Rauch. They proved in [3] that it implies the existence of two positive constants $M$ and $\lambda$ such that

$$\forall t \ge 0, \ \|e^{At}\|_{\mathcal{L}(X)} \le M e^{-\lambda t} \ . \tag{2.5}$$

It is classical to prove that (2.5) and (2.1) (with $\alpha < \frac{2}{d-2}$) imply that $S(t)$ is an asymptotically smooth system (see [9]).

Let $\Phi$ be the functional

$$\Phi \ : \ \begin{pmatrix} X & \longrightarrow & \mathbb{R} \\ (u,v) & \longmapsto & \int_\Omega \frac{1}{2}(|\overrightarrow{\nabla} u|^2 + |v|^2) - F(x,u) \end{pmatrix} \ , \tag{2.6}$$

where $F(x,u) = \int_0^u f(x,\xi)d\xi$. Obviously, $\Phi$ is non-increasing along the trajectories $U(t) = (u,u_t)(t) = S(t)U_0$ since

$$\Phi(U_0) - \Phi(S(t)U_0) = \int_0^t \int_\Omega \gamma(x)|u_t(x,s)|^2 dxds \ .$$

We recall that the decay of $\Phi$ and Hypothesis (2.2) imply that the trajectories of bounded sets are bounded.

Here we are interested in the gradient-like structure of (2.4).

**Definition 2.1.** *Let $S(t)$ be a dynamical system on a Banach space $X$.*
*We say that $S(t)$ is a* **gradient** *dynamical system if there exists a functional $\Phi \in \mathcal{C}^0(X)$, called the Lyapounov functional, such that, for all $t \geq 0$ and $U_0 \in X$, $\Phi(S(t)U_0) \leq \Phi(U_0)$ and if $\Phi(S(t)U_0) = \Phi(U_0)$ for all $t \geq 0$ then $U_0$ is an equilibrium point that is that $S(t)U_0 = U_0$ for all $t \geq 0$.*
*We say that $S(t)$ is a* **gradient-like** *dynamical system if, for all $U_0 \in X$, the $\alpha-$ and $\omega-$limit sets of $U_0$ only contain equilibrium points and if their intersection is empty.*

Different geometric conditions on the support of the dissipation $\gamma$ are known to imply that $\Phi$ is a Lyapounov function for $S(t)$ and so to ensure that $S(t)$ is a gradient-like system (see [22] and [16]). All of them satisfy the geometric control condition of [3]. Thus, it seems natural to wonder if the effectiveness of $\gamma$ is sufficient for $S(t)$ to be a gradient-like dynamical system. The first answer to this question is given in [12] (see also [15]). Its proof is based on a regularity result of [11] and a unique continuation property of [21].

**Theorem 2.2.** *Let $\gamma$ be an effective dissipation. If $f(x,u) \in \mathfrak{G}^k$ is a function of class $\mathcal{C}^\infty(\Omega \times \mathbb{R}, \mathbb{R})$ which is analytic with respect to $u$, then the dynamical system $S(t)$ generated by Equation (2.4) is gradient-like.*

The purpose of this section is to prove the following result.

**Theorem 2.3.** *Let $\gamma$ be an effective dissipation. The dynamical system $S(t)$ generated by Equation (2.4) is gradient-like, generically with respect to $f(x,u) \in \mathfrak{G}^k$ or with respect to $f(u) \in \tilde{\mathfrak{G}}^k$ where $\tilde{\mathfrak{G}}^k$ is the subset of $\mathfrak{G}^k$ consisting of the functions $f(u)$ depending of $u$ only.*

**Remark 2.4.** Actually, the above result is true for any subset $\tilde{\mathfrak{G}}^k \subset \mathfrak{G}^k$ such that, for all $M > 0$, any open set of $\tilde{\mathfrak{G}}^k$ contains a function $f(x,u)$ whose restriction to $\Omega \times [-M,M]$ is of class $\mathcal{C}^\infty$ and is analytic with respect to $u$.

## 2.2 Proof of Theorem 2.3

Let $k \geq 1$ and $f \in \tilde{\mathfrak{G}}^k$, where $\tilde{\mathfrak{G}}^k \subset \mathfrak{G}^k$ is a subset of $\mathfrak{G}^k$ satisfying the hypothesis given in Remark 2.4. For all $M > 0$, we set

$$\begin{aligned}\mathbb{W}_f^M = &\big\{ U(t) \in \mathcal{C}^0(\mathbb{R}, X) \ / \ U(t) = (u, u_t)(t) \text{ is a solution of (2.4)}, \ \Phi(U(t)) \text{ is} \\ &\text{constant with respect to } t, \ \sup_{t\in\mathbb{R}} \|U(t)\|_X \leq M \text{ and } \sup_{t\in\mathbb{R}} \|u_t(t)\|_{\mathbb{L}^2} \geq \frac{1}{M} \big\}\end{aligned} \tag{2.7}$$

Clearly, $\cup_{M>0}\mathbb{W}_f^M$ contains all the trajectories which are not equilibrium points and are included in the $\alpha-$ and $\omega-$limit sets of $S(t)$ and also contains the homoclinic orbits. Thus, $S(t)$ is gradient-like if $\cup_{M>0}\mathbb{W}_f^M = \emptyset$. To show Theorem 2.3, we will prove that the set of functions $f \in \tilde{\mathfrak{G}}^k$ such that $\mathbb{W}_f^M = \emptyset$ contains a dense open set of $\tilde{\mathfrak{G}}^k$.
We begin with a classical regularity lemma.

**Lemma 2.5.** *Let $M > 0$ and $f \in \tilde{\mathfrak{G}}^k$. There exist a neighborhood $\mathcal{V}$ of $f$ in $\tilde{\mathfrak{G}}^k$ and a positive constant $K$ such that for any $g \in \mathcal{V}$ and any $U(t) = (u, u_t)(t) \in \mathbb{W}_g^M$,*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{\mathbb{L}^\infty} \leq K \ .$$

***Proof :*** See [11] and also [9] and [13] □

As a consequence, we obtain the following result.

**Lemma 2.6.** *The set of functions $f \in \tilde{\mathfrak{G}}^k$ such that $\mathbb{W}_f^M = \emptyset$ is dense in $\tilde{\mathfrak{G}}^k$.*

***Proof :*** Let $\mathcal{O}$ be a neighborhood of a function $f$ in $\tilde{\mathfrak{G}}^k$. Let $\mathcal{V}$ and $K > 0$ be the neighborhood and the constant introduced in Lemma 2.5. By hypothesis, we can find a function $g(x, u) \in \mathcal{V} \cap \mathcal{O}$ whose restriction to $\Omega \times [-K, K]$ is of class $\mathcal{C}^\infty$ and is analytic with respect to $u$. With the same arguments as the ones used in the proof of Theorem 2.2, $\mathbb{W}_g^M = \emptyset$ (see [12]). □

To prove that the set of functions $f \in \tilde{\mathfrak{G}}^k$ such that $\mathbb{W}_f^M = \emptyset$ is open in $\tilde{\mathfrak{G}}^k$, we fix a sequence of functions $(f_n) \in \tilde{\mathfrak{G}}^k$ such that $(f_n)$ converges to a limit function $f$ in the Whitney topology. We denote by $S_n(t)$ (resp. $S(t)$) the dynamical system generated by Equation (2.4) corresponding to the non-linearity $f_n$ (resp. $f$). We assume that $\mathbb{W}_n^M := \mathbb{W}_{f_n}^M$ is not empty and we have to show that $\mathbb{W}_f^M$ is also not empty.
We first notice that the following convergence result holds.

**Lemma 2.7.** *The dynamical system $S_n(t)$ converges to $S(t)$ in the sense that for all bounded set $\mathcal{B} \subset X$, and all time $T \geq 0$,*

$$\sup_{U_0 \in \mathcal{B}} \sup_{t \in [0,T]} \|S(t)U_0 - S_n(t)U_0\|_X \xrightarrow[n \longrightarrow +\infty]{} 0 \ .$$

***Proof :*** The proof is very classical. Let $\varepsilon > 0$. For $n$ large enough, $f_n$ belongs to the neighborhood $\mathcal{V}$ of $f$ defined by (2.3) with $\delta(u) = \varepsilon$. The assumption (2.2) implies that the trajectories $\cup_{t \geq 0} S_n(t)\mathcal{B}$ are uniformly bounded with respect to $n$. Moreover, (2.1) implies that $(u, v) \mapsto (0, f(x, u))$ is Lipschitz-continuous on the bounded sets of $X$, in particular has a Lipschitz constant $C_{lip}$ on $\cup_n \cup_{t \geq 0} S_n(t)\mathcal{B}$. We set $U_n = (u_n, \partial_t u_n) = S_n(t)U_0$ and $U = (u, \partial_t u) = S(t)U_0$. For all $t \in [0, T]$, we find, for $n$ large enough,

$$\begin{aligned} \|S(t)U_0 - S_n(t)U_0\|_X &\leq \int_0^t \|e^{A(t-s)}(0, f(x, u_n(x,s)) - f_n(x, u_n(x,s)))\|_X ds \\ &\quad + \int_0^t \|e^{A(t-s)}(0, f(x, u(x,s)) - f(x, u_n(x,s)))\|_X ds \\ &\leq \varepsilon M \sqrt{|\Omega|} + \int_0^t M e^{-\lambda(t-s)} C_{lip} \|U(s) - U_n(s)\|_X ds \\ &\leq \varepsilon M \sqrt{|\Omega|} + M C_{lip} \int_0^t \|U(s) - U_n(s)\|_X ds \ , \end{aligned}$$

where $|\Omega| = \int_\Omega dx$ is the volume of $\Omega$.
Then, we conclude with Gronwall's lemma. □

We will also need the convergence of the Lyapounov functions.

**Lemma 2.8.** *Let $\Phi_n$ be the functional defined by (2.6) with $f$ replaced by $f_n$. Then, we have*

$$\sup_{U \in X} |\Phi(U) - \Phi_n(U)| \xrightarrow[n \longrightarrow 0]{} 0 \ .$$

***Proof :*** Let $\eta > 0$. We define a neighborhood $\mathcal{V}_\eta$ of $f$ by setting $\delta(u) = \eta e^{-|u|}$ in (2.3). As $(f_n)$ converges to $f$ in the Whitney topology, for $n$ large enough, $f_n$ belongs to $\mathcal{V}_\eta$. Let $F(x, u) = \int_0^u f(x, \xi) d\xi$, $F_n(x, u) = \int_0^u f_n(x, \xi) d\xi$ and $\Lambda(u) = \int_0^{|u|} \eta e^{-|u|}$. Clearly, $\Lambda$ is bounded in $\mathbb{L}^\infty(\Omega)$ by $\eta$. Thus, for $n$ large enough, we have, for all $U = (u, v) \in X$,

$$|\Phi(U) - \Phi_n(U)| \leq \int_\Omega |F(x, u) - F_n(x, u)| dx \leq \int_\Omega \Lambda(u) dx \leq \eta |\Omega| \ .$$

□

Since $\Phi$ is a Lyapounov function, the trajectory $S(t)B(0,M)$ of any ball is bounded in $X$. As $S(t)$ is asymptotically smooth, there exists a compact bounded invariant set $\mathcal{A}_M$ such that $S(t)B(0,M)$ is attracted by $\mathcal{A}_M$, that is

$$\sup_{\|U\|_X \leq M} \operatorname{dist}_X(S(t)U, \mathcal{A}_M) \xrightarrow[t \longrightarrow +\infty]{} 0 \ .$$

It is well-known that Lemma 2.7 implies that

$$\sup_{U_n \in \mathbb{W}_n^M} \operatorname{dist}_X(U_n, \mathcal{A}_M) \xrightarrow[n \longrightarrow +\infty]{} 0 \ , \tag{2.8}$$

where $\operatorname{dist}_X$ is the distance between a point $U \in X$ and a set $\mathcal{S} \subset X$ defined by

$$\operatorname{dist}_X(U, \mathcal{S}) = \inf_{U' \in \mathcal{S}} \|U - U'\|_X \ . \tag{2.9}$$

**Lemma 2.9.** *Let $U_n(t) \in \mathcal{C}^0(\mathbb{R}, X)$ be a trajectory of $S_n(t)$ belonging to $\mathbb{W}_n^M$. Let $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence of times. There exists a globally defined and bounded trajectory $U(t) \subset \mathcal{A}_M$ for the system $S(t)$ and subsequences $S_{\varphi(n)}(t)$, $t_{\varphi(n)}$ and $U_{\varphi(n)}$, such that, for every positive time $T$, we have*

$$\sup_{t \in ]-T,T[} \|U(t) - U_{\varphi(n)}(t_{\varphi(n)} + t)\|_X \longrightarrow 0 \quad \textit{when} \;\; n \longrightarrow +\infty \ . \tag{2.10}$$

***Proof :*** Using (2.8), we know that there exists $V_n \in \mathcal{A}_M$ such that $\|U_n(t_n) - V_n\|_X \longrightarrow 0$. As $\mathcal{A}_M$ is compact, there exists $U \in \mathcal{A}_M$ and an extraction $\varphi_1 : \mathbb{N} \mapsto \mathbb{N}$, such that $U_{\varphi_1(n)}(t_{\varphi_1(n)}) \longrightarrow U$. We set $T = 1$. Using the same arguments as above, we find a point $V_2 \in \mathcal{A}_M$ and an extraction $\varphi_2$ such that $U_{\varphi_1\circ\varphi_2(n)}(t_{\varphi_1\circ\varphi_2(n)} - T) \longrightarrow V_2$. Lemma 2.7 implies that

$$\sup_{t \in ]-T,T[} \|U_{\varphi_1\circ\varphi_2(n)}(t_{\varphi_1\circ\varphi_2(n)} + t) - S(t+T)V_2\|_X \longrightarrow 0 \ .$$

For all $t \in ]-T,T[$, we set $U(t) = S(t+T)V_2$. Notice that $U(0) = U$.
Then, we repeat the same arguments : let $T = 2$, there exists $V_3 \in \mathcal{A}_M$ and an extraction $\varphi_3$ such that

$$\sup_{t \in ]-T,T[} \|U_{\varphi_1\circ\varphi_2\circ\varphi_3(n)}(t_{\varphi_1\circ\varphi_2\circ\varphi_3(n)} + t) - S(t+T)V_3\|_X \longrightarrow 0 \ ,$$

and for all $t \in ]-T,T[$, we set $U(t) = S(t+T)V_3$, and so on...
Then, Lemma 2.9 follows from the classical diagonal extraction $\varphi(n) = \varphi_1 \circ \varphi_2 \circ \ldots \circ \varphi_n(n)$.
$\square$

We recall that we want to show that the fact that $\mathbb{W}_n^M$ is not empty implies that $\mathbb{W}_f^M$ is also not empty. Let $U_n(t) = (u_n, \partial_t u_n)(t) \in \mathcal{C}^0(\mathbb{R}, X)$ be a trajectory of $S_n(t)$ belonging to $\mathbb{W}_n^M$. By definition of $\mathbb{W}_n^M$, we can find a sequence of times $(t_n) \subset \mathbb{R}$ such that

$$\liminf_{n \longrightarrow +\infty} \|\partial_t u_n(t_n)\|_{\mathbb{L}^2} \geq \frac{1}{M} \ .$$

Lemma 2.9 implies that, up to the extraction of a subsequence, there exists a globally defined and bounded trajectory $U(t) \subset \mathcal{A}_M$ for the system $S(t)$ such that, for all $T > 0$,

$$\sup_{t \in ]-T,T[} \|U(t) - U_n(t_n + t)\|_X \xrightarrow[n \longrightarrow +\infty]{} 0 \ .$$

Of course, by continuity, we have that

$$\sup_{t \in \mathbb{R}} \|U(t)\|_X \leq M \text{ and } \sup_{t \in \mathbb{R}} \|u_t(t)\|_{\mathbb{L}^2} \geq \frac{1}{M} \ .$$

Thus, it remains to show that $\Phi$ is constant on $U$. For each $n$, the Lyapounov function $\Phi_n$ is constant on $U_n(t)$. Moreover, the sequence of values $\Phi_n(U_n)$ is bounded. Therefore, up to the extraction of a subsequence, we can assume that $\Phi_n(U_n)$ converges and Lemma 2.8 implies that $\Phi$ is constant on $U$. This shows that $U$ belongs to $\mathbb{W}_f^M$, which is therefore not empty. Thus, the set of functions $f \in \tilde{\mathfrak{G}}^k$ such that $\mathbb{W}_f^M = \emptyset$ is open in $\tilde{\mathfrak{G}}^k$, and Theorem 2.3 is proved.

# 3 Gradient-like structure for a wave equation with indefinite damping

Let $\Omega$ be a bounded domain of $\mathbb{R}^d$ $(d = 1, 2, 3)$. Let $\gamma \in \mathbb{L}^\infty(\Omega, \mathbb{R}_+)$ be chosen so that there exists $x_0 \in \mathbb{R}^d$ such that the support of $\gamma$ contains a neighborhood of the set $\{x \in \partial\Omega \ / \ (x - x_0).\nu > 0\}$, where $\nu$ is the exterior unit normal vector. It is well-known that this hypothesis implies the exponential decay property (2.5). Moreover, it is proved in [16] that it also implies the following unique continuation property.

**Proposition 3.1.** *There exists a time $T > 0$ such that, for all $w \in \mathbb{H}^1(\Omega \times ]0, T[)$ and $h \in \mathbb{L}^\infty(\Omega \times ]0, T[)$ satisfying*

$$\begin{cases} w_{tt} = \Delta w + h(x,t)w \\ w_{|\partial\Omega} = 0 \\ w = 0 \text{ on the support of } \gamma(x) \end{cases}$$

*then $w$ vanishes everywhere in $\Omega \times ]0, T[$.*

Let $f \in \mathcal{C}^2(\Omega \times \mathbb{R}, \mathbb{R})$. We assume that $f$ can be written as follows :

$$f(x,u) = a(u) + \lambda u + b(x,u) \ , \tag{3.1}$$

with $\lambda < \lambda_1$, where $\lambda_1$ is the first eigenvalue of the Laplacian operator $-\Delta_D$ with Dirichlet boundary conditions.
We assume that $b \in \mathcal{C}^1(\Omega \times \mathbb{R}, \mathbb{R})$ and that $b(x,u)$ and $\overrightarrow{\nabla}_x b(x,u)$ are sublinear in the sense that $D_u b(x,u)$ and $D_u \overrightarrow{\nabla}_x b(x,u)$ are globally bounded and satisfy

$$\lim_{u \to \pm\infty} \ \sup_{x \in \Omega} \ |D_u b(x,u)| + |D_u \overrightarrow{\nabla}_x b(x,u)| = 0 \ . \tag{3.2}$$

We assume that $a \in \mathcal{C}^1(\mathbb{R})$ and that, for all $u \in \mathbb{R}$, $a(u)u \leq 0$. We also assume that $a$ satisfies the growth condition (2.1) and one of the following properties :

- either $a'$ is globally bounded and the limits $\lim_{u \to \pm\infty} a'(u) = a'(\pm\infty)$ exist,
- or $a$ is superlinear in the sense that there exists $\delta > 0$ such that $-a(u)u \geq -(2+\delta) \int_0^u a(\zeta) d\zeta$, for all $u \in \mathbb{R}$.

**Remark 3.2.** As $a(u)u \leq 0$ for every $u \in \mathbb{R}$, the positive constant $\delta$ can be chosen as small as needed.

Let $\Phi$ be the functional defined by (2.6). The hypotheses on $a$ were introduced by E.Zuazua in [24] to obtain the exponential decay of the energy $\Phi$ along the trajectories of the damped wave equation if $b = 0$. We also notice that the assumptions on $f$ imply that

$$\exists C > 0, \ \forall U \in X, \quad \|U\|_X^2 \leq C(\Phi(U) + C) \ . \tag{3.3}$$

Let $g \in \mathbb{L}^\infty(\Omega, \mathbb{R})$ be a damping with indefinite sign. We set $g_+ = \max(g, 0)$ and $g_- = \max(-g, 0)$. For all $\varepsilon \geq 0$, we denote $S_\varepsilon(t)$ the semigroup generated on $X = \mathbb{H}_0^1(\Omega) \times \mathbb{L}^2(\Omega)$ by the equation

$$\begin{cases} u_{tt} + (\gamma(x) + \varepsilon g(x)) u_t = \Delta u + f(x,u) & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

Our assumption on the support of $\gamma$ implies, via Proposition 3.1, that the functional $\Phi$ is a Lyapounov function for $S_0(t)$ and that $S_0(t)$ is a gradient dynamical system. In fact, the result is stronger.

**Proposition 3.3.** *Let $T$ be the time introduced in Proposition 3.1. If $U_0 \in X$ is such that $\Phi(S_0(T)U_0) = \Phi(U_0)$, then $U_0$ is an equilibrium point.*

It is also well-known that $S_0(t)$ admits a compact global attractor $\mathcal{A}_0$ (see for example [9]).

In this section, we study the gradient-like structure of $S_\varepsilon(t)$ (see Definition 2.1). We enhance that, if the support of $g_-$ is not included in the support of $\gamma$, the classical Lyapounov functional $\Phi$ is even not non-increasing along the trajectories of $S_\varepsilon(t)$, as soon as $\varepsilon$ is positive. Thus, the gradient-like structure of $S_\varepsilon(t)$ is not immediate.
We recall that $\text{dist}_X(U, \mathcal{S})$ denotes the distance between a point $U \in X$ and a set $\mathcal{S} \subset X$, see (2.9). We also introduce the Hausdorff distance between two sets $\mathcal{S}_1 \subset X$ and $\mathcal{S}_2 \subset X$ by

$$d_X(\mathcal{S}_1, \mathcal{S}_2) = \max\left( \sup_{U \in \mathcal{S}_1} \text{dist}_X(U, \mathcal{S}_2) \ ; \ \sup_{U \in \mathcal{S}_2} \text{dist}_X(U, \mathcal{S}_1))\right) \ . \tag{3.5}$$

The first result of this section is the following.

**Theorem 3.4.** *There exists a positive number $\varepsilon_0$ such that, for all $\varepsilon \in ]0, \varepsilon_0[$, $S_\varepsilon(t)$ admits a compact global attractor $\mathcal{A}_\varepsilon$. Moreover, the family of attractors is uniformly bounded in $X$ and upper-semicontinuous with respect to $\varepsilon$ :*

$$\sup_{U \in \mathcal{A}_\varepsilon} dist_X(U, \mathcal{A}_0) \xrightarrow[\varepsilon \to 0]{} 0 \ .$$

The main result of this section is the gradient-like structure for the wave equation (3.4).

**Theorem 3.5.** *Assume that all the equilibria of $S_0(t)$ are hyperbolic. Then, there exists a positive number $\varepsilon_0$ such that, for all $\varepsilon \in ]0, \varepsilon_0[$, $S_\varepsilon(t)$ is gradient-like. Moreover, for $\varepsilon$ small enough, the dynamics on the attractor $\mathcal{A}_\varepsilon$ of $S_\varepsilon(t)$ respect the order of equilibria induced by the energy functional $\Phi$ in the sense that, if $E_-$ and $E_+$ are two equilibria such that there exists a heteroclinic orbit $U_\varepsilon(t)$ for $S_\varepsilon(t)$ with* $\lim_{t \to \pm\infty} U_\varepsilon(t) = E_\pm$*, then $\Phi(E_-) > \Phi(E_+)$. In addition, the attractors $A_\varepsilon$ are continuous :*

$$d_X(\mathcal{A}_\varepsilon, \mathcal{A}_0) \xrightarrow[\varepsilon \longrightarrow 0]{} 0 \ . \tag{3.6}$$

We recall that the hypothesis that all the equilibria of $S_0(t)$ are hyperbolic is a generic hypothesis (see [23] and [4]). We point out that the dynamics on $\mathcal{A}_\varepsilon$ can be different from the ones on $\mathcal{A}_0$. Indeed, we do not assume any Morse-Smale property for $S_0(t)$.

## 3.1 Proof of Theorem 3.4

The existence of a compact global attractor for the dynamical system $S_\varepsilon(t)$ is equivalent to the fact that $S_\varepsilon(t)$ is asymptotically compact, point dissipative and is such that the trajectories of the bounded sets of $X$ are bounded (see [9] or [19] for example).
The last two properties are direct consequences of (3.3) and of the following proposition.

**Proposition 3.6.** *There exist a bounded set $\mathcal{B} \subset X$ and a constant $\varepsilon_0 > 0$ such that, for all $\varepsilon \in [0, \varepsilon_0[$, the dynamical system $S_\varepsilon(t)$ is exponentially decreasing outside $\mathcal{B}$ in the following sense. There exist two positive constants $K$ and $\mu$, independent of $\varepsilon$, such that, for all $\varepsilon \in [0, \varepsilon_0[$, if $U_0 \in X$ and $t \geq 0$ satisfy $S_\varepsilon(s)U_0 \notin \mathcal{B}$ for all $s \in [0,t]$, then*

$$\Phi(S_\varepsilon(t)U_0) \leq Ke^{-\mu t}\Phi(U_0) ,$$

*where $\Phi$ is the energy defined in* (2.6).

The proof is mainly based on the arguments of [24] for a non-linearity $f(u)$ satisfying the same assumptions as $a(u)$. Thus, we only give here the outline of the proof and the main differences with [24]. We refer to [24] for the missing details.
Let $U_0 \in X$ and $\varepsilon \in [0, \varepsilon_0[$. We set $S_\varepsilon(t)U_0 = U_\varepsilon(t) = (u_\varepsilon, \partial_t u_\varepsilon)(t)$. The first step of the proof is the following estimate.

**Lemma 3.7.** *Let $T$ be the time such that the unique continuation property stated in Proposition 3.1 holds. There exist a time $T' \geq T$ and a positive constant $C$ such that, for all $U_0 \in X$,*

$$\Phi(U_\varepsilon(T')) \leq C\left(\int_0^{T'}\int_\Omega(\gamma(x) + \varepsilon|g(x)|)|\partial_t u_\varepsilon|^2 dxdt + \int_0^{T'}\int_\Omega |u_\varepsilon|^2 dxdt + 1\right) . \quad (3.7)$$

***Proof :*** As $b$ and $\overrightarrow{\nabla}_x b$ are sublinear, there exists a positive constant $C$ such that

$$\int_\Omega |b(x,u)u|dx + \int_\Omega |\overrightarrow{\nabla}_x b(x,u)u|dx \leq C(\int_\Omega |u|^2 + 1) . \quad (3.8)$$

Let $m(x) = (x - x_0)$ where $x_0 \in \mathbb{R}^d$ is chosen so that the support of $\gamma$ contains a neighborhood of the set $\Gamma(x_0) = \{x \in \partial\Omega \ / \ (x - x_0).\nu > 0\}$. We set

$$\gamma_\varepsilon(x) = \gamma(x) + \varepsilon|g(x)|$$

Using (3.8) and the arguments of [24], as well as multipliers technics, one shows that there exists a positive constant $C$ such that, for all $\eta > 0$,

$$\begin{aligned}
&\int_0^{T'}\int_\Omega (\frac{d}{2} - \eta)|\partial_t u_\varepsilon|^2 + (1 + \eta - \frac{d}{2})|\overrightarrow{\nabla} u_\varepsilon|^2 dxdt \\
&-\eta\int_0^{T'}\int_\Omega a(u_\varepsilon)u_\varepsilon dxdt + d\int_0^{T'}\int_\Omega \left(\int_0^{u_\varepsilon} a(\zeta)d\zeta\right) dxdt \\
&\qquad \leq \frac{1}{2}\int_0^{T'}\int_{\Gamma(x_0)} (m.\nu)|\frac{\partial u_\varepsilon}{\partial \nu}|^2 d\sigma dt + \int_0^{T'}\int_\Omega \gamma_\varepsilon(x)|\partial_t u_\varepsilon m.\overrightarrow{\nabla} u_\varepsilon| dxdt \\
&\qquad + \left[\int_\Omega \left|\partial_t u_\varepsilon m.\overrightarrow{\nabla} u_\varepsilon + \eta u_\varepsilon\left(\partial_t u_\varepsilon + \frac{1}{2}\gamma_\varepsilon(x)u_\varepsilon\right)\right| dx\right]_0^{T'} + C\eta\left(\int_0^{T'}\int_\Omega |u|^2 dxdt + T'\right)
\end{aligned} \quad (3.9)$$

Using (3.3), we obtain the estimate

$$\left[\int \left|\partial_t u_\varepsilon m.\overrightarrow{\nabla} u_\varepsilon + \eta u_\varepsilon \left(\partial_t u_\varepsilon + \frac{1}{2}\gamma_\varepsilon u_\varepsilon\right)\right|\right]_0^{T'} \leq C\left(\Phi(U_\varepsilon(0)) + \Phi(U_\varepsilon(T') + C)\right)$$

and for all positive number $\kappa$

$$\int_\Omega \gamma_\varepsilon(x)|\partial_t u_\varepsilon m.\overrightarrow{\nabla} u_\varepsilon| dx \leq \kappa \|m\|_{\mathbb{L}^\infty}^2 \int_\Omega |\overrightarrow{\nabla} u_\varepsilon| dx + \frac{1}{2\kappa}\|\gamma_\varepsilon(x)\|_{\mathbb{L}^\infty} \int_\Omega \gamma_\varepsilon(x)|\partial_t u_\varepsilon|^2 dx\ .$$

Moreover, if $a$ is globally Lipschitz-continuous, we have

$$\int_\Omega |a(u_\varepsilon)u_\varepsilon| dx + \int_\Omega \left|\int_0^{u_\varepsilon} a(\zeta)d\zeta\right| dx \leq C\left(\int_\Omega |u|^2 dx + 1\right)\ ,$$

and if $a$ is superlinear, then, for all $\eta \in ]\frac{d}{2}-1, \frac{d}{2}[$, we have, for $\delta$ as small as needed,

$$-\eta\delta \int_\Omega \left(\int_0^{u_\varepsilon} a(\zeta)d\zeta\right) dx \leq -\eta \int_\Omega a(u_\varepsilon)u_\varepsilon dx + d\int_\Omega \left(\int_0^{u_\varepsilon} a(\zeta)d\zeta\right) dx\ .$$

In both cases, the above inequalities together with (3.9) show that there exists a positive constant $C$ such that

$$\begin{aligned} C\int_0^{T'} \Phi(U_\varepsilon(t))dt \ &\leq\ \frac{1}{2}\int_0^{T'} \int_{\Gamma(x_0)} (m.\nu)|\frac{\partial u_\varepsilon}{\partial \nu}|^2 dxdt + \int_0^{T'} \int_\Omega \gamma_\varepsilon(x)|\partial_t u_\varepsilon|^2 dxdt \\ &\quad + \Phi(U_\varepsilon(0)) + \Phi(U_\varepsilon(T')) + \int_0^{T'} \int_\Omega |u_\varepsilon|^2 dx + CT' \end{aligned} \tag{3.10}$$

We follow again the method of [24] and use (3.8) to get

$$\begin{aligned} C\int_0^{T'} \int_{\Gamma(x_0)} (m.\nu)|\frac{\partial u_\varepsilon}{\partial \nu}|^2 dxdt \ &\leq\ \int_0^{T'} \int_\Omega \gamma_\varepsilon(x)|\partial_t u_\varepsilon|^2 dxdt + \int_0^{T'} \int_\Omega |u_\varepsilon|^2 \\ &\quad + \Phi(U_\varepsilon(0)) + \Phi(U_\varepsilon(T') + CT' \end{aligned} \tag{3.11}$$

Since for all $t \in [0, T']$,

$$\Phi(U_\varepsilon(0)) - \int_0^t \int_\Omega \gamma_\varepsilon(x)|\partial_t u_\varepsilon|^2 dxdt \leq \Phi(U_\varepsilon(t)) \leq \Phi(U_\varepsilon(0)) + \varepsilon \int_0^t \int_\Omega g_-(x)|\partial_t u_\varepsilon|^2 dxdt\ , \tag{3.12}$$

(3.10) and (3.11) show that

$$CT' \min_{t\in[0,T']} \Phi(U_\varepsilon(t)) \leq \int_0^{T'} \int_\Omega \gamma_\varepsilon(x)|\partial_t u_\varepsilon|^2 dxdt + \int_0^{T'} \int_\Omega |u_\varepsilon|^2 dxdt + 2\Phi(U_\varepsilon(T')) + CT'\ .$$

Using (3.12) and the fact that the constant $C$ is independent of $T'$, we find that (3.7) is satisfied for $T'$ large enough. □

**Lemma 3.8.** *Let $T$ be the time such that the unique continuation property stated in Proposition 3.1 holds. For all $T' \geq T$, there exist positive constants $\varepsilon_0$ and $C$ such that, for all $\varepsilon \in [0, \varepsilon_0[$, and all $U_0 \in X$,*

$$\int_0^{T'} \int_\Omega |u_\varepsilon|^2 dxdt \leq C \left( \int_0^{T'} \int_\Omega \gamma(x) |\partial_t u_\varepsilon|^2 dxdt + 1 \right) . \tag{3.13}$$

***Proof :*** Assume that (3.13) is not satisfied, then there exist a sequence $\varepsilon_n \longrightarrow 0$ and a sequence of initial data $U_0^n$ such that

$$\int_0^{T'} \int_\Omega |u_{\varepsilon_n}^n|^2 dxdt \geq n \left( \int_0^{T'} \int_\Omega \gamma(x) |\partial_t u_{\varepsilon_n}^n|^2 dxdt + 1 \right) . \tag{3.14}$$

We set $\tilde{u}_{\varepsilon_n}^n = u_{\varepsilon_n}^n / \lambda_n$ where $\lambda_n = \int \int |u_{\varepsilon_n}^n|^2 \longrightarrow +\infty$. Then, we get a contradiction with the same arguments as the ones of [24]. We only emphasize the slight modifications. If $a(u)$ is superlinear, we have $-F(x,u) \geq C(|u|^{2+\delta} - C)$ which implies that (3.14) is incompatible with the estimate (3.7). If $a(u)$ is globally Lipschitz-continuous, the arguments consist in finding a limit $v$ for the sequence $u_{\varepsilon_n}^n / \lambda_n$ in $\mathbb{L}^2(\Omega \times ]0, T[)$ and a limit equation satisfied by $v_t$. Notice that, due to (3.2), the term $b(x,u)$ does not appear in this limit process. Then, we apply Proposition 3.1 to get $v_t = 0$ in $\Omega \times ]0, 1[$. Finally, the contradiction comes from the sign conditions on $a$ and $\lambda$ and the elliptic equation satisfied by $v(x,t) = v(x)$. □

**Proof of Proposition 3.6 :** Using Lemma 3.8, the inequality (3.7) becomes

$$\Phi(U_\varepsilon(T')) \leq C \left( \Phi(U_\varepsilon(0)) - \Phi(U_\varepsilon(T')) + \varepsilon \int_0^{T'} \int_\Omega |g|(x) |\partial_t u_\varepsilon|^2 + 1 \right) . \tag{3.15}$$

Using (3.3) and (3.12), we obtain

$$\begin{aligned}
\int_0^{T'} \int_\Omega |g|(x) |\partial_t u_\varepsilon|^2 &\leq \|g\|_\infty T' \sup_{t \in [0,T']} \|\partial_t u_\varepsilon(t)\|_{\mathbb{L}^2}^2 \\
&\leq C \sup_{t \in [0,T']} \|U_\varepsilon(t)\|_X^2 \\
&\leq C \left( \sup_{t \in [0,T']} \Phi(U_\varepsilon(t)) + C \right) \\
&\leq C \left( \Phi(U_\varepsilon(0)) + \varepsilon \int_0^{T'} \int_\Omega g_-(x) |\partial_t u_\varepsilon|^2 dxdt + C \right)
\end{aligned}$$

So, for $\varepsilon$ small enough, $\int_0^{T'} \int_\Omega g(x)|\partial_t u_\varepsilon|^2 \leq C(\Phi(U_\varepsilon(0)) + C)$. Thus, for $\Phi(U_\varepsilon(0))$ large enough and $\varepsilon$ small enough, (3.15) implies that there exists a positive constant $C$, with $C < 1$, such that

$$\Phi(U_\varepsilon(T')) \leq C\Phi(U_\varepsilon(0)) \ .$$

This is well-known to imply the exponential decay of the energy. Notice that the fact that the above estimate is only true for $\Phi(U_\varepsilon(0))$ large enough implies that the exponential decay only occurs outside a ball of $X$. Moreover, we enhance that all the constants of Lemma 3.7 and 3.8 are uniform with respect to $\varepsilon \in [0, \varepsilon_0[$. □

In Proposition 3.6, we have shown that the energy $\Phi$ decreases exponentially outside a ball, with constants $K$ and $\mu$ independent of $\varepsilon \in [0, \varepsilon_0[$. Choosing $f = 0$, the same result for the linear semigroups immediately proves the following corollary.

**Corollary 3.9.** *Let*

$$A_\varepsilon = \begin{pmatrix} 0 & Id \\ \Delta & -\gamma(x) - \varepsilon g(x) \end{pmatrix} \ .$$

*There exist three positive constants $\varepsilon_1$, $K$ and $\mu$ such that, for all $\varepsilon \in [0, \varepsilon_1[$, and $t \geq 0$,*

$$\|e^{A_\varepsilon t}\|_{\mathcal{L}(X)} \leq K e^{-\mu t} \ .$$

Notice that the exponential decay of the semigroup $e^{A_\varepsilon t}$ had already been investigated. In particular, in [18], using Carleman estimates, an explicit value for $\varepsilon_1$ is given.

Corollary 3.9 and the fact that $f$ is subcritical imply that the dynamical systems $S_\varepsilon(t)$ are asymptotically smooth (see [9]). Together with the fact that $S_\varepsilon(t)$ is point dissipative and is such that the trajectories of the bounded sets of $X$ are bounded, this implies that $S_\varepsilon(t)$ admits a compact global attractor $\mathcal{A}_\varepsilon$. Moreover, since we have proved in Proposition 3.6 that one can choose the same absorbing set for all the systems $S_\varepsilon(t)$, the union $\cup_\varepsilon \mathcal{A}_\varepsilon$ is bounded in $X$. The upper-semicontinuity of the attractors then follows from the convergence of the trajectories (see [19] for example).

**Lemma 3.10.** *There exists a positive constant $C$ such that, for all $\varepsilon \in [0, \varepsilon_1[$ and $t \geq 0$,*

$$\|e^{A_\varepsilon t} - e^{A_0 t}\|_{\mathcal{L}(X)} \leq C\varepsilon \ . \tag{3.16}$$

*Moreover, for any bounded set $\mathcal{B} \subset X$, there exists a positive constant $C(\mathcal{B})$ such that, for all $\varepsilon \in [0, \varepsilon_1[$, $U_0 \in \mathcal{B}$ and $t \geq 0$,*

$$\|S_\varepsilon(t)U_0 - S_0(t)U_0\|_X \leq C(\mathcal{B})e^{C(\mathcal{B})t}\varepsilon \ . \tag{3.17}$$

***Proof :*** Let $U_0 \in X$ and $\varepsilon \in [0, \varepsilon_1[$. We set $U(t) = (u, u_t)(t) = S_\varepsilon(t)U_0$. The estimate (3.16) immediatly follows from Corollary 3.9 and the equality

$$(e^{A_\varepsilon t} - e^{A_0 t})U_0 = \int_0^t e^{A_0(t-s)}(0, \varepsilon g(x)u_t(x,s))ds \ .$$

Then, we classically deduce (3.17) from Duhamel's formula and Gronwall's lemma. □

## 3.2 Proof of Theorem 3.5

First, we state a preliminary result concerning the local stable and unstable manifolds of hyperbolic equilibria and their convergence when $\varepsilon \longrightarrow 0$. Let $E$ be an equilibrium point of $S_0(t)$. For all $r > 0$, we denote by $B(E, r)$ the ball of center $E$ and radius $r$.

**Theorem 3.11.** *There exists a positive constant $\varepsilon_0$ such that the following properties hold. If $E$ is a hyperbolic equilibrium point of $S_0(t)$, then for all $\varepsilon \in [0, \varepsilon_0[$, $E$ is also a hyperbolic equilibrium point of $S_\varepsilon(t)$. Moreover, there exists a radius $r > 0$ small enough such that the sets $W^s_\varepsilon(E, r) = \{U_0 \in B(E, r) \mid \forall t \geq 0,\ S_\varepsilon(t)U_0 \in B(E, r)\}$ and $W^u_\varepsilon(E, r) = \{U_0 \in B(E, r) \mid$ there exists a negative trajectory $U(t) \in \mathcal{C}^0(] - \infty, 0], X)$ for $S_\varepsilon(t)$ such that $U(0) = U_0$ and $\forall t \leq 0,\ U(t) \in B(E, r)\}$, are embedded manifolds of $X$. In addition, any $U_0 \in W^s_\varepsilon(E, r)$ satisfies $S_\varepsilon(t)U_0 \longrightarrow E$ when $t \longrightarrow +\infty$ and any trajectory $U(t) \in \mathcal{C}^0(] - \infty, 0], X)$ included in $W^u_\varepsilon(E, r)$ satisfies $U(t) \longrightarrow E$ when $t \longrightarrow -\infty$. Finally, the local stable and unstable manifolds are continuous at $\varepsilon = 0$, that is*

$$d_X(W^u_\varepsilon(E, r), W^u_0(E, r)) \leq \varepsilon \quad \text{and} \quad d_X(W^s_\varepsilon(E, r), W^s_0(E, r)) \leq \varepsilon \ ,$$

*where $d_X$ is the Hausdorff distance defined in* (3.5).

***Proof :*** The existence of local stable and unstable manifolds is classical, see for example the Appendix of [9] or [2]. The outline of the proof of the result of convergence is also well-known, see [19], [10] or [14]. We only want to enhance that the construction of the local stable and unstable manifolds is done in a neighborhood of $E$ uniform with respect to $\varepsilon \in [0, \varepsilon_0[$ due to Corollary 3.9. □

We assume in the remaining part of this section that all the equilibria of $S_0(t)$ are hyperbolic. We have to show that, for $\varepsilon$ small enough, the dynamical system $S_\varepsilon(t)$ has the property that any $\omega-$limit set of a point $U \in X$ is an equilibrium point. As the proof of the same property for the $\alpha-$limit sets of globally bounded trajectories is similar, we omit it.
We argue by contradiction. Let $(\varepsilon_n)$ be a sequence of positive numbers converging to 0

and let $(U_n)$ be a sequence of points of $X$ such that the $\omega-$limit set of $U_n$ for $S_{\varepsilon_n}(t)$ does not only contain one equilibrium point. Due to Theorem 3.4, there exists a bounded set $\mathcal{B}$ such that the $\omega-$limit set of $U_n$ for $S_{\varepsilon_n}(t)$ belongs to $\mathcal{B}$. In particular, we can assume without loss of generality that $S_{\varepsilon_n}(t)U_n \in \mathcal{B}$ for all $n \geq 0$ and $t \geq 0$. For all $t \geq 0$, we set $U_n(t) = S_{\varepsilon_n}(t)U_n$. Using the same arguments as in Lemma 2.9, we obtain the following result.

**Lemma 3.12.** *Let $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence of times with $t_n \longrightarrow +\infty$. There exists a globally defined and bounded trajectory $U(t) \subset \mathcal{A}$ for the system $S_0(t)$ and subsequences $t_{\varphi(n)}$ and $U_{\varphi(n)}(t)$, such that, for all positive time $\theta$, and for all $n$ such that $t_{\varphi(n)} \geq \theta$, we have*

$$\sup_{t\in]-\theta,\theta[} \|U(t) - U_{\varphi(n)}(t_{\varphi(n)} + t)\|_X \xrightarrow[n\longrightarrow 0]{} 0 . \tag{3.18}$$

We recall that $\Phi$ denotes the Lyapounov functional of $S_0(t)$ defined by (2.6) and that this functional is no longer a Lyapounov functional for $S_\varepsilon(t)$ when $\varepsilon$ is positive. We set

$$l = \inf\{\zeta \in \mathbb{R} \ / \ \exists (t_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+,\ t_n \xrightarrow[n\to+\infty]{} +\infty \text{ and } \Phi(U_n(t_n)) \xrightarrow[n\to+\infty]{} \zeta\} . \tag{3.19}$$

This infimum exists since $\Phi$ is bounded from below in $\mathcal{B}$. Let $(t_n) \subset \mathbb{R}_+$ be a sequence such that $t_n \longrightarrow +\infty$ and $\Phi(U_n(t_n)) \longrightarrow l$ when $n \longrightarrow +\infty$. Lemma 3.12 implies that, up to the extraction of a subsequence, there exists a complete trajectory $U(t)$ for $S_0(t)$ such that, for all $\theta > 0$

$$\sup_{t\in]-\theta,\theta[} \|U(t) - U_n(t_n + t)\|_X \xrightarrow[n\longrightarrow 0]{} 0 . \tag{3.20}$$

Moreover, by continuity of $\Phi$ and by the definition of $l$, we have that

$$\forall t \geq 0,\ \Phi(U(t)) \geq \Phi(U(0)) .$$

As $\Phi$ is a Lyapounov functional for $S_0(t)$, this implies that for all $t \geq 0$, $U(t) = E$ where $E$ is an equilibrium point of the dynamical systems $S_\varepsilon(t)$.
Let $r > 0$ be the radius introduced in Theorem 3.11. We next prove that, for $\varepsilon_n$ small enough, $U_n(t)$ belongs to $B(E, r)$ for all $t \geq t_n$. As $E$ is hyperbolic for $S_{\varepsilon_n}(t)$, if $U_n(t)$ belongs to $B(E, r)$ for all $t \geq t_n$, then Theorem 3.11 implies that $U_n(t) \longrightarrow E$ when $t \longrightarrow +\infty$. This contradicts the fact that the $\omega-$limit set of $U_n$ for $S_{\varepsilon_n}(t)$ does not only contain a equilibrium point and finishes the proof of the gradient-like structure of $S_\varepsilon(t)$ stated in Theorem 3.5.
Once again, we argue by contradiction : assume that there exists a time $t \geq t_n$ such that $U_n(t) \notin B(E, r)$, where $t_n \longrightarrow +\infty$ is a sequence of times satisfying $\Phi(U_n(t_n)) \longrightarrow l$. We set

$$\tau_n = \inf\{t \geq t_n \ / \ U_n(t) \notin B(E, r)\} .$$

Lemma 3.12 shows that, up to the extraction of a subsequence, there exists a complete trajectory $V(t)$ for $S_0(t)$ such that, for all $T > 0$,

$$\sup_{t \in ]-T,T[} \|V(t) - U_n(\tau_n + t)\|_X \xrightarrow[n \longrightarrow 0]{} 0 \ . \tag{3.21}$$

Since Property (3.20) holds for $U(t) = E$, $\tau_n - t_n$ has to go to $+\infty$. Therefore, $V(t)$ belongs to $B(E,r)$ for all $t \leq 0$, that is that $V(t)$ belongs to the local unstable manifold $W_0^u(E,r)$ for all $t \leq 0$. Moreover, the definition (3.19) of $l$ implies that $\Phi(V(0)) \geq \Phi(E)$. This is impossible according to Proposition 3.3 and the fact that $V(t) \longrightarrow E$ when $t \longrightarrow -\infty$.

We just have proved that, for $\varepsilon$ small enough, $S_\varepsilon(t)$ is a gradient-like dynamical system. Then, the hyperbolicity of the equilibrium points implies that

$$\mathcal{A}_\varepsilon = \cup_{E \in \mathcal{E}} W_\varepsilon^u(E) = \cup_{E \in \mathcal{E}} \cup_{t \geq 0} S_\varepsilon(t) W_\varepsilon^u(E,r) \ ,$$

where $\mathcal{E}$ is the set of equilibrium points of $S_\varepsilon(t)$ and $W_\varepsilon^u(E)$ is the global unstable manifold of $E$. Then, the continuity of the attractors (3.6) classically follows from Theorem 3.11 (see [10], [2] or [19] for example).

We finish the proof of Theorem 3.5 by showing that $S_\varepsilon(t)$ preserves the order on the equilibrium points induced by $\Phi$. Assume that it is not the case and that there exist a sequence $(\varepsilon_n)$ converging to 0 and a sequence $(U_n(t)) \subset \mathcal{C}^0(\mathbb{R}, X)$ such that $U_n(t)$ is a trajectory of $S_{\varepsilon_n}(t)$ and $U_n(t) \longrightarrow E_\pm$ when $t \longrightarrow \pm\infty$, with $\Phi(E_-) \leq \Phi(E_+)$. Let $r > 0$ be such that the balls $B(E,r)$, $E \in \mathcal{E}$, are disjoint. We use some technics close to the ones of the *finite-dimensional combined trajectory* introduced in [1] (see [2] and also [14] for similar methods). Let $\sigma_n^1$ be the first time such that $U_n(\sigma_n^1) \in \partial B(E_-, r)$. Using the same arguments as above, we know that, up to the extraction of a subsequence, there exists $V_1 \in W_0^u(E_-, r)$ such that $U_n(\sigma_n^1)$ converges to $V_1$. There exists an equilibrium point $E_1$ such that $S_0(t)V_1 \to E_1$ when $t \to +\infty$. As $S_0(t)$ is a gradient dynamical system and $V_1$ connects $E_-$ to $E_1$, $\Phi(E_-) > \Phi(E_1)$. Using Lemma 3.10 and the fact that $\cup_n \mathcal{A}_{\varepsilon_n}$ is bounded, we obtain a sequence of times $t_n^1$ such that $S_{\varepsilon_n}(t_n^1) \to E_1$ when $n \to +\infty$. Then, up to the extraction of a subsequence, either $U_n(t)$ belongs to $B(E_1, r)$ for all $t \geq t_n^1$, that is that $E_1 = E_+$, or there exists a first time $\sigma_n^2 > t_n^1$ such that $U_n(\sigma_n^1) \in \partial B(E_-, r)$. In the last case, there exists $V_2 \in W_0^u(E_1, r)$ such that $U_n(\sigma_n^2)$ converges to $V_2$, and so on... We apply the same arguments until $E_k = E_+$. This needs only a finite number of iterations since, at each step $\Phi(E_k)$ decreases and there is only a finite number of equilibrium points. Finally, this yields a sequence of equilibria $E_- = E_0, E_1, ..., E_p = E_+$ and a sequence of heteroclinic orbits $V_k(t)$ for $S_0(t)$, connecting $E_k$ to $E_{k+1}$. As $\Phi$ is a Lyapounov functional for $S_0(t)$, we have $\Phi(E_-) > \Phi(E_1) > ... > \Phi(E_+)$, which lieds to a contradiction and finishes the proof of Theorem 3.5.

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